Fractional parts and their relations to the values of the Riemann zeta function

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Abstract A well-known result, due to Dirichlet and later generalized by de la Vallée–Poussin, expresses a relationship between the sum of fractional parts and the Euler–Mascheroni constant. In this paper, we prove an asymptotic relationship between the summation of the products of fractional parts with powers of integers on the one hand, and the values of the Riemann zeta function, on the other hand. Dirichlet's classical result falls as a particular case of this more general theorem.

Mathematics Subject Classification 11M06

الملخص

نتيجة معروفة، ترجع أساسا إلى ديريشلت، ثم عمّمت بواسطة دولافالي-بوسان، تعبر عن علاقة بين مجموع أجزاء كسور وثابت أيلور-ماشروني. في هذا البحث، نثبت علاقة تقاربية بين مجموع مضروبات أجزاء كسور مع قوى الأعداد الصحيحة من جهة و قيم دالة زيتا لريمان من جهة أخرى. وعليه، فإن نتيجة ديريشلت القديمة تصبح حالة خاصة من هذه النظرية العامة.

1 Background

In 1849, Dirichlet established a relationship between the Euler–Mascheroni constant $\gamma = 0.5772\cdots$ and the average of fractional parts. More specifically, writing $[x]$ for the integral (floor) part of the number $x \in \mathbb{R}$ and $\{x\} = x - [x]$ for its fractional part, Dirichlet [3,5] proved that

$$\frac{1}{n}\sum_{k=1}^{n}\left\{\frac{n}{k}\right\} = 1 - \gamma + O\left(\frac{1}{\sqrt{n}}\right). \tag{1.1}$$

This surprising connection between γ and the average of fractional parts was, in turn, used by Dirichlet to prove that the number of divisors of an integer n is of the order $\log n$. The technique introduced by Dirichlet to prove these results is often called the hyperbola method, which is a counting argument to the number of lattice points that lie beneath a curve [3,6].

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The error term in (1.1) is known to be pessimistic. Finding the optimal exponent $\theta > 0$ such that

$$\sum_{k=1}^{n} \left\{ \frac{n}{k} \right\} - (1-\gamma)n = O(n^{\theta+\epsilon}),$$

for any $\epsilon > 0$ is known as the Dirichlet divisor problem, which remains unsolved to this date. A well-known result of Hardy is that $\theta \geq \frac{1}{4}$, which is conjectured to be the true answer to this problem [3].

In 1898, de la Vallée–Poussin generalized (1.1). He showed that for any integer $w \in \mathbb{N}$,

$$\frac{w}{n} \sum_{k=1}^{\frac{n-1}{w}} \left\{ \frac{n}{wk+1} \right\} = 1 - \gamma + O\Big(\frac{1}{\sqrt{n}}\Big). \tag{1.2}$$

As noted by de la Vallée–Poussin, this result is quite remarkable because the limiting average of the fractional parts remains unchanged regardless of the arithmetic progression that one wishes to use [3].

More recently, Pillichshammer obtained a different generalization of Dirichlet's result. He showed that for any $\beta > 1$,

$$\sum_{k=1}^{\sqrt[\beta]{n}} \left\{ \frac{n}{k^\beta} \right\} = (1-\gamma_{1/\beta})\sqrt[\beta]{n} + O\Big(n^{\frac{1}{\beta+1}}\Big), \tag{1.3}$$

where $\gamma_{1/\beta}$ is a family of constants whose first term is $\gamma_1 = \gamma$ [5].

