Experiments in Robotic Self-Repair: A 3D Printer Repairs Its Own Timing Pulley

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Abstract—We observe an experiment for self-repair in robots that can fabricate their own parts. A challenge emerges when imperfections or degradations in the robot impact its ability to fabricate ideal components to guarantee self-repair. In the proposed experiment, we start with a defective or degraded component. We do not fabricate an ideal part initially, but only after a sequence of increasing-in-quality parts. We construct and validate two mathematical models to describe the experiment and match the observations. We observe convergence to the ideal part in all experiments and models, thus restoring the capability for self-repair to the robot.

I. INTRODUCTION

The latest achievements in space exploration including Perseverance Rover, the first drone flight on Mars [1] and confirmation of water molecules on the Moon’s surface [2], have made the idea of space colonies more appealing than ever before. However, human lifespans are insignificant compared to astronomic time scales, and the human body is adapted exclusively to conditions on Earth. Additionally, the idea of terraforming Mars is difficult, since the planet’s overall CO₂ resources are not large enough to make it habitable for humans [3]. For these reasons, a self-replicating, self-repairing robot colony represents a reasonable alternative for space exploration. Although studies on Mars are currently underway, research focused on self-replicating systems for the Moon have been conducted for many years [4]–[8]. Other applications for exponentially growing, self-replicating robot colonies on Earth could include reduction of greenhouse gases such as CO₂, or colonization of hazardous zones like the ocean, north and south poles, and deserts. For instance, Saudi Arabia is home to The Empty Quarter, a 650,000 km² desert where humans can barely survive without the correct equipment [9].

John von Neumann was the first to formalize a theory of self-reproducing automata [10]. He approached the subject from a logical point of view and introduced the essential question: How can reliable systems be constructed from unreliable components? This question was partially answered by von Neumann from the probabilistic point of view in his work [11], where he assumed each component had a fixed probability of failing and he developed strategies to bound the overall failing probability.

Many schools of thought on self-reproducing automata have emerged since von Neumann’s works. On the one hand, the cellular automata theory proposes that a group of cells with a set of instructions can achieve self-replication. Over the years, cellular automata with self-replicating capability that require fewer instructions were developed [12], [13]. A compilation of works on cellular automata during the previous century can be found in [14]. On the other hand, kinetic self-replicating machines are physical objects in a physical space. Of course, this school of automata involves several challenges related to fabrication and assembly [15] that do not apply to cellular automata [16].

Advances in fused deposition modeling (FDM) and other types of 3D printing have become very promising for the future of self-replicating, self-repair machines. Experimental work has been conducted with non-standard printing materials [17], [18], or with optimized printing methods to maximize some characteristics [19], [20]. In principle, there is no limitation on a 3D printer printing its own pieces [21]–[23]. However, fabrication and assembly of kinetic machines pose several practical challenges. For instance, a fundamental challenge, aligned with the essential question introduced by von Neumann, is the lack or degeneration of precision after replication or at the moment of repair. Richard Feynman mentioned this problem in his talk There’s Plenty of Room at the Bottom [24]. He suggested an iterative method to miniaturize machines where, at each step, each machine fabricates a more miniature copy of itself. However, Feynman also mentioned that it is necessary to improve the precision of the apparatus at each stage to keep this process going.

The goal of this work is to build a theory for robotic self-repair. We aim to achieve self-repair in cases where we have imperfect or degraded components. We hypothesize that a 3D printer with a worn or damaged pulley that affects its performance can still repair itself by 3D printing a new pulley. Inspired by Feynman’s idea, we introduce an experiment where we start with an imperfect or degraded component, and we perform a recursive series of successively printing the same part. To the best of our knowledge, this experiment to achieve self-repair has not been proposed before. We study the necessary and sufficient conditions for the series to converge. In particular, we analyze how the printer repairs one of its timing pulleys after only a few recursive iterations.

