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NON-UNIQUENESS OF WEAK SOLUTIONS OF THE QUANTUM-HYDRODYNAMIC SYSTEM

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ABSTRACT. We investigate the non-uniqueness of weak solutions of the Quantum-Hydrodynamic system. This form of ill-posedness is related to the change of the number of connected components of the support of the position density (called nodal domains) of the weak solution throughout its time evolution. We start by considering a scenario consisting of initial and final time, showing that if there is a decrease in the number of connected components, then we have non-uniqueness. This result relies on the Brouwer invariance of domain theorem. Then we consider the case in which the results involve a time interval and a full trajectory (position-current densities). We introduce the concept of trajectory-uniqueness and its characterization.

1. Introduction. In this paper, we study the non-uniqueness of (so-called Schrödinger-generated) bounded energy weak solutions of the Quantum Hydrodynamic (QHD) system [13]

$$J_t + \operatorname{div} J = 0, \\ J_t + \operatorname{div} \left[\frac{J \otimes J}{\varrho} \right] + \varrho \nabla V - \frac{1}{2} \varrho \nabla \left(\frac{\Delta \sqrt{\varrho}}{\sqrt{\varrho}} \right) = 0, \quad x \in \mathbb{R}^d, t \in \mathbb{R},$$

along with appropriate initial data, as discussed below. We define the quantities ϱ , J of the QHD system through their connection with the Schrödinger equation

$$i\psi_t = -\frac{1}{2}\Delta\psi + V(x)\psi, \quad x \in \mathbb{R}^d, t \in \mathbb{R}, \quad (1)$$

subject to the initial condition

$$\psi(x, t = 0) = \psi_0(x), \quad x \in \mathbb{R}^d.$$

Hence, $\varrho = \varrho(x, t) := |\psi(x, t)|^2$ and $J = J(x, t) := \operatorname{Im}(\bar{\psi}(x, t) \nabla \psi(x, t))$. We call ϱ the position density and J the current density generated by the wave function ψ . For a detailed derivation of the QHD system see, e.g., [1, 4, 5, 6, 12].

The Schrödinger energy is given by

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \psi(x, t)|^2 dx + \int_{\mathbb{R}^d} V(x) |\psi(x, t)|^2 dx.$$

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It is a constant of the motion, i.e., $E(t) = E(t=0) \forall t \in \mathbb{R}$ (under the assumptions made below).

Henceforth we assume that the potential V is in $C_{loc}^{1,1}(\mathbb{R}^d)$ and bounded from below, such that the Hamiltonian $H := -\frac{1}{2}\Delta + V$ is essentially self-adjoint on $L^2(\mathbb{R}^d)$.

Gasser and Markowich [10] showed that if $E(t=0) < \infty$ and $\psi_0 \in L^2(\mathbb{R}^d)$, then the position and current densities corresponding to the uniquely defined energy-conserving solution $\psi \in C(\mathbb{R}_t; L^2(\mathbb{R}_x^d))$ satisfy the QHD system in the sense of distributions with initial data $\varrho_0(x) = \varrho(x, t=0) = |\psi_0(x)|^2$ and $J_0(x) = J(x, t=0) = \text{Im}(\bar{\psi}_0(x) \nabla \psi_0(x))$. Clearly we have $\varrho \in C(\mathbb{R}_t; L^1(\mathbb{R}_x^d))$ and $J \in C(\mathbb{R}_t; \mathcal{D}'(\mathbb{R}_x^d)^d)$, $J(t) \in L^1(\mathbb{R}^d)^d \forall t \in \mathbb{R}$ (uniformly) because of the energy conservation.

Note that the QHD system is formulated above in conservative form, namely, as compressible Euler equations where the enthalpy is given by the sum of the so-called Bohm potential

$$Q := -\frac{1}{2} \frac{\Delta \sqrt{\varrho}}{\sqrt{\varrho}}$$

and the external potential V . In this form, vacuum states $\varrho = 0$ do not have to be dealt with particularly (see [10]), which is not the case for the associated non-conservative form. The reason for this lies in the fact that the velocity $u := J/\varrho$ cannot be reasonably defined for wave-functions ψ which exhibit nodes, i.e., vacuum states where $\psi = 0$. Contrary to this, the internal energy tensor $(J \otimes J)/\varrho$ makes perfect sense when J and ϱ are defined through any bounded energy wave-function (see [2, 3]). However, as we shall point out in the sequel, the conservative form of QHD is prone to an ill-posedness in the form of non-uniqueness of finite energy weak solutions, which is generated precisely by the occurrence of vacuum states.

On the other hand, the QHD equations are the zeroth and first order moment equations of the Bohm equation with the mono kinetic closure. See [9, 15, 16] for a detailed treatment of this topic.

