Supplementary Material for ”Scattering properties of acoustic beams off spinning objects: Induced radiation force and torque”

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Abstract

This Supplementary Material gives details for both Mie theory and the finite-element modeling (implemented in COMSOL Multiphysics) of the spinning flow interacting with acoustic waves and compares their results. It also details the scattering and beam-shape coefficients of the spinning objects interacting with acoustic beams, as well as geometrical and material tunability.

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I. MIE THEORY FOR ACOUSTICS IN SPINNING MEDIA

Figure S.1 is a schematic of the problem that we plan to study in this Letter. We choose cylindrical coordinates \((r, \varphi, z)\). The fields depend on the azimuthal angle \(\varphi\) via \(e^{i\varphi}\). By using the convenient approximations [1–4] and by ignoring the role of viscosity in this system, which is an acceptable approximation for water and taking into account the size of the cylinder \((\approx 0.5 \text{ m})\), and by denoting \(p, v_r, v_\varphi\), the pressure field, the normal, and the azimuthal components of the velocity field, respectively, in the cylindrical coordinate system.

FIG. S.1: Schematic representation of an acoustic beam incident from left to right. This beam impinges into an acoustical object of azimuthal symmetry (and infinitely extended in the \(z\)-direction, i.e., the normal to the plane of the figure). The object can be a single cylinder (what is mainly discussed in this accompanying letter) of radius \(a\) and spinning with an angular rotation \(\Omega \mathbf{e}_z\), where \(\mathbf{e}_z\) is the unit-vector perpendicular to the plane of the figure (i.e., symmetry axis of the cylinder). The object is assumed to have the same physical properties as the surrounding medium, i.e., \(\rho/\rho_0 = \beta/\beta_0 = 1\). The black arrows indicate acoustic scattering due to spinning.
\((r, \varphi, z)\). It can be shown that the system obeys the linear system as function of these scalar fields. This system of coupled partially differential equations can be expressed as [2–4]

\[
\begin{align*}
\zeta v_r - 2\Omega v_\varphi + \frac{1}{\rho} \frac{\partial p}{\partial r} &= 0, \\
2\Omega v_r + \zeta v_\varphi + \frac{il}{\rho r} p &= 0, \\
\left\{ \frac{1}{r} + \frac{\partial}{\partial r} \right\} v_r + \frac{il}{r} v_\varphi + \frac{\zeta}{\rho c^2} p &= 0,
\end{align*}
\]  

(S.1)

with \(\zeta = -i(\omega - l\Omega)\) the Doppler-modified angular frequency.

Furthermore, this equation [or equivalently Eq. (1)] of the manuscript (MS) has to be accompanied with the appropriate boundary conditions. For a non-moving substance, one has to enforce the continuity of pressure field \(p\) and of the normal displacement field (proportional to the normal component of the velocity \(v_r \propto (1/\rho)\partial p/\partial r\)). For the case of a media undergoing a spinning motion, one has to ensure as before the continuity of \(p\) and of the normal displacement \(\psi_r\) [1, 2], i.e.,

\[
\psi_r = \frac{\zeta v_r + \Omega v_\varphi}{\zeta^2 + \Omega^2} = \frac{(2\Omega^2 - \zeta^2) \partial p/\partial r - 3i\zeta \Omega np/r}{\rho (4\Omega^2 + \zeta^2) (\Omega^2 + \zeta^2)}. 
\]  

(S.2)

By letting \(\Omega = 0\) in Eq. (S.2), one gets a displacement proportional to \((1/\rho)\partial p/\partial r\) as for sound waves in non-moving media.

In the standard framework of Mie theory, one expand the \(p\) in terms of Bessel and Hankel functions, and solving the expansion coefficients by taking boundary condition into account. Details are available in the literature such as Refs. [1–4].

II. FINITE-ELEMENT MODEL FOR ACOUSTICS IN SPINNING MEDIA

In order to model the complex interaction of a flow (spinning fluid in our case) with acoustics, we make use of the finite element method (FEM) implemented in the commercial software COMSOL and its acoustic module of "linearized Navier-Stokes model" interface [5]. This is a robust model that describes the interaction of a stationary background flow with an acoustic field, as in our case. This interface is very general and takes into account several effects such as turbulent and non-isothermal flows. We implement an acoustic background (plane wave) field, use perfectly matched layers (PMLs) and proper boundary conditions (slip boundary conditions for the velocity, see Section
FIG. S.2: Scattering cross-section from a fixed cylinder (no spinning) of radius 0.35 m and (a) relative density of 10 and (b) a relative compressibility of 10. The surrounding medium is water.

