Exact Outage Probability of Dual-Hop CSI-Assisted AF Relaying Over Nakagami-m Fading Channels

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Exact Outage Probability of Dual-Hop CSI-Assisted AF Relaying over Nakagami-

Minghua Xia, Yik-Chung Wu, and Sonia Aissa, Senior Member, IEEE

Abstract—In this paper, considering dual-hop channel state information (CSI)-assisted amplify-and-forward (AF) relaying over Nakagami-

I. INTRODUCTION

For dual-hop amplify-and-forward (AF) relaying systems, the gain of relaying node aims to invert the first-hop (source-torelay) channel and is determined by the channel state information (CSI) of the first hop. According to the amount of CSI obtained at the relay, there are three different AF schemes that can be implemented: blind, semi-blind, and CSI-assisted relaying. Among them, the theoretical analysis of CSI-assisted relaying is extremely important since this scheme characterizes the best performance.

Let γ1 and γ2 be the instantaneous signal-to-noise ratios (SNRs) of two consecutive hops, respectively, the end-to-end SNR of CSI-assisted relaying can be derived as γend = \( \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2 + 1} \) [1]. However, due to the existence of the unity in the denominator, which corresponds to the additive white Gaussian noise (AWGN) at the relay, exact performance analysis was usually considered to be intractable over arbitrary Nakagami-

For the CSI-assisted relaying over arbitrary Nakagami-

II. CDF OF THE END-TO-END SNR

For the CSI-assisted dual-hop AF relaying system, the end-to-end SNR from the source to the destination was shown to be exactly given by [1]

\[ \gamma_{\text{end}} = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2 + 1} \]  

(1)

where γ1 and γ2 refer to the instantaneous SNRs at the first and second hops, respectively.

Assuming that the channels at two consecutive hops are subject to Nakagami-

\[ f_{\gamma_i}(\gamma_i) = \frac{m_i^{m_i}}{\Gamma(m_i)} \gamma_i^{m_i-1} \exp\left(-\frac{m_i}{\gamma_i}\right), \quad i = 1, 2 \]  

(2)

For the second bound, a number of analyses have been performed [2], [6]–[10]. For the second bound, it is assumed that γ1 and γ2 are non-symmetric and the distribution of γend is replaced by the distribution of the minimum between γ1 and γ2. This bound is usually exploited to analyze the average symbol error probability (ASEP) [11], [12], which has the intuitive meaning that the ASEP of the whole link (source-relaydestination) is dominated by the worst link between the source-to-relay and relay-to-destination channels. Recently, the authors of [13] derive the exact probability density function (PDF) of the end-to-end SNR in a single-integral form, considering multi-hop CSI-assisted AF relaying scenario over general Nakagami fading channels.

Although the above bounds seem reasonable at high SNR, they are loose in the low and medium SNR regions. In this paper, the exact cumulative density function (CDF) of the end-to-end SNR of CSI-assisted dual-hop AF relaying is derived in an analytical form, considering transmission over arbitrary Nakagami-

M. Xia is with the Division of Physical Sciences and Engineering, KAUST, Thuwal, Saudi Arabia (e-mail: minghua.xia@ieee.org).

Y.-C. Wu is with the Department of Electrical and Electronic Engineering, The University of Hong Kong, Hong Kong (e-mail: ycwu@eee.hku.hk).

S. Aissa is with INRS, University of Quebec, Montreal, QC, Canada and with KAUST, Thuwal, Saudi Arabia (e-mail: sonia.aissa@ieee.org).

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where \( m_i \) denotes the Nakagami fading parameter at the \( i \)-th hop, \( \Gamma(\cdot) \) stands for the Gamma function [15, Eq.(8.310)], \( \gamma_i \triangleq E\{ h_{RX}^2 \} / \sigma_i^2 = m_1(m_1 + 1) E_2 / \sigma_i^2 \), and \( \gamma_2 \triangleq E\{ h_{LX}^2 \} / \sigma_2^2 = m_2(m_2 + 1) E_2 / \sigma_2^2 \) with E\{ \} being the statistical expectation operator. Also, the CDF of \( \gamma_i \) is given by [2]

\[
F_{\gamma_i}(\gamma_i) = 1 - \frac{1}{\Gamma(m_i)} \Gamma \left( m_i, \frac{\gamma_i}{\gamma_i} \right), \quad i = 1, 2
\]

where \( \Gamma(\cdot, \cdot) \) stands for the upper incomplete Gamma function [15, Eq.(8.350.2)]. Finally, by integrating the conditional CDF of \( \gamma_{\text{end}} \) with respect to \( \gamma_i \) over the PDF of \( \gamma_i \), the CDF of \( \gamma_{\text{end}} \) is given by