We aim to analyze the experiment as completely as possible, using observations to build simulation and analytical models of the process. In Section II, we describe the experimental setup. Section III contains the observations. In Section IV, we introduce two mathematical models to explain the experiment; one of them is a computational simulation,
while the other is purely analytical. We also provide a Git repository [25] with all of the scripts used in this work. Section VI presents the validations for the models. Finally, we provide a closing conclusion and future work in Section VII.

II. EXPERIMENT DESCRIPTION

We utilize a Creality3D Ender-5 DIY 3D Printer [26] as experimental research device. As the printer can print some of its parts, we choose and intentionally damage one of these parts to simulate a degradation in the printer. We aim to utilize the damaged printer to print a perfect replica of the damaged part and, in that way, achieve self-repair experimentally.

The 3D printer includes three timing pulleys, three 200 steps-per-revolution motors, and three timing belts with three corresponding bearings. This mechanism allows the nozzle to move within the printing area, which is a three-dimensional cube. Each motor displaces the nozzle along an axis, i.e., x-axis, y-axis, and z-axis, defining the printing area.

In Fig. 1 we provide a graphic representation of the utilized printer, which indicates the modified pulley. The printer’s ideal pulley is a 20 teeth pulley with an outer radius of 6 mm. We test the printer’s capability to self-repair by focusing on its own x-axis timing pulley. We chose the timing pulley because it directly affects the geometry of the printed part, and its influence on the printer can be modeled utilizing standard geometry and calculus. The pulley is held by the motor, which controls the nozzle’s movement along the x-axis— as can be seen in Fig. 1. Thus, we actually refer to timing pulleys when we mention a 3D model or 3D part. No more non-idealities are added to the printer during the experiment.

![Fig. 1: Graphical representation of the utilized 3D printer (Creality3D Ender-5) with the axis references and the modified timing pulley, circled in red. CAD model downloaded from [27].](image)

In Fig. 2 we provide a block diagram of the utilized 3D printer. A 20 teeth pulley with an outer radius of 6 mm is utilized as reference. The iteration consists of successive steps of printing the reference pulley. The superindex on C represents the second step in the first experimental recursion. The subindex indicates the variables’ instance (experimental or mathematical model, and the recursion number). The subindex indicates the step in the iteration. E.g., \( C_{ref}^{E,1} \) represents the second step in the first experimental recursion.

![Fig. 2: Block diagram of the utilized 3D printer.](image)

We utilize the circular and triangular pulley from Fig. 3 as the reference model \( C_{ref}^{E,1} \) and starting point \( C_{0}^{E,1} \), respectively. We figured out that two steps are enough since

\[
C_{ref}; C_{ref} ; \quad / / \text{Initialization} \\
\text{for} \ (i = 0; i < n; i = i + 1) \ do \\
\ \\
\end \\
\text{Algorithm 1: Algorithmic representation of the experiment.}

\]

The expression \( F(\cdot, \cdot) \) is a mathematical description of the 3D printer. Elements \( C_i \) and \( C_{i+1} \) are in the set of physical 3D objects, while element \( C_{ref} \) lives in the set of computational 3D designs.

During the experiment, we have as starting point the pulley \( C_0 \), and we utilize the pulley \( C_{ref} \) as reference. The iteration consists of successive steps of printing the reference \( C_{ref} \). In the first step, the x-axis pulley is \( C_0 \), and after the printing process is finished, we manually replace the pulley in the x-axis motor by the printed output \( C_1 \). We repeat this procedure at the end of each step. Before the replacement, it is necessary to utilize a drill to ensure that the inside radius of the printer pulley is sufficiently large to fit the printer. After the replacement, we also adjust the belt’s length to avoid a loose belt-pulley contact. Algorithm 1 shows the experiment procedure for \( n \) steps.

\[
C_{0}^{E,1}; C_{ref}^{E,1}; \quad / / \text{Initialization} \\
\text{for} \ (i = 0; i < n; i = i + 1) \ do \\
\ \\
\end \\
\text{Algorithm 1: Algorithmic representation of the experiment.}

\]

We introduce the following notation: The superindex on C indicates the variables’ instance (experimental or mathematical model, and the recursion number). The subindex indicates the step in the iteration. E.g., \( C_{ref}^{E,1} \) represents the second step in the first experimental recursion.
we converged to a fixed point, so the complete experiment involves three pulleys: $C_{0,1}^E \rightarrow C_{1,1}^E \rightarrow C_{2,1}^E$. Pictures of the setup can be seen in Fig. 4 where we can observe the triangular initial pulley $C_{0,1}^E$ being held by the $x$-axis motor.