"Linearized Navier-Stokes Model" of Ref. [5]), and finally compute the scattering cross-section (SCS) of the objects within the spinning flow. The equations of this model treat a linearization of the full compressible, non-isothermal, and viscous flow. These equations may be found in https://www.comsol.com/blogs/modeling-aeroacoustics-with-the-linearized-navier-stokes-equations/[5]. Moreover, in COMSOL, a material is defined by the following properties: density, dynamic viscosity, bulk viscosity, thermal conductivity, and heat capacity at constant pressure, denoted by $\rho_0$, $\mu$, $\mu_B$, $k$, and $C_P$, respectively, showing the degree of generality of this model. We also add a background acoustic field to the domain, defined by pressure, velocity, and temperature $p_b$, $u_b$, and $T_b$, respectively.

In addition to all these definitions, we implement the perfectly matched layers (PMLs) for the spinning flow and integrate the parameters of the study such as the scattering cross-section. First we validate our model for a non-spinning scenario, where the results are analytically known from Mie theory for acoustics. The agreement between the COMSOL simulations and the Mie theory show that our model solves the actual scattering problem as shown in Fig. S.2(a) for a cylinder of relative density of 10 and in Fig. S.2(b) for the same cylinder of relative compressibility of 10, embedded in water. These extreme values of relative density and compressibility are chosen to test the code in an extreme scenario.

The first spinning case that we consider [Fig. S.3(a)] consists of a cylinder of radius 0.35 m, made of a medium with relative density of 10 inside water (extreme scenario to test the
FIG. S.3: (a) Scattering cross-section for the object of Fig. S.2 for a relative density of 10 and a spinning frequency of 10 Hz (top panel) and 25 Hz (bottom panel). Same as in (a) but for a relative compressibility of 10. The dashed curves correspond to the analytical (Mie) model while the black plus curves correspond to the linearized Navier-Stokes full-wave simulations, using COMSOL Multiphysics [5].

For this situation, two spinning frequencies are analyzed, i.e., 10 Hz (top panel) and 25 Hz (bottom panel). The second scenario consists in the same object but with a relative compressibility of 10 in Fig. S.3(b). The FEM-based numerical experiment using linearized Navier-Stokes (black plus curves) fits well with the Mie-based calculations (red dashed lines). This validates our model and shows that the resonant scattering due to spinning stems from first principles and exists when the full-wave simulations are employed.
FIG. S.4: Scattering cross-section for the object of Fig. S.2, i.e., for a relative density of 10 and at rest in (a) and an object at spinning frequency of 25 Hz in (b). The plots show the various multipoles making up the scattering cross-section. Scattering cross-section from a fixed cylinder of radius 0.1 m and relative density of 10 (to test the model in an extreme scenario), in the (c) low and (d) high frequency regime, i.e., with spinning frequencies of $\Omega = 2\pi \times 10$ rad/s and $\Omega = 2\pi \times 200$ rad/s, respectively. The inset in (c) shows the scattered pressure field from both FEM and Mie model at the resonance.

Interestingly, we could use the Mie formalism in order to compute the scattering coefficients separately for both the object at rest (same as in Fig. S.2(a)) and the spinning one (of Fig. S.3(a)). The plots are shown in Fig. S.4(a) and Fig. S.4(b) respectively in addition to the total SCS for comparison. For the object at rest, it is clear that multipoles of order $\pm l$
contribute exactly the same amount to the SCS and are resonant at the same frequencies. When spinning is induced, the situation dramatically changes and ±l multipoles are split owing to the acoustic analog of the Zeeman effect [6]. These figures undoubtedly show the effect on spinning on the interaction of sound with objects in motion.