\[
F_{\gamma_{\text{end}}}(\gamma) = 1 - \frac{C_1}{\Gamma(m_2)} \int_0^\infty \Gamma \left( m_2, \frac{m_2}{\gamma_2} \left( 1 + \frac{\gamma + 1}{x} \right) \right) \times (x + \gamma)^{m_1-1} \exp \left( - \frac{m_1}{\gamma_1} (x + \gamma) \right) dx,
\]

where \( C_1 \triangleq m_1^{m_1} / (\Gamma(m_1) \gamma_1^{m_1}) \). The integral term in (4) cannot be calculated directly, since the incomplete Gamma function is involved. In order to proceed, two different series expansions of the incomplete Gamma function are exploited and thus two different cases are discussed in the following, depending on the values of the fading parameters \( m_1 \) and \( m_2 \).

A. Scheme with integer values for \( m_1 \) and \( m_2 \)

When \( m_2 \) takes integer values, the expansion of the incomplete Gamma function in (4) is a finite series [15, Eq.(8.352.7)] and thus this scheme can be easily analyzed. For the completeness of exposition, the CDF of the end-to-end SNR is reproduced and it is given by [3]

\[
F_{\gamma_{\text{end}}}(\gamma) = 1 - 2C_1 \sum_{n=0}^{m_2-1} \frac{1}{n!} \left( \frac{m_2}{\gamma_2} \right)^n \exp \left[ \left( \frac{m_1}{\gamma_1} + \frac{m_2}{\gamma_2} \right) \gamma \right]
\times \sum_{p=0}^{m_2-1} \sum_{q=0}^{m_1} \frac{(m_1 - 1)}{p} \frac{1}{q} \gamma_1^{m_1+n-p} \gamma_2^{m_2-q} \times (\gamma + 1)^{\frac{m_2}{\gamma_2}} \gamma_2 \times K_{\gamma_2} \left( 2\sqrt{\gamma} \right),
\]

where \( \binom{n}{p} \) denotes the binomial coefficient, \( K_{\gamma}(x) \) is the \( \nu \)-th order modified Bessel function of the second kind [15, Eq.(8.342.6)], \( v_1 \triangleq p + q + 1, v_2 \triangleq p - q + 1, \) and \( \gamma \triangleq \gamma_2 \gamma_1^{m_2}(1 + 12) \).

It is known that the Nakagami-\( m \) fading reduces to the Rayleigh fading when \( m_1 = m_2 = 1 \). Accordingly, putting \( m_1 = m_2 = 1 \) into (5) reduces it to the result previously reported in [16, Eq.(2)].

B. Scheme with non-integer values for \( m_1 \) and \( m_2 \)

When \( m_2 \) takes non-integer values, the expansion of the incomplete Gamma function in (4) is an infinite series [15, Eq.(8.354.2)]. Substituting the infinite series into (4) and performing some algebraic manipulations yield (6) at the top of the next page.

Although the infinite series representation for the incomplete Gamma function is involved, this series is absolutely convergent for \( m_2 \geq 0.5 \) and converges rapidly because of the factorial term \( n! \) in the denominator. Moreover, the integral term \( I_1 \) in (6) can be calculated as [15, Eq.(8.350.2)]

\[
I_1 = C_1 \frac{\gamma_1}{m_1} \Gamma \left( m_1, \frac{m_2}{\gamma_1} \right) = \frac{1}{\Gamma(m_1)} \Gamma \left( m_1, \frac{m_1}{\gamma_1} \right).
\]

For the integral term \( I_2 \) in (6), the Newton’s generalized binomial theorem \((1 + x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n \) cannot be applied, since the infinite series on the right-hand side converges only for \( |x| < 1 \) [17, p.28]. Clearly, this condition is not satisfied for the binomials \((1 + \frac{x}{\gamma_1})^{m_1-1} + (1 + \frac{x+1}{\gamma_1})^{m_2+1} \) in (6), where \( 0 < x \) and \( \gamma < \infty \). Therefore, how to calculate \( I_2 \) becomes challenging, which explains why no analytical CDF for \( \gamma_{\text{end}} \) has been reported in the open literature till now. In the sequel, we exploit the Fox’s \( H \)-function and the generalized Laplace transform of the product of two \( H \)-functions to tackle this problem, such that an analytical expression for \( I_2 \) is obtained.

