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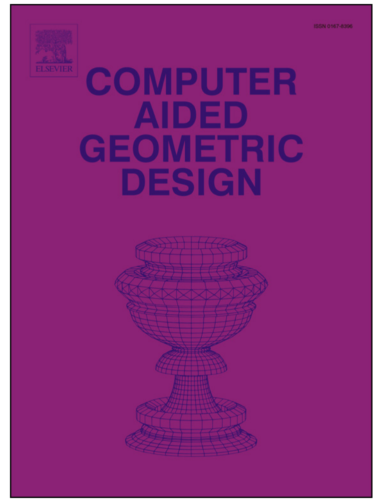

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Rachid Ait-Haddou, Michael Bartoň

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Highlights

- We study the problem of constrained degree reduction with respect to Jacobi norms.
- Our results generalize several previous findings on polynomial degree reduction.
- We explore the space of Jacobi parameters on the reduced polynomial approximation.

Constrained multi-degree reduction with respect to Jacobi norms

Rachid Ait-Haddou[a,*], Michael Bartoň[b]

[a]Visual Computing Center, King Abdullah University of Science and Technology, Thuwal 23955-6900, Kingdom of Saudi Arabia
[b]Numerical Porous Media Center, King Abdullah University of Science and Technology, Thuwal 23955-6900, Kingdom of Saudi Arabia

Abstract

We show that a weighted least squares approximation of Bézier coefficients with factored Hahn weights provides the best constrained polynomial degree reduction with respect to the Jacobi L_2-norm. This result affords generalizations to many previous findings in the field of polynomial degree reduction. A solution method to the constrained multi-degree reduction with respect to the Jacobi L_2-norm is presented.

Keywords: Degree reduction, weighted least squares, Jacobi norm, Hahn orthogonal polynomials

1. Introduction

Optimal *degree reduction* is one of the fundamental tasks in Computer Aided Geometric Design (CAGD) and therefore has attracted researchers' attention for several decades [4, 17, 2, 1, 10, 13, 15]. Used not only for data compression, CAD/CAM software typically requires algorithms capable of converting a curve (surface) of a high degree to a curve (surface) of a lower degree. Considering the problem coordinate-wise, the goal is formulated as follows: given a *univariate* polynomial p of degree n, find its best polynomial approximation q of degree m, $m < n$, with respect to a certain given norm.

The degree reduction can be seen as an inverse operation to the *degree elevation*. Whereas elevating polynomial degree from m to n is always possible, see e.g. [8], because it is equivalent to expressing a polynomial $q \in \mathbb{P}_m$ in the basis of a larger linear space $\mathbb{P}_n$, $\mathbb{P}_m \subset \mathbb{P}_n$, the degree reduction is in general not. A natural alternative is then finding the best approximation that minimizes a certain error. This can be interpreted as projecting $p \in \mathbb{P}_n$ into $\mathbb{P}_m$. Depending on a particular norm defined on $\mathbb{P}_n$, various schemes for degree reduction were derived [7, 14, 12, 11, 6, 5].

An elegant resemblance between the L_2-norm and the Euclidean norm acting on the vector of Bernstein coefficients was revealed by Lutterkort et al. in [13]. They proved that the least squares approximation of Bézier coefficients provides the best polynomial degree reduction in the L_2-norm. Two interesting generalizations of this result were achieved by Ahn et al. in [2] and by Ait-Haddou in [3]. Ahn et al. in [2] showed

*Corresponding author
Email addresses: rachid.aithaddou@kaust.edu.sa (Rachid Ait-Haddou), Michael.Barton@kaust.edu.sa (Michael Bartoň)

that a weighted least squares approximation of Bézier coefficients provides the best *constrained* polynomial degree reduction in the L_2-norm. By constrained we understand that the original polynomial and its reduced-degree approximation match at the boundaries up to a specific continuity order. Ait-Haddou in [3] shows that the weighted least squares approximation of Bézier coefficients with Hahn weights provides the best polynomial degree reduction with respect to the Jacobi L_2-norm. In view of these two generalizations, it is natural to ask the following question:

(Q) – *Is there an analogue to the result of Lutterkort et al. [13] for the constrained degree reduction with respect to the Jacobi L_2-norm?*

The Jacobi L_2-norm depends on two real parameters and a partial answer to the question (Q) is given in [9] for specific values of the parameters of the Jacobi L_2-norm.