In this paper, we look into a different line of generalizing (1.1). Specifically, we address the question of deriving the asymptotic expressions to summations of the form

$$f_s(n) = \sum_{k=1}^{n} \left\{ \frac{n}{k} \right\} k^s, \tag{1.4}$$

for positive real numbers $s > 0$. This is the summation of the products of fractional parts and powers of integers. Interestingly, we will show that the asymptotic behavior of this summation is connected to the values of the Riemann zeta function $\zeta(s)$, and we will recover Dirichlet's result in (1.1) as a particular case. More specifically, we prove that for any real number $s > 0$,

$$\frac{1}{n^{s+1}} \sum_{k=1}^{n} \left\{ \frac{n}{k} \right\} k^s = \frac{1}{s} - \frac{\zeta(s+1)}{s+1} + O\Big(\frac{1}{\sqrt{n}}\Big). \tag{1.5}$$

We conclude this section with two classical theorems that we will rely on in our proofs:

Theorem 1.1 (Abel summation formula) *Let a_k be a sequence of complex numbers and $\phi(x)$ be a function of class C^1. Then,*

$$\sum_{1 \leq k \leq n} a_k \phi(k) = A(n)\phi(n) - \int_1^n A(t)\phi'(t)\mathrm{d}t, \tag{1.6}$$

where $A(x) = \sum_{k=1}^{[x]} a_x$.

Theorem 1.2 (Euler–Maclaurin summation formula) *We have*

$$\sum_{k=1}^{n} \phi(k) = A + \int_1^n \phi(t)\mathrm{d}t + \sum_{r=1}^{s} \frac{B_r}{r!}\phi^{(r-1)}(n) + O(\phi^{(s)}(n)), \tag{1.7}$$

for some constant A, where $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0, \ldots$ are the Bernoulli numbers.

These results can be found in many places, such as [2,4].

2 Notation

We will use the following notations:

- $[x]$ denotes the integral (floor) part of x and $\{x\} = x - [x]$ denotes the fractional part.
- $\mathbb{N}$ denotes the set of positive integers, often called the *natural* numbers; $\mathbb{Z}^+$ is the set of non-negative integers; $\mathbb{R}$ is the set of real numbers; $\mathbb{C}$ is the set of complex numbers.
- $\mathcal{R}(s)$ denotes the real part of $s \in \mathbb{C}$.

3 The fractional transform

3.1 Overview

The key insight we will employ to derive the asymptotic expansion of the function $f_s(n)$ in (1.4) is that we can solve this problem indirectly by answering a *different* question. Specifically, we will be interested in the following function:

$$\Phi_s(n) = \sum_{k=1}^{n-1} \left[\left\{ \frac{n}{k} \right\} - \left\{ \frac{n}{k+1} \right\} \right] k^s. \tag{3.1}$$

More generally, when

$$\Phi(n) = \sum_{k=1}^{n-1} \left[\left\{ \frac{n}{k} \right\} - \left\{ \frac{n}{k+1} \right\} \right] \phi(k), \tag{3.2}$$

we will call $\Phi(n)$ the *fractional transform* of $\phi(n)$. $\Phi_s(n)$ allows us to answer our original question because

$$\Phi_s(n) = \sum_{k=1}^{n-1} \left[\left\{ \frac{n}{k} \right\} - \left\{ \frac{n}{k+1} \right\} \right] k^s = \sum_{k=1}^{n} \left\{ \frac{n}{k} \right\} \Big(k^s - (k-1)^s \Big),$$

where we have used the fact that $\{n/n\} = \{n\} = 0$. By expanding the right-hand side using the binomial theorem, we obtain a method of solving our original question.

3.2 Preliminary

We present a few useful lemmas related to the fractional transform defined above. Before we do this, we introduce the following symbol:

$$\partial(n) = \left\{ k \in \mathbb{N} : \left[\frac{n}{k+1}, \frac{n}{k} \right] \cap \mathbb{N} \neq \emptyset \right\}. \tag{3.3}$$

In other words, $\partial(n)$ is the set of positive integers $k \in \mathbb{N}$ that are less than n, and for which the interval $[n/(k+1), n/k]$ contains, at least, one integer. For instance, $2 \in \partial(5)$ because the interval $[5/3, 5/2]$ contains the integer two, whereas $3 \notin \partial(5)$ because the interval $[5/4, 5/3]$ lies strictly between 1 and 2.