Moreover, for the purely spinning-induced resonance, it can be seen in Fig. S.4(c) that the first (dipolar) resonance almost replicates the results from the Mie model. To be more specific, the FEM simulation identifies a resonant frequency at 9.07 Hz, while the Mie model predicts the resonant frequency at 9.09 Hz, i.e., a relative error of 0.2%. The near field patterns obtained from both methods are also shown in the inset of Fig. S.4(c), which are identical as well. However, the other resonance, at 10.12 Hz cannot be excited by the current FEM code. From the Mie model, we found it is a monopolar resonance. Exciting such mode seems to be tricky. It is difficult to couple with the plane wave, which is assumed in our FEM code. Nevertheless, both methods indicate the existence of the dipolar resonance, which is a direct consequence of the spinning as can be further verified for the higher frequency regime in Fig. S.4(d).

These results serve as a numerical experiment that is closer to the reality compared with simple Mie theory models that help in getting a physical picture of the interaction between sound and spinning flows.

III. BEAM-SHAPE COEFFICIENTS

By making use of the orthogonality of the functions $e^{il\phi}$, it can be shown that the expression of the beam-shape coefficients (BSC) is given by [7–15]

$$b_l = \frac{1}{2\pi J_l(k_0r)} \int_0^{2\pi} d\phi' e^{-il\phi'} p_{\text{inc}}(r, \phi') .$$  \hspace{1cm} (S.3)

In the case where the incident beam is simply a plane wave, Eq. (S.3) can be simplified and is shown to lead to $b_l = i^l$, with $i^2 = -1$ [see Fig. S.5(a)].

Here we first consider a cylindrical diverging (converging) wave that is described by the pressure field [see Fig. S.5(b)-(c), respectively] [16, 17]

$$p_{\text{inc}} = \frac{H_0^{(1,2)}(k_0R_0)}{H_0^{(1,2)}(k_0r_0)} ,$$  \hspace{1cm} (S.4)

with $r_0$ the distance between the source and the cylinder’s center and $R_0 = (r^2 + r_0^2 + 2rr_0 \cos \varphi)^{1/2}$ the distance between the source’s position and the observation point $(r, \varphi)$.
FIG. S.5: Amplitude $|p|$ and phase $\arg(p)/\pi$ of the total acoustic pressure, i.e., $p^{\text{scatt}} + p^{\text{inc}}$ in water of density $\rho_0 = 1000 \text{ kg/m}^3$ and bulk modulus $1/\beta_0 = 2.22 \text{ GPa}$, for different incident signals: (a) A plane wave of frequency 170.4 Hz, (b) a converging, (c) a diverging cylindrical beam located at $r_0 = 6 \text{ m}$ from the origin of the coordinates (see Fig. 1 of the MS), and (d) a quasi-Gaussian beam of order $n = 1$. The amplitude and phase of each of these signals depicts some particular behaviors.

The expression of Eq. (S.4) is valid for $a \leq r \leq r_0$, with $a$ the cylinder’s radius. In this realm the BSC is [8, 18–20]

$$b^{d,c}_i = (-1)^i \frac{H^{(1,2)}_i (k_0 r_0)}{H^{(1,2)}_0 (k_0 r_0)}.$$  

(S.5)

The second kind of acoustic source that we will analyze in our study is the quasi-Gaussian cylindrical beam [21, 22] [see Fig. S.5(d)], whose BSC are given as [8, 18, 19]

$$b^{qG}_{i,s} = i^{l+s} \frac{I_{l+s}(k_0 \xi)}{I_0(k_0 \xi)},$$

(S.6)
FIG. S.6: Scattered pressure amplitude (upper panel) and normalized phase (lower panel) for the quasi-Gaussian beam of order 1 (a), 2 (b), and 3 (c), showing the vortex behavior.

where \( \xi \) is a parameter of the beam related to its waist (i.e., Rayleigh length). Now the field inside the cylinder shall be given by taking into account the modified Helmholtz equation, i.e.,

\[
p(r, \varphi) = \sum_{l=\infty}^{+\infty} b_l a_l J_l (k_l r) e^{il\varphi},
\]

with \( k_l \) the quasi-wavenumber in the spinning cylinder, \( b_l \) the BSC, and \( a_l \) the unknown coefficients of the internal pressure field.