To motivate our subsequent analysis, take $\psi_0 = \psi_0(x)$ smooth in $L^2(\mathbb{R}^d)$ and define $\varrho_0 = \varrho_0(x)$ and $J_0 = J_0(x)$ as before. Moreover, consider a second L^2 -wave function, $\varphi_0 = \varphi_0(x)$. We want to determine the conditions under which

$$|\varphi_0|^2 = \varrho_0, \quad \text{Im}(\bar{\varphi}_0 \nabla \varphi_0) = J_0 \text{ a.e. on } \mathbb{R}^d. \quad (2)$$

To this end, define $D_0 := \{x \in \mathbb{R}^d : \varrho_0 \neq 0\}$ and let $\Lambda_1^0, \Lambda_2^0, \dots$ be the (countably many) connected components of D_0 . Furthermore, let ψ_0 be given by

$$\psi_0(x) = \sqrt{\varrho_0(x)} \exp(iS_0(x)),$$

where $\varrho_0(x)$ and $S_0(x)$ are smooth real valued functions. Then,

$$J_0(x) = \varrho_0(x) \nabla S_0(x).$$

Set $\varphi_0(x) = \sqrt{\sigma_0(x)} \exp(iR_0(x))$ with $\sigma_0(x) = |\varphi_0(x)|^2$. Thus, (2) becomes

$$\begin{aligned} \sigma_0 &= \varrho_0, \\ \varrho_0 \nabla R_0 &= \varrho_0 \nabla S_0, \end{aligned}$$

which implies $\nabla R_0 = \nabla S_0$ on D_0 . Therefore, we find that due to the connectivity of Λ_k^0 , $k \in \mathbb{N}$,

$$R_0(x) = S_0(x) + C_k, \quad x \in \Lambda_k^0,$$

where $C_k \in \mathbb{R}$.

Hence, (2) is satisfied if and only if

$$\varphi_0(x) = \psi_0(x) e^{iC_k}, \quad x \in \Lambda_k^0, \quad k = 1, 2, \dots \text{ a.e. in } \mathbb{R}^d,$$

for some $C_k \in \mathbb{R}$. Note that, under appropriate smoothness and geometric assumptions, the energy associated to φ_0 is finite for all choices of the constants C_k (see Section 2).

We shall argue that, under certain assumptions, the constants C_k can be chosen such that the initial wave-function φ_0 generates a Schrödinger solution whose position and current densities differ from those generated by the Schrödinger evolution of ψ_0 (the original initial wave-function) at a time $T \neq 0$. To be more specific, consider the following scenario. Let $d = 1$ and $\psi_0 = \psi_0(x)$ smooth with $\psi_0(0) = 0$, $\psi_0(x) \neq 0$ for $x \neq 0$, such that at some $T > 0$ we have $\psi(x, T) \neq 0$ for all $x \in \mathbb{R}$ (we shall show an example below). Then $D_0 = \mathbb{R} - \{0\}$ and $\Lambda_1^0 = \{x < 0\}$, $\Lambda_2^0 = \{x > 0\}$. Define

$$\varphi_0(x) := \begin{cases} \psi_0(x), & x < 0, \\ \psi_0(x) e^{i\alpha}, & x > 0, \end{cases} \quad \text{for some } \alpha \in \mathbb{R}, \quad \alpha \neq 0.$$

It is straightforward to check that $\varphi_0 \in H^1(\mathbb{R}^d)$.

Assume that the position and current densities of $\psi(T)$ and $\varphi(T)$ coincide. Since $\{|\psi(\cdot, T)|^2 \neq 0\} = \mathbb{R}$, we conclude that there is $\beta \in \mathbb{R}$ such that

$$e^{i\beta} \psi(x, T) = \varphi(x, T), \quad \forall x \in \mathbb{R}.$$

Now solve the Schrödinger equation

$$\begin{aligned} i\varphi_t &= -\frac{1}{2}\varphi_{xx} + V(x)\varphi, \\ \varphi(T) &\text{ given } (= e^{i\beta}\psi(x, T)), \end{aligned}$$

back to $t = 0$ and find that $\varphi(x, t = 0) = e^{i\beta}\psi(x, t = 0)$ on \mathbb{R} , i.e., $\varphi_0(x) = e^{i\beta}\psi_0(x)$ on \mathbb{R} , which is a contradiction. Therefore, we conclude the non-uniqueness of the corresponding initial value problem (IVP) for the QHD system.