Lemma 3.1

$$\left\{ \frac{n}{k} \right\} - \left\{ \frac{n}{k+1} \right\} = \frac{n}{k(k+1)} - \left| \left[\frac{n}{k+1}, \frac{n}{k} \right] \cap \mathbb{N} \right|, \tag{3.4}$$

where $|\mathbb{S}|$ denotes the size (cardinality) of the set $\mathbb{S}$.

Proof We have:

$$\begin{aligned} \left\{ \frac{n}{k} \right\} - \left\{ \frac{n}{k+1} \right\} &= \left(\frac{n}{k} - \frac{n}{k+1} \right) - \left(\left[\frac{n}{k} \right] - \left[\frac{n}{k+1} \right] \right) \\ &= \frac{n}{k(k+1)} - \left[\frac{n}{k} \right] + \left[\frac{n}{k+1} \right] = \frac{n}{k(k+1)} - \left| \left[\frac{n}{k+1}, \frac{n}{k} \right] \cap \mathbb{N} \right|. \end{aligned}$$

□

Lemma 3.2 *If $k \geq \sqrt{n}$ and $k \in \partial(n)$, then*

$$\left\{\frac{n}{k}\right\} - \left\{\frac{n}{k+1}\right\} = \frac{n}{k(k+1)} - 1.$$

Proof The interval $[n/(k+1), n/k]$ can contain, at most, a unique integer since:

$$\frac{n}{k} - \frac{n}{k+1} = \frac{n}{k(k+1)} \leq \frac{n}{\sqrt{n}(\sqrt{n}+1)} < 1.$$

This fact and Lemma 3.1 both imply the statement of the lemma. □

4 Main results

We begin with the following lemma:

Lemma 4.1 *For any real number $s > 0$,*

$$\sum_{k=1}^{[n^\epsilon]} \left[\left\{\frac{n}{k}\right\} - \left\{\frac{n}{k+1}\right\}\right] k^s = O(n^{\epsilon s}), \tag{4.1}$$

where the constant in $O(\cdot)$ depends on s.

Proof First, let us consider the following function:

$$g_n(w) = \sum_{k=1}^{w} \left[\left\{\frac{n}{k}\right\} - \left\{\frac{n}{k+1}\right\}\right] k$$

Since

$$\sum_{k=1}^{w} \left\{\frac{n}{k}\right\} = \sum_{k=1}^{w} \left\{\frac{n}{k}\right\} k - \sum_{k=1}^{w} \left\{\frac{n}{k}\right\}(k-1) = g_n(w) + \left\{\frac{n}{w+1}\right\} w,$$

we obtain

$$g_n(w) = -\left\{\frac{n}{w+1}\right\} w + \sum_{k=1}^{w} \left\{\frac{n}{k}\right\}$$

Consequently, we conclude that $-w \leq g_n(w) \leq w$. Using Theorem 1.1,

$$\begin{aligned}
\sum_{k=1}^{[n^\epsilon]} \left[\left\{\frac{n}{k}\right\} - \left\{\frac{n}{k+1}\right\}\right] k^s &= \sum_{k=1}^{[n^\epsilon]} \left[\left\{\frac{n}{k}\right\} - \left\{\frac{n}{k+1}\right\}\right] k k^{s-1} \\
&= [n^\epsilon]^{s-1} g_n([n^\epsilon]) - (s-1) \int_1^{[n^\epsilon]} g_n(t)\, t^{s-2} \mathrm{d}t \\
&\leq [n^\epsilon]^s - (s-1) \int_1^{[n^\epsilon]} g_n(t)\, t^{s-2} \mathrm{d}t \\
&\leq [n^\epsilon]^s + |s-1| \int_1^{[n^\epsilon]} t^{s-1} \mathrm{d}t \leq \left(1 + \frac{|s-1|}{s}\right)[n^\epsilon]^s.
\end{aligned}$$

Here, we used the fact that $|g(w)| \leq w$. Similarly,

$$\sum_{k=1}^{[n^\epsilon]} \left[\left\{\frac{n}{k}\right\} - \left\{\frac{n}{k+1}\right\}\right] k^s \geq -\left(1 + \frac{|s-1|}{s}\right)[n^\epsilon]^s.$$

Therefore, the statement of the lemma follows. □

Now, we are ready to prove our first main result.

