Study of Fixed-Points in the Self-Repair Process of a 3D Printer
Renzo Caballero†, Eric Feron†

Abstract—We present and prove a theorem guaranteeing global stability in a non-linear system representing the iterative self-repair process where a 3D printer repairs its timing pulley. The process consists of gradually improving the broken part in the 3D printer until the printer reaches its repaired state. To prove global stability, we verify that the limit of the self-repair sequence does not depend on the initial condition, and always converges to the repaired state. Even though the convergence of this process has been analyzed under strong assumptions, in the present work, the convergence is proven for a more general case.

Index Terms—Automata, Discrete event systems, Fault accomodation, Modeling, Stability of nonlinear systems.

I. INTRODUCTION

The theory of self-repairing systems was first formalized by John von Neumann [1], [2]. He approached the subject from a logical point of view and introduced the essential question: How can reliable systems be constructed from unreliable components? The self-repair theory for physical—from centimeter to meter scale—machines have been evolving since then, but at a slow rate and with few successes, more than some proof of concept [3]–[12].

In our previous work [13], we analyzed an experiment where a 3D printer with a damaged pulley achieved self-repair. The experiment was conducted as follows: the 3D printer had an initial (non-ideal) condition where its x-axis timing pulley was broken or damaged in some way, which affects the pulley geometry. Utilizing the non-ideal printer, we attempted to print an ideal version of the damaged pulley. Since the printer was not working under ideal conditions, this resulted in a pulley that was not ideal, but was still better than the initial pulley on the x-axis. Once the printing process was completed, we substituted the non-ideal pulley in the x-axis with the just printed (possibly) non-ideal timing pulley, and then we repeated this process. After few iterations, the printer achieved self-repair by printing an identical copy of the ideal timing pulley. In Fig. 1, we see an example of an experiment with two iterations where the initial condition (a) has a triangular shape, and the second iteration output (c) is already ideal.

This experiment was inspired by Richard Feynman’s talk There is Plenty of Room at the Bottom [14], where Feynman suggested a mental experiment where a machine fabricates a copy of itself, but at a smaller scale. He mentioned that each consecutive smaller machine somehow must increase its precision as a necessary condition to keep the process going. Analogously, a 3D printer with unreliable components in our experiment becomes closer to the ideal printer at each step, resulting in a reliable system capable of recovering its functionality.

In [13], we conducted this experiment for different initial conditions in the x-axis. In addition, under some assumptions, we built a computational model and a mathematical model to model the self-repair process in the printer and to understand the transition from broken to repaired. All experiments converged to the ideal pulley after few iterations and all models predicted the same behavior for all initial conditions.

In this work, we relax the assumptions utilized to build the mathematical model in [13], and we establish a model for the printing process. In particular, whenever we proposed a zero-order approximation in [13] to model the pulley-belt interaction for non-ideal pulleys, in this work, we utilize bounds to conclude the convergence to the ideal pulley, leaving the pulley-belt relationship untouched. Section II describes the mathematical model and introduces the main theorem. Section III contains the proof of the theorem. Finally, Section IV discusses the implications of the theorem.

II. MATHEMATICAL MODEL AND THEOREM

We model the pulley-belt mechanism and, once with the nozzle dynamics for the non-ideal printer, we model the printing process.

A. Timing Belt - Timing Pulley Mechanism

Assuming the correct tension on the timing belt, when the timing pulley C in the x-axis motor rotates an angular displacement $\Delta \theta \in \mathbb{R}$, the nozzle experiences a displacement $\Delta x \in \mathbb{R}$ in the x-axis direction; this statement indicates the
existence of a \( f(\cdot) \) function, such that \( \Delta x = f(\Delta \theta) \). Notice that a timing pulley is a three-dimensional object; however, in this work, we consider a timing belt with zero-thickness, and from the timing pulley, we only consider the two-dimensional section defined by the timing belt contact, i.e., \( C \subset \mathbb{R}^2 \).

Fig. 2. Basic diagram of the pulley-belt system.