In the present work, we give an affirmative answer to question (Q); namely we show that there exists a weighted inner product on the Bézier coefficients for which the problem of constrained degree reduction with respect to the Jacobi L_2-norm is equivalent to the problem of weighted least squares approximation of the Bézier coefficients.

Our methodology for answering question (Q) is very similar to Lutterkort et al. [13] and its extension by Ahn et al. in [2]. The main challenge lies in the construction of the adequate inner product of Bézier coefficients.

We note that a general solution to the problem of constrained degree reduction with respect to the Jacobi L_2-norm is derived in [16]. Although their solution does not require matrix inversion, the derivation is rather complicated because it requires an explicit computation of the dual bases of the discrete Bernstein bases. Moreover, their methodology does not involve the approach taken in [13] and [2]. In contrast, our solution, even though it requires the computation of a single Moore-Penrose inverse, is simple and fits to the framework of [13, 2].

The rest of the paper is organized as follows. In section 2, we prove that the best constrained polynomial degree reduction with respect to the Jacobi L_2-norm is equivalent to a weighted least squares approximation of Bézier coefficients with factored Hahn weights. We demonstrate how to compute the degree-reduced polynomials in Section 3, present several examples in Section 4, and finally conclude the paper in Section 5.

2. Constrained polynomial degree reduction with Jacobi norms

Denote by $\mathbb{P}_n$ the linear space of polynomials of degree at most n and let $\mathbf{B}^n$ be its Bernstein-Bézier (BB) basis and $\mathbf{Q}^n$ be its Lagrange basis with respect to the nodes $(0,1,\ldots,n)$, i.e.,

$$\mathbf{B}^n := [B_0^n,\ldots,B_n^n], \quad \text{where} \quad B_i^n(t) = \tbinom{n}{i}(1-t)^{n-i}t^i, \quad t\in[0,1], \quad \text{and}$$

$$\mathbf{Q}^n := [Q_0^n,\ldots,Q_n^n], \quad \text{where} \quad Q_i^n(t) = \prod_{j=0, j\neq i}^{n} \tfrac{t-j}{i-j}.$$

Let $\mathbb{P}_m$ be a subspace of $\mathbb{P}_n$, $m<n$ and let k and l be two non-negative integers such that $k+l\le m+1$, we define $\mathbb{P}_m^{k,l}$ as:

$$\mathbb{P}_m^{k,l} = \{f\in\mathbb{P}_m : f^{(i)}(0)=0,\ i=0,1,\ldots,k-1;\ f^{(j)}(1)=0,\ j=0,\ldots,l-1\}.$$

That is, $\mathbb{P}_m^{k,l}$ is a linear space of polynomials of degree at most m with k vanishing derivatives at $t=0$ and l vanishing derivatives at $t=1$. Moreover, we define

$$\mathbb{Q}_m^{k,l} = \{f\in\mathbb{P}_m : f(i)=0,\ i=0,1,\ldots,k-1 \text{ and } i=n-l+1,\ldots,n\}.$$

Let $\alpha > -1$ and $\beta > -1$ be two real numbers and define the Jacobi inner product in $\mathbb{P}_n$ by

$$< p,q >_{L_2} = \int_0^1 t^\alpha (1-t)^\beta p(t)q(t)\mathrm{d}t. \quad (1)$$

Considering the vectors of BB coefficients of p and q, $\mathbf{p} = [p_0,\ldots,p_n]^T$ and $\mathbf{q} = [q_0,\ldots,q_n]^T$, respectively, we define the following weighted Euclidean inner product of the BB coefficients

$$< p,q >_{E_2} = < \mathbf{B}^n\mathbf{p}, \mathbf{B}^n\mathbf{q} >_{E_2} = \sum_{i=k}^{n-l} w_i p_i q_i, \quad (2)$$

with the weights

$$w_i = \frac{\binom{n}{i}}{\binom{i}{i-k}\binom{n-i}{n-i-l}} (\alpha+1)_{k+i}(\beta+1)_{n-i+l}, \quad (3)$$

where $(a)_s = a(a+1)\ldots(a+s-1)$ denotes the Pochhammer symbol with the convention that $(a)_0 := 1$.