As an example, consider the harmonic oscillator in 1D, i.e., equation (1) with $V(x) = x^2/2$, $x \in \mathbb{R}$. In this case there is a family of solutions (with appropriate initial conditions) for $n = 0, 1, 2, \dots$ and energy levels given by $E_n = (n + 1/2)$. The first three members of the family are

$$\psi_0(x, t) = e^{-iE_0 t} e^{-x^2/2} = e^{-it/2} e^{-x^2/2},$$

$$\psi_1(x, t) = x e^{-\frac{3}{2}it} e^{-x^2/2},$$

$$\psi_2(x, t) = (1 - 2x^2) e^{-\frac{5}{2}it} e^{-x^2/2}.$$

Due to the linearity of the Schrödinger equation, any linear combination of the previous solutions will also be a solution. Hence, consider

$$\psi(x, t) = \psi_0(x, t) - \psi_2(x, t) = e^{-i\frac{t}{2} - \frac{x^2}{2}} (1 - (1 - 2x^2) e^{-2it}).$$

Then

$$\psi(x, t) = e^{-i\frac{t}{2} - \frac{x^2}{2}} (1 - (1 - 2x^2) \cos 2t + i(1 - 2x^2) \sin 2t)$$

with

$$|\psi(x, t)|^2 = e^{-x^2} \left((1 - (1 - 2x^2) \cos 2t)^2 + (1 - 2x^2)^2 (\sin 2t)^2 \right).$$

Clearly, $|\psi(x, t = 0)|^2 = 4e^{-x^2}x^4 = 0$ if and only if $x = 0$. In general, $|\psi(x, t)|^2 = 0$ if and only if $(1 - (1 - 2x^2)\cos 2t)^2 = 0$ and $(1 - 2x^2)^2(\sin 2t)^2 = 0$. Therefore, $|\psi(x, t)|^2 = 0$ if and only if

$$x = 0 \text{ and } t = \frac{l\pi}{2}, \text{ } l \text{ even,}$$

or

$$x = \pm 1 \text{ and } t = \frac{l\pi}{2}, \text{ } l \text{ odd.}$$

Thus, we have our required scenario if we choose $\psi_0 = \psi(x, t = 0)$ and any $0 < T < \pi/2$.

The primary goal of this paper is to give rather explicit (sufficient) conditions on a bounded energy Schrödinger solution which guarantee that the QHD trajectory (position-current densities) generated by such wave-function is not unique in the sense that a different QHD trajectory intersects it at some $t \in \mathbb{R}$. It turns out that we can state such conditions in connection with the topological structure of the so-called nodal domains of the wave function, defined as the connected components of the set where the wave function does not vanish (in other words, the connected domains of the non-vacuum set of the quantum flow).

The rest of the paper is organized as follows. In Section 2 we generalize the previous result to wave function solution of the Schrödinger IVP (QHD-IVP) with less regularity, arbitrary dimension, and an arbitrary number of connected components. In Section 3, we consider the case in which the results involve a time interval and a full QHD trajectory.

2. (Non) Uniqueness of the IVP.

Proposition 1. *Let ψ be a realization of an $H^1(\mathbb{R}^d)$ -function, $\varrho := |\psi|^2$, and $J = \text{Im}(\bar{\psi}\nabla\psi)$. Let Λ be a connected component of the (set-theoretic) support of ϱ , defined by $D_0 := \{x \in \mathbb{R}^d : \varrho(x) > 0\}$, and assume that Λ is open in \mathbb{R}^d . Let $\varphi \in H^1(\mathbb{R}^d)$, $\sigma := |\varphi|^2$ and $I := \text{Im}(\bar{\varphi}\nabla\varphi)$. If*

$$\varphi = \sigma, \quad J = I, \quad \text{a.e. on } \Lambda$$

then there exists a real constant, C , such that

$$\psi(x) = \varphi(x) e^{iC}, \quad x \in \Lambda \text{ a.e.}$$

Proof. Clearly, $\varrho, \sigma, I, J \in L^1(\mathbb{R}^d)$. Consider, for $\omega \in C_0^\infty(\Lambda)$, $\omega \geq 0$:

$$\begin{aligned} & \int_\Lambda |I - J| \omega dx \\ &= \int_\Lambda \left| \text{Im}(\bar{\varphi}\nabla\varphi - \bar{\psi}\nabla\psi) \right| \omega dx \\ &= \int_\Lambda \left| \text{Im}\left(\frac{\nabla\varphi}{\varphi} - \frac{\nabla\psi}{\psi}\right) \right| \omega \varrho dx \\ &= \int_\Lambda \left| \text{Im}\left(\nabla \ln \frac{\varphi}{\psi}\right) \right| \omega \varrho dx \\ &= \int_\Lambda \left| \nabla \text{Im}\left(\ln \frac{\varphi}{\psi}\right) \right| \omega \varrho dx. \end{aligned}$$

Hence,

$$0 = \nabla \text{Im}\left(\ln\left(\frac{\varphi}{\psi}\right)\right) \quad \text{a.e. on } \Lambda.$$