Fig. 3 shows a simplified model of the pulley-belt system, in which the physical relation between \( \Delta x \) and \( \Delta \theta \) can be observed. When the pulley \( C \) is ideal—an ideal circle with radius \( r \)—, this relationship is given by

\[
\Delta x = f(\Delta \theta) = r \Delta \theta. \tag{1}
\]

However, a single-variable linear function is not sufficient to describe the behavior of non-ideal timing pulleys, since they might not be symmetric. To expand the definition of \( f(\cdot) \), we need to formalize the following three assumptions:

**Assumption 1:** All timing pulleys with the same convex hull influence the 3D printer equally.

**Assumption 2:** The tension in the timing belt is enough to avoid slips but—at the same time—is not large enough to affect the pulley rotation.

**Assumption 3:** The timing pulley only rotates in the clockwise direction.

Fig. 3 shows an example of a non-ideal timing pulley covered by the timing belt. Notice that the contact of the pulley with the belt depends only on the two-dimensional convex hull of the timing pulley (reinforcing Assumption 1).

Assumption 2 allows us to ignore any possible slip or random mismatch between the teeth in the pulley and the belt. Finally, Assumption 3 allows us to consider the top contact point between the belt and the pulley as the main driver of displacement of the nozzle (see Fig. 3 and Appendix A in [13]).

Fig. 3. Example of a non-ideal timing pulley covered by the timing belt. Notice that the belt only sees the 2D convex hull of the timing pulley.

Fig. 4. The bottom plot shows the normalized instant displacement for: (a) the ideal pulley, (b) a squared timing pulley, and (c) a triangular timing pulley. The plot with the corresponding normalized instant displacements is constructed utilizing the computational model from [13] available in [15]; the normalization is taken with respect to \( ID \) of the ideal timing pulley \( P_{\text{Ideal}} \), which follows \( P_{\text{Ideal}}(\theta) = r \) for all \( \theta \in [0, 2\pi] \). This computational model was also constructed under assumptions 1, 2, and 3.

**B. Printing Process Model**

We model the dynamics of the nozzle utilizing (2) and assuming a quasistatic system. Given a complete rotation of the timing pulley \( C \) from \( \theta_0 \) to \( \theta_0 + 2\pi \), and an initial position...
for the nozzle \( x(\theta_0) = x_0 \), the dynamics of the nozzle are given by

\[
x(\theta) = x_0 + \int_{\theta_0}^{\theta} P_C(s) \, ds
\]  

(3)

for \( \theta \in [\theta_0, \theta_0 + 2\pi) \). In the ideal case, we have that \( x(\theta) = x_0 + r(\theta - \theta_0) \), which represents a linear relation between \( x \) and \( \theta \). When the \( x \)-axis timing pulley \( C \) is non-ideal, the relation might not be linear and it follows [3].

For the ideal printer, when we want to print an object included in the subset \([x_0, x_1] \times [y_0, y_1]\) of the printing area, and \( x_0 = x(\theta_0) \), then \( \theta_1 \) such that \( x(\theta_1) = x_1 \) is calculated as \( \theta_1 = (x_1 - x_0)/r \), and the total needed rotation for the \( x \)-axis timing pulley to cover the object \( x \)-axis length is \( \theta_1 - \theta_0 \). Since the ideal pulley has a diameter of \( 2r \), it fits into a square in the printing area with edge \( 2r \), which ideally requires a rotation of \( 2\pi \) rad in the \( x \)-axis pulley.

Fig. 5 shows the shapes of the printed pulleys when we attempt to print the ideal timing pulley \( C_{\text{ideal}} \) utilizing a 3D printer with (a) the ideal pulley, (b) a squared pulley, and (c) a triangular pulley on the \( x \)-axis motor. Whenever the value of the normalized ID in Fig. 4 for each figure is larger than one or smaller than one, the printing process has a stretching or contracting effect on the model \( C_{\text{ideal}} \), respectively.

![Fig. 5. Output of the simulated printing process when we aim to print \( C_{\text{ideal}} \) utilizing (a) an ideal pulley, (b) a square pulley, and (c) a triangular pulley as the \( x \)-axis timing pulley. Green represents expansion, and red represents contraction.](image)

C. Mathematical Results and Main Theorem

Remark 1: By Assumption 1 if two timing pulleys \( C_1 \) and \( C_2 \) share the same convex hull, with \( P_1 \) and \( P_2 \) the corresponding instant displacements, then \( P_1 = P_2 \).