![Fig. 4: Initial triangular pulley $C_{0,1}^E$ utilized during the experiment.](image)

Fig. 4: Initial triangular pulley $C_{0,1}^E$ utilized during the experiment.

### III. Observations

In Fig. 5 we can see all pulleys involved in the iteration: The triangular starting pulley $C_{0,1}^E$, and the output from the two steps, $C_{1,1}^E$ and $C_{2,1}^E$. We cannot observe any difference between pulley $C_{2,1}^E$ and the ideal reference $C_{ref}^E$ by visual examination. Moreover, the relative difference between $C_{ref}^E$ and $C_{2,1}^E$ radii is $<0.5\%$.

![Fig. 5: Initial pulley $C_{0,1}^E$ with outputs $C_{1,1}^E$ and $C_{2,1}^E$ where the reference $C_{ref}^E$ is the ideal circular pulley.](image)

Fig. 5: Initial pulley $C_{0,1}^E$ with outputs $C_{1,1}^E$ and $C_{2,1}^E$ where the reference $C_{ref}^E$ is the ideal circular pulley.

### IV. Mathematical Model

This experiment motivates us to construct mathematical theory to explain the observations. We start by introducing the mathematics behind the belt-pulley interaction in an ideal and a non-ideal system, and we (under different hypotheses) construct two models to explain the experiment in Section III.

#### A. Relation Belt-Pulley

In Fig. 6 we observe the relationship between displacement of the belt $\Delta u$ (in mm) as a function of rotations in the pulley $\Delta w$ (in rad), i.e.,

$$\Delta u = \Delta w r,$$

where $r$ is the radius of the pulley in mm. However, (2) only holds when the timing pulley, timing belt, and their contact are ideal. If we relax the first two conditions – i.e., we utilize a non-ideal timing pulley and a non-ideal timing belt – the belt-pulley motion is characterized by directional-dependent instant displacement

$$\frac{du}{dw} (w, \text{sign}(w)),$$

where $w \in [0, 2\pi)$ and $\text{sign}(w) \in \{-1, 1\}$. The input $\text{sign}(w)$ is necessary to distinguish the pulley rotation’s orientation.

**Remark 1:** If the timing belt’s length is fixed, only a few non-ideal pulleys can be used to keep this condition since they have different perimeters and non-symmetric shapes. To avoid this inconsistency, we assume the belt’s elasticity is enough to adapt to different pulley shapes while the tension is enough to prevent slipperiness.

**Remark 2:** Expression (3) only is valid once the timing pulley is rotating and only while its rotation direction does not change. When the pulley is at rest, or when it changes its rotation direction, the sudden change in the belt’s tension invalidates (3).

In the following subsections, where we construct the models, we utilize the same assumptions used to formulate (3).

![Fig. 6: Simple x-axis dynamic representation of an ideal FDM 3D printer. CAD model downloaded from [27].](image)

Fig. 6: Simple $x$-axis dynamic representation of an ideal FDM 3D printer. CAD model downloaded from [27].

#### B. Computational Model

We construct a mathematical model for the experiment described in Section III, and we utilize a computer to simulate it. All scripts and additional details can be found in the project’s Git repository [25].

Since we assume ideal belt-pulley contact, we formulate the following principle to simplify the mathematical models: *Ideally, all pulleys with the same convex hull influence the printer equally.* This principle motivates the use of convex sets to describe and simplify the pulley geometry, and defines an equivalence class for the timing pulleys.