Therefore,

$$\arg \psi = \arg \varphi + C + 2k\pi.$$

We have

$$\psi = \sqrt{\varrho} e^{i \arg \psi}, \quad \varphi = \sqrt{\sigma} e^{i \arg \varphi},$$

and since $\varrho = \sigma$, we conclude that

$$\psi = \varphi e^{i C_\alpha} \text{ a.e. on } \Lambda.$$

□

Remark 1. Assume that all the connected components Λ_α of D_0 are of locally finite perimeter (Λ_α are Caccioppoli sets), that ψ is continuous on \mathbb{R}^d and in $H^1(\mathbb{R}^d)$. Then the characteristic function $\mathbf{1}_{\Lambda_\alpha}$ of Λ_α has locally bounded total variation, which in turn implies that its distributional gradient is a vector valued (signed) Radon measure supported on the boundary of Λ_α . Thus the function φ , defined by

$$\varphi = \psi \sum_{\alpha \in A} e^{-i C_\alpha} \mathbf{1}_{\Lambda_\alpha},$$

is in $H^1(\mathbb{R}^d)$, and we have $\varrho = \sigma$ and $J = I$ on \mathbb{R}^d . Specifically, we compute

$$\nabla \varphi = \nabla \psi \sum_{\alpha \in A} e^{-i C_\alpha} \mathbf{1}_{\Lambda_\alpha},$$

taking into account that $\psi|_{\partial \Lambda_\alpha} = 0$. Note that the index set $A \subseteq \mathbb{N}$.

Proposition 2. Let $\psi = \psi(x, t)$ be a bounded energy solution of the Schrödinger equation on $[0, T]$ with ψ continuous (in x) for $t = 0, T$. Define

$$\begin{aligned} N_0 \in \mathbb{N} &:= \text{Number of connected components of } \{\varrho(\cdot, t = 0) \neq 0\} := D_0, \\ N_T \in \mathbb{N} &:= \text{Number of connected components of } \{\varrho(\cdot, t = T) \neq 0\} := D_T, \end{aligned}$$

and assume $N_T < N_0 \leq \infty$. Furthermore, assume that all the connected components of D_0 are of locally finite perimeter. Then there is non-uniqueness of Schrödinger-generated bounded energy solutions of the QHD system on $[0, T]$, with $\varrho(\cdot, t = 0)$, $J(\cdot, t = 0)$ given.

Proof. To simplify the presentation, assume that $N_0 < \infty$. Let $\Lambda_1^0, \dots, \Lambda_{N_0}^0$ be the connected components of D_0 and let $\Lambda_1^T, \dots, \Lambda_{N_T}^T$ be the connected components of D_T .

Note that $\psi(\cdot, t = 0)$ and

$$\varphi(x, t = 0) := \psi(x, t = 0) e^{i \alpha_l}, \quad x \in \Lambda_l^0, \quad l = 1, \dots, N_0$$

are in $H^1(\mathbb{R}^d)$ and generate the same position and current densities at $t = 0$ for all choices $\alpha_l \in \mathbb{R}$.

Assume that the QHD system has a unique Schrödinger-generated bounded energy solution on $[0, T]$. Then, at $t = T$, we must have

$$\varphi(x, t = T) = \psi(x, t = T) e^{i \beta_k}, \quad x \in \Lambda_k^T, \quad k = 1, \dots, N_T,$$

where $\varphi(t)$ is the Schrödinger solution with initial datum $\varphi(\cdot, t = 0)$ and $\varrho = |\varphi|^2$, $J = \text{Im}(\bar{\varphi} \nabla \varphi)$. This defines a map $\mathbb{T}^{N_0} \mapsto \mathbb{T}^{N_T}$ (\mathbb{T}^n is the n -dimensional torus)

$$F : (\alpha_1, \dots, \alpha_{N_0}) \rightarrow (\beta_1, \dots, \beta_{N_T}).$$

This map is injective since the backward Schrödinger IVP has the uniqueness property. Continuity of F is a consequence the $L^2(\mathbb{R}^d)$ -continuity of the Schrödinger

evolution in the following way. Take a sequence $(\alpha_1^{(m)}, \dots, \alpha_{N_0}^{(m)}) \rightarrow (\alpha_1^{(0)}, \dots, \alpha_{N_0}^{(0)})$ as $m \rightarrow \infty$. Hence,

$$\begin{aligned} & \sum_{l=1}^{N_0} \int_{\Lambda_l^0} \left| \psi(x, t=0) e^{i\alpha_l^{(m)}} - \psi(x, t=0) e^{i\alpha_l^{(0)}} \right|^2 dx \\ &= \sum_{l=1}^{N_0} \int_{\Lambda_l^0} |\psi(x, t=0)|^2 \left| 1 - e^{i(\alpha_l^{(0)} - \alpha_l^{(m)})} \right|^2 dx \rightarrow 0. \end{aligned}$$