Finally, by substituting (7), (11), and \( C_1 \) into (6) and performing some algebraic manipulations, we obtain the CDF of the end-to-end SNR of dual-hop CSI-assisted relaying systems as (14) in the middle of the next page, where \( C_2 \triangleq 1 / (\Gamma(m_1) \Gamma(m_2)) \).

The infinite series in (14) is absolutely convergent. This is demonstrated as follows. Firstly, the bivariate \( G \)-function \( G^\nu_{m_1, m_2} \) is defined in terms of double Mellin-Barnes type integrals, and it converges if the following conditions are satisfied [21, p.62]:

\[
p + q + s + t < 2(n + v_1 + w_1), \quad p + q + s + t < 2(n + v_2 + w_2),
\]

\[
| \arg(x) | < \pi [n + v_1 + w_1 - (p + q + s + t_1) / 2], \quad | \arg(y) | < \pi [n + v_2 + w_2 - (p + q + s + t_2) / 2].
\]

It is easy to show that the parameters of the \( G \)-function (13) satisfy these sufficient conditions above and, therefore, the \( G \)-function converges in the sense of finite value. Secondly, it is clear that the infinite series in (14) is an alternating series and thus, by use of the Leibnitz’s test [17, Theorem 1.5], it is known that this infinite series is conditionally convergent because of the factorial term \( n! \) in the denominator of the summation. Furthermore, applying the ratio test to the associated series of positive terms yields the infinite series in (14) to be absolutely convergent.
\[ F_{\text{end}}(\gamma) = 1 - C_1 \int_0^\infty (x + \gamma)^{m_1-1} \exp \left(-\frac{m_1}{\gamma_1} (x + \gamma)\right) \, dx + \frac{C_1}{\Gamma(m_2)} \sum_{n=0}^\infty \frac{(-1)^n}{n! (m_2 + n)} \left(\frac{m_2}{\gamma_2}\right)^{m_2+n} \times \int_0^\infty (x + \gamma)^{m_1-1} \left(1 + \frac{x + \gamma}{x}\right)^{m_2+n} \exp \left(-\frac{m_1}{\gamma_1} (x + \gamma)\right) \, dx. \]  

(6)

\[ I_2 = \gamma^{m_1-1}(\gamma + 1)^{m_2+n} \exp \left(-\frac{m_1}{\gamma_1}\right) \int_0^\infty x^{-(m_2+n)} \exp \left(-\frac{m_1}{\gamma_1} x\right) \left(1 + \frac{x}{\gamma}\right)^{m_1-1} \left(1 + \frac{x}{\gamma + 1}\right)^{m_2+n} \, dx \]

\[ = \gamma^{m_1-1}(\gamma + 1)^{m_2+n} \exp \left(-\frac{m_1}{\gamma_1}\right) \int_0^\infty x^{-(m_2+n)} \exp \left(-\frac{m_1}{\gamma_1} x\right) \left(1 + \frac{x}{\gamma}\right)^{m_1-1} \left(1 + \frac{x}{\gamma + 1}\right)^{m_2+n} \, dx \]

\[ = \exp \left(-\frac{m_1}{\gamma_1}\right) \sum_{p=0}^{m_1-1} \sum_{q=0}^{m_2+n} \binom{m_1-1}{p} \binom{m_2+n}{q} \gamma^{m_1-p-1}(\gamma + 1)^{m_2+n-q} \times \int_0^\infty x^{p+q-(m_2+n)} \exp \left(-\frac{m_1}{\gamma_1} x\right) \left(1 + \frac{x}{\gamma}\right)^{-m_1} \left(1 + \frac{x}{\gamma + 1}\right)^{-m_2} \, dx. \]

(8)

\[ I_2 = \exp \left(-\frac{m_1}{\gamma_1}\right) \sum_{p=0}^{m_1-1} \sum_{q=0}^{m_2+n} \binom{m_1-1}{p} \binom{m_2+n}{q} \gamma^{m_1-p-1}(\gamma + 1)^{m_2+n-q} \times \frac{1}{\Gamma(m_1)\Gamma(m_2)} \int_0^\infty x^{p+q-(m_2+n)} \exp \left(-\frac{m_1}{\gamma_1} x\right) H_{11}^1 \left[\frac{x}{\gamma}\right] (1 - \bar{m}_1, 1) \, dx. \]

(9)

\[ I_2 = \exp \left(-\frac{m_1}{\gamma_1}\right) \sum_{p=0}^{m_1-1} \sum_{q=0}^{m_2+n} \binom{m_1-1}{p} \binom{m_2+n}{q} \gamma^{m_1-p-1}(\gamma + 1)^{m_2+n-q} \times \frac{1}{\Gamma(m_1)\Gamma(m_2)} \int_0^\infty x^{p+q-(m_2+n)} \exp \left(-\frac{m_1}{\gamma_1} x\right) H_{11}^1 \left[\frac{x}{\gamma}\right] (1 - \bar{m}_1, 1) \, dx. \]

