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ON ITERATIVE IMPES FORMULATION FOR TWO-PHASE FLOW WITH CAPILLARITY IN HETEROGENEOUS POROUS MEDIA

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Abstract. This work is a continuation of Kou and Sun [36] where we present an efficient improvement on the IMplicit Pressure Explicit Saturation (IMPES) method for two-phase immiscible fluid flow in porous media with different capillarity pressures. In the previous work, we present an implicit treatment of capillary pressure appearing in the pressure equation. A linear approximation of capillary function is used to couple the implicit saturation equation into the pressure equation that is solved implicitly. In this paper, we present an iterative version of this method. It is well-known that the fully implicit scheme has unconditional stability. The new method can be used for solving the coupled system of nonlinear equations arisen after the fully implicit scheme. We follow the idea of the previous work, and use the linear approximation of capillary function at the current iteration. This is different from iterative IMPES that computes capillary pressure by the saturations at the previous iteration. From this approximation, we couple the saturation equation into the pressure equation, and establish the coupling relation between the pressure and saturation. We employ the relaxation technique to control the convergence of the new method, and we give a choice of relaxation factor. The convergence theorem of our method is established under the natural conditions. Numerical examples are provided to demonstrate the performance of our approach, and the results show that our method is efficient and stable.

Key words. Two-phase flow, IMPES, Heterogeneous media, Capillary pressure.

1. Introduction

Two-phase fluid flow model in porous media is a coupled system of nonlinear time-dependent partial differential equations. We often use two different types of time discretization schemes: the fully implicit and the IMplicit-EXplicit (IMEX). The fully implicit scheme [5, 21, 23, 54, 66] implicitly treats with all terms including capillary pressure, and hence has unconditional stability and maintains the inherent coupling of two-phase flow model. This scheme results in a system of nonlinear equations. IMEX [4, 7, 31, 35, 37] generally treats the linear terms implicitly and evaluates the others explicitly, and consequently, it is conditionally stable. One advantage of IMEX is to eliminate the nonlinearity of original equations.

There are two different approaches [41,60] used for solving the coupled system of nonlinear equations arisen after the fully implicit scheme. One is the fully coupled approach that simultaneously solves all variables and equations by a Newton-type method. Consequently, the pressures and saturations can be easily coupled at each iterative step. However, the computational cost and memory requirement will be particularly expensive, especially when the size of problems becomes large. This restricts the applications of the fully coupled approach to a certain extent. The other approach is the simulator coupling scheme that splits the entire problem into some sub-problems. These sub-problems may be solved independently by different approaches, and they are coupled by data exchanges at each iteration. Thus, the simulator coupling approach is more flexible than the fully coupled approach. If the simulator coupling approach reaches the full convergence of the iterative algorithm,

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the computed solution is the same as that of the fully coupled approach. The main advantage of the simulator coupling approach is to reduce the computational cost and memory requirement compared to the fully coupled approach [47].

To reduce the computational cost of Newton-type simultaneous methods, a number of improved approaches are presented and discussed in [3, 44, 48, 67, 72]. The reduced degree of freedom method presented in [67] uses an approximation of the Jacobian matrix in the Newton-Raphson iteration to partition the coupled system of equations into a solution of some selected primary variables plus a back substitution procedure for the solution of the other variables. A phase-based potential ordering is presented in [44] to reduce the nonlinear algebraic system arisen from the fully implicit scheme into one with only pressure dependence, and then Newton's method is applied to solve the reduced system. On the other hand, various precondition techniques are used to to increase the convergence of the Newton-type methods, for example, [9, 10, 41].

Operator splitting [1, 25, 30, 40, 46, 58] can reduce a complex time-dependent physical problem into some simpler problems based on the time-lag of dimension or physics. By operator-splitting approach, we can construct iterative operator-splitting methods [30, 32, 42], which may be used to solve the nonlinear system arisen after the fully implicit scheme.

The IMplicit Pressure Explicit Saturation (IMPES) approach is viewed as an IMEX method, which employs a splitting approach based on physics. IMPES solves the pressure equation implicitly and updates the saturation explicitly. In IMPES, we substitute the saturation constraint and Darcy's law into the sum of the two mass conservation laws to obtain the pressure equation, and explicitly treat all other variables in the pressure equation to eliminate its nonlinearity. After the pressure is obtained, we explicitly compute Darcy's velocity and two-phase saturations. As a time discretization scheme, the IMPES method is conditionally stable, and hence it must take very small time step size, especially for highly heterogeneous permeable media where the capillary pressure affects substantially on the path of fluid flow. The instability of the IMPES method [20] results from the explicit treatment of the capillary pressure and the decoupling between the pressure equation and the saturation equation.

There are numerous improved versions of IMPES for two-phase flow, for example, [16, 52, 68, 70]. Iterative IMPES is to use IMPES as an iterative scheme for full implicit systems instead of Newton iteration. This approach splits the whole equation system into a pressure and a saturation equation that are solved in the sequence as IMPES. An iterative scheme developed in [45, 50, 51] solves a pressure implicitly and an implicit saturation equation in each iteration. This implicit saturation equation is derived from the implicit capillary pressure introduced in the original saturation equation. As an iterative method, the computational cost and memory required by iterative IMPES method is smaller than the fully coupled approach at each iterative step, which is more pronounced for very large size computational problems. Iterative coupling is also popular in the simulation of single-phase and two-phase flow and reactive transport [2, 24, 27, 63–65, 71].

In this work, we pay attention to two-phase flow in heterogeneous media. Heterogeneity in capillary pressure is a computational chanllege as it may have a significant influence on flow paths [6, 8, 22, 26, 34, 38, 39, 43, 53, 56, 59, 69]. For the rocks with different permeability types, we employ the different capillary pressure functions; that is, the capillary pressure functions are discontinuous on the interfaces of rocks. In this case, the discontinuity of saturation results from the continuity of capillary

pressure [38]. The classical fractional flow formulation used in [11–14, 28] may not be suitable for highly heterogeneous media because of its inconsistency [38]. The wetting-phase pressure is always continuous as long as none of the phases is immobile, and can correctly describe discontinuities in saturation due to continuity of capillary pressure and discontinuities of different capillary pressure functions on the interface of regions. Based on this conception, a two-phase flow formulation has been proposed in [38,39], which will be employed in this paper. Because of the discontinuity of saturation across rock interface, the methods that need the gradient of saturation in spacial dimension can not be well used to the case of different capillary pressure functions for multiple rock types.

Theoretically, the full implicit scheme is unconditionally stable and allows a large time step size, but practically a nonlinear solver may restrict the size of time step to guarantee the convergence. The main disadvantage of iterative IMPES method is the decoupling of pressure and saturation equations, which results from the explicit treatment for capillary pressure. In highly heterogenous media, the capillary pressure forces may change the saturation distributions of two phase in a very short time, and hence for stability, iterative IMPES still requires a much smaller time step size. One approach to improve the stability is to treat the capillary pressure implicitly. In our previous work [36], we present an implicit treatment of capillary pressure in the pressure equation. A linear approximation of capillary function is introduced to couple the implicit saturation equation into pressure equation. We solve the coupled pressure equation implicitly and then explicitly update the saturation. Comparison shows that our proposed method is more efficient and stable than the classical IMPES approach.

In this paper, we present an iterative version of our previous method proposed in [36]. The new method is viewed as a simulator coupling approach for solving the coupled system of nonlinear equations arisen after the fully implicit scheme. Unlike iterative IMPES, capillary pressure is not computed by the saturations at the previous iteration, and we use the linear approximation of capillary function at the current iteration, which is constructed by the saturations at the current and previous iterations. From this, we couple the saturation equation into pressure equation, and establish the coupling relation between the pressure and saturation. The used approximation of capillary pressure is always well-behaved, our method is suitable not only to homogenous but also to heterogenous media. A relaxation approach is used to control the convergence of our method, and we introduce a choice of relaxation factor. We prove the convergence theorem of our method under the natural conditions.

This paper is organized as follows. In section 2, we describe the mathematical model of two-phase incompressible flow. In section 3, we present our new method and describe the implement based on the cell-centered finite difference (CCFD) method for spatial discretization. In section 4, we analyze the convergence of the new method. In section 5, we discuss the choice of relaxation factor. In Section 6, we give some numerical examples to show the performance of the proposed method and compare it with iterative IMPES. Finally, we summarize this work.