Remark 1. The choice of the weights w_i in (3) is not arbitrary but follows a certain logic that combines the work of Ahn et al. in [2] and the work of Ait-Haddou in [3]. First recall that Hahn orthogonal polynomials $H_i, i = 0,1,\ldots,n$, are orthogonal polynomials with respect to the Hahn inner product

$$< p,q >_H = \sum_{i=0}^{n} \binom{n}{i} (\alpha+1)_i(\beta+1)_{n-i} p(i)q(i). \quad (4)$$

When $k = l = 0$ (the unconstrained case) the weights w_i in (3) coincide with the Hahn weights $\binom{n}{i}(\alpha+1)_i(\beta+1)_{n-i}$ given in (4). This is what is naturally expected from the results in [3]. Moreover, in the constrained case, i.e, when $k \neq 0$ or $l \neq 0$, and when $\alpha = \beta = 0$, we recover from (3) the inner product given in [2]. The weights in (3) are essentially the Hahn weights in (4) multiplied by certain factors that depend on the order of the endpoint constraints.

We need the following lemma given in [2].

Lemma 1. *(Ahn et al. [2]) A polynomial $\mathbf{B}^n\mathbf{p}$ is of degree m, $m \leq n$, with $p(i) = 0$ for $i = 0,\ldots,k-1$ and $i = n-l+1,\ldots,n$ if and only if the vector of coefficients is a polynomial of degree m with zeros at $i = 0,\ldots,k-1$ and $i = n-l+1,\ldots,n$ in its index, i.e.,*

$$\mathbf{B}^n\mathbf{p} \in \mathbb{P}_m^{k,l} \Leftrightarrow \mathbf{Q}^n\mathbf{p} \in \mathbb{Q}_m^{k,l}.$$

Now we are in a position to state the main result of this work.

Theorem 1. *The orthogonal complements of $\mathbb{P}_m^{k,l}$ in $\mathbb{P}_n^{k,l}$ with respect to the Jacobi inner product (1) and the weighted Euclidean inner product (2) are equal.*

Proof. Denote by $\mathbb{P}_{m,n}^{k,l}$ the orthogonal complement of $\mathbb{P}_m^{k,l}$ in $\mathbb{P}_n^{k,l}$ with respect to the Euclidean inner product (2). Let $\mathbf{B}^n\mathbf{q}$ be an element of $\mathbb{P}_{m,n}^{k,l}$. Thus we have

$$< \mathbf{B}^n\mathbf{q}, \mathbf{B}^n\mathbf{p} >_{E_2} = 0$$

for any element $\mathbf{B}^n\mathbf{p} \in \mathbb{P}_m^{k,l}$. Let s be an integer smaller or equal than $m-k-l$. We have

$$< \mathbf{B}^n\mathbf{q}, t^{k+s}(1-t)^l >_{L_2} = \int_0^1 t^{\alpha+s+k}(1-t)^{\beta+l}\mathbf{B}^n\mathbf{q}\,\mathrm{d}t$$
$$= \sum_{i=0}^{n} q_i \int_0^1 t^{\alpha+s+k}(1-t)^{\beta+l}B_i^n(t)\,\mathrm{d}t.$$

Therefore,

$$< \mathbf{B}^n\mathbf{q}, t^{k+s}(1-t)^l >_{L_2} = < \mathbf{B}^n\mathbf{q}, \mathbf{B}^n\phi >_{E_2}, \tag{5}$$

where $\phi = [\phi_0, \phi_1, \ldots, \phi_n]$ with

$$\phi_i = \frac{\binom{i}{i-k}\binom{n-i}{n-i-l}}{\binom{n}{i}} \frac{1}{(\alpha+1)_{k+i}(\beta+1)_{n-i+l}} \int_0^1 t^{\alpha+s+k}(1-t)^{\beta+l}B_i^n(t)\,\mathrm{d}t.$$