Therefore,

$$\psi^{(m)}(x, t=0) := \psi(x, t=0) e^{i\alpha_l^{(m)}}, \quad x \in \Lambda_l^0, \quad l = 1, \dots, N_0$$

converges to

$$\psi^{(0)}(x, t=0) := \psi(x, t=0) e^{i\alpha_l^{(0)}}, \quad x \in \Lambda_l^0, \quad l = 1, \dots, N_0$$

in $L^2(\mathbb{R}^d)$. By L^2 -continuity of the Schrödinger evolution, we have that

$$\psi^{(m)}(x, t=T) := \psi(x, t=T) e^{i\beta_k^{(m)}}, \quad x \in \Lambda_k^T, \quad k = 1, \dots, N_T$$

converges in $L^2(\mathbb{R}^d)$ to

$$\psi^{(0)}(x, t=T) := \psi(x, t=T) e^{i\beta_k^{(0)}}, \quad x \in \Lambda_k^T, \quad k = 1, \dots, N_T.$$

By the same computation, we find that

$$\begin{aligned} & \int_{\mathbb{R}^d} |\psi^{(0)}(x, T) - \psi^{(m)}(x, T)|^2 dx \\ &= \sum_{k=1}^{N_T} \int_{\Lambda_k^T} |\psi(x, T)|^2 dx \left| 1 - e^{i(\beta_k^{(0)} - \beta_k^{(m)})} \right|^2 \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$ if and only if $\beta_k^{(m)} \rightarrow \beta_k^{(0)}$, $k = 1, \dots, N_T$. Hence, F is continuous.

Since by assumption $N_0 > N_T$, we have a contradiction due to the fact that, as a consequence of the Brouwer invariance of domain theorem (see, e.g., [7, 11, 17]), there is no continuous injective mapping from \mathbb{R}^n to \mathbb{R}^m when $n > m$.

It is straightforward to show that the result also holds for $N_0 = \infty > N_T$. \square

3. Trajectory case. Let $\psi \in C(\mathbb{R}_t; L^2(\mathbb{R}_x^d)) \cap L^\infty(\mathbb{R}_t; H^1(\mathbb{R}_x^d))$ be a bounded energy solution of the Schrödinger equation (1). Let $\varrho := |\psi|^2$ and $J := \text{Im}(\bar{\psi} \nabla \psi)$, and assume that $\varrho \in C(\mathbb{R}_x^d \times \mathbb{R}_t)$. Denote

$$\Omega := \{(x, t) \in \mathbb{R}^{d+1} : \varrho(x, t) \neq 0\},$$

and let $\Omega_\alpha \subseteq \Omega$, $\alpha \in B \subseteq \mathbb{N}$, be the connected components of Ω . Note that each set Ω_α is open in \mathbb{R}^{d+1} and

$$\partial\Omega_\alpha \subseteq \{(x, t) \in \mathbb{R}^{d+1} : \varrho(x, t) = 0\}.$$

Proposition 3. *Let $\varphi \in C(\mathbb{R}_t; L^2(\mathbb{R}_x^d)) \cap L^\infty(\mathbb{R}_t; H^1(\mathbb{R}_x^d))$ be another bounded energy solution of the Schrödinger equation (1) and denote $\sigma := |\varphi|^2$, $I := \text{Im}(\bar{\varphi} \nabla \varphi)$. If*

$$\varrho = \sigma, \quad J = I \quad \forall t \text{ a.e. in } \mathbb{R}_x^d,$$

then there exists a constant, $C_\alpha \in \mathbb{R}$, for every $\alpha \in B$ such that $\varphi(x, t) = \psi(x, t) e^{iC_\alpha}$ a.e. in Ω_α .

Proof. Let $\omega \in C_0^\infty(\Omega_\alpha)$ and consider

$$\begin{aligned} & \int_{\mathbb{R}_t} \int_{\mathbb{R}_x^d} |J - I| \omega dx \\ &= \int_{\mathbb{R}_t} \int_{\mathbb{R}_x^d} \left| \operatorname{Im} (\bar{\psi} \nabla \psi - \bar{\varphi} \nabla \varphi) \right| \omega dx \\ &= \int_{\mathbb{R}_t} \int_{\mathbb{R}_x^d} \left| \operatorname{Im} \left(\frac{\nabla \psi}{\psi} - \frac{\nabla \varphi}{\varphi} \right) \right| \omega \varrho dx \\ &= \int_{\mathbb{R}_t} \int_{\mathbb{R}_x^d} \left| \nabla_x \operatorname{Im} \ln \left(\frac{\psi}{\varphi} \right) \right| \omega \varrho dx = 0. \end{aligned}$$