2. Two-phase incompressible flow

In this paper, we follow the two-phase flow formulation [38, 43, 53]. The fluids considered here are incompressible and immiscible. In the following model, we indicate the wetting phase and non-wetting phase by the subscripts w and n,

respectively. The mass conservation within each phase is described by

$$\phi\frac{\partial S_\alpha}{\partial t}+\nabla\cdot\mathbf{u}_\alpha=q_\alpha,\ \alpha=w,n, \tag{1}$$

where ϕ is the porosity of the medium and S_α, q_α, and $\mathbf{u}_\alpha$ are the saturation, the external mass flow rate, and Darcy's velocity of each phase α, respectively. Darcy's velocity for each phase α is determined by Darcy's law:

$$\mathbf{u}_\alpha=-\frac{k_{r\alpha}}{\mu_\alpha}\mathbf{K}(\nabla p_\alpha+\rho_\alpha g\nabla z),\ \alpha=w,n, \tag{2}$$

where $\mathbf{K}$ is the absolute permeability tensor in the porous medium, g is the gravity acceleration, z is the depth, and each phase has its own relative permeability $k_{r\alpha}$, viscosity μ_α, pressure p_α and density ρ_α.

The saturations of the two fluids are constrained by the relation

$$S_w+S_n=1. \tag{3}$$

The capillary pressure is the deference between the wetting phase and non-wetting phase pressures, i.e.

$$p_c(S_w)=p_n-p_w. \tag{4}$$

In this work, the absolute permeability tensor is chosen as $\mathbf{K}=k\mathbf{I}$, where $\mathbf{I}$ is the identity matrix and k is a positive real number. One typical capillary pressure function [38] is given by

$$p_c(S_w)=-B_c\log(S_e), \tag{5}$$

where B_c is a positive parameter and it is inversely proportional to $\sqrt{k}$. Here, the normalized saturation S_e is given by

$$S_e=\frac{S_w-S_{rw}}{1-S_{rw}-S_{rn}}, \tag{6}$$

where S_{rw} and S_{rn} are the residual saturations for the wetting and non-wetting phases, respectively. In numerical simulation, we may set the minimum S_{emin} for the normalized saturation S_e; that is, S_e will be taken equal to S_{emin} if $S_e\le S_{emin}$. This treatment not only accords with the physical property, but also makes the capillary pressure function sufficiently regular for helping to carry out the theoretical analysis. Besides, we can see that the values of B_c vary with the different absolute permeabilities, and as a result, the different capillary pressure functions are applied for the heterogeneous porous media.

One typical formulation for relative permeability of each phase is given by

$$k_{rw}=S_e^\beta, \tag{7}$$

$$k_{rn}=(1-S_e)^\beta, \tag{8}$$

where β is a positive integer number. Here, the relative permeabilities of two phases are functions of the wetting-phase saturation, and $0\le k_{rw}\le 1, 0\le k_{rn}\le 1$.

We now describe the governing equations used in [38, 43, 53] as

$$\nabla\cdot(\mathbf{u}_a+\mathbf{u}_c)\equiv-\nabla\cdot\lambda_t\mathbf{K}\nabla\Phi_w-\nabla\cdot\lambda_n\mathbf{K}\nabla\Phi_c=q_w+q_n, \tag{9}$$

and

$$\phi\frac{\partial S_w}{\partial t}-q_w=-\nabla\cdot(f_w\mathbf{u}_a)\equiv\nabla\cdot\lambda_w\mathbf{K}\nabla\Phi_w, \tag{10}$$

where $\Phi_w = p_w + \rho_w gz$, $\Phi_c = p_c + (\rho_n - \rho_w)gz$, $\lambda_\alpha = \frac{k_{r\alpha}}{\mu_\alpha}$, $\lambda_t = \lambda_w + \lambda_n$, and $f_w = \lambda_w/\lambda_t$. The two velocity variables $\mathbf{u}_a$ and $\mathbf{u}_c$ are given by

$$\mathbf{u}_a = -\lambda_t \mathbf{K}\nabla\Phi_w, \tag{11}$$

and

$$\mathbf{u}_c = -\lambda_n \mathbf{K}\nabla\Phi_c. \tag{12}$$

By the definition of $\mathbf{u}_a$, the wetting-phase velocity is expressed as

$$\mathbf{u}_w = f_w \mathbf{u}_a. \tag{13}$$

To complete the formulation of the model, the boundary and initial conditions are required. Let the boundary $\partial\Omega$ of the computational domain Ω be decomposed into the Dirichelt part Γ^D and Newmann part Γ^N where $\partial\Omega = \Gamma^D \bigcup \Gamma^N$ and $\Gamma^D \bigcap \Gamma^N = \phi$. We use the following boundary conditions

$$p_w(\text{or } p_n) = p^D \text{ on } \Gamma^D, \tag{14}$$

$$(\mathbf{u}_a + \mathbf{u}_c)\cdot\mathbf{n} = q^N \text{ on } \Gamma^N, \tag{15}$$

where $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$, p^D the pressure on Γ^D and q^N the imposed inflow rate on Γ^N, respectively. The saturations on the boundary are subject to

$$S_w(\text{or } S_n) = S^N \text{ on } \Gamma^N. \tag{16}$$

At the same time, the saturation of the wetting phase at the beginning of the flow displacing process is initially defined by

$$S_w = S_w^0 \text{ in } \Omega. \tag{17}$$

3. New iterative IMPES method

Let the total time interval $[0, T]$ be divided into N time steps as $0 = t^0 < t^1 < \cdots < t^N = T$. Define the time step length $\triangle t^i = t^{i+1} - t^i$. We use the backward Euler time discretization for the pressure and saturation equations, and obtain

$$-\nabla\cdot\lambda_t(S_w^{i+1})\mathbf{K}\nabla\Phi_w^{i+1} - \nabla\cdot\lambda_n(S_w^{i+1})\mathbf{K}\nabla\Phi_c(S_w^{i+1}) = q_w^{i+1} + q_n^{i+1}, \tag{18}$$

and

$$\phi\frac{S_w^{i+1} - S_w^i}{\Delta t^i} - q_w^{i+1} = -\nabla\cdot(f_w^{i+1}\mathbf{u}_a^{i+1}) \equiv \nabla\cdot\lambda_w(S_w^{i+1})\mathbf{K}\nabla\Phi_w^{i+1}, \tag{19}$$

where the superscript $(i+1)$ represents the current time step. The above system is fully implicit and coupled, but it can not be solved directly because of its nonlinearity. Hence, iterative methods are often employed to solve such systems.

3.1. Classical iterative IMPES method. The classical iterative IMPES formulation for solving the equations (18) and (19) is given by

$$-\nabla\cdot\lambda_t(S_w^{i+1,j})\mathbf{K}\nabla\Phi_w^{i+1,j+1} - \nabla\cdot\lambda_n(S_w^{i+1,j})\mathbf{K}\nabla\Phi_c(S_w^{i+1,j}) = q_w^{i+1} + q_n^{i+1}, \tag{20}$$

and

$$\phi\frac{S_w^{i+1,j+1} - S_w^i}{\Delta t^i} - q_w^{i+1} = -\nabla\cdot(f_w^{i+1,j}\mathbf{u}_a^{i+1,j+1}) \equiv \nabla\cdot\lambda_w(S_w^{i+1,j})\mathbf{K}\nabla\Phi_w^{i+1,j+1}, \tag{21}$$

where the superscript $(i+1)$ represents the current time step, and the superscripts j and $(j+1)$ represent the iterative steps within the current time step.

In each iteration of the classical iterative IMPES method, the capillary potential Φ_c is calculated using the saturation from the previous iteration and the capillary pressure functions. The other variables λ_w, λ_n, λ_t and f_w in the pressure and

saturation equations are also computed by the saturation at the previous iteration. These treatments can linearize the pressure and saturation equations.

In each iterative procedure, we firstly solve the pressure equation (20) to compute the wetting-phase pressure at the current iteration, and once the pressure is obtained, Darcy's velocity can be evaluated and then the saturation of the current iteration is calculated explicitly from (21). This procedure is repeated until the convergence criterion of material balance errors has been satisfied. From the procedure, we can see that iterative IMPES is a splitting approach. When computing the pressure, we do not use the information of the saturation at the current iteration; that is, the pressure is not coupled with the saturation in each iteration step.

3.2. New iterative IMPES formulation. As mentioned previously, in the classical iterative IMPES formulation, the pressure and saturation are not coupled within each iteration step. Therefore this approach weakens the inherent coupling of the system given by (18) and (19). In this work, we attempt to overcome this disadvantage and speedup the convergence of iterative IMPES.

We now derive our method. It is like iterative IMPES that our method uses the saturation from the previous iteration to compute the variables λ_w, λ_n, λ_t and f_w in the pressure and saturation equations. The capillary potentials Φ_c, however, employs the saturation at the current iteration step instead of the previous iteration, and then its nonlinearity can be reduced by a linear approximation

$$\Phi_c(S_w^{i+1,j+1}) \simeq \Phi_c(S_w^{i+1,j}) + \Phi_c'(S_w^{i+1,j})(S_w^{i+1,j+1} - S_w^{i+1,j}). \tag{22}$$

These treatments can linearize the pressure and saturation equations and keep their coupling relationship. Note that in the above approximation, the derivative of Φ_c is in fact viewed as the changing quantity of Φ_c when the saturation at each spatial point is changing in response to changes along with iterations. Therefore this approximation of capillary pressure is always well-behaved.