(10)

\[ F_{\text{end}}(\gamma) = 1 - \frac{1}{\Gamma(m_1)} \Gamma \left(m_1, \frac{m_1}{\gamma_1}\right) + C_2 \exp \left(-\frac{m_1}{\gamma_1}\right) \sum_{n=0}^\infty \frac{(-1)^n}{n! (m_2 + n)} \left(\frac{m_2}{\gamma_2}\right)^{m_2+n} \times \sum_{p=0}^{m_1-1} \sum_{q=0}^{m_2+n} \binom{m_1-1}{p} \binom{m_2+n}{q} \gamma^{m_1-p-1} \left(\frac{m_1}{\gamma_1}\right)^{m_1-p-1} \left(\frac{m_1}{\gamma_1}(\gamma + 1)\right)^{m_2+n-q} \, G_{11}. \]

(14)

Note that the bivariate $G$-function cannot be directly computed by popular mathematical softwares such as Matlab and Mathematica. In general, it has to be computed by its definition in terms of the double Mellin-Barnes type integrals [19], such as the Mathematica code [20], which is satisfied when non-integer $m_1$ and $m_2$ are applied to (13). Although it seems complicated, (13) involves only common special functions and it can be easily evaluated in a numerical way. The accuracy of (13) is corroborated by simulation results in the next section.

Remark II.1. (The PDF of the end-to-end SNR) After obtaining the CDF of the end-to-end SNR of CSI-assisted relaying over arbitrary Nakagami-$m$ fading channels, its corresponding PDF can be readily obtained by taking the derivative of $F_{\text{end}}(\gamma)$ with respect to $\gamma$. More specifically, the derivative of $K_0(\gamma)$ in (5) with respect to $\gamma$ can be obtained by using [15, Eq.(8.486.12)]. On the other hand, as a special case of bivariate $H$-function [20, Eq.(6.4.1)], the derivative of $G_1$ in (14) can be obtained by exploiting the derivative of bivariate $H$-function shown in [20, Eq.(6.5.7)].
III. OUTAGE PROBABILITY

Once the CDF and PDF of the end-to-end SNR have been obtained, they can be widely applied to evaluate the system performance in terms of different performance metrics. Based on the CDF, for example, the outage probability, outage capacity, and codeword error probability can be analytically obtained. On the other hand, by exploiting the PDF, the ergodic capacity and the output statistics such as the moments of the output SNR can be numerically evaluated. Herein, due to page limitation, we only demonstrate the accuracy of our main result (14) using outage probability. In all simulations, without loss of generality, the variances of the AWGN at the relay and at the destination are assumed to be identical, that is, $\sigma_1^2 = \sigma_2^2$. Furthermore, the values of (14) are computed with the first 9 terms of the infinite series, in which the function $G_1$ is also computed with its first 9 terms as per (20). Further test results show that, when more than 9 terms from the infinite series in (20) are involved, both the proposed method and that in [22] yield almost the same output.

Outage probability, $P(\gamma_{th})$, is defined as the probability that the instantaneous output SNR falls below a pre-defined threshold $\gamma_{th}$. Hence, evaluating the CDF (5) or (14) at $\gamma_{th}$, we obtain

$$P(\gamma_{th}) = \text{Pr} \{ \gamma_{end} < \gamma_{th} \} = F_{\gamma_{end}}(\gamma_{th}).$$

(21)

For comparison purposes, the outage probability based on the two upper bounds discussed in Section I are also reproduced here. For the first bound $\gamma_{end} < \frac{\gamma_{th}^2}{\gamma_{th} + \gamma_2}$ with the symmetric fading shape factors $m_1 = m_2 = m$, the outage probability is given by [6, Eq.(18)]

$$P^{b1}(\gamma_{th}) = \frac{\sqrt{\pi}}{2(m-\frac{1}{2})^2} \left( \frac{m}{\gamma_{th}} \right) \times G_{2,1}^{1,1} \left[ \frac{4m}{\gamma_{th}} \right] \left( (m-1, 2m-1, -1) \right)$$

(22)

where the superscript $b_1$ of $P^{b1}(\gamma_{th})$ refers to the first bound, and $G_{[\cdot,\cdot]}$ denotes the Meijer’s $G$-function [15, Eq.(9,301)]. For the non-symmetric case with non-integer values $m_1 \neq m_2$, to the best of our knowledge, no result was ever reported.