Theorem 4.2 *For any* $s > 1$,

$$\frac{1}{n^s}\sum_{k=1}^{n-1}\left[\left\{\frac{n}{k}\right\}-\left\{\frac{n}{k+1}\right\}\right]k^s=\frac{1}{s-1}-1+\zeta(s)+O\left(\frac{1}{\sqrt{n}}\right).$$

Proof We split the sum into two parts:

$$\Phi_s(n)=\sum_{k=1}^{[\sqrt{n}]}\left[\left\{\frac{n}{k}\right\}-\left\{\frac{n}{k+1}\right\}\right]k^s+\sum_{k=[\sqrt{n}]+1}^{n-1}\left[\left\{\frac{n}{k}\right\}-\left\{\frac{n}{k+1}\right\}\right]k^s. \tag{4.2}$$

The first term is $O(n^{\frac{s}{2}})$ as proved in the previous lemma, which is $O(n^{s-\frac{1}{2}})$ when $s > 1$. Next, we examine the second term. We have by Lemma 3.2,

$$\begin{aligned}\sum_{k=[\sqrt{n}]+1}^{n-1}\left[\left\{\frac{n}{k}\right\}-\left\{\frac{n}{k+1}\right\}\right]k^s&=\sum_{k=[\sqrt{n}]+1}^{n-1}\frac{n\,k^s}{k(k+1)}-\sum_{k=[\sqrt{n}]+1}^{n-1}\left|\left[\frac{n}{k+1},\frac{n}{k}\right]\cap\mathbb{N}\right|k^s\\&=\sum_{k=[\sqrt{n}]+1}^{n-1}\frac{n\,k^s}{k(k+1)}-\sum_{k=2}^{[\sqrt{n}]-1}\left[\frac{n}{k}\right]^s\\&=\sum_{k=[\sqrt{n}]+1}^{n-1}\frac{n\,k^s}{k(k+1)}-\sum_{k=2}^{[\sqrt{n}]-1}\left(\frac{n}{k}\right)^s+\sum_{k=2}^{\sqrt{n}-1}\left\{\frac{n}{k}\right\}^s\\&=\sum_{k=[\sqrt{n}]+1}^{n-1}\frac{n\,k^s}{k(k+1)}-n^s\sum_{k=2}^{[\sqrt{n}]-1}\frac{1}{k^s}+O(\sqrt{n}).\end{aligned}$$

Using the fact that for $\mathcal{R}(s) > 0$ (see [1]),

$$\zeta(s)=\sum_{k=1}^{w}\frac{1}{k^s}+\frac{w^{1-s}}{s-1}-s\int_w^\infty\frac{\{t\}}{t^{s+1}}\mathrm{d}t,$$

we conclude that

$$\sum_{k=1}^{w}\frac{1}{k^s}=\frac{w^{1-s}}{1-s}+\zeta(s)+O(w^{-s}).$$

Alternatively, the error term $O(w^{-s})$ in the above expression can be derived from Theorem 1.2. Hence,

$$\begin{aligned}\sum_{k=2}^{[\sqrt{n}]-1}\frac{1}{k^s}&=\sum_{k=1}^{[\sqrt{n}]}\frac{1}{k^s}-1-\frac{1}{[\sqrt{n}]^s}\\&=\frac{[\sqrt{n}]^{1-s}}{1-s}+\zeta(s)-1+O\left(n^{-\frac{s}{2}}\right)\\&=\frac{(\sqrt{n}-\{\sqrt{n}\})^{1-s}}{1-s}+\zeta(s)-1+O\left(n^{-\frac{s}{2}}\right)\\&=\frac{n^{\frac{1-s}{2}}}{1-s}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)+\zeta(s)-1+O\left(n^{-\frac{s}{2}}\right)\\&=\frac{n^{\frac{1-s}{2}}}{1-s}+\zeta(s)-1+O\left(n^{-\frac{s}{2}}\right).\end{aligned}$$