The relaxation approach is often applied to control the convergence of nonlinear iterative solvers. This approach is also introduced in our method. We now give the formulation of the new iterative method as

$$-\nabla\cdot\lambda_t(S_w^{i+1,j})\mathbf{K}\nabla\Phi_w^{i+1,j+1} - \nabla\cdot\lambda_n(S_w^{i+1,j})\mathbf{K}\nabla\widetilde{\Phi}_c(\widetilde{S}_w^{i+1,j+1}) = q_w^{i+1} + q_n^{i+1}, \tag{23}$$

$$\widetilde{\Phi}_c(\widetilde{S}_w^{i+1,j+1}) = \Phi_c(S_w^{i+1,j}) + \Phi_c'(S_w^{i+1,j})(\widetilde{S}_w^{i+1,j+1} - S_w^{i+1,j}), \tag{24}$$

$$\phi\frac{\widetilde{S}_w^{i+1,j+1} - S_w^i}{\Delta t^i} - q_w^{i+1} = -\nabla\cdot(f_w^{i+1,j}\mathbf{u}_a^{i+1,j+1}) \equiv \nabla\cdot\lambda_w(S_w^{i+1,j})\mathbf{K}\nabla\Phi_w^{i+1,j+1}, \tag{25}$$

and

$$S_w^{i+1,j+1} = S_w^{i+1,j} + \theta(\widetilde{S}_w^{i+1,j+1} - S_w^{i+1,j}), \tag{26}$$

where $\theta \in (0,1]$ is a relaxation factor. The choice of relaxation factor will be discussed in Section 5.

3.3. Spatial discretization. In this work, the cell-centered finite difference (CCFD) method [33,49,54] is employed for the spatial discretization, although our approach can be also extended to the systems arisen from the other spatial discretization schemes.

Applying CCFD scheme to (18), we obtain the discretization of the fully implicit pressure equation given by

$$\mathbf{A}_a(\mathbf{S}_w^{i+1})\Phi_w^{i+1} + \mathbf{A}_c(\mathbf{S}_w^{i+1})\Phi_c(\mathbf{S}_w^{i+1}) = \mathbf{Q}_{ac}^{i+1}. \tag{27}$$

We apply the upwind CCFD method for the spatial discretization of the fully implicit saturation equation

$$\phi\frac{S_w^{i+1}-S_w^i}{\Delta t^i}+\nabla\cdot(f_w^{i+1}\mathbf{u}_a^{i+1})=q_w^{i+1}, \tag{28}$$

and obtain that

$$\mathbf{M}\frac{\mathbf{S}_w^{i+1}-\mathbf{S}_w^i}{\Delta t^i}+\mathbf{A}_s(\mathbf{S}_w^{i+1},\Phi_w^{i+1})\mathbf{f}_w(\mathbf{S}_w^{i+1})=\mathbf{Q}_s^{i+1}. \tag{29}$$

Note that $\mathbf{A}_s$ depends on $\mathbf{S}_w$ and Φ_w by the definition of Darcy's velocity.

The discretization form of classical iterative IMPES is written as

$$\mathbf{A}_a(\mathbf{S}_w^{i+1,j})\Phi_w^{i+1,j+1}+\mathbf{A}_c(\mathbf{S}_w^{i+1,j})\Phi_c(\mathbf{S}_w^{i+1,j})=\mathbf{Q}_{ac}^{i+1}. \tag{30}$$

and

$$\mathbf{M}\frac{\mathbf{S}_w^{i+1,j+1}-\mathbf{S}_w^i}{\Delta t^i}+\mathbf{A}_s(\mathbf{S}_w^{i+1,j},\Phi_w^{i+1,j+1})\mathbf{f}_w(\mathbf{S}_w^{i+1,j})=\mathbf{Q}_s^{i+1}. \tag{31}$$

We now describe the formulation of our method. The discretization form of pressure equation in the new method is given by

$$\mathbf{A}_a(\mathbf{S}_w^{i+1,j})\Phi_w^{i+1,j+1}+\mathbf{A}_c(\mathbf{S}_w^{i+1,j})\widetilde{\Phi}_c(\widetilde{\mathbf{S}}_w^{i+1,j+1})=\mathbf{Q}_{ac}^{i+1}, \tag{32}$$

where $\widetilde{\Phi}_c(\widetilde{\mathbf{S}}_w^{i+1,j+1})$ is the approximation of capillary pressure $\Phi_c(\mathbf{S}_w^{i+1,j+1})$ at $\mathbf{S}_w^{i+1,j}$, i.e.

$$\widetilde{\Phi}_c(\widetilde{\mathbf{S}}_w^{i+1,j+1})=\Phi_c(\mathbf{S}_w^{i+1,j})+\Phi_c'(\mathbf{S}_w^{i+1,j})(\widetilde{\mathbf{S}}_w^{i+1,j+1}-\mathbf{S}_w^{i+1,j}). \tag{33}$$

Here, $\Phi_c'(\mathbf{S}_w)=\mathbf{diag}(\Phi_c'(S_{w,k})), k=1,2,\cdots,N_c$, and N_c is the total number of all cells. The relation between the wetting-phase saturation and pressure is described by the saturation equation

$$\phi\frac{\widetilde{S}_w^{i+1,j+1}-S_w^i}{\Delta t^i}-\nabla\cdot\lambda_w(S_w^{i+1,j})\mathbf{K}\nabla\Phi_w^{i+1,j+1}=q_w^{i+1}, \tag{34}$$

which may be approximated by CCFD method as

$$\mathbf{M}\frac{\widetilde{\mathbf{S}}_w^{i+1,j+1}-\mathbf{S}_w^i}{\Delta t^i}+\mathbf{A}_w(\mathbf{S}_w^{i+1,j})\Phi_w^{i+1,j+1}=\mathbf{Q}_w^{i+1}. \tag{35}$$

Note that this form of the saturation equation will be coupled into the pressure equation, but not to be used to update the wetting-phase saturation.

Substituting (33) and (35) into (32), we obtain the coupled pressure equation

$$\mathbf{A}_t(\mathbf{S}_w^{i+1,j})\Phi_w^{i+1,j+1}=\mathbf{Q}_t(\mathbf{S}_w^{i+1,j}), \tag{36}$$

where

$$\mathbf{A}_t(\mathbf{S}_w^{i+1,j})=\mathbf{A}_a(\mathbf{S}_w^{i+1,j})-\Delta t^i\mathbf{A}_c(\mathbf{S}_w^{i+1,j})\Phi_c'(\mathbf{S}_w^{i+1,j})\mathbf{M}^{-1}\mathbf{A}_w(\mathbf{S}_w^{i+1,j}), \tag{37}$$

and

$$\begin{aligned}\mathbf{Q}_t(\mathbf{S}_w^{i+1,j}) &= \mathbf{Q}_{ac}^{i+1}-\mathbf{A}_c(\mathbf{S}_w^{i+1,j})\Big(\Phi_c(\mathbf{S}_w^{i+1,j})+\Phi_c'(\mathbf{S}_w^{i+1,j})(\mathbf{S}_w^i-\mathbf{S}_w^{i+1,j})\Big)\\ &\quad -\Delta t^i\mathbf{A}_c(\mathbf{S}_w^{i+1,j})\Phi_c'(\mathbf{S}_w^{i+1,j})\mathbf{M}^{-1}\mathbf{Q}_w^{i+1}.\end{aligned} \tag{38}$$

Note that in the above processes, $\mathbf{M}$ is a diagonal matrix arisen from the porosity, and hence its inverse is not expensive.

In order to update the saturation, we consider the following form of saturation equation

$$\phi\frac{\widetilde{S}_w^{i+1,j+1}-S_w^i}{\Delta t^i}+\nabla\cdot(f_w^{i+1,j}\mathbf{u}_a^{i+1,j+1})=q_w^{i+1}. \tag{39}$$

We apply the upwind CCFD method for the spatial discretization of the saturation equation (39) and obtain the matrix-vector form

$$\mathbf{M}\frac{\widetilde{\mathbf{S}}_w^{i+1,j+1}-\mathbf{S}_w^i}{\Delta t^i}+\mathbf{A}_s(\mathbf{S}_w^{i+1,j},\Phi_w^{i+1,j+1})\mathbf{f}_w(\mathbf{S}_w^{i+1,j})=\mathbf{Q}_s^{i+1}. \tag{40}$$

Each iteration step of our method consists of three substeps. In the first substep of each iteration, we solve the linear system (36), along with (37) and (38), to obtain the pressure $\Phi_w^{i+1,j+1}$ and then compute the the velocity $\mathbf{u}_a^{i+1,j+1}$ by Darcy's Law. The pressure obtained from (36) contains the change of saturation by the coupling of pressure and saturation. In the second substep, we compute the saturations by (40), but need not to solve any linear system. In the third substep, the updated saturation at each iteration is obtained from (26), which requires a suitable relaxation factor. From computational procedure, we only solve one linear system with the same size to the classical iterative IMPES at each iteration, and the computational cost and size are much less than the simultaneous methods.