Since we are modeling non-ideal pulleys, the value of the instant displacement $\frac{du}{dw}$ depends on both the pulley’s angular position $w$ and the rotation orientation. In this model, we only consider the clockwise direction to simplify the calculations. To model the pulleys, we describe each pulley as the convex hull of a finite number of points in a 2D plane where the following property holds: For each angular position, one (and only one) vertex controls the belt’s movement as a function of the pulley’s rotation. Using this property, we find an exact expression for $\frac{du}{dw}$ given the pulley’s geometry and the proposed model. Fig. 7 shows (on the left) four pulleys with their corresponding convex hulls, and (on the right) the plot for their instant derivatives $\frac{du}{dw}$.

The three blocks in Fig. 8 illustrate the simulation flow. Since the printer’s $z$-axis works ideally, the influence of
the damaged pulley only affects the \( xy \)-plane. Then, the 3D timing pulleys \( C_{i-1}, C_{\text{ref}} \) and \( C_i \) are modeled utilizing 2000 \( \times \) 2000 two-dimensional \((0, 1)\)-matrices, in which the entries with value 1 represent the physical body of the pulley. Four examples are presented in Fig. 7 left, where we observe a graphical representation of the binary matrices for different pulleys. The input \( w_0 \in [0, 2\pi) \) in Fig. 8 is the initial angular position of the printer’s timing pulley, which is relevant since non-ideal pulleys might not be entirely symmetric. We utilize the same initial angular position for each step in the recursion.

Block \( \text{CH} \) calculates the convex hull for pulley \( C_{i-1} \) and expresses the convex hull in Cartesian coordinates, i.e., \( \{x_1, x_2, \ldots, x_n\} \), and \( \{y_1, y_2, \ldots, y_n\} \) are the sets of coordinates for the \( n \) vertices in the convex hull. In Fig. 7 left, we observe four plots with the matrix representation for the pulleys and their corresponding convex sets.

Block \( \text{DE} \) utilizes the convex hull vertices to construct an exact expression for \( \frac{du}{dw}(w) \), where \( w \in [0, 2\pi) \). A detailed explanation of this block’s construction can be found in the Appendix, Subsection VII-A.

Finally, block \( \text{Sim. Printer} \) simulates the printing process of the reference \( C_{\text{ref}} \), where the \( z \)-axis timing pulley utilized by the virtual printer is \( C_{i-1} \), and the printed object is \( C_i \). Notice that, in the ideal case, when \( C_{i-1} \) is the ideal circular pulley and \( \frac{du}{dw}(w) = 1 \) for all \( w \in [0, 2\pi) \), we find that both matrices, \( C_{i-1} \) and \( C_i \), are equal. Since we utilize a 200 steps-per-revolution motor, the effect of using a non-uniform pulley is averaged over each step, e.g., given the angular position \( w_0 + i\Delta \) after the \( i \)-th step with \( i \in \{0, \ldots, 199\} \),

where \( \Delta \) is the pulley’s angular displacement per motor’s step, the \( (i+1) \)-th belt’s change of position after a following step is calculated as

\[
\Delta u_{i+1} = \int_{w_0 + i\Delta}^{w_0 + (i+1)\Delta} \frac{du}{dw}(s) \, ds. \tag{4}
\]

An example with illustrations for the simulation process involving the three blocks can be found in the Git repository [25].

C. Analytic Model

We construct an entirely analytic model for the experiment described in Section II and we utilize this model to find the iteration’s limits. We also verify whether these limits are stable points. To do so, we model the experiment using recursive sequences and utilize the Staircase Theorem [28].
where $r$ is the ideal pulley’s outer radius. If the belt and its contact with the timing pulley are ideal and we only consider one rotation orientation, the integral in (5) is equal to the perimeter of the utilized pulley’s convex hull. Therefore, in this section we model the belt-pulley relationship with (2) instead of utilizing $r$ to relate $\Delta u$ with $\Delta w$, we utilize the value of $[5]$. To study global stability, we utilize the following theorem:

**Theorem 1 (Staircase Theorem [28]):** Let $h(v)$ be a continuous real-valued function on the open interval $(a, b)$. Let $p$ represent an inner point of $(a, b)$. If $v < h(v) \leq p$ on $(a, p)$ and $p \leq h(v) < v$ on $(p, b)$, then the recursive sequence $x_i = h(x_{i-1})$ converges to the fixed point $p$ for all initial point $x_0 \in (a, b)$. 