As a result,

$$n^s \sum_{k=2}^{[\sqrt{n}]-1} \frac{1}{k^s} = n^s \left(\frac{n^{\frac{1-s}{2}}}{1-s} + \zeta(s) - 1 + O(n^{-\frac{s}{2}}) \right) = (\zeta(s)-1)\, n^s + \frac{n^{\frac{1+s}{2}}}{1-s} + O(n^{\frac{s}{2}}).$$

Finally, we look into the remaining term. Using Theorem 1.2, we have for any $s > 1$,

$$\begin{aligned} n \sum_{k=[\sqrt{n}]+1}^{n-1} \frac{k^{s-1}}{k+1} &= n \left[\sum_{k=[\sqrt{n}]+1}^{n-1} \frac{k^{s-2}}{1+\frac{1}{k}} \right] \\ &= n \left[\sum_{k=[\sqrt{n}]+1}^{n-1} k^{s-2} \left(1 - \frac{1}{k} + \frac{1}{k^2} - \cdots \right) \right] \\ &= n \left[\sum_{k=[\sqrt{n}]+1}^{n-1} (k^{s-2} - k^{s-3} + \cdots) \right] \\ &= \frac{n^s}{s-1} + \frac{n^{\frac{1}{2}(s+1)}}{1-s} + O(n^{s-\frac{1}{2}}). \end{aligned}$$

Here, we used the fact that for any $u \geq 2$, we have by Theorem 1.2,

$$\begin{aligned} \sum_{k=1}^{n-1} k^{s-u} &= C(u) + \frac{n^{s-u+1}}{s-u+1} - \frac{n^{s-u}}{2} + \sum_{k=2}^{m} \binom{s-u}{k-1} \frac{B_k}{k}\, n^{s-u-k+1} + O(n^{s-u-m}) \\ \sum_{k=1}^{[\sqrt{n}]} k^{s-u} &= C(u) + \frac{n^{\frac{s-u+1}{2}}}{s-u+1} + \frac{n^{\frac{s-u}{2}}}{2} + \sum_{k=2}^{m} \binom{s-u}{k-1} \frac{B_k}{k}\, n^{\frac{s-u-k+1}{2}} + O(n^{\frac{s-u-m}{2}}), \end{aligned}$$

for some constant $C(u)$ that is independent of n.

Putting everything together, we conclude that for any real number $s > 1$,

$$\sum_{k=1}^{n-1} \left[\left\{ \frac{n}{k} \right\} - \left\{ \frac{n}{k+1} \right\} \right] k^s = \left[\frac{1}{s-1} - 1 + \zeta(s) \right] n^s + O\left(n^{s-\frac{1}{2}} \right),$$

which is the statement of the theorem. □

Theorem 4.2 is illustrated in Fig. 1. Clearly, this theorem generalizes Dirichlet's result, as promised earlier, because

$$\lim_{s \to 1} \left\{ \frac{1}{s-1} - \zeta(s) \right\} = -\gamma$$

and the fact that

$$\sum_{k=1}^{n-1} \left[\left\{ \frac{n}{k} \right\} - \left\{ \frac{n}{k+1} \right\} \right] k = \sum_{k=1}^{n-1} \left\{ \frac{n}{k} \right\} \Big(k - (k-1) \Big) = \sum_{k=1}^{n} \left\{ \frac{n}{k} \right\}.$$

Now, we are ready to derive the asymptotic expression of the function $f_s(n)$ given in (1.4).