4. Convergence analysis

In this section, we consider the convergence of our new method based on the system arisen after the cell-centered finite difference scheme for spatial discretization, but it may be also extended to the other discretization schemes.

Let $M_m = \|\mathbf{M}^{-1}\|$. From the physical property, each element of saturation vector $\mathbf{S}_w$ lies in $[0,1]$. The terms $\mathbf{Q}_w, \mathbf{Q}_{ac}$ and $\mathbf{Q}_s$ are naturally bounded, which are arisen from the boundary conditions and the injection/extraction to the interior of the domain. The absolute permeabilities have upper and lower bounds. The equations we consider are all nonlinear, and therefore we assume that the following conditions hold.

(A1) The matrix function $\mathbf{A}_a(\mathbf{S}_w)$ is invertible for any $\mathbf{S}_w$, and its inverse is bounded

$$\left\|\mathbf{A}_a(\mathbf{S}_w)^{-1}\right\| \le M_a, \tag{41}$$

where $M_a > 0$.

(A2) The matrix functions $\mathbf{A}_a(\mathbf{S}_w)$, $\mathbf{A}_w(\mathbf{S}_w)$ and $\mathbf{A}_w(\mathbf{S}_w)$ are Lipschitz continuous; that is, there exist three positive real numbers C_a, C_w and C_c such that for any $\mathbf{S}_w, \overline{\mathbf{S}}_w$, we have

$$\left\|\mathbf{A}_a(\mathbf{S}_w)-\mathbf{A}_a(\overline{\mathbf{S}}_w)\right\| \le C_a\left\|\mathbf{S}_w-\overline{\mathbf{S}}_w\right\|, \tag{42}$$

$$\left\|\mathbf{A}_w(\mathbf{S}_w)-\mathbf{A}_w(\overline{\mathbf{S}}_w)\right\| \le C_w\left\|\mathbf{S}_w-\overline{\mathbf{S}}_w\right\|, \tag{43}$$

$$\left\|\mathbf{A}_c(\mathbf{S}_w)-\mathbf{A}_c(\overline{\mathbf{S}}_w)\right\| \le C_c\left\|\mathbf{S}_w-\overline{\mathbf{S}}_w\right\|. \tag{44}$$

(A3) The matrix function $\mathbf{A}_s(\mathbf{S}_w,\Phi_w)$ and the vector function $\mathbf{f}_w$ are Lipschitz continuous, i.e. for any saturations $\mathbf{S}_w, \overline{\mathbf{S}}_w$ and any pressures $\Phi_w, \overline{\Phi}_w$,

$$\left\|\mathbf{A}_s(\mathbf{S}_w,\Phi_w)-\mathbf{A}_s(\overline{\mathbf{S}}_w,\Phi_w)\right\| \le C_s\left\|\mathbf{S}_w-\overline{\mathbf{S}}_w\right\|, \tag{45}$$

$$\left\|\mathbf{A}_s(\mathbf{S}_w,\Phi_w)-\mathbf{A}_s(\mathbf{S}_w,\overline{\Phi}_w)\right\| \le C_p\left\|\Phi_w-\overline{\Phi}_w\right\|, \tag{46}$$

$$\|\mathbf{f}_w(\mathbf{S}_w)-\mathbf{f}_w(\overline{\mathbf{S}}_w)\| \le C_f\|\mathbf{S}_w-\overline{\mathbf{S}}_w\|, \tag{47}$$

where C_s, C_p, C_f are positive real numbers.

(A4) The capillary potential is bounded for $S_w \in [0,1]$

$$\|\Phi_c(S_w)\| \le M_{pc}. \tag{48}$$

Moreover, its derivative is bounded and Lipschitz continuous, i.e.,

$$\|\Phi_c'(S_w)\| \leq N_{pc}, \tag{49}$$

$$\left\|\Phi_c'(S_w) - \Phi_c'(\overline{S}_w)\right\| \leq C_{pc}\left\|S_w - \overline{S}_w\right\|, \ S_w, \overline{S}_w \in [0,1]. \tag{50}$$

We now give some remarks on the above assumptions. It is obtained by physical property that for any $S_w \in [0,1]$, the total motion λ_t has the positive supremum and infimum, and the functions $\lambda_t, \lambda_w, \lambda_n$ are very smooth in general. Consequently, the assumptions (A1)-(A3) are not restricted, and in fact, they are natural. As mentioned in Section 2, if the minimum S_{emin} is set for S_e, the capillary pressure function is sufficiently smooth and regular, and hence it satisfies assumption (A4).

From the above assumptions, we have the following lemmas.

Lemma 1. *The matrices $\boldsymbol{A}_w(\boldsymbol{S}_w)$ and $\boldsymbol{A}_c(\boldsymbol{S}_w)$ are bounded for any saturation vector $\boldsymbol{S}_w$, i.e.*

$$\|\boldsymbol{A}_w(\boldsymbol{S}_w)\| \leq M_w, \tag{51}$$

$$\|\boldsymbol{A}_c(\boldsymbol{S}_w)\| \leq M_c, \tag{52}$$

where M_w and M_c are all positive real numbers. For any $\boldsymbol{S}_w$, the matrix $\boldsymbol{A}_t(\boldsymbol{S}_w)$ is invertible if $\Delta t^i < 1/(M_a M_w M_c M_m N_{pc})$, and in this case, we have

$$\left\|\boldsymbol{A}_t(\boldsymbol{S}_w)^{-1}\right\| \leq \frac{M_a}{1 - \Delta t^i M_a M_w M_c M_m N_{pc}}. \tag{53}$$

Proof. By the assumption (A2), it follows that

$$\begin{aligned} \|\mathbf{A}_w(\mathbf{S}_w)\| &\leq \|\mathbf{A}_w(0)\| + \|\mathbf{A}_w(\mathbf{S}_w) - \mathbf{A}_w(0)\| \\ &\leq \|\mathbf{A}_w(0)\| + C_w\|\mathbf{S}_w\| \\ &\leq \|\mathbf{A}_w(0)\| + C_w, \end{aligned} \tag{54}$$

Letting $M_w = \|\mathbf{A}_w(0)\| + C_w$, we obtain (51). In the same way, we can get (52).

If $\Delta t^i < 1/(M_a M_w M_c M_m N_{pc})$, then we have $\|\mathbf{A}_t(\mathbf{S}_w) - \mathbf{A}_a(\mathbf{S}_w)\|\|\mathbf{A}_a(\mathbf{S}_w)^{-1}\| < 1$, and it follows by Banach Lemma that $\mathbf{A}_t(\mathbf{S}_w)$ is invertible and satisfies (53). □

Lemma 2. *$\boldsymbol{A}_t(\boldsymbol{S}_w)$ and $\boldsymbol{Q}_t(\boldsymbol{S}_w)$ are Lipschitz continuous, i.e. for any $\boldsymbol{S}_w, \overline{\boldsymbol{S}}_w$,*

$$\left\|\boldsymbol{A}_t(\boldsymbol{S}_w) - \boldsymbol{A}_t(\overline{\boldsymbol{S}}_w)\right\| \leq (C_a + \Delta t^i C_t)\|\boldsymbol{S}_w - \overline{\boldsymbol{S}}_w\|, \tag{55}$$

$$\|\boldsymbol{Q}_t(\boldsymbol{S}_w) - \boldsymbol{Q}_t(\overline{\boldsymbol{S}}_w)\| \leq (C_{q1} + \Delta t^i C_{q1})\|\boldsymbol{S}_w - \overline{\boldsymbol{S}}_w\|, \tag{56}$$

where C_t, C_{q1} and C_{q2} are all positive real numbers.

