Title
A robust seismic tomography framework via physics-informed machine learning with hard constrained data

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Main objectives
Introduce a data hard-constrained seismic tomography framework using physics-informed neural networks (PINNs).

Novelty
Previous PINNs-based inversion schemes are slow to converge and are highly sensitive towards the weighting of its multi-term loss function. Here we propose a new formulation to overcome these major challenges. To mitigate the instability problem, we reformulate the eikonal such that the data fitting term becomes a hard constraint. To mitigate the convergence problem, we resort to the use of a non-standard additive factorization of the eikonal (almost all of the previous studies mainly utilize the multiplicative version).

Summary
Accurate traveltime modeling and inversion play an important role across geophysics. Specifically, traveltime inversion is used to locate microseismic events and image the Earth’s interior. Considered to be a relatively mature field, most of the conventional algorithms, however, still suffer from the so-called first-order convergence error and face a significant challenge in dealing with irregular computational grids. On the other hand, employing physics-informed neural networks (PINNs) to solve the eikonal equation has shown promising results in addressing these issues. Previous PINNs-based eikonal inversion and modeling schemes, however, suffer from slow convergence. We develop a new formulation for the isotropic eikonal equation by imposing the boundary conditions as hard constraints (HC). We implement the theory of functional connections (TFC) into the eikonal-based tomography, which admits a single loss term for training the PINN model. We demonstrate that this formulation leads to a robust inversion framework. More importantly, its ability to handle uneven acquisition geometry and topography providing an alternative answer towards the call for an energy-efficient acquisition setup.
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Introduction

Accurate traveltime modeling and inversion play an important role across geophysics. In global seismology, traveltime inversion is used to locate earthquakes and image the Earth’s interior. On the exploration front, it is heavily involved in near-surface imaging, velocity model building for migration and full-waveform inversion, and reservoir monitoring and characterization. These applications are mainly driven by the need to either directly solve the eikonal equation or its characteristics equation (via ray theory). In all cases, traveltime tomography requires an efficient, and preferably flexible, way to compute accurate traveltimes as well as the ability to handle the non-linear nature of the eikonal equation. Moreover, the emerging need to account for less energy footprint measurement brings another important challenge to the problem: uneven, sometimes sparse, acquisition geometry.

To handle the non-linearity of the travetime tomography problem, several approaches have been developed. Two broad categories of such approaches are by means of linearizing the inversion operator (e.g. via ray theory (Červený, 2000) or via the eikonal (Alkhalifah, 2002)) and using the adjoint-state method (Taillandier et al., 2009). To compute the traveltime efficiently, most of the conventional approaches resort to either the ray-based approaches (Červený, 2000) or the grid-based approaches (Vidale, 1988). Most of these algorithms, however, still suffer from the so-called first-order convergence error and face a significant challenge in dealing with an irregular computational grid. On the other hand, employing physics-informed neural networks (PINNs) to solve the eikonal equation has shown promising results in addressing these issues. Bin Waheed et al. (2021) demonstrate a framework for treating the ill-posed cross-well tomography problem using PINNs via the factored eikonal equation.

Although proven to mitigate the source-singularity problem, when applied with PINNs (Bin Waheed et al., 2021; Taufik et al., 2022), the factorization implicitly drives the PINNs’ loss function to include at least two terms, namely the partial differential equation (PDE) loss and the boundary-related condition loss. During traveltime modeling, the boundary condition (source location) needs to be included in the loss function to make the training stable. While in the application of traveltime tomography, both the boundary condition and data mismatch need to be explicitly imposed during the training of the neural network. The multi-term nature of the PINNs’ training under this regime often induces instability due to the need to properly balance the loss terms with respect to their weight and number of samples.

Here, we develop a new formulation for the isotropic eikonal equation by imposing the boundary conditions as hard constraints (HC). We implement the theory of functional connections (TFC) (Schissi et al., 2020) into the eikonal-based tomography, which admits a single loss term for training the PINN model (Taufik et al., 2023). Trained in this fashion, not only our formulation still inherits all the nice properties of PINNs, but it also makes the training more robust and accurate. These properties are the result of the natural inclusion of boundary conditions in the optimization problem. Combining these with a data interpolation NN makes the proposed scheme handle sparse measurement well and thus provides an energy-efficient and accurate framework for the seismic tomography problem.