Theorem 4.3 *For any real number* $s > 0$,

$$\frac{1}{n^{s+1}} \sum_{k=1}^{n} \left\{ \frac{n}{k} \right\} k^s = \frac{1}{s} - \frac{\zeta(s+1)}{s+1} + O\left(\frac{1}{\sqrt{n}} \right).$$

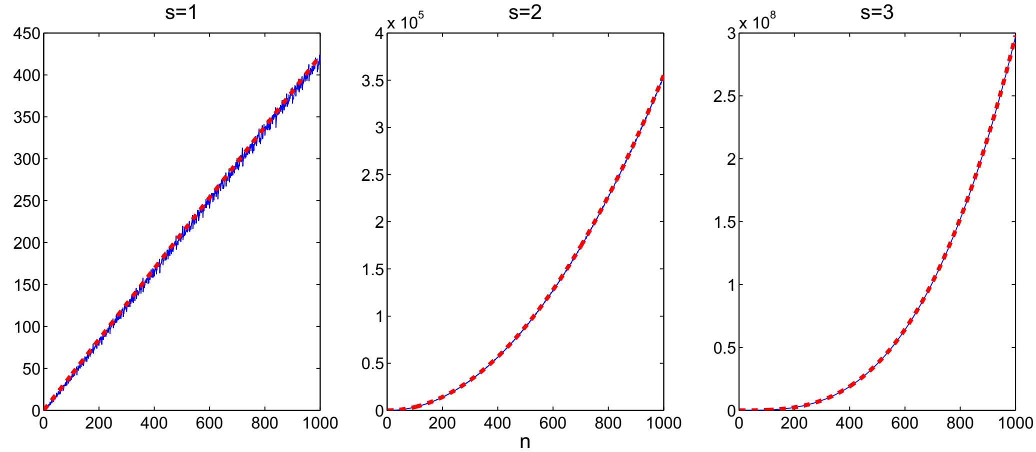

Fig. 1 A comparison between the values of $\Phi_s(n)$ marked in *blue* and the asymptotic expression derived in Theorem 4.2 marked in *red*. The x-axis is n while the y-axis is $\Phi_s(n)$. The *left*, *middle*, and *right figures* correspond to $s = 1$, $s = 2$ and $s = 3$, respectively

Proof Let $f_s(n)$ be as defined in Eq. (1.4). Then, writing by Theorem 4.2,

$$\begin{aligned}\left[\frac{1}{s} - 1 - \zeta(s+1)\right] n^{s+1} + O(n^{s+\frac{1}{2}}) &= \sum_{k=1}^{n-1} \left[\left\{\frac{n}{k}\right\} - \left\{\frac{n}{k+1}\right\}\right] k^{s+1} \\ &= \sum_{k=1}^{n-1} \left\{\frac{n}{k}\right\} \left(k^{s+1} - (k-1)^{s+1}\right) \\ &= \sum_{k=1}^{n-1} \left\{\frac{n}{k}\right\} \sum_{k=1}^{s+1} (-1)^{k+1} \binom{s+1}{k} k^{s+1-k} \\ &= \sum_{k=1}^{s+1} (-1)^{k+1} \binom{s+1}{k} f_{s+1-k}(n) \\ &= (s+1)\, f_s(n) + \sum_{k=2}^{s+1} (-1)^{k+1} \binom{s}{k} f_{s+1-k}(n).\end{aligned}$$

However,

$$0 \le f_s(n) = \sum_{k=1}^{n} \left\{\frac{n}{k}\right\} k^s \le \sum_{k=1}^{n} k^s = O(n^{s+1})$$

Therefore,

$$\sum_{k=2}^{s+1} (-1)^{k+1} \binom{s+1}{k} f_{s+1-k}(n) = O(n^s).$$

Hence for $s > 0$,

$$\sum_{k=1}^{n} \left\{\frac{n}{k}\right\} k^{s+1} = f_s(n) = \left[\frac{1}{s} - \frac{\zeta(s+1)}{s+1}\right] n^{s+1} + O(n^{s+\frac{1}{2}}),$$

which implies the statement of the theorem. □

5 Conclusion

In this paper, we generalized Dirichlet's classical result on the connection between the Euler–Mascheroni constant and the average of fractional parts. Our theorem reveals that the fractional parts are, in general, connected to the values of the Riemann zeta function $\zeta(s)$. Hence, $\zeta(s)$ with $s > 1$ can be expressed as a limiting average of the products of fractional parts with powers of positive integers.

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