As $\mathbf{B}^n\mathbf{q}$ is an element of $\mathbb{P}_{m,n}^{k,l}$ and by Lemma 1, the inner product (5) vanishes if and only if ϕ_i is a polynomial in i of degree less or equal to m that vanishes at $0,1,\ldots,k-1$ and at $n-l+1,\ldots,n$. We have

$$\int_0^1 t^{\alpha+s+k}(1-t)^{\beta+l}B_i^n(t)\,\mathrm{d}t = \binom{n}{i}\frac{\Gamma(\alpha+s+k+i+1)\Gamma(\beta+l+n-i+1)}{\Gamma(\alpha+\beta+s+k+l+n+2)},$$

where $\Gamma(.)$ is the Gamma function. Thus

$$\int_0^1 t^{\alpha+s+k}(1-t)^{\beta+l}B_i^n(t)\,\mathrm{d}t = \binom{n}{i}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\frac{(\alpha+1)_{i+k+s}(\beta+1)_{n-i+l}}{(\alpha+\beta+2)_{s+n+k+l}}.$$

Therefore,

$$\phi_i = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\frac{(\alpha+i+1)(\alpha+i+2)\ldots(\alpha+i+s)}{(\alpha+\beta+2)_{s+n+k+l}}\binom{i}{i-k}\binom{n-i}{n-i-l},$$

which shows that ϕ_i is a polynomial of degree at most m in the variable i that vanishes at $0,1,\ldots,k-1$ and at $n-l+1,\ldots,n$. Therefore, $\mathbb{P}_{m,n}^{k,l}$ is contained in the orthogonal complement of $\mathbb{P}_m^{k,l}$ in $\mathbb{P}_n^{k,l}$ with respect to the weighted Euclidean inner product (2). The fact of equal dimensions of both orthogonal complements completes the proof. □

A consequence of Theorem 1 for the constrained degree reduction with respect to the Jacobi L_2-norm is the following Corollary; for which the proof goes along the same lines as the proof of Corollary 4.1 in [2].

Corollary 1. *Given a polynomial p of degree n, the approximation problem*

$$\min_{q\in\mathbb{P}_m}\{||p-q|| : p^{(i)}(0) = q^{(i)}(0) \text{ for } i=0,\ldots,k-1, \text{ and } p^{(j)}(1) = q^{(j)}(1) \text{ for } j=0,\ldots,l-1\} \tag{6}$$

has the same minimizer for the norm induced either by the Jacobi L_2-inner product (1) or the weighted Euclidean inner product (2).

Remark 2. The following factorization for the constrained degree reduction holds: Denote by $\mathscr{P}_{m,n}^{k,l}$ the linear operator that maps polynomials of degree n to their best constrained Jacobi L_2-approximations; then we have $\mathscr{P}_{m,n}^{k,l} = \mathscr{P}_{m,h}^{k,l}\mathscr{P}_{h,n}^{k,l}$ with $m \le h \le n$. That is, reducing the degree sequentially by one from n to m gives the same result as projecting p directly from $\mathbb{P}_n$ to $\mathbb{P}_m$ while preserving the endpoint constraints.

Remark 3. Our result is a generalization of both unconstrained and constrained L_2-degree reduction schemes [13, 2, 3]. For $\alpha = \beta = 0$, we recover the results given in [2] and if additionally $k = l = 0$, we obtain the results in [13]. For $k = l = 0$ and α, β arbitrary, we recover the results given in [3].

3. A solution method to the constrained degree reduction

We now introduce computational tools to solve the constrained polynomial degree reduction with respect to the Jacobi L_2-norm. Apart from the expression of the weight matrices, that accommodates the expression of the inner product (2), the method is similar to the one given in Ahn et al. [2].