Now let ω be supported in an arbitrary convex subset, Σ_α , of Ω_α . Then

$$\arg \varphi = \arg \psi + C_{\Sigma_\alpha}(t) \text{ in } \Sigma_\alpha$$

for some measurable function $C_{\Sigma_\alpha} = C_{\Sigma_\alpha}(t)$ on \mathbb{T}^d . This gives

$$\varphi = \psi e^{iC_{\Sigma_\alpha}(t)} \text{ in } \Sigma_\alpha.$$

We insert into the Schrödinger equation and find ($\psi \neq 0, \varphi \neq 0$ in Ω_α)

$$\dot{C}_{\Sigma_\alpha}(t) = 0 \text{ for all } t \in \Sigma'_\alpha = \Sigma_\alpha \cap \{(y, t) : y \in \mathbb{R}^d\} \neq \emptyset.$$

Therefore for every convex open subset Σ_α of Ω_α there exists $C_{\Sigma_\alpha} \in \mathbb{R}$ such that

$$\varphi = \psi e^{iC_{\Sigma_\alpha}} \text{ in } \Sigma_\alpha.$$

To prove that C_{Σ_α} depends only on α and not on the convex subset of Ω_α we take any two points $(x_1, t_1), (x_2, t_2) \in \Omega_\alpha$ and connect them by a continuous curve, $\Gamma \subseteq \Omega_\alpha$. Now cover Γ by (finitely many, since Γ is compact) open balls $B_{\alpha,1}, \dots, B_{\alpha,L} \subseteq \Omega_\alpha$. In each ball $B_{\alpha,l}$ we have

$$\varphi = \psi e^{iC_{B_{\alpha,l}}}.$$

Since for each B_{α,l_1} there exists B_{α,l_2} with $l_1 \neq l_2$ such that

$$B_{\alpha,l_1} \cap B_{\alpha,l_2} \neq \emptyset,$$

we conclude that $C_{B_{\alpha,1}} = \dots = C_{B_{\alpha,l}}$. Therefore, a unique constant $C = C_{\Omega_\alpha}$ exists such that

$$\varphi = \psi e^{iC_{\Omega_\alpha}} \text{ on } \Omega_\alpha \text{ a.e.}$$

□

Remark 2. Let $C_\alpha \in \mathbb{R}$ for $\alpha \in B$, $\psi \in C(\mathbb{R}_x^d \times \mathbb{R}_t) \cap L^\infty(\mathbb{R}_t; H^1(\mathbb{R}^d))$. Then the function

$$\varphi(x, t) = \psi(x, t) \sum_{\alpha \in B} e^{iC_\alpha} \mathbf{1}_{\Omega_\alpha}$$

is in $L^\infty(\mathbb{R}_t; H^1(\mathbb{R}^d))$ if all sets Ω_α have locally finite perimeter. Actually, it suffices to ask that $\nabla_x \mathbf{1}_{\Omega_\alpha} \in L_{loc}^\infty(\mathbb{R}_t; \mathcal{M}(\mathbb{R}_x^d)^d)$, where $\mathcal{M}(\mathbb{R}_x^d)$ is the set of (scalar signed) Radon measures on \mathbb{R}_x^d .

Definition 3.1. Let

$$\{(\varrho(t), J(t)) : t \in \mathbb{R}\} \in C(\mathbb{R}_t; L^1(\mathbb{R}^d)) \times \left(L^\infty(\mathbb{R}_t; L^1(\mathbb{R}^d))^d \cap C(\mathbb{R}_t; \mathcal{D}'(\mathbb{R}_x^d))^d \right)$$

be a solution-curve of the QHD system for $-\infty < t < \infty$. It is called trajectory-unique if for all other such solution curves $\{(\sigma(t), I(t)) : t \in \mathbb{R}\}$ we have $(\varrho, J)(t) \neq (\sigma, I)(t) \forall t \in \mathbb{R}$ or $(\varrho, J)(t) = (\sigma, I)(t) \forall t \in \mathbb{R}$.

In simple terms, this means that different trajectories do not intersect, neither forward nor backward in time.

Proposition 4. Consider a QHD-trajectory, $S = \{(\varrho(t), J(t)) : t \in \mathbb{R}\}$, with $\psi \in C(\mathbb{R}_x^d \times \mathbb{R}_t)$ and bounded energy. If for some $T \in \mathbb{R}$ the number N of connected components of

$$\mathbb{R}^{d+1} \supseteq \Omega := \{\varrho(x, t) \neq 0 : (x, t) \in \mathbb{R}^{d+1}\}$$

is smaller than the number N_T of connected components of

$$\mathbb{R}^d \supseteq \Lambda_T := \{\varrho(x, T) \neq 0 : x \in \mathbb{R}^d\},$$

then S is not trajectory-unique, provided that all connected components have locally finite perimeter.