**Proof:** [Staircase Theorem] Let $x_0 \in (a, p)$ be an initial point for the recursive sequence $h$. Assume $\lim_{i \to \infty} h(x_i) \neq p$. Since $x_i = h(x_{i-1})$ is an increasing and bounded sequence (bounded by $p$) in $(a, p)$, there must exist $q \in (x_0, p)$, such that $\lim_{i \to \infty} h(x_i) = q$; however, by hypothesis, since $q \in (a, p)$, we have that $h(q) < q$ which contradicts the continuity of $h$. The proof for interval $(p, b)$ is analogous.

We use a triangular pulley as the starting point $C_{i,1}^A$ and we iterate with ideal circular pulleys as reference $C_{i,1}^A$ – see Section II and Fig. 3. Since we utilize (5) to approximate the relationship belt-pulley, any printed body $C_{i,1}^A$ is essentially a stretched or compressed version of the reference $C_{i,1}^A$ in the x-axis direction, and then, since the reference has a circular shape, every analytic printed object $C_{i,1}^A$ will have an elliptical shape. We summarize this statement as follows:

**Assumption 1:** When the reference $C_{i,1}^A$ is the ideal circular pulley, all printed pulleys $C_{i,1}^A$ will have an elliptical shape.

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Since we approximate the printed pulleys $C_{i,1}^A$ with ellipses, and the integral in $[5]$ is approximated by the perimeter of the x-axes pulley’s convex hull, we need an expression for an ellipse’s perimeter. We utilize Ramanujan’s approximation [29] to estimate the ellipse perimeter

$$p \approx \pi \left[3(k + l) - \sqrt{(3k + l)(k + 3l)}\right],$$

where the semi-axis lengths $k$ and $l$ are visible in Fig. 9 and correspond to the ellipse $x^2/k^2 + y^2/l^2 = 1$.

We denote with $H$ and $W$ the ellipse’s height and width, respectively. Since the printer’s y-axis works ideally, any printed ellipse $C_{i,1}^A$ will have height $H_i$ equal to the ideal pulley’s height, i.e., $H_i = 2r$. Hence, we are interested in the evolution of the width $W_i(C_{i,1}^A)$, where $C_{i,1}^A$ is the i-th element of the sequence that we are modeling.

- First iteration: $W_1 = W_i \alpha p_0 = \frac{p_0}{2\pi}$, where $p_0$ is the convex hull’s perimeter for the initial arbitrary timing pulley $C_{0,1}^A$ and $W_0 = 2r$ is the ideal pulley width.
- i-th iteration: $W_i = W_i \alpha p_{i-1} = \frac{p_{i-1}}{2\pi}$, where $p_{i-1}$ is the approximate diameter of the $(i-1)$-th printed elliptical pulley.

From the previous i-th iteration equation and [9], we define the continuous function $f(v)$ with $v \in (0, \infty)$, such that $W_i = f(W_{i-1})$ as

$$W_i = f(W_{i-1}) = \frac{1}{2} \left[3(2r + W_{i-1}) - \sqrt{12r^2 + 20rW_{i-1} + 3W_{i-1}^2}\right].$$

1) **Fixed points:** We now look for fixed points the recursion (7). Let $\alpha$ be a fixed point. Such fixed point solutions $\alpha$ must satisfy

$$\alpha^2 \left(-\frac{1}{2}\right) + \alpha(-2r) + 6r^2 = 0. $$

Since the width is a positive quantity, the only admissible solution and fixed point is $\alpha = 2r$.

2) **Global stability:** We use Theorem 1 since it provides necessary and sufficient conditions for global stability. We verify the hypothesis of Theorem 1 on the domain of $f$, and the fixed point $\alpha = 2r$. Since the possible widths are positive, we verify for all $v \in (0, \infty)$.