Denote by $A_{n,m}$ the degree raising matrix that maps the BB coefficients $\mathbf{q}$ of a polynomial q of degree m to BB coefficients $\overline{\mathbf{q}} = [\overline{q}_0, \ldots, \overline{q}_n]^T$ when raised to degree n, i.e., $\overline{\mathbf{q}} = A_{n,m}\mathbf{q}$. The matrix $A_{n,m}$ is of order $(n+1)\times(m+1)$ and can be decomposed into elementary degree raising matrices as $A_{n,m} = A_{n,n-1}A_{n-1,n-2}\ldots A_{m+1,m}$ where

$$A_{h,h-1}(i,j) = \begin{cases} i/h & \text{if} \quad j = i-1, \\ 1-i/h & \text{if} \quad j = i, \\ 0 & \text{else.} \end{cases}$$

Denote by W (resp. $\sqrt{W}$) the diagonal $(n+1)\times(n+1)$ *weight* matrix where the diagonal elements are the weights w_i (resp. $\sqrt{w_i}$) from (3).

According to Corollary 1, the constrained degree reduction problem reduces to solving the least squares problem

$$\min_{\mathbf{q}\in\mathbb{R}^{m+1}} ||\mathbf{p} - A_{n,m}\mathbf{q}||_{E_2}. \tag{7}$$

with k boundary constraints at $t = 0$ and l boundary constraints at $t = 1$. Imposing the boundary constraints equals expressing the first k (the last l) coefficients of $\mathbf{q}$ in terms of the first k (the last l) coefficients of $\mathbf{p}$. This yields a $(k\times k)$ (an $(l\times l)$) linear system $p_i = \overline{q}_i$, $i = 0,\ldots,k-1$ ($p_i = \overline{q}_i$, $i = n-l+1,\ldots,n$) with the coefficients of the vector $\mathbf{q}$ as unknowns, see e.g. [8]. We denote by

$$\hat{\mathbf{q}} = [q_0,\ldots,q_{k-1},0,\ldots,0,q_{n-l+1},\ldots q_n]^T, \quad \hat{\mathbf{q}} \in \mathbb{R}^{m+1},$$

the vector that contains the solutions of the two boundary linear systems. That is, there is $m+1-k-l$ coefficients of $\mathbf{q}$ that are not affected by the boundary constraints, we denote them by $\widetilde{\mathbf{q}} = [q_k,\ldots,q_{m-l}] \in \mathbb{R}^{m+1-k-l}$. These coefficients are the free parameters for the least squares minimization described as follows.

Let $W_n^{k,l}$ be a diagonal $(n+1-k-l)$ matrix obtained from W_n by skipping the first k and the last l entries and define $\widetilde{\mathbf{p}}$ as

$$[\underbrace{0,\ldots,0}_{k},\widetilde{\mathbf{p}},\underbrace{0,\ldots,0}_{l}]^T = \mathbf{p} - A_{m,n}\hat{\mathbf{q}}.$$

Then the least squares problem (7) transforms to

$$\min_{\widetilde{\mathbf{q}}\in\mathbb{R}^{m+1-k-l}} ||\sqrt{W_n^{k,l}}\widetilde{\mathbf{p}} - \sqrt{W_n^{k,l}}A_{n,m}^{k,l}\widetilde{\mathbf{q}}||_2, \tag{8}$$

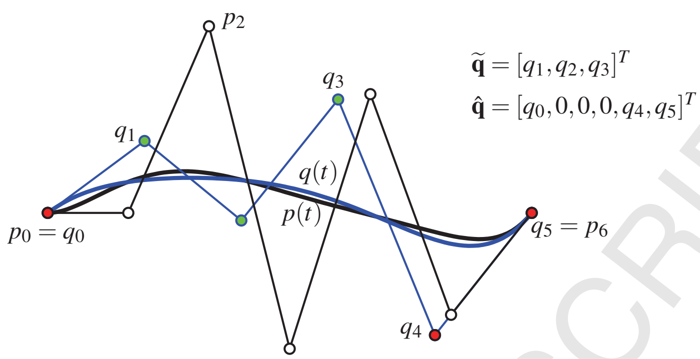

Figure 1: Constrained degree reduction for $n = 6$, $m = 5$, $k = 1$, $l = 2$, $\alpha = \beta = 0$. The input polynomial p with the control polygon $\mathbf{p} = [p_0, \dots, p_6]^T = [0, 0, 22, -16, 14, -12, 0]^T$ (black) and its best constrained approximation with respect to the L_2-norm, q, with the control polygon $\mathbf{q} = [q_0, \dots, q_5]^T$ (blue) are shown. The constrained control points (red) are computed directly from $\mathbf{p}$, the free parameters that intervene in the least squares optimization and minimize (8) are $\widetilde{\mathbf{q}}$ (green).