Proof. By the assumptions (A2), (A4) and Lemma 1, we have

$$\begin{aligned}
&\left\|\mathbf{A}_c(\mathbf{S}_w)\Phi_c'(\mathbf{S}_w)\mathbf{M}^{-1}\mathbf{A}_w(\mathbf{S}_w) - \mathbf{A}_c(\overline{\mathbf{S}}_w)\Phi_c'(\overline{\mathbf{S}}_w)\mathbf{M}^{-1}\mathbf{A}_w(\overline{\mathbf{S}}_w)\right\| \\
\leq\ & \left\|\mathbf{A}_c(\mathbf{S}_w) - \mathbf{A}_c(\overline{\mathbf{S}}_w)\right\| \|\Phi_c'(\mathbf{S}_w)\mathbf{M}^{-1}\mathbf{A}_w(\mathbf{S}_w)\| \\
&+ \left\|\mathbf{A}_c(\overline{\mathbf{S}}_w)\right\| \|\Phi_c'(\mathbf{S}_w)\mathbf{M}^{-1}\mathbf{A}_w(\mathbf{S}_w) - \Phi_c'(\overline{\mathbf{S}}_w)\mathbf{M}^{-1}\mathbf{A}_w(\overline{\mathbf{S}}_w)\| \\
\leq\ & M_w M_m N_{pc} C_c \|\mathbf{S}_w - \overline{\mathbf{S}}_w\| + M_c\|\Phi_c'(\mathbf{S}_w)\mathbf{M}^{-1}\mathbf{A}_w(\mathbf{S}_w) \\
&-\Phi_c'(\overline{\mathbf{S}}_w)\mathbf{M}^{-1}\mathbf{A}_w(\overline{\mathbf{S}}_w)\| \\
\leq\ & M_w M_m N_{pc} C_c \|\mathbf{S}_w - \overline{\mathbf{S}}_w\| + M_c\Big[\|\Phi_c'(\mathbf{S}_w) - \Phi_c'(\overline{\mathbf{S}}_w)\|\|\mathbf{M}^{-1}\mathbf{A}_w(\mathbf{S}_w)\| \\
&+\|\Phi_c'(\overline{\mathbf{S}}_w)\|\|\mathbf{M}^{-1}\|\|\mathbf{A}_w(\mathbf{S}_w) - \mathbf{A}_w(\overline{\mathbf{S}}_w)\|\Big] \\
\leq\ & M_w M_m (N_{pc}C_c + M_c C_{pc})\|\mathbf{S}_w - \overline{\mathbf{S}}_w\| \\
&+M_m M_c N_{pc}\|\mathbf{A}_w(\mathbf{S}_w) - \mathbf{A}_w(\overline{\mathbf{S}}_w)\| \\
\leq\ & [M_w M_m (N_{pc}C_c + M_c C_{pc}) + M_m M_c N_{pc} C_w]\|\mathbf{S}_w - \overline{\mathbf{S}}_w\|. \qquad (57)
\end{aligned}$$

We obtain (55) by substituting (57) into

$$\begin{aligned}
\left\|\mathbf{A}_t(\mathbf{S}_w) - \mathbf{A}_t(\overline{\mathbf{S}}_w)\right\| \leq\ & \left\|\mathbf{A}_a(\mathbf{S}_w) - \mathbf{A}_a(\overline{\mathbf{S}}_w)\right\| + \Delta t^i \|\mathbf{A}_c(\mathbf{S}_w)\Phi_c'(\mathbf{S}_w)\mathbf{M}^{-1}\mathbf{A}_w(\mathbf{S}_w) \\
&-\mathbf{A}_c(\overline{\mathbf{S}}_w)\Phi_c'(\overline{\mathbf{S}}_w)\mathbf{M}^{-1}\mathbf{A}_w(\overline{\mathbf{S}}_w)\|, \qquad (58)
\end{aligned}$$

and letting $C_t = M_w M_m (N_{pc}C_c + M_c C_{pc}) + M_m M_c N_{pc} C_w$.

It is obtained from (50) that [57]

$$\begin{aligned}
&\|\Phi_c(\mathbf{S}_w) - \Phi_c(\overline{\mathbf{S}}_w) - \Phi_c'(\overline{\mathbf{S}}_w)(\mathbf{S}_w - \overline{\mathbf{S}}_w)\| \\
\leq\ & \frac{1}{2}C_{pc}\|\mathbf{S}_w - \overline{\mathbf{S}}_w\|^2 \leq \frac{1}{2}C_{pc}\|\mathbf{S}_w - \overline{\mathbf{S}}_w\|. \qquad (59)
\end{aligned}$$

Therefore, it follows from the assumptions (A2), (A4) and Lemma 1 that

$$\begin{aligned}
&\|\mathbf{Q}_t(\mathbf{S}_w) - \mathbf{Q}_t(\overline{\mathbf{S}}_w)\| \\
=\ & \Big\|\mathbf{A}_c(\mathbf{S}_w)\Big(\Phi_c(\mathbf{S}_w) + \Phi_c'(\mathbf{S}_w)(\mathbf{S}_w^i - \mathbf{S}_w) + \Delta t^i \Phi_c'(\mathbf{S}_w)\mathbf{M}^{-1}\mathbf{Q}_w^{i+1}\Big) \\
&-\mathbf{A}_c(\overline{\mathbf{S}}_w)\Big(\Phi_c(\overline{\mathbf{S}}_w) + \Phi_c'(\overline{\mathbf{S}}_w)(\mathbf{S}_w^i - \overline{\mathbf{S}}_w) + \Delta t^i \Phi_c'(\overline{\mathbf{S}}_w)\mathbf{M}^{-1}\mathbf{Q}_w^{i+1}\Big)\Big\| \\
\leq\ & \|\mathbf{A}_c(\mathbf{S}_w) - \mathbf{A}_c(\overline{\mathbf{S}}_w)\|\|\Phi_c(\mathbf{S}_w) + \Phi_c'(\mathbf{S}_w)(\mathbf{S}_w^i - \mathbf{S}_w) + \Delta t^i \Phi_c'(\mathbf{S}_w)\mathbf{M}^{-1}\mathbf{Q}_w^{i+1}\| \\
&+\|\mathbf{A}_c(\overline{\mathbf{S}}_w)\|\Big[\|\Phi_c(\mathbf{S}_w) - \Phi_c(\overline{\mathbf{S}}_w) - \Phi_c'(\overline{\mathbf{S}}_w)(\mathbf{S}_w - \overline{\mathbf{S}}_w)\| \\
&+\|\Phi_c'(\mathbf{S}_w) - \Phi_c'(\overline{\mathbf{S}}_w)\|(\|\mathbf{S}_w^i - \mathbf{S}_w\| + \Delta t^i\|\mathbf{M}^{-1}\mathbf{Q}_w^{i+1}\|)\Big] \\
\leq\ & (C_{q1} + \Delta t^i C_{q1})\|\mathbf{S}_w - \overline{\mathbf{S}}_w\|,
\end{aligned}$$

where $C_{q1} = C_c(M_{pc} + N_{pc}) + \frac{3}{2}M_c C_{pc}$ and $C_{q2} \geq (C_c N_{pc} + M_c C_{pc})M_m\|\mathbf{Q}_w^{i+1}\|$. □

Lemma 3. *The pressure potential Φ_w and the matrix $\boldsymbol{A}_s$ at the time step $(i+1)$ are all bounded, i.e.*

$$\|\Phi_w^{i+1}\| \leq M_{pw}, \qquad (60)$$

$$\|\boldsymbol{A}_s(\boldsymbol{S}_w^{i+1}, \Phi_w^{i+1})\| \leq M_s, \qquad (61)$$

where $M_{pw} > 0, M_s > 0$.

Proof. It is obtained by (A1), (A4), (27) and Lemma 1 that

$$\begin{aligned}
\|\Phi_w^{i+1}\| \leq\ & \left\|\mathbf{A}_a(\mathbf{S}_w^{i+1})^{-1}\right\|\left[\|\mathbf{A}_c(\mathbf{S}_w^{i+1})\Phi_c(\mathbf{S}_w^{i+1})\| + \|\mathbf{Q}_{ac}^{i+1}\|\right] \\
\leq\ & M_a(M_c M_{pc} + \|\mathbf{Q}_{ac}^{i+1}\|). \qquad (62)
\end{aligned}$$

Letting $M_{pw} \geq M_a(M_c M_{pc} + \|\mathbf{Q}_{ac}^{i+1}\|)$, we obtain (60). Moreover, we have (61) from the Lipschitz continuation. □

Based on the above lemmas, we can prove the following convergence theorem.