Proof. Let $N < N_T < \infty$ and assume that S is generated by the wave-function $\psi = \psi(x, t)$, i.e., $\varrho = |\psi|^2$, $J = \text{Im}(\bar{\psi} \nabla \psi)$ and ψ solves the Schrödinger equation.

Note that

$$\varphi(x, t) = \psi(x, t) e^{iC_{\Omega_\alpha}}, \quad (x, t) \in \Omega_\alpha,$$

where Ω_α , $\alpha = 1, \dots, N$, are the connected components of Ω , generates the same trajectory S (sufficient and necessary). Now, for $C_{\Lambda_{T,1}}, \dots, C_{\Lambda_{T,N_T}} \in \mathbb{R}$ set

$$\sigma(x, T) = \psi(x, T) e^{iC_{\Lambda_{T,\beta}}}, \quad x \in \Lambda_{T,\beta},$$

where $\Lambda_{T,1}, \dots, \Lambda_{T,N_T}$ are the connected components of Λ_T . Solve the IVP for the Schrödinger equation starting from $\sigma(\cdot, T)$ backwards and forwards. If all the possible choices of $\sigma(T)$ gave the same trajectory S , there would have to exist a continuous and injective map from $\mathbb{R}^{N_T} \mapsto \mathbb{R}^N$, which is a contradiction since $N_T > N$. The proof extends easily to $N_T = \infty$. \square

Proposition 5. Let $\psi = \psi(x, t)$ be a bounded energy solution of the Schrödinger equation with $\varrho(\cdot, t = T_1)$ and $\varrho(\cdot, t = T_2)$ continuous on \mathbb{R}^d ($T_1 < T_2$) and let all connected components of $\{\varrho(\cdot, t = T_1) \neq 0\}$ and $\{\varrho(\cdot, t = T_2) \neq 0\}$ be Caccioppoli sets. If $N_{T_1} \neq N_{T_2}$, then $\{(\varrho, J) : t \in [T_1, T_2]\}$ (QHD-trajectory) is not trajectory-unique.

Proof. Already established for $N_{T_1} > N_{T_2}$ (Proposition 2). If $N_{T_2} > N_{T_1}$ apply the same argument calculating backwards in time. \square

Now let $\Omega_1 \neq \Omega_2$ be two adjacent connected components of $\Omega \subseteq \mathbb{R}_x^d \times \mathbb{R}_t$, with the smooth interface surface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$.

Let $\psi = \psi(x, t)$ be a smooth solution of the Schrödinger equation, with obviously $\psi|_\Gamma = 0$. Denote by \mathcal{Y} the unit normal of Γ , pointing (for definiteness sake) into Ω_1 and set $\mathcal{Y} = \begin{pmatrix} \mathcal{Y}_x \\ \mathcal{Y}_t \end{pmatrix}$ according to the coordinate ordering $\begin{pmatrix} x \\ t \end{pmatrix}$.

Proposition 6. Let $C_1 \neq C_2$ be real constants and set

$$\varphi(x, t) = \psi(x, t) \cdot \begin{cases} e^{iC_1}, & (x, t) \in \Omega_1, \\ e^{iC_2}, & (x, t) \in \Omega_2. \end{cases}$$

Then φ is a solution of the Schrödinger equation in $\Omega_1 \cup \Gamma \cup \Omega_2$ if and only if

$$\nabla \psi \cdot \mathcal{Y}_x|_\Gamma = 0.$$

Proof. Set $\alpha_1 = e^{iC_1}$, $\alpha_2 = e^{iC_2}$. Then

$$\varphi = \alpha_1 \psi \mathbf{1}_{\Omega_1} + \alpha_2 \psi \mathbf{1}_{\Omega_2} \text{ in } \Omega_1 \cup \Gamma \cup \Omega_2.$$

Compute, in $\Omega_1 \cup \Gamma \cup \Omega_2$

$$\varphi_t = \alpha_1 \psi_t \mathbf{1}_{\Omega_1} + \alpha_2 \psi_t \mathbf{1}_{\Omega_2} + \alpha_1 \psi \frac{\partial}{\partial t} \mathbf{1}_{\Omega_1} + \alpha_2 \psi \frac{\partial}{\partial t} \mathbf{1}_{\Omega_2}.$$

Note that $(\frac{\partial}{\partial t}\mathbf{1}_{\Omega_i})$ is supported on $\partial\Omega_i$, where ψ vanishes continuously. Thus

$$\psi \frac{\partial}{\partial t} \mathbf{1}_{\Omega_i} = 0$$

and

$$\varphi_t = \alpha_1 \psi_t \mathbf{1}_{\Omega_1} + \alpha_2 \psi_t \mathbf{1}_{\Omega_2}.$$