where $A_{n,m}^{k,l}$ is a submatrix of $A_{n,m}$ obtained from $A_{n,m}$ by skipping the first k and the last l rows and columns. Using Moore-Penrose inverse, the unique minimizer to (8) is

$$\widetilde{\mathbf{q}} = M_{n,m}^{k,l} \widetilde{\mathbf{p}} = \left((A_{n,m}^{k,l})^T W_n^{k,l} A_{n,m}^{k,l} \right)^{-1} (A_{n,m}^{k,l})^T W_n^{k,l} \widetilde{\mathbf{p}}. \tag{9}$$

Example 1. *Consider degree reduction from $n = 6$ to $m = 5$ with the C^0 continuity constraint at $t = 0$ and C^1 continuity constraint at $t = 1$, i.e., $k = 1$, $l = 2$, see Fig. 1. Then*

$$\mathbf{p} - A_{m,n}\mathbf{q} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix}.$$

The first k and the last l control points of $\mathbf{q}$ are determined by the boundary constraints. Incorporating them we obtain

$$\mathbf{p} - A_{m,n}\mathbf{q} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & \mathbf{5} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{2} & \mathbf{4} & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{3} & \mathbf{3} & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{4} & 2 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} p_0 \\ q_1 \\ q_2 \\ q_3 \\ \frac{6}{5}p_5 - \frac{1}{5}p_6 \\ p_6 \end{bmatrix}.$$

Splitting $\mathbf{q}$ into boundary-determined and boundary-independent vectors,

$$\hat{\mathbf{q}} = [p_0, 0, 0, 0, \frac{6}{5}p_5 - \frac{1}{5}p_6, p_6]^T \quad \text{and} \quad \widetilde{\mathbf{q}} = [q_1, q_2, q_3]^T,$$

