A Derivation of Identifiable Condition for Non-Uniform Linear Array DOA Estimation

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Hui Chen, Tarig Ballal, and Tareq Y. Al-Naffouri

***Abstract*—Phase ambiguity happens in uniform linear arrays (ULAs) when the sensor distance is greater than $\lambda/2$. This problem in direction of arrival (DOA) estimation and can be solved by designing a proper sensor configuration. In this work, we derive the identifiable condition for ULA DOA estimation.**

***Index Terms*—Direction of arrival, phase difference, disambiguity, non-uniform linear arrays.**

I. Observation Model

We consider a complex sinusoidal source signal, with a frequency f and amplitude A, $s(t) = Ae^{-j2\pi ft}$ in the *far field* [1] of a non-uniform linear array of N sensors. The source impinges on the array from a direction $\theta_0 \in (-\pi/2, \pi/2)$ rad. By using d_{uv} to denote the distance between two sensors (u and v) normalized by $\lambda/2$, where λ is the signal wavelength, the received signal (vector) at time t can be modelled as [2]

$$\mathbf{x}(t) = \mathbf{a}(\theta_0)s(t) + \mathbf{w}(t), \tag{1}$$

where $\mathbf{a}(\theta_0) = [1, \;\; e^{-j\pi d_{12}\sin(\theta_0)}, \; ..., \; e^{-j\pi d_{1N}\sin(\theta_0)}]^T$ is the array steering vector, and $\mathbf{w}(t)$ is vector of the additive noise.

The *principal* phase difference across a sensor pair, u and v, can be estimated from the u-th and v-th elements of $\mathbf{x}$ as

$$\hat{\psi}_{uv}(t) = \text{angle}(x_u(t) \cdot x_v^*(t)) \in [-\pi, \pi), \tag{2}$$

where $(\cdot)^*$ is the complex conjugate operation. Without loss of generality, we will focus on single-snapshot scenarios. Hence, we will subsequently drop the time variable t.

To develop our proposed method, we start from noise-free principal phase observations, ψ_{uv}. These observations are related to the actual phase difference, $\phi_{uv} = \pi d_{uv}\sin(\theta_0)$ as

$$\psi_{uv} = \text{mod}(\phi_{uv} + \pi, 2\pi) - \pi = \pi d_{uv}\sin(\theta_0) - 2\pi q_{uv}, \tag{3}$$

where $\text{mod}(\cdot,\cdot)$ is the modulus operation, and q_{uv} is an integer value given by the rounding operation

$$q_{uv} = \text{round}\left(\frac{\pi d_{uv}\sin(\theta_0)}{2\pi}\right). \tag{4}$$

Based on (3), we observe that estimating the DOA from ψ_{uv} requires knowledge of the integer q_{uv}, which may not be available if a methods such as (2) is used to estimate ψ_{uv}. For $d_{uv} \le 1$, $q_{uv} = 0$ for any θ. For $d_{uv} > 1$, the latter result is not guaranteed, except for a specific range of θ. Since θ is unknown, ψ_{uv} will always be *ambiguous* for $d_{uv} > 1$.

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II. Identifiable Condition

The concept of wrapped phase-difference pattern (WPDP) is introduced in [3] to visualize phase-difference and estimate DOA.

From the WPDP, we can see that the sufficient and necessary condition is that there do not exist two DOAs that have the same WPD vectors. It is obvious that if two points have the same WPD vector, they cannot be differenciated from each other. Thus, we can have

$$\pi\text{sin}(\theta_1)\mathbf{d} \neq \pi\text{sin}(\theta_2)\mathbf{d} + 2\pi\mathbf{q} \tag{5}$$

where $\theta_1 > \theta_2$, $\mathbf{q} = [q_1, q_2, ..., q_M]$ is a nonnegative integer vector indicating the possible wrapping cycle for each sensor pair. Define $q_{i,max}$ as the maximum integer that q_i might be, $q_{i,max}$ can be calculated as

$$q_{i,max} = \text{floor}\left(\frac{\pi(\sin(\theta_1) - \sin(\theta_2))d_i}{2\pi}\right) \le \text{floor}(d_i). \tag{6}$$