Remark 2 is a consequence of the timing belt only seeing the convex hull of \( C \), and the fact that \( P_C(\theta) \) only depends on the top contact point between the convex hull of the pulley and the belt, where \( \theta \) is the angular position of the timing pulley.

Remark 2 is a consequence of (1), and Assumption 1. Also, in [13], the mathematical construction of instant displacement guarantees an injective relation between a discrete convex cover \( H \supset C \) and its associated instant displacement \( P_H \).

We are in conditions to describe the experiment utilizing the introduced notation: let a sequence of \((n + 1) \in \mathbb{N}\) timing pulleys \( \{C_i\}_{i=0} \) represent the initial condition \( C_0 \) and all of the following outputs for the iterative 3D printer self-repair experiment. Let \( \{P_i\}_{i=0} \) represent the corresponding sequence of instant displacements and the two real and positive sequences \( \{\overline{P}_i\}_{i=0} \) and \( \{\underline{P}_i\}_{i=0} \) such that:

\[
\overline{P}_i = \max_{\theta \in [0, 2\pi]} P_i(\theta) \quad \text{and} \quad \underline{P}_i = \min_{\theta \in [0, 2\pi]} P_i(\theta).
\]  

(4)

Both \( \overline{P}_i \) and \( \underline{P}_i \) are well defined for all \( i \in \{0, 1, \ldots, n\} \), since \( P_i \) is continuous and periodic [13]. We now have the conditions to introduce:

Theorem 1: Under assumptions [1] 2] and [3] and for any initial pulley in the 3D printer \( x \)-axis \( C_0 \), the sequence \( \{C_i\}_{i=0} \) consists of successive attempts to print the ideal pulley \( C_{\text{ideal}} \) and replace the pulley in the \( x \)-axis with the printed pulley, as has its limit the ideal pulley \( C_{\text{ideal}} \), i.e.,

\[
\lim_{i \to \infty} C_i = C_{\text{ideal}}.
\]  

(5)

Corollary 1: The ideal timing pulley \( C_{\text{ideal}} \) is an attractive fixed point for the sequence of pulleys \( \{C_i\}_{i=0} \).

III. PROOF

We aim to construct the proof for Theorem 1 in a logical way, with figures complementing the mathematical notation. In particular, even when we can abstract ourselves from the real physical meaning of an instant displacement \( P_C \), we still mention the corresponding timing pulleys \( C \) during the proof to remind the reader of the ultimate purpose of this paper.

We utilize the symbols \( \overline{E} \) and \( \underline{E} \) to refer to ellipses and the symbols \( \overline{C} \) and \( \underline{C} \) to refer to circles with different diameters. The mentioned ellipses and circles are useful to bound the timing pulley during the iterations since they are convex shapes.

To prove Theorem 1, we prove the equivalent statement consequence of Remark 2, i.e., \( \lim_{i \to \infty} P_i(\theta) = 1 \) for all \( \theta \in [0, 2\pi) \). The starting point is understanding the behavior of \( P_i \) for \( i \in \{0, 1, 2, 3\} \) and subsequently, for the general case \( i > 3 \). Also, we distinguish three different cases for \( i = 0 \) (which we call (a), (b), and (c)), and another three cases for \( i > 0 \) (which we call (a’), (b’), and (c’)). Such cases cover all possible scenarios and are useful to visualize the proof. Finally, the proof is independent of (i) the initial position \( \theta_0 \) of the timing pulley on the printer \( x \)-axis and (ii) the position over the printing bed where we are printing.

A. Initial Step: \( i \in \{0, 1, 2, 3\} \)

1) From \( P_0 \) to \( P_1 \): Let \( P_0 \) be the instant displacement for an ideal or non-ideal timing pulley \( C_0 \). By definition, \( P_0 \geq P_0 \) holds, and the initial instant displacement \( P_0 \) is non-negative, continuous, and periodic with period \( 2\pi \) (or a divisor of \( 2\pi \)) [13]. No more information is given about \( P_0 \) in order to guarantee generality. Fig. 6 illustrates examples of regions containing the graphs of \( G(P_0) \) i.e., \( G(P_0) \subset [0, 2\pi) \times [P_0, P_0] \) in cases where: (a) \( P_0 \geq 1 \) and \( P_0 < 1 \), (b) \( \overline{P}_0 \geq P_0 \geq \underline{P}_0 \geq 1 \), and (c) \( P_0 \leq \overline{P}_0 \leq \underline{P}_0 \). Notice that the square and triangular examples from Fig. 4 correspond to case (a). The subsequent figures in the paper, despite being examples, are consistent with the numerical values introduced in Fig.