For the second bound $\gamma_{end} < \min\{\gamma_1, \gamma_2\}$, the outage probability is clearly given by

$$P^{b2}(\gamma_{th}) = F_{\gamma_1}(\gamma_{th}) + F_{\gamma_2}(\gamma_{th}) - F_{\gamma_1}(\gamma_{th})F_{\gamma_2}(\gamma_{th}),$$

(23)

where the superscript $b_2$ of $P^{b2}(\gamma_{th})$ refers to the second bound.

Figure 1 shows the outage probability of the systems with non-symmetric non-integer fading parameters $(m_1, m_2) = (4.3, 1.5)$, and the threshold $\gamma_{th} = 0.5$ dB. It is observed that the analytical results based on (21) almost perfectly match with the simulation results whereas the second bound (23) yields a small gap at low SNR. Notice that, since the first bound (22) holds only in the symmetric fading case (i.e., the fading parameters $m_1 = m_2 = m$), it cannot be applied to this non-symmetric fading scenario ($m_1 \neq m_2$).

Figure 2 presents the outage probability with symmetric non-integer fading parameters $m_1 = m_2 = 1.5$, and the threshold $\gamma_{th} = 0.5$ dB. It is observed that the bound (23) performs worst and it is very loose in the whole SNR region under consideration, since this bound is derived with the assumption that the SNRs at consecutive hops are non-symmetric. The first bound (22) performs a little better than the second bound (23) but it is still loose in the low SNR region, since the effect of AWGN is ignored. On the other hand, our analytical result in (21) is always consistent with the simulation results.

Note that the CDF expression of the non-integer case in (14) cannot reduce to that of the integer case in (5). This is because we exploited two exclusive series expansions of the incomplete Gamma functions $\Gamma(m, x)$ with respect to the integer and non-integer values of $m$ [15, Eqs.(8.352.7) & (8.354.2)], respectively. However, when a non-integer value of $m$ closely approaches an integer value, these two expansions should have almost the same numerical value; in other words, the result of (14) should be almost the same as that of (5). This is illustrated in Fig. 3. This figure shows the outage probability of different non-integer fading scenarios, compared with the integer fading scenario where $(m_1, m_2) = (1, 3)$. For the non-integer cases, the fading parameter $m_2$ at the second hop is set to $m_2 = 2.999$, which is almost identical to the integer case with $m_2 = 3$. On the other hand, the fading parameter $m_1$ at the first hop varies from the worst case $m_1 = 0.5$ to $m_1 = 0.9$ and finally $m_1 = 0.98$. It is observed that, the worst fading parameter $m_1 = 0.5$ results in the highest outage probability. When $m_1$ increases, the outage probability of non-integer cases decreases and it becomes closer and closer to that of the integer case. Also, the analytical results coincide perfectly with the simulation results. This demonstrates the effectiveness of our derivations.

Finally, comparing Fig. 1 with Fig. 2, we observe that the slopes of all curves are identical at high SNR. Moreover, the slopes of the curves in Fig. 3 improve with $m_1$. These observations are in accordance with the well-known result that the diversity order of dual-hop AF relaying systems is given by $\min\{m_1, m_2\}$ [4], [24].

IV. CONCLUSION

Due to the difficulty of mathematical derivation, analyzing the performance of CSI-assisted AF relaying transmission in an exact way is very challenging, especially when the transmission is performed over the general Nakagami-$m$ fading channels. In this paper, exact expression for the distribution function of the end-to-end SNR was derived. In particular, when $m$ takes non-integer values, the Fox’s $H$-function, bivariate $H$-function and $G$-function were exploited. Simulation results of outage probability corroborated all analytical results and these special functions were shown to be efficient tools for system performance evaluation.

REFERENCES

Fig. 1. Outage probability of dual-hop CSI-assisted AF relaying systems with non-symmetric non-integer Nakagami-m fading parameters.

Symmetric fading with $m_1=m_2=1.5$

Non-symmetric with $m_1=4.3, m_2=1.5$

Outage Probability

Average SNR at Relaying Node (dB)

Simulation
Analysis, Eq.(21)
2nd Bound, Eq.(23)

Fig. 2. Outage probability of dual-hop CSI-assisted AF relaying systems with symmetric non-integer Nakagami-m fading parameters.

Outage threshold=0dB

Outage Probability

Average SNR at Relaying Node (dB)

Simulation
Analysis, Eq.(21)
1st Bound, Eq.(22)
2nd Bound, Eq.(23)

Fig. 3. Outage probability of dual-hop CSI-assisted AF relaying systems (non-integer versus integer fading parameters).