There are two situations:

1) If there is no phase-wrapping in any sensor pair i, the inequation (5) holds because $q_i = q_{i,max} = 0$ and $\theta_1 \neq \theta_2$. This is the case that the distance between one of the sensor pair is smaller than half-wavelength.
2) If phase-wrapping happens for all the sensor pair, q_i can be an integer from set $\{1, 2, ..., q_{i,max}\}$. Then, (5) can be reformulated as equation (7) **does not hold** for all the possible value of q_i.

$$\frac{d_1}{q_1} = \frac{d_2}{q_2} = ... = \frac{d_M}{q_M}\left(= \frac{2}{\sin(\theta_1) - \sin(\theta_2)} \ge 1\right). \tag{7}$$

Note that $q_{i,max} \le$ floor(d_i), the content inside the bracelet can be ignored.

Let us take two examples:

(a). An unidentifiable case with $\Delta = 1.2, \delta = 4$ provided by the reviewer.

In this case, $\mathbf{r} = [0, 1.2, 6]$, $\mathbf{d} = [d_{12}, d_{13}, d_{23}] = [1.2, 6, 4.8]$, and $\mathbf{q}_{max} = [1, 6, 4]$. if integer vector $\mathbf{q}$ is chosen as $[1, 5, 4]$, equation (7) holds and hence it is an unidentifiable case.

(b). An identifiable case with $\Delta = 3.6, \delta = 1.25$ provided in Fig. 2.(a).

In this case, $\mathbf{r} = [0, 3.6, 8.1]$, $\mathbf{d} = [d_{12}, d_{13}, d_{23}] = [3.6, 8.1, 4.5]$, and $\mathbf{q}_{max} = [3, 8, 4]$. Whatever we choose the integer vector $\mathbf{q}$, equation (7) cannot be satisfied and hence it is an identifiable case.

III. Quick Check of the Identifiability

There is a quick way to check the condition in (7) is satisfied or not for a certain layout.

1) Find a positive real number I, which makes $D_i = Id_i$ an integer for all the $i \in (1, 2, ..., M)$ and the greatest common divisor for $D_1, D_2, ...D_M$ is 1;
2) Since q_i is an integer and $D_1, D_2, ...D_M$ have the greatest common divisor 1, the only way to make $\frac{D_1}{q_1} = \frac{D_2}{q_2} = ... = \frac{D_M}{q_M}$ is to choose q_i equals to D_i or equals to multiple times of D_i.
3) If $D_i \leq p_{i,max}$ for $i = 1, 2, ..., M$, equation (7) is satisfied.

Let us take the same two examples:

(a). An unidentifiable case with $\Delta = 1.2, \delta = 4$ provided by the reviewer.

Multiply $\mathbf{d}$ by 10 to obtain $[12, 60, 48]$, then divided by the greatest common divisor to obtain $\mathbf{D} = [1, 5, 4]$ ($I = \frac{5}{6}$). Because $D_i \leq q_{i,max}$ for $i = 1, 2, ..., M$, equation (7) is satisfied and hence this configuration is unidentifiable.

(b). An identifiable case with $\Delta = 3.6, \delta = 1.25$ provided in Fig. 2.(a).

Multiply $\mathbf{d}$ by 10 to obtain $[36, 81, 45]$, then divided by the greatest common divisor to obtain $\mathbf{D} = [4, 9, 5]$ ($I = \frac{10}{9}$). Because $D_i > q_{i,max}$ for some i, equation (7) is not satisfied and hence this configuration is identifiable.

IV. Conclusion

In this work, we briefly described the DOA estimation model in a far field scenario. An identifiable condition is derived based on the wrapped phase-difference pattern (WPDP), and a quick check approach is provided.

References

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