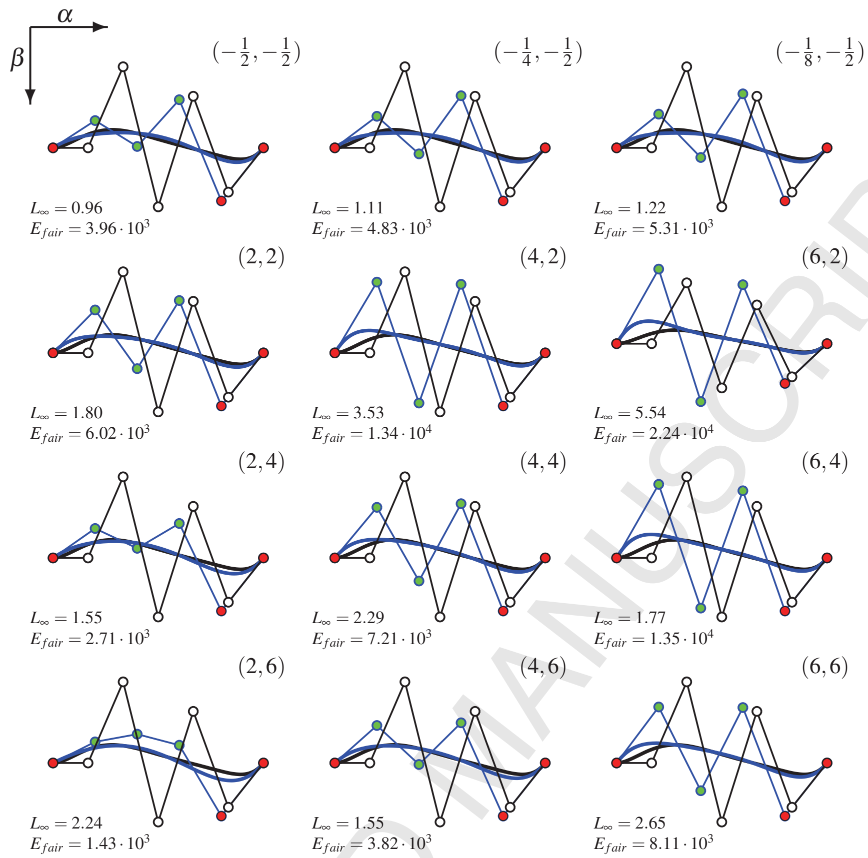

Figure 2: Optimal constrained degree reduction for the data from Fig. 1 for various pairs of Jacobi parameters (α,β) are shown. The initial polynomial p of degree $n = 6$ (black) is reduced to q of degree $m = 5$ (blue) with C^0 and C^1 boundary constraints (red), respectively. The L_∞ error between p and q is displayed, together with the fairness energy E_{fair} (10). For $\alpha = \beta = 0$ (shown in Fig. 1), $L_\infty = 1.11$ and $E_{fair} = 4.42\cdot 10^3$.

and grouping the p-dependent terms together, the system reduces to

$$\widetilde{\mathbf{p}} - A^{k,l}_{m,n}\widetilde{\mathbf{q}} = \begin{bmatrix} p_1 - \frac{1}{6}p_0 \\ p_2 \\ p_3 \\ p_4 - \frac{2}{5}p_5 + \frac{1}{15}p_6 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} \mathbf{5} & \mathbf{0} & \mathbf{0} \\ \mathbf{2} & \mathbf{4} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} & \mathbf{3} \\ \mathbf{0} & \mathbf{0} & \mathbf{4} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

and (9) gives $\widetilde{\mathbf{q}}$.

4. Examples

In this section, we show several numerical examples of constrained degree reduction. Fig. 2 shows the effect of the parameters α and β on the final approximant q. In this figure, p and q are both univariate polynomials (graphs). One can see that the shape of the degree-reduced curve changes considerably for various α and β and these Jacobi weights can be seen as two free *shape parameters*. Their influence on the degree-reduced curve can be interpreted as follows. For example, if $\alpha \ll \beta$, the final L_2 error is forced to be smaller in the neighbourhood of $p(0)$ because the weight

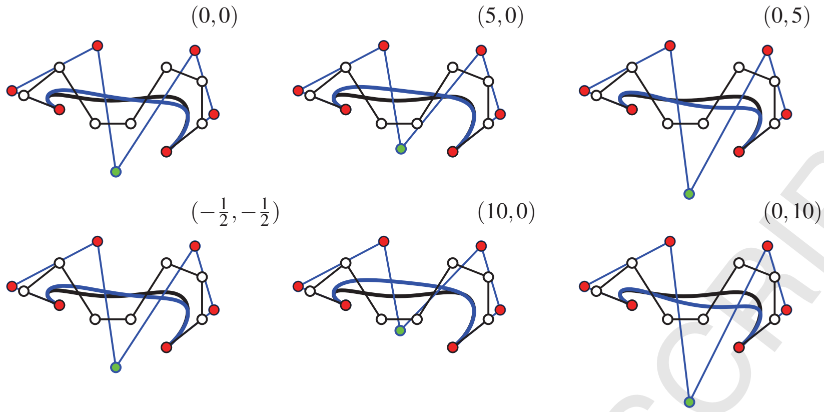

Figure 3: Curvature constrained degree reduction of planar curves. Bézier curve of degree $n = 8$ (black) is reduced to $m = 6$ (blue) with C^2 constraints at both ends (red). The shape of the degree-reduced curve is governed by the only free control point (green). The results for various pairs of (α, β) are shown.

magnifies the L_2 error to be minimized, see (1). This results in a curve that is closely aligned to the initial curve closer to its startpoint $p(0)$.