Theorem 1. *Assume that the conditions* A(1)-A(4) *hold. Suppose that for any time step* $(i+1)$ *(*$i \geq 0$*), there exist a solution of saturation* $\boldsymbol{S}_w^{i+1}$ *and pressure potential* Φ_w^{i+1} *such that the pressure and saturation equations hold. If the saturation of previous time step* $\boldsymbol{S}_w^i$ *is chosen as the initial approximation* $\boldsymbol{S}_w^{i+1,0}$ *of the exact saturation* $\boldsymbol{S}_w^{i+1}$*, then there exists a suitable time step size* Δt^i *such that the sequences* $\boldsymbol{S}_w^{i+1,j}$ *and* $\Phi_w^{i+1,j}$ *generated by the new method converges to* $\boldsymbol{S}_w^{i+1}$ *and* Φ_w^{i+1}*, respectively.*

Proof. We now choose the saturations of previous time step $\mathbf{S}_w^i$ as the initial approximation $\mathbf{S}_w^{i+1,0}$ of the exact saturations $\mathbf{S}_w^{i+1}$. Since the values of the saturation lie in $[0,1]$, we have

$$\|\mathbf{S}_w^{i+1,0} - \mathbf{S}_w^{i+1}\| = \|\mathbf{S}_w^i - \mathbf{S}_w^{i+1}\| \leq 1. \tag{63}$$

Let

$$\gamma = 1 - \theta + \theta \Delta t^i M_m \left[M_s C_f + C_s + \frac{C_p M_a \big[M_{pw} C_a + C_{q1} + \Delta t^i (M_{pw} C_t + C_{q2}) \big]}{1 - \Delta t^i M_a M_w M_c M_m N_{pc}} \right].$$

For any given $\theta \in (0,1]$, it is easy to obtain that $0 \leq \gamma < 1$ as $\Delta t^i = 0$, and γ is increasing with respect to $\Delta t^i \in (0, 1/(M_a M_w M_c M_m N_{pc}))$. Therefore there exists at least a time step size $\Delta t^i > 0$ such that $0 \leq \gamma < 1$. Now we take such a time step size $\Delta t^i > 0$ such that $0 < \gamma < 1$.

In the following, we will prove that

$$\big\|\mathbf{S}_w^{i+1,j} - \mathbf{S}_w^{i+1}\big\| \leq \gamma^j \|\mathbf{S}_w^i - \mathbf{S}_w^{i+1}\| \leq 1, \ j \geq 0. \tag{64}$$

The case of $j = 0$ is justified by the choice of initial approximation of saturation. We suppose that (64) holds for $j \geq 0$, and now we prove the case $j+1$.

Apparently, the exact saturation $\mathbf{S}_w^{i+1}$ and pressure potential Φ_w^{i+1} satisfy

$$\mathbf{A}_t(\mathbf{S}_w^{i+1}) \Phi_w^{i+1} = \mathbf{Q}_t(\mathbf{S}_w^{i+1}). \tag{65}$$

Subtracting (65) from (36), we obtain

$$\mathbf{A}_t(\mathbf{S}_w^{i+1,j}) \Phi_w^{i+1,j+1} - \mathbf{A}_t(\mathbf{S}_w^{i+1}) \Phi_w^{i+1} = \mathbf{Q}_t(\mathbf{S}_w^{i+1,j}) - \mathbf{Q}_t(\mathbf{S}_w^{i+1}), \tag{66}$$

which is equivalent to

$$\begin{aligned} \mathbf{A}_t(\mathbf{S}_w^{i+1,j})(\Phi_w^{i+1,j+1} - \Phi_w^{i+1}) &= (\mathbf{A}_t(\mathbf{S}_w^{i+1}) - \mathbf{A}_t(\mathbf{S}_w^{i+1,j})) \Phi_w^{i+1} \\ &\quad + \mathbf{Q}_t(\mathbf{S}_w^{i+1,j}) - \mathbf{Q}_t(\mathbf{S}_w^{i+1}). \end{aligned} \tag{67}$$

By the assumptions and Lemmas 1-3, we have

$$\begin{aligned} & \big\|\Phi_w^{i+1,j+1} - \Phi_w^{i+1}\big\| \\ \leq\ & \|\mathbf{A}_t(\mathbf{S}_w^{i+1,j})^{-1}\| \Big[\|\mathbf{A}_t(\mathbf{S}_w^{i+1}) - \mathbf{A}_t(\mathbf{S}_w^{i+1,j})\| \|\Phi_w^{i+1}\| \\ & + \|\mathbf{Q}_t(\mathbf{S}_w^{i+1,j}) - \mathbf{Q}_t(\mathbf{S}_w^{i+1})\| \Big] \\ \leq\ & \frac{M_a \big[M_{pw} C_a + C_{q1} + \Delta t^i (M_{pw} C_t + C_{q2}) \big]}{1 - \Delta t^i M_a M_w M_c M_m N_{pc}} \|\mathbf{S}_w^{i+1,j} - \mathbf{S}_w^{i+1}\|. \end{aligned} \tag{68}$$

On the other hand, in the same way to obtain (66), we get

$$\begin{aligned}\mathbf{M}\frac{\widetilde{\mathbf{S}}_w^{i+1,j+1}-\mathbf{S}_w^{i+1}}{\Delta t^i} &= \mathbf{A}_s(\mathbf{S}_w^{i+1},\Phi_w^{i+1})\mathbf{f}_w(\mathbf{S}_w^{i+1})\\ &\quad -\mathbf{A}_s(\mathbf{S}_w^{i+1,j},\Phi_w^{i+1,j+1})\mathbf{f}_w(\mathbf{S}_w^{i+1,j}),\end{aligned}\tag{69}$$

and then it follows that

$$\begin{aligned}&\left\|\widetilde{\mathbf{S}}_w^{i+1,j+1}-\mathbf{S}_w^{i+1}\right\|\\ \le\ & \Delta t^i M_m\left\|\mathbf{A}_s(\mathbf{S}_w^{i+1},\Phi_w^{i+1})\mathbf{f}_w(\mathbf{S}_w^{i+1})-\mathbf{A}_s(\mathbf{S}_w^{i+1,j},\Phi_w^{i+1,j+1})\mathbf{f}_w(\mathbf{S}_w^{i+1,j})\right\|\\ \le\ & \Delta t^i M_m\Big[\|\mathbf{A}_s(\mathbf{S}_w^{i+1},\Phi_w^{i+1})\|\|\mathbf{f}_w(\mathbf{S}_w^{i+1})-\mathbf{f}_w(\mathbf{S}_w^{i+1,j})\|\\ &+\|\mathbf{A}_s(\mathbf{S}_w^{i+1},\Phi_w^{i+1})-\mathbf{A}_s(\mathbf{S}_w^{i+1,j},\Phi_w^{i+1,j+1})\|\|\mathbf{f}_w(\mathbf{S}_w^{i+1,j})\|\Big]\\ \le\ & \Delta t^i M_m\Big[(M_sC_f+C_s)\|\mathbf{S}_w^{i+1,j}-\mathbf{S}_w^{i+1}\|+C_p\left\|\Phi_w^{i+1,j+1}-\Phi_w^{i+1}\right\|\Big].\end{aligned}\tag{70}$$

From (68) and (70), we obtain

$$\begin{aligned}\left\|\widetilde{\mathbf{S}}_w^{i+1,j+1}-\mathbf{S}_w^{i+1}\right\| &\le \Delta t^i M_m\|\mathbf{S}_w^{i+1,j}-\mathbf{S}_w^{i+1}\|\Big[M_sC_f+C_s\\ &\quad +\frac{C_pM_a\big[M_{pw}C_a+C_{q1}+\Delta t^i(M_{pw}C_t+C_{q2})\big]}{1-\Delta t^iM_aM_wM_cM_mN_{pc}}\Big].\end{aligned}\tag{71}$$

It follows that

$$\begin{aligned}\left\|\mathbf{S}_w^{i+1,j+1}-\mathbf{S}_w^{i+1}\right\| &\le (1-\theta)\left\|\mathbf{S}_w^{i+1,j}-\mathbf{S}_w^{i+1}\right\|+\theta\left\|\widetilde{\mathbf{S}}_w^{i+1,j+1}-\mathbf{S}_w^{i+1}\right\|\\ &\le \gamma\|\mathbf{S}_w^{i+1,j}-\mathbf{S}_w^{i+1}\|.\end{aligned}\tag{72}$$

Because of the supposition that (64) holds for the case j, we further obtain

$$\left\|\mathbf{S}_w^{i+1,j+1}-\mathbf{S}_w^{i+1}\right\|\le\gamma^{j+1}\|\mathbf{S}_w^{i}-\mathbf{S}_w^{i+1}\|\le 1.\tag{73}$$

From this, it is induced that the inequality (64) is true for any $j\ge 0$.

Therefore it is concluded by $\gamma<1$ that as $j\to\infty$, the sequence $\mathbf{S}_w^{i+1,j}$ converges to $\mathbf{S}_w^{i+1}$, and by (68), $\Phi_w^{i+1,j}$ converges to Φ_w^{i+1}. □

5. Choice of relaxation factor

In practical computation, we usually make the relaxation factor large to speedup the convergence when the approximate solutions are sufficiently close to the exact solutions, and otherwise, we need a small relaxation factor to guarantee that the iterative process can be convergent. However, it is difficult to choose a best relaxation factor for a given problem. In the following, we will introduce a practical choice strategy for relaxation factor.