In Fig. 7 where the dimensions are normalized with respect to the radius \( r \) of the ideal timing pulley \( C_{\text{ideal}} \), we see the area in which \( C_1 \) is included for the three cases. For example, if we consider case (a), Fig. 6(a) shows the region in which
the graph of $P_0$ exists, i.e., $G(P_0) \subset [0, 2\pi) \times [P_0, \overline{P}_0]$, and Fig. 6(a) shows the area where $C_1$ exists, i.e., the closed set within the inner ellipse $E_1$ and outer ellipse $E_1$, both with the same height equal to 2, and with widths $2\overline{P}_0$ and $2\overline{P}_0$, respectively. Such ellipses are constructed by printing $C_{\text{Ideal}}$ utilizing a pulley with constant instant displacement in the $x$-axis equal to $\overline{P}_0$ (for $E_1$) and equal to $\overline{P}_0$ (for $E_1$); by (3), the construction consists in a linear expansion or contraction on the $x$-axis of the ideal timing pulley $C_{\text{Ideal}}$ of a factor $\overline{P}_0$ (for $E_1$) or $\overline{P}_0$ (for $E_1$).

Fig. 7. Regions including $C_1$ for cases (a) (top-left), (b) (top-right), and (c) (bottom-left). The dimensions are normalized with respect to the radius $r$ of the ideal pulley timing $C_{\text{Ideal}}$.

To understand the instant displacement $P_1$ from pulley $C_1$, we notice the bounds that exist in Fig. 7. Let $E_1$ and $E_1$ be the ellipses defining the region for $C_1$ and let $C_1$ and $C_1$ be the circles with radius $\overline{P}_0$ and $\overline{P}_0$, respectively. The instant displacement of a body (either an ellipse or $C_1$) between two consecutive circles is bounded by the instant displacement of those circles, e.g., in the case of Fig. 7(a), the $ID$ of $E_1$ is upper bounded by the $ID$ of $C_1$ and lower bounded by the $ID$ of $C_{\text{Ideal}}$ (see Subsection V-C of the Appendix). Notice that $C_1$ might not necessarily be convex (see Fig. 5), but is bounded by $\overline{C}_1$.

Fig. 8 shows the plots for the instant displacements of $C_1$, $C_1$, $E_1$, and $E_1$ for cases (a), (b), and (c). Since we aim to print $C_{\text{Ideal}}$ and we only affect the $x$-axis, all outputs $C_i$ with $i > 0$ have a fixed height equal to 2. As a consequence, $P_i \geq 1$ and $P_i \leq 1$ hold for all $i > 0$ (observe that all possible $C_i$ in Fig. 7 have a fixed height equal to 2, and that effect is reflected by forcing $P_i$ to take the value 1 at least two times in $[0, 2\pi]$) and we define the cases for $i > 0$ ($a^*$) $P_0 > 1$ and $P_0 < 1$, ($b^*$) $P_1 \geq 1$ and $P_0 = 1$, and ($c^*$) $P_1 \leq 1$ and $P_0 \leq 1$.

Fig. 8. Instant displacements of figures in Fig. 7 corresponding to cases ($a^*$) (top-left), ($b^*$) (top-right), and ($c^*$) (bottom-left).

By our model of the printing process in Subsection II-B when $\overline{P}_0 > 1$, since $2\pi$ is enough to cover the diameter of the ideal timing pulley $C_{\text{Ideal}}$, a necessary condition for $\overline{P}_1 = \overline{P}_0$ is the existence of some $\phi_0 \in [0, 2\pi)$, such that $[\phi_0, \phi_0 + 2] \subset [0, 4\pi)$, and

$$\int_{\phi_0}^{\phi_0 + 2} \overline{P}_0(s) \, ds \leq \overline{P}_0$$

which implies that, by continuity, $P_0 = \overline{P}_0$ for all values of $\theta$ in $[\phi_0, \phi_0 + 1]$ and / or $[\phi_0 + 1, \phi_0 + 2]$. The reasoning for $P_1$ is analogous. Notice that when $\overline{P}_0 > 1$ and $P_0 < 1$, since these two equalities $\overline{P}_1 = \overline{P}_0$ and $P_1 = P_0$ cannot hold simultaneously (see Subsection V-A of the Appendix), then we have that $|\overline{P}_1 - \overline{P}_0| < |\overline{P}_0 - P_0|$. In Table 2 we observe the upper and under bounds for $P_1$ as function of the extremes of $P_0$.