Theory

The eikonal equation for an isotropic two-dimensional (2-D) medium can be written in the form of (Červený, 2000)

$$|\nabla T(x, z)|^2 = \frac{1}{v^2(x, z)},$$  \hspace{1cm} (1)

where $T$ denotes the traveltime field and $v$ denotes the medium velocity, both as a function of position vector $\{x, z\}$. To mitigate the source singularity problem, the traveltime field can also be decomposed into an additive form, which yields the so-called factored eikonal equation given by

$$T(x, z) = \tau(x, z) + T_0(x, z).$$  \hspace{1cm} (2)
where a scalar $\tau$ is introduced to map the background traveltime $T_0$ to the actual traveltime $T$. Specifically, the background traveltime is defined as:

$$T_0(x, z) = \sqrt{\left(\frac{x-x_s}{v_0}\right)^2 + \left(\frac{z-z_s}{v_0}\right)^2},$$  

(3)

which is given by the distance from the source location $\{x_s, z_s\}$ divided over some background velocity, e.g. a constant velocity ($v_0$).

To incorporate the measured traveltime in the eikonal as a hard constraint, we use the theory of connection functions (Schiassi et al., 2020). For this derivation, we only consider a typical surface tomography experiment in which the traveltime measurements $T_d$ took place along a constant-depth surface at $z_r$. In this case, we suggest the following representation of traveltime

$$T(x, z) = \zeta(z)\hat{T}(x, z) + T_d(x) + T_0(x, z),$$  

(4)

where

$$\zeta(z) = z - z_r,$$  

(5)

and

$$T_d(x) = T_d(x) - T_0(x, z = z_r).$$  

(6)

Here, $x$ and $z$ represent the spatial coordinate. The $\hat{T}(x, z)$ is parameterized by a neural network (NN) functional. Alternatively, for a typical cross-well measurement, we can easily swap equation 5 with $\zeta(x) = x - x_r$. More generally, we can even further impose a topography-dependent recording surface in which the $z_r$ in equation 5 becomes a function of $x$ (for a surface tomography problem). Compared to the original factored eikonal equation, here the new factor $\hat{T}$ is not guaranteed to be positive everywhere in the domain of interest. Moreover, several previous works on PINN-based eikonal formulations have used multiplicative factorization (i.e., changing the addition operation into multiplication in equation 2). Here, we argue for a hard-constraint PINN tomography, additive factorization is a better choice as it results in $\hat{T}(x, z)$ being a smoother function compared to the multiplicative case.

![Figure 1](image_url)

**Figure 1** The proposed workflow for a topographic-dependent surface tomography problem.

The overall workflow of our proposed PINN-based tomography is demonstrated in Figure 1. The core of the inversion is shown by the left blue dashed box while the orange dashed box corresponds to the data interpolation NN ($NN_{T_d}$). The PINNs consist of two NNs, trained simultaneously, for inverting the velocity ($NN_{\hat{v}}$) and the traveltime ($NN_{\hat{T}}$). The inputs to the traveltime NN function are the location of the source, $\{x_s, z_s\}$ and the position in space $\{x, z\}$, whereas $\{x, z\}$ only are inputs to the velocity NN function. The data NN function of the source and sensor locations is trained prior to the PINNs using a loss function, $L$, that measures the misfit between the predicted data and the measured data $T_d$. The data network takes as input the location of the source $\{x_s, z_s\}$ and receiver $\{x_r, z_r\}$.

Using equation 4, the traveltime gradient components of the eikonal can be formulated as

$$\frac{\partial T(x, z)}{\partial x} = \zeta(z)\frac{\partial \hat{T}(x, z)}{\partial x} + \frac{\partial T_d(x)}{\partial x} + \frac{\partial T_0(x, z)}{\partial x},$$  

(7)
and

\[
\frac{\partial T(x,z)}{\partial z} = \xi(z) \frac{\partial \hat{\tau}(x,z)}{\partial z} + \frac{\partial \xi(z)}{\partial z} \hat{\tau}(x,z) + \frac{\partial T_0(x,z)}{\partial z}.
\]

(8)

Hence, the proposed loss function for the PINN, \( \mathcal{J} \), can be constructed by plugging in the gradients from the traveltime network (equations 7 and 8) and the velocity \( \hat{v} \) from the velocity network into equation 1.