Due to the two degrees of freedom, additional criteria on the quality of the fit can be considered. Fig. 2 shows the L_∞ error between p and q and also the fairness energy defined as

$$E_{fair}(q) = \sum_{i=1}^{m-1} \|q_{i-1} - 2q_i + q_{i+1}\|^2 \tag{10}$$

which reflects the variation of the control points and consequently the fairness of the curve. It is difficult to estimate *a-priori* the optimal pair (α, β) that minimizes either L_∞ or E_{fair}. However, since the minimization of (6) is achieved by (9) efficiently, one may explore the space of shape parameters to seek for an approximation that meets additional criteria, e.g. low L_∞ and/or E_{fair}.

Curvature constrained degree reduction of planar Bézier curves is shown in Fig. 3. The problem is reduced component-wise and, for the particular setting $n = 8$, $m = 6$, $k = l = 3$, the coordinates of a single free control point (green) are computed from (9). Observe the impact of (α, β) on the shape of the approximation.

An example of a C^1-constrained degree reduction is shown in Fig. 4. A planar curve is considered as a half-meridian of a surface of revolution and the two parameter family of the optimal approximations with respect to Jacobi L_2-norm (1) is explored. Again, one can see how the weights α and β influence the shape by increasing (decreasing) the importance of the L_2 error in the vicinity of the corresponding endpoint.

5. Conclusion

We have derived an analogous result to Lutterkort et al. in [13] for solving the problem of multi-degree reduction of polynomials with boundary constraints with respect to Jacobi inner product. We have proved that the best constrained degree-reduced approximation is equal to the weighted least squares fit of the Bézier coefficients with factored Hahn weights. We have shown on several examples that the two additional parameters of the Jacobi L_2-norm, when compared to the classical L_2 inner product, can serve as supplementary shape parameters for constrained degree reduction.

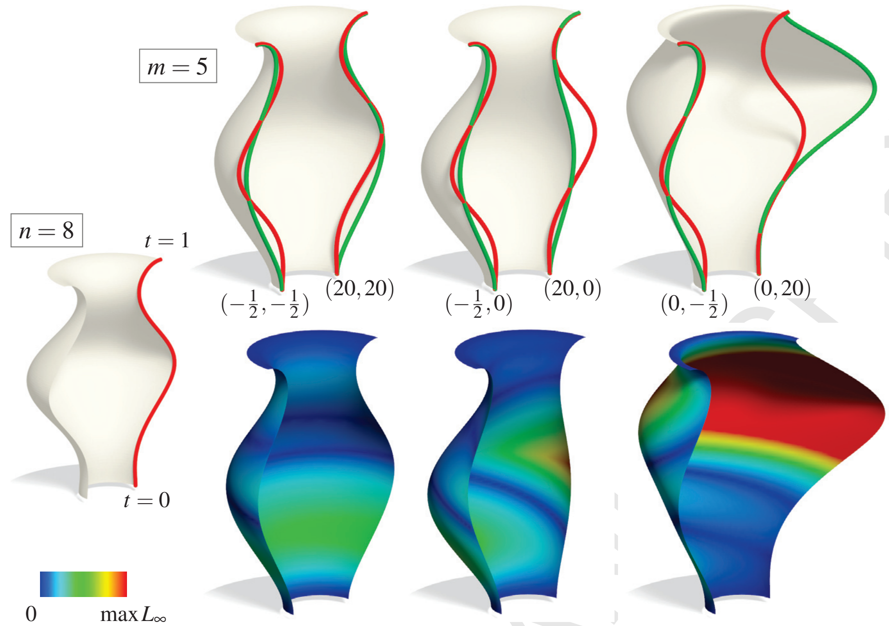

Figure 4: Exploring a space of free parameters α and β. Left: an initial planar curve of degree $n = 8$ (red) is taken as a half-meridian for a surface of revolution. Top: the optimally degree reduced curves (green) of degree $m = 5$ with respect to Jacobi L_2-norm (1) are shown for various pairs (α,β). The curves are required to preserve C^1 constraints at both endpoints. Three lines in the parameter (α,β)-space are explored: the diagonal, α-axis, and β-axis. The surface of revolution is deformed according to the corresponding degree reduced fit; the revolution angles correspond to uniformly sampled points on the lines in the (α,β)-space. Bottom: the color coding reflects the L_∞ error from the original surface of revolution (left).

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