The transition from $P_0$ to $P_1$ can be considered a special case, since $C_0$ can have virtually any shape and the only trivial bounds for $P_0$ are its extreme values. However, the following instant displacements have bounds that depend on the previous instant displacement (see Fig. 8 for cases ($a^*$), ($b^*$), and ($c^*$)). To help visualize this concept, Fig. 9 shows a possible graph for $P_1$ in case ($a^*$), where we added a marker over each one of the extremes.

2) From $P_0$ to $P_2$ and $P_3$: With the values of $\overline{P}_1$ and $P_1$, we can repeat the process described in this subsection to bound $P_0$ from $P_0$, but this time we will bound $P_2$ and $P_3$. If there

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**Fig. 6.** Different examples of regions for $P_0$ depending on the three cases: (a) $\overline{P}_0 > 1$ and $P_0 < 1$ (top-left), (b) $\overline{P}_0 \geq P_0 \geq 1$ (top-right), and (c) $P_0 \leq P_0 \leq 1$ (bottom-left).

**Fig. 7.** Regions including $C_1$ for cases (a) (top-left), (b) (top-right), and (c) (bottom-left). The dimensions are normalized with respect to the radius $r$ of the ideal pulley timing $C_{\text{Ideal}}$.
exists $\phi_1 \in [0, 2\pi)$ such that $[\phi_1, \phi_1 + 2] \subset [0, 4\pi)$ with
\[
\int_{\phi_1}^{\phi_1+2} P_1(s) \, ds = \overrightarrow{P_1}
\]
and also the printing position of the ideal pulley $C_{\text{ideal}}$ is exactly over $[\phi_1, \phi_1 + 2]$, then the equality $\overrightarrow{P_2} = \overrightarrow{P_1}$ will hold. We recall the three cases (a), (b), and (c)) of Fig. 9. In case (a), there is a compression in the extremes due to the continuity of $P_1$. Case (c) is analogous to case (b'), but considers condition (7) with $\overrightarrow{P_1}$ instead of $\overrightarrow{P_2}$. We need to analyze case (b') where since $\overrightarrow{P_2} = \overrightarrow{P_1} = 1$, then if condition (7) holds we have that $\overrightarrow{P_2} = \overrightarrow{P_1} \neq 1$, which implies $|\overrightarrow{P_2} - \overrightarrow{P_1}| = |\overrightarrow{P_1} - \overrightarrow{P_1}| \neq 0$. However, condition (7) cannot hold in two consecutive iterations, as can be seen in Subsection V-B of the Appendix. Since case (a') always results in a contraction and the extremes equal to one are equilibrium points (see Subsection V-D of the Appendix), we conclude that $|\overrightarrow{P_3} - \overrightarrow{P_3}| \leq |\overrightarrow{P_1} - \overrightarrow{P_1}|$ where the equality only holds when $\overrightarrow{P_1} = \overrightarrow{P_1} = 1$.

**B. Inductive Step: $i > 3$**

In Subsection III-A we analyzed the first four elements of the sequence $\{P_i\}_{i=0}^3$ and concluded that $|\overrightarrow{P_3} - \overrightarrow{P_3}| \leq |\overrightarrow{P_1} - \overrightarrow{P_1}|$, with the equality only holding when $\overrightarrow{P_1} = \overrightarrow{P_1} = 1$. Since the only special case is the transition from $P_0$ to $P_1$, if we trivially substitute the three subindexes $\{1, 2, 3\}$ in $\overrightarrow{P_3}$ with the new subindexes $\{i-2, i-1, i\}$ (respecting the order) with $i \in \{4, 5, \ldots, n\}$, we obtain $|\overrightarrow{P_i} - \overrightarrow{P_i}| \leq |\overrightarrow{P_{i-2}} - \overrightarrow{P_{i-2}}|$, with the equality only holding when $\overrightarrow{P_{i-2}} = \overrightarrow{P_{i-2}} = 1$, which concludes the proof.