Define the residual of the wetting-phase mass conservative equation by

$$\begin{aligned}\left\|\mathbf{R}_w^{i+1,j+1}\right\| &= \Big\|\mathbf{S}_w^{i+1,j}-\mathbf{S}_w^{i}\\ &\quad +\Delta t^i\mathbf{M}^{-1}\Big[\mathbf{A}_s(\mathbf{S}_w^{i+1,j},\Phi_w^{i+1,j+1})\mathbf{f}_w(\mathbf{S}_w^{i+1,j})-\mathbf{Q}_s^{i+1}\Big]\Big\|,\end{aligned}\tag{74}$$

where $j\ge 0$. In order to guarantee the convergence of the iterative process, the relaxation factor θ should be chosen such that the residual defined by (74) is decreasing with respect to iterations, i.e.

$$\left\|\mathbf{R}_w^{i+1,j+1}\right\|<\left\|\mathbf{R}_w^{i+1,j}\right\|.\tag{75}$$

It is obtained from (40) and (74) that

$$\left\|\mathbf{R}_w^{i+1,j+1}\right\| = \left\|\widetilde{\mathbf{S}}_w^{i+1,j+1} - \mathbf{S}_w^{i+1,j}\right\| = \frac{1}{\theta}\left\|\mathbf{S}_w^{i+1,j+1} - \mathbf{S}_w^{i+1,j}\right\|,\ j \geq 0. \tag{76}$$

Therefore, from (75) and (76), we need to choose a suitable relaxation such that

$$\left\|\mathbf{S}_w^{i+1,j+1} - \mathbf{S}_w^{i+1,j}\right\| \leq \rho\|\mathbf{S}_w^{i+1,j} - \mathbf{S}_w^{i+1,j-1}\|,\ j \geq 1, \tag{77}$$

where $\rho \in (0,1)$.

Theoretically, it is a nonlinear problem to find such relaxation factor, and hence it is practically feasible to compute the relaxation factor adaptively in the iterative process. We denote the relaxation factor at jth iteration step by θ_j, and in this case, we have

$$\mathbf{S}_w^{i+1,j+1} = \mathbf{S}_w^{i+1,j} + \theta_{j+1}\left(\widetilde{\mathbf{S}}_w^{i+1,j+1} - \mathbf{S}_w^{i+1,j}\right). \tag{78}$$

For the first iteration, θ_1 needs to be given in advance. We can determine θ_{j+1}, $j \geq 1$, by

$$\max\left\{\rho\frac{\left\|\mathbf{S}_w^{i+1,j} - \mathbf{S}_w^{i+1,j-1}\right\|}{\left\|\widetilde{\mathbf{S}}_w^{i+1,j+1} - \mathbf{S}_w^{i+1,j}\right\|}, \theta_{min}\right\} \leq \theta_{j+1} \leq \min\left\{\rho\frac{\left\|\mathbf{S}_w^{i+1,j} - \mathbf{S}_w^{i+1,j-1}\right\|}{\left\|\widetilde{\mathbf{S}}_w^{i+1,j+1} - \mathbf{S}_w^{i+1,j}\right\|}, \theta_{max}\right\}, \tag{79}$$

where $\theta_{min}, \theta_{max} \in (0,1]$ are the minimum and maximum of relaxation factor θ set in advance, respectively.

Finally, we give a theoretical remark on the above choice of relaxation factor. If (77) holds, then we can obtain that

$$\left\|\mathbf{S}_w^{i+1,j+1} - \mathbf{S}_w^{i+1,j}\right\| \leq \rho^j\|\mathbf{S}_w^{i+1,1} - \mathbf{S}_w^{i}\|,\ j \geq 1. \tag{80}$$

It follows that for any $n, m > 1$,

$$\begin{aligned}
\left\|\mathbf{S}_w^{i+1,n+m} - \mathbf{S}_w^{i+1,n}\right\| &\leq \sum_{j=n}^{n+m-1}\left\|\mathbf{S}_w^{i+1,j+1} - \mathbf{S}_w^{i+1,j}\right\| \\
&\leq \|\mathbf{S}_w^{i+1,1} - \mathbf{S}_w^{i}\|\sum_{j=n}^{n+m-1}\rho^j, \\
&\leq \frac{\rho^n}{1-\rho}\|\mathbf{S}_w^{i+1,1} - \mathbf{S}_w^{i}\|. \qquad (81)
\end{aligned}$$

This indicates that the sequence $\{\mathbf{S}_w^{i+1,j}\}$ is a Cauchy sequence and hence it will converges to $\{\mathbf{S}_w^{i+1}\}$.

6. Numerical tests

In this section, we test some examples to show the performance of the new method presented in this work. We also carry out the comparison to the classical iterative IMPES scheme.

6.1. Physical and computational data used in numerical tests. In this work, we present two numerical examples. In these examples, we consider two-phase fluid flow in a horizontal layer with heterogeneous permeabilities. With media being horizontal, the effect of gravity is neglected in all examples. The void of medium is initially fully saturated with oil and then we flood the system by water at the left end. The production end is the right-hand side. The other boundaries

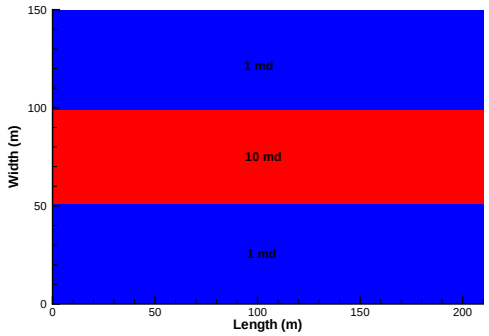

FIGURE 1. Heterogeneous permeabilities: Example 1.

are impermeable; that is, the normal component of the Darcy velocity on these boundaries vanishes. There is no injection/extraction to the interior of the domain.

We use the absolute permeability tensor as $\mathbf{K} = k\mathbf{I}$, where $\mathbf{I}$ is the identity matrix and k is a positive real number. We employ the capillary pressure function and relative permeabilities given in Section 2. It is chosen to be zero for the residual saturations of water and oil; that is, $S_e = S_w$. In computation, we take the minimum of saturation as $S_{wmin} = 10^{-4}$.

For both the classical iterative IMPES method and our new method, we choose the saturation at previous time step as the saturation approximation of the first iteration. The relaxation factor has some effects on the stability and efficiency of iterative method. The classical iterative IMPES method without relaxation factor may often require more iterations. In the following tests, the relaxation approach, along with the choice strategy of relaxation factor mentioned in Section 5, is applied for both methods. For two methods, θ_1 is computed from (79) where we take $\|\mathbf{S}_w^{i+1,0} - \mathbf{S}_w^{i+1,-1}\| = 1$. The iteration loop of both methods continues until $\|\mathbf{S}_w^{i+1,j+1} - \mathbf{S}_w^{i+1,j}\| < S_{wmin}$. In our tests, we use 2-norm for vectors and matrices.

6.2. Example 1: Regular heterogenous permeabilities. In example 1, the tested medium with the domain dimensions 210 m $\times$ 150 m $\times$ 1 m consists of three subdomains with different configurations for the distribution of permeability, which is shown in Figure 1. The porosity of this medium is 0.2. We use the quadratic relative permeabilities, namely, $\beta = 2$. The viscosities of water and oil are 1 cP and 0.35 cP, respectively. The capillary pressure parameter B_c is taken as 70 bar when the absolute permeability k is equal to 1 md. The injection rate is 0.2 PV/year.

The total number of mesh elements is 3500 uniform rectangles for example 1. For the choice of relaxation factor discussed in Section 5, we take three parameters in (79) as $\theta_{min} = 0.2, \theta_{max} = 0.9$ and $\rho = 0.5$. This choice of relaxation factor is applied for both the classical iterative IMPES method and New method.

We calculate the saturation until 0.5 pore volume injection (PVI). We test two different time step sizes. One time step size (called Case 1 here) is taken as the maximum time step size required by iterative IMPES to guarantee the convergence. The other (Case 2) is a larger time step size, in which iterative IMPES can not converge. Our new method can converge for the two time step sizes. The computational results are displayed in Table 1, in which "Total iterations" represents

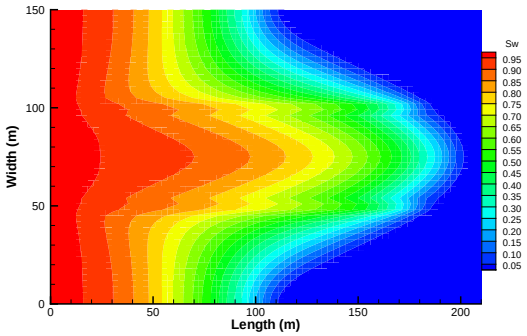

FIGURE 2. Wetting-phase saturation profiles at 0.5 PVI by iterative IMPES (Case 1): Example 1.

the sum of iterations in all time steps, and "Average iterations" represents the average iteration number in each time step. Figures 2,3 and 4 show the distributions for water saturation at $T = 0.5$ PVI computed by the iterative IMPES and New method, respectively.