The boundary conditions at the cylinder’s outer radius \( r = a \) are (i) the continuity of total pressure and (ii) normal displacement, that is [1, 2]

\[
\frac{1}{\rho_0 \omega^2} \frac{\partial}{\partial r} \left[ p^{\text{inc}} + p^{\text{scatt}} \right] \bigg|_{r=a} = \frac{2 \Omega^2 - \zeta_l^2}{\rho (4 \Omega^2 + \zeta_l^2) (\Omega^2 + \zeta_l^2)} \frac{\partial, p - 3 i \zeta_l \Omega l p}{r} \bigg|_{r=a}.
\]

By enforcing the boundary conditions (\( p \) and \( \zeta_r \)) across the boundary \( r = a \), we can get both the internal coefficients \( a_l \) the scattering coefficients \( s_l \) of order \( l \), i.e.,

\[
a_l = \begin{vmatrix}
J_l (k_0 a) & -H_l^{(1)} (k_0 a) \\
\frac{1}{\rho_0 k_0 c_0} J_l' (k_0 a) & -\frac{1}{\rho_0 k_0 c_0} H_l^{(1)'} (k_0 a) \\
J_l (k_l a) & -H_l^{(1)} (k_0 a) \\
\Pi J_l & -\frac{1}{\rho_0 k_0 c_0} H_l^{(1)'} (k_0 a)
\end{vmatrix}.
\]
FIG. S.7: SCS, axial and transverse force for the quasi-Gaussian beam of order 2 (higher row, i.e., (a), (c), and (e)), order 4 (middle row, i.e., (b), (d), and (f)), and order 5 (lower row, i.e., (g), (h), and (i))

The field in the region $a < r < r_0$ is $p^0 = p^{\text{inc}} + p^{\text{scatt}}$, and hence the scattering coefficient is given by

$$s_l = \frac{\begin{vmatrix} J_l(k_1a) & J_l(k_0a) \\ \frac{1}{\rho_0k_0c_0}J'_l(k_0a) & 0 \end{vmatrix}}{\begin{vmatrix} J_l(k_1a) & -H_l^{(1)}(k_0a) \\ \frac{1}{\rho_0k_0c_0}H'_l(k_0a) & 0 \end{vmatrix}}, \quad (S.10)$$
FIG. S.8: SCS, axial and transverse force (left, middle, and right panel, respectively) for the quasi-Gaussian beam of order 0 for different radii and spinning frequencies of the scatterer. The first row corresponds to Figs. 2(b),(e),(h) of the main manuscript.

with $|M|$ the determinant of any matrix $M$ and

$$\Pi_{H_i} = \frac{(2\Omega^2 - \zeta_i^2) k_i a J'_i (k_i a) - 3 \zeta_i \Omega \imath J_i (k_i a)}{\rho a (4\Omega^2 + \zeta_i^2) (\Omega^2 + \zeta_i^2)}.$$  \hspace{1cm} (S.11)

By solving the system of Eq. (S.8), we get the full information on the interaction between the beam and the scatterer. For instance, the scattering cross-section (SCS) $\sigma_{\text{scatt}}$ represents a quantitative measure of the visibility of an object in the far field. If we further define the scattering amplitude $f(\varphi)$, it is straightforward to obtain that $\sigma_{\text{scatt}} = 1/2 \int_0^{2\pi} d\varphi |f(\varphi)|^2$.
FIG. S.9: Normalized torque for the quasi-Gaussian beam of order 0 for different amount of loss. The first subplot (starting from top-left corner) corresponds to that of Fig. 4(e) of the main manuscript.

Its form in terms of the scattering coefficients is

\[ \sigma^{\text{scatt}} = \frac{4}{k_0} \sum_{l=-\infty}^{l=+\infty} |b_l|^2 |s_l|^2. \]  

\[ (S.12) \]

Obviously, from Eq. (S.12) it is clear that the nature of the incident signal (plane wave or beam) influences the scattering behavior [presence of the BSC in the expression of the SCS].

Figure S.6 gives the snapshot of the amplitude and phase of the pressure field in presence of spinning cylinders and shows the vortex nature of this interaction. Figure S.7 analyzes the SCS and radiation force for quasi-Gaussian beams of higher order and shows that when the order of the beam is 5 or higher, the force vanishes.

In order to analyze the robustness of our concept, we analyze the effect of changing the size of the scatterer on the radiation force in Fig. S.8 showing that even for smaller radii the effect is present. Last but not least, we characterize the effect of loss in density of the scatterer on the torque in Fig. S.9. The results show that the smaller the loss, the higher
the torque and the narrower the resonance (high quality factor).