**Numerical Tests**

**Figure 2** The true velocity model (a) and the inverted velocity profiles for four different acquisition scenarios (b to e). The sources and receivers are denoted by the white and yellow dots, respectively.

In this section, we discuss some numerical experiments showing the empirical proof of the proposed formulation. We consider four distinct scenarios to showcase the flexibility of our proposed approach. These include the typical regularly sampled shot-receiver geometry, the same sampling scheme as in the first scenario with a gap, sparse shot distribution, and topography-dependent recordings in a passive seismic recording setup. In all of these tests, we use an NN with 10 hidden layers containing 10 neurons in each layer for the velocity network and the data interpolation network. An NN with 20 hidden layers having 20 neurons in each layer is used for the higher dimensional traveltime NN. Both of these two networks (\( \text{NN}_v \) and \( \text{NN}_\hat{\tau} \)) are trained simultaneously for 1000 epochs starting from random initialization. Since the loss function depends on the data and their lateral derivatives, we first train the data NN using 3000 epochs. All of the NNs use the locally adaptive exponential linear unit (l-ELU) and the Adam optimizer. We utilize a portion of the Marmousi model (Figure 2a) with a maximum offset of 8.6 km and maximum depth of 1 km.

Having the most shot-receiver pairs regularly sampled at a constant-depth acquisition setup, we consider the tomogram from the first scenario (Figure 2b) to be the reference solution for the other three scenarios. The sources and receivers are sampled with a sampling rate of 200 m and 20 m, respectively. As shown in Figure 2b, the inversion manages to capture the lateral variation accurately. We then use a sparser receiver sampling of 300 m and introduce a gap (on the source and receiver domain) that extends from 1.2 to 5.8 km offset in the model. Depicted in Figure 2c is the inverted velocity. We see degradation in terms of the lateral velocity resolution as we reduce almost half of the recording surface. The degraded tomogram, however, might still be considered a reasonable initial velocity model for further imaging (e.g. FWI). Using receivers sampled like the first scenario, we reduce the shot sampling interval to 2.7 km in the third scenario. From the inverted velocity profile (Figure 2d), we can see that even with only four shots, the inverted velocity model captures lateral variation of the actual model with high fidelity.
Although some deep structures are obviously not well resolved, compared to the dense shot (Figure 2b), the two tomograms are in good agreement.

To highlight the beauty of the new PINNs formulation, we test the same problem with a topography-dependent surface recording. It is straightforward to derive the PINNs’ objective function by introducing a different $\zeta$ in equation 5, i.e. $\zeta(x, z)$. We also demonstrate the framework ability to handle uneven source distributions by considering a sampled earthquake locations from the southern part of California. The overall improvement from the tomogram can be attributed to the fact that now we image the transmission from the source as opposed to the diving waves in the previous three scenarios. Finally, vertical velocity profiles are drawn to further analyze the reconstructed velocity models. Although in general, the vertical profiles are identical (at least at the given offset location), we can see that the 4-source experiments (Figures 2d and 3c) produce identical results when compared to the dense measurement (Figures 2b and 3a).

Figure 3 The vertical velocity profiles extracted at location 6.84 km from the inversion results in Figures 2b, 2c, 2d, and 2e.

Conclusion

We proposed a new formulation for the isotropic eikonal equation, which is more suitable for PINN-based tomography. The new formulation allows us to rely on a stable single-term loss function. The stability of the PINNs inversion can also be attributed to the additive factorization used to decompose the traveltime field. We found that this factorization will result in a more stable PINN-based tomography compared to the multiplicative version. We have demonstrated that it leads to a robust inversion framework. More importantly, its ability to handle uneven acquisition geometry and topography-dependent measurement provide an alternative answer towards the call for an energy-efficient acquisition setup.

References