Table 1. Comparison for Example 1

	Iterative IMPES (Case 1)	New (Case 1)	New (Case 2)
Time step size (days)	2.2812	2.2812	6.0833
Total iterations	17771	2504	1704
Average iterations	44.4275	6.26	11.36

In this example, the domain is highly heterogenous and hence the two different capillary pressure functions are made on the different subdomains. Capillary pressure forces may push the water flowing into the areas with the low permeability. The distributions for water saturation computed by the two methods are close and accurate, but the computational performance of new method is distinguished. We can see from Table 1 that the average iteration number of the new method is far less than that of iterative IMPES in the same time step size. Moreover, we can take a larger time step size for the new method, and in this case, the total iterations may be further reduced, although the average iterations increase. This indicates the new method has better stability and efficiency than iterative IMPES.

6.3. Example 2: Random permeabilities. In example 2, the domain dimension of tested medium is 150 m $\times$ 150 m $\times$ 1 m. This medium has random distribution of permeability as shown in Figure 5. The porosity of this medium is 0.2. The viscosities of water and oil are 1 cP and 0.3 cP, respectively. The relative permeabilities are quadratic. We take the capillary pressure parameter as $B_c = 35$ bar when the absolute permeability k is equal to 1 md. The injection rate is 0.15 PV/year.

The computational domain is divided into 2500 uniform rectangles in this example. The choice of relaxation factor given by (79) is applied for both iterative IMPES and New Method, and the three parameters are taken as $\theta_{min} = 0.2, \theta_{max} = 0.9$ and $\rho = 0.2$.

We calculate the saturation until 0.5 PVI. It is like example 1 that we take two different time step sizes; that is, Case 1 is taken as the maximum time step size required for the convergence of iterative IMPES, and Case 2 is a larger time step

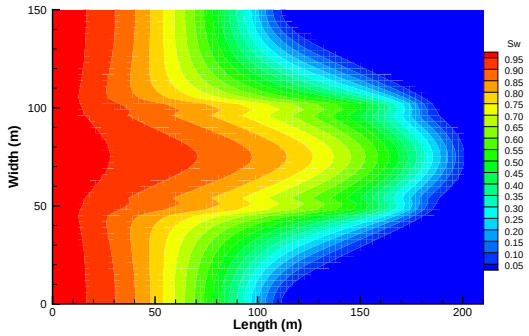

FIGURE 3. Wetting-phase saturation profiles at 0.5 PVI by New Method (Case 1): Example 1.

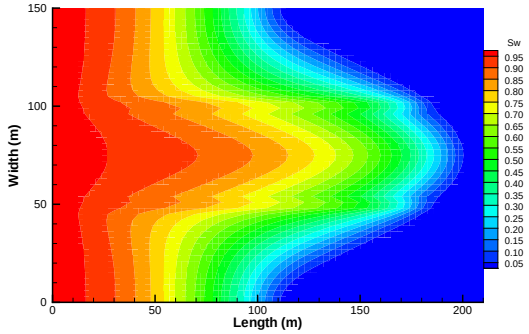

FIGURE 4. Wetting-phase saturation profiles at 0.5 PVI by New Method (Case 2): Example 1.

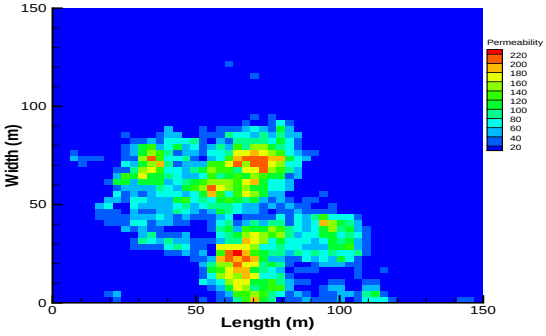

FIGURE 5. Heterogeneous permeabilities (the unit is md)): Example 2.

size. Our new method can converge for the two time step sizes, while iterative IMPES can not converge in Case 2. The computational results are displayed in Table

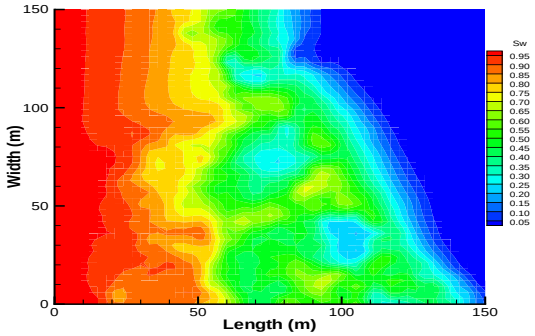

FIGURE 6. Wetting-phase saturation profiles at 0.5 PVI by iterative IMPES (Case 1): Example 2.

2, in which the representations of the row names, "Total iterations" and "Average iterations", are the same to Table 1. Figures 6,7 and 8 show the distributions for water saturation at $T = 0.5$ PVI computed by the iterative IMPES and New method, respectively.

The permeabilities of this example are random distributed, and vary in a large scope, as shown in Fig. 5. In this case, there are multiple different capillary pressure functions made on the domain. The contrast arisen from capillary pressure functions may distribute in the whole domain, and hence we can see the discontinuity of saturations almost everywhere. In this case, capillary pressure forces may have a great influence on the flow path of the above fluids. We can see from the distributions for water saturation that the two methods can simulate two-phase flow well. The average iteration number of the new method is far less than that of iterative IMPES in the same time step size. This performance conforms with Example 1. When we double the time step size for the new method, the average iterations increase only more than 1, and hence the total iterations may be reduced largely. This strongly indicates that our method can achieve excellent performance for random distributed permeability.

Table 2. Comparison for Example 2

	Iterative IMPES (Case 1)	New (Case 1)	New (Case 2)
Time step size (days)	1.2167	1.2167	2.4333
Total iterations	11909	4097	2719
Average iterations	11.909	4.097	5.438

7. Conclusions

Based on our previous method proposed in [36], we have developed an iterative scheme to solve the coupled system of nonlinear equations arisen after the fully implicit scheme for two-phase fluid flow in porous media. In our method, we use the linear approximation of capillary function at the current iteration, which is constructed by the saturations at the current and previous iterations. This treatment allows us to couple the saturation equation into pressure equation, and also maintains the inherent coupling relation between the pressure and saturation. The proposed method is suitable not only to homogenous but also to heterogenous media.

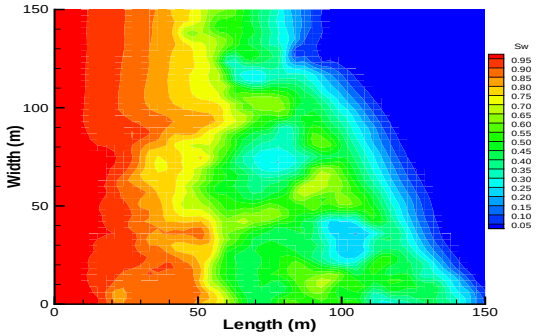

FIGURE 7. Wetting-phase saturation profiles at 0.5 PVI by New Method (Case 1): Example 2.

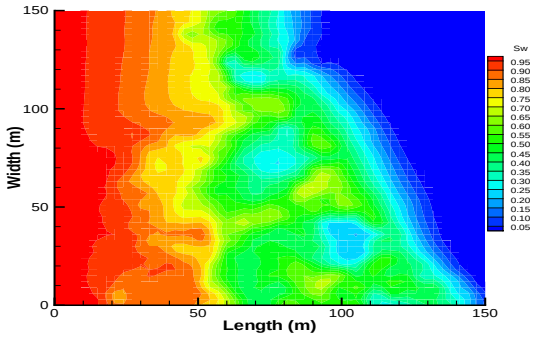

FIGURE 8. Wetting-phase saturation profiles at 0.5 PVI by New Method (Case 2): Example 2.

We use the relaxation approach to control the convergence of our method, and give a practical choice of relaxation factor that is suitable for two-phase flow model. We prove the convergence theorem of our method under the natural conditions.

We have tested two examples to shows the stability and efficiency of the new method. In our tested cases, for the same time step sizes, our method requires much less iteration steps than iterative IMPES. Moreover, the new method is able to converge for the time step sizes that are larger than the maximum one by iterative IMPES. When a large time step size is employed, the computational cost of our method may be further reduced.

In future work, we will extend the proposed approach to the compressible fluid flow, three-phase flow and compositional flow in porous media [55], and study the convergence of the corresponding methods.

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