**IV. CONCLUSION AND DISCUSSION**

We stated and proved a theorem that guarantees convergence to the ideal timing pulley $C_{\text{ideal}}$ for any initial condition $C_0$ on the 3D printer $x$-axis. As a direct consequence, we observe that the ideal timing pulley is a globally stable attracting point.

In this work, we modeled the timing pulleys as two-dimensional convex hulls. However, the theory can be extended to more complex spaces which consider more characteristics of the timing pulleys where still the ideal timing pulley is a global stable point.

Since little work has been conducted to model and analyze the self-repair process of physical machines; the guarantee self-repair, even in a particular case, will assist the development of the field and opens the door for future applications.

**V. APPENDIX**

We utilize the Appendix to expand some mathematical notions that might break the proof flow in the main text.

**A. Special Case for $P_i > 1 > P_{i+1}$**

The following reasoning is relevant in case (a) for $C_0$, and case (a') for $C_i$ with $i > 0$. Suppose that for some $i \in \{0, 1, \ldots, n\}$, the instant displacement $P_i$ corresponds to the timing pulley $C_i$ and $P_{i+1} > 1 > P_{i+1}$ (either case (a) or case (a')). From conditions (6) and (7), in order to have $P_{i+1} = P_i$ and $P_{i+1} = P_i$, there must exist $\phi_i \in [0, 2\pi)$ with $[\phi_i, \phi_i + 2] \subset [0, 4\pi)$, such that either:

\[
\int_{\phi_i}^{\phi_i+2} P_i(s) \, ds = \overrightarrow{P_i}
\]

or

\[
\int_{\phi_i}^{\phi_i+2} P_i(s) \, ds = \overrightarrow{P_i}
\]

holds. However, by continuity of $P_i$, both conditions (8) and (9) cannot hold since, for instance, (8) requires that $P_i(\theta) = \overrightarrow{P_i}$ for $\theta \in (\phi_i, \phi_i + 1)$ and $P_i(\theta) = \overrightarrow{P_i}$ for $\theta \in (\phi_i + 1, \phi_i + 2)$, which violates the continuity of $P_i$ (condition (9) is analogous).

**B. Special Case for Condition (7)**

Assume that $P_i$ is like in case (b') from Fig. 8 i.e., $P_i \geq 1$ and $P_{i+1} = 1$. If there exists $\phi_i \in [0, 2\pi)$ such that $[\phi_i, \phi_i + 2] \subset [0, 4\pi)$ with

\[
\int_{\phi_i}^{\phi_i+2} P_i(s) \, ds = \overrightarrow{P_i}
\]

and also the printing area for the ideal pulley coincides with $[\phi_i, \phi_i + 2]$, then the equality $\overrightarrow{P_{i+1}} = \overrightarrow{P_i}$ holds. In such a case, $P_{i+1}$ is equal to $\overrightarrow{P_{i+1}}$ in at most two zero-measure points and smaller in the rest of the domain, which implies that $\overrightarrow{P_{i+2}} = \overrightarrow{P_{i+1}}$ cannot hold since the integrals from (7) will integrate less than $\overrightarrow{P_{i+1}}$. 

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**TABLE I**

**RELATIONS BETWEEN EXTREMES OF $P_0$ AND EXTREMES OF $P_1$.**

<table>
<thead>
<tr>
<th>(i)</th>
<th>(ii)</th>
</tr>
</thead>
<tbody>
<tr>
<td>If: $\overrightarrow{P_0} &gt; 1$</td>
<td>$\overrightarrow{P_0} &lt; 1$</td>
</tr>
<tr>
<td>Then: $1 \leq \overrightarrow{P_1} \leq \overrightarrow{P_0}$</td>
<td>$\overrightarrow{P_1} \leq \overrightarrow{P_0}$</td>
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<th>(iii)</th>
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<td>Then: $1 \geq \overrightarrow{P_1} \geq \overrightarrow{P_0}$</td>
<td>$\overrightarrow{P_1} = 1$</td>
</tr>
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</table>

**Fig. 9.** Possible graph for $P_1$ in case (a').
C. Boundedness by Circles

Circular pulleys are ideal bounds since they are completely symmetric, and as a consequence—for any position of the bearing with respect to the pulley or any length for the timing belt—the circular pulleys have constant instant displacement.