The analogous computation holds with $\nabla \mathbf{1}_{\Omega_i}$. Then

$$\nabla \varphi = \alpha_1 \nabla \psi \mathbf{1}_{\Omega_1} + \alpha_2 \nabla \psi \mathbf{1}_{\Omega_2}$$

and

$$\begin{aligned} \Delta \varphi = & \alpha_1 \Delta \psi \mathbf{1}_{\Omega_1} + \alpha_2 \Delta \psi \mathbf{1}_{\Omega_2} \\ & + \alpha_1 \nabla \psi \cdot \nabla \mathbf{1}_{\Omega_1} + \alpha_2 \nabla \psi \cdot \nabla \mathbf{1}_{\Omega_2}. \end{aligned}$$

Plug into the Schrödinger equation to find (since ψ solves the Schrödinger equation)

$$0 = \nabla \psi \cdot (\alpha_1 \nabla \mathbf{1}_{\Omega_1} + \alpha_2 \nabla \mathbf{1}_{\Omega_2}).$$

Take a test-function, $\omega = \omega(x, t) \in C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_t)$:

$$\begin{aligned} 0 &= \langle \alpha_1 \nabla \psi \cdot \nabla \mathbf{1}_{\Omega_1} + \alpha_2 \nabla \psi \cdot \nabla \mathbf{1}_{\Omega_2}, \omega \rangle_{x,t} \\ &= -\alpha_1 \int_{\Omega_1} \operatorname{div}_x (\omega \nabla \psi) dx dt - \alpha_2 \int_{\Omega_2} \operatorname{div}_x (\omega \nabla \psi) dx dt \\ &= \alpha_1 \int_{\Gamma} \omega \nabla \psi \cdot \Upsilon_x ds - \alpha_2 \int_{\Gamma} \omega \nabla \psi \cdot \Upsilon_x ds \\ &= (\alpha_1 - \alpha_2) \int_{\Gamma} \nabla \psi \cdot \Upsilon_x \omega ds. \end{aligned}$$

Therefore,

$$\nabla \psi \cdot \Upsilon_x|_{\Gamma} = 0.$$

□

For the following results, we use the generalized Green's formula by De Giorgi-Federer:

$$\int_E \operatorname{div} F dx = \int_{\partial^* E} F \cdot \Upsilon d\mathcal{H}^{n-1},$$

where F is any locally Lipschitz function, $E \subseteq \mathbb{R}^n$ is of finite perimeter, Υ is the normal of ∂E , and $\partial^* E$ is the reduced boundary (see, e.g., [8, 14]).

Proposition 7. *Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}_x^d \times \mathbb{R}_t$ be of locally finite perimeter. Let $\nabla \psi \in C_{loc}^{0,1}(\mathbb{R}_x^d \times \mathbb{R}_t)$. Then φ (defined as in Proposition 6) solves the Schrödinger equation if and only if*

$$\nabla \psi \cdot \Upsilon_x = 0 \quad \mathcal{H}^d - \text{a.e. on } (\partial\Omega_1 \cap \partial\Omega_2)^*.$$

The consequence is:

Proposition 8. *Let all connected components Ω_α of Ω ($\alpha \in B$) be of locally finite perimeter and let*

$$\nabla \psi \in C_{loc}^{0,1}(\mathbb{R}_x^d \times \mathbb{R}_t).$$

Assume that there is a non-space-like interface segment, Γ , between two connected components such that

$$\nabla \psi \cdot \Upsilon_x \neq 0 \quad \mathcal{H}^d - \text{a.e. on } \Gamma^*.$$

Then the QHD-trajectory generated by $\psi = \psi(x, t)$ is not trajectory-unique on \mathbb{R}_t .

Proof. Choose $T \in \mathbb{R}$ such that

$$\Gamma \cap \{(y, T) : y \in \mathbb{R}^d\} \neq \emptyset.$$

Choose $C_1 \neq C_2$ real numbers and set

$$\varphi(x, T) := \begin{cases} \psi(x, T) e^{iC_1} & \text{in } \Omega_1 \cap \{t = T\}, \\ \psi(x, T) e^{iC_2} & \text{in } \Omega_2 \cap \{t = T\}, \\ \psi(x, T) & \text{elsewhere,} \end{cases}$$

and solve the Schrödinger IVP with $\varphi(\cdot, T)$ as initial datum. By the above

$$\varphi(x, t) \neq \psi(x, t) \cdot \begin{cases} e^{iC_1}, & (x, t) \in \Omega_1, \\ e^{iC_2}, & (x, t) \in \Omega_2, \end{cases}$$

and hence trajectory non-uniqueness follows. \square

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