1) Outer Circle: For any pulley-belt contact point at distance $d_o > 0$ from the center of rotation, the circular pulleys (with radius $d_i$, in this case) present the maximum possible tangential translation of the contact point, which upper bounds any other tangential translation of pulleys with a different geometry that still have a contact point at distance $d_o$ from the center of rotation.

For instance, in Fig. 10(a), the outer circular pulley $C_1$ completely bounds $E_1$, and the same effect occurs with the corresponding instant displacements. However, observe that both $C_1$ and $E_1$ share two common points, and since the tangential speeds are equal for these points, both instant displacements have the same value at such points.

2) Inner Circle: Let the inner circle have radius $d_i > 0$. By definition, the inner circle $C_i$ is included into the body of timing pulley $C_i$. Any contact point between the timing belt and the timing pulley $C_i$ must be at a distance $d \geq d_i$ from the center of rotation.

Fig. 10. Inner circle with radius $d_i$. We observe three possible contact locations (A, B, and C) between the timing belt and a point with a distance $d$ from the rotation center, corresponding to a point in the perimeter of the pulley $C_i$. D represents the contact point between the timing belt and the inner circle $C_i$.

Fig. 10 shows the trajectory of an arbitrary point over the perimeter of the convex hull of $C_i$, moving from position A to position C. Points A and C are defined as the two intersections between the top tangential line to $C_i$, which also passes by the bearing point O, and the arc with radius $d$. The presiding analyses can be applied to each point on the perimeter of the convex hull of $C_i$.

Let D be the contact point, until the pulley rotation makes the contact to transit from D to A. By continuity of the instant displacement, if we consider the transition A→D, both contact points A and D shows the same value of instant displacement. The same argument is applied to D and C during the transition C→D. Finally, we observe that the tangent speed of the contact point with radius $d$ has constant module from A to C, but points A and C show the maximum misalignment of the tangential speed with respect to the timing belt. Then, the instant displacement of the contact point at distance $d$ of the center of rotation, over the trajectory from A to C, is lower bounded by the value of the instant displacement at the extremes A and C.

D. Equilibrium Points for the Sequence

The ideal case, where $P_i = P_{i+1} = 1$, corresponds only to the ideal timing pulley $C_{ideal}$. Then, by definition, if $i > 0$, the ideal case can correspond to both ($b^*$) and ($c^*$). Let an instant displacement $P_i$ be used to print the ideal timing pulley in the interval $[\phi_i, \phi_i + 2]$ and consider the right-side of the printing process, i.e., $[\phi_i + 1, \phi_i + 2]$. At each point $\theta$ of the interval $[\phi_i + 1, \phi_i + 2]$, we compress or expand $C_{ideal}$ depending on the value of $P_i(\theta)$, where the accumulated effect for point $\theta^*$ in the interval $[\phi_i + 1, \phi_i + 2]$ is given by

$$\int_{\phi_i + 1}^{\theta^*} P_i(s) \, ds$$

(11)

If the value of (11) is larger than 1 for some $\theta^*$ on the interval $[\phi_i + 1, \phi_i + 2]$, then the posterior $P_{i+1}$ will be larger than 1 in a non-zero measure interval included in $[0, 2\pi]$. On the other hand, if (11) is less than or equal to 1 for all points in $[\phi_i + 1, \phi_i + 2]$, then $P_{i+1}$ will be also less than or equal to 1 in at least half of its period.

Thus, we conclude that if $P_i(\theta) \leq 1$ for $\theta \in [0, 2\pi]$, then $P_{i+1}(\theta) \leq 1$ (the case where $P_i(\theta) \geq 1$ implies $P_{i+1}(\theta) \geq 1$ is analogous). As a consequence, cases (a) and ($a^*$) can transition to either ($a^*$), ($b^*$), or ($c^*$), but cases (b) and ($b^*$) can only transition to case ($b^*$) and cases (c) and ($c^*$) can only transition to case ($c^*$).

REFERENCES