We start by verifying if $v < f(v)$ when $v \in (0, \alpha)$:

$$v < f(v) = \frac{1}{2} \left[3(2r + v) - \sqrt{12r^2 + 20rv + 3v^2}\right]$$

leads to $2v^2 + 8rv - 24r^2 < 0$, which is true when $v \in (0, \alpha)$. The second condition is $v > f(v)$ when $v \in (\alpha, \infty)$:

$$v > f(v) = \frac{1}{2} \left[3(2r + v) - \sqrt{12r^2 + 20rv + 3v^2}\right]$$

leads to $2v^2 + 8rv - 24r^2 > 0$, which is true when $v \in (\alpha, \infty)$.

Then, by Theorem 1 $\alpha = 2r$ is a globally stable point in the interval $(0, \infty)$ for the recursive sequence $W_i = f(W_{i-1})$, and then the ellipse $x^2/r^2 + y^2/r^2 = 1$, which is the circular ideal pulley, is a globally stable shape.
V. Models Validation

We present the results from the models, and we compare them with the experimental observations. We validate by choosing another two initial conditions for the experiment and comparing the experiments’ outputs with the models’ outputs.

The main result is the actual convergence and self-repair of the printer in the experimental approach for all the different initial conditions, and how the simulation and analytic approaches predicted the same results under simplified models.

In Fig. 10 we present the experimental observations –first row–, the output from the computational model –second row–, and the output from the analytic model –third row–. Notice that we added a representation for the initial condition in the analytical model ($C_1^{A,1}$), which does not look like a triangular pulley. By construction, the analytic model only utilizes the convex hull’s perimeter of the initial condition. Even when we use a triangular pulley as the initial condition, in Fig. 10 we added the equivalent ellipse with the same perimeter to complete the visualization.

Figures 11 and 12 shows the results for the validations. In Fig. 11, the initial condition is an 8 teeth pulley with an outer radius of 4 mm while in Fig. 12 the initial condition is a rectangular pulley with dimensions 2.4 mm × 12 mm.

We observe convergence to the ideal circular pulley in the experiment and validations. Both models predict the same behavior. Regarding the step-wise comparison between experiment and models, we observe that the computational model approximates the experimental outputs better than the analytic model, which is expected since the computational model utilized fewer approximations.

VI. Conclusions and Future Work

We introduced and modeled an experiment to achieve self-repair in robots. We utilize mathematical modeling to support the experimental observations. The experiment, simulations and analytical approaches show convergence after few iterations to the needed part to repair the robot when the correct reference is utilized.

We conclude that 3D printing is very promising in the area of self-replicating and self-repair machines. We expect this experiment to motivate this research area and helps to
solve problems in situations where autonomous robots need to survive challenging environments or scenarios such as space exploration.

The field of self-replicating, self-repair machines is up-and-coming and remains to be developed. For instance, the following questions emerge as a consequence of this study:

- We assumed that the printer has enough precision to print the ideal pulley, but what if the precision is not sufficient? We recall Feynman’s talk [24], where he suggests the need to increase the precision at each step.
- We proved global stability in the analytic approach. However, it is reasonable to think that, in reality, some particular pulleys—or, for instance, the complete absence of a timing pulley in the printer—will nullify the hypotheses and break the global stability for the ideal timing pulley. We would like to find what are the minimum hypotheses to guarantee global stability.
- The experiment and models are open-loop systems, where the 3D printer has no information about which part or parts are imperfect or degraded. How about printers learning to adapt to their imperfections via feedback?
- In the same line, we can study the best feedback channel (vision sensor, encoders, etc.) and answer questions like which feedback?
- How about if several elements are imperfect (e.g., more pulleys and beams)? At what point does the self-repair process break down? Also, the order in which the hypotheses and break the global stability for the ideal timing pulley.
- Finally, we want to explore how this work can be extended to more generic robots.

VII. APPENDIX

A. Computation of $\frac{db}{dm}$ for the Computational Model

In the following construction, we will only consider the clockwise rotation for the $x$-axis timing pulley.

Fig. 13: Convergence plot for all the three simulation. The plot shows the relative difference of the output pulley $C_i$ with respect to the ideal reference $C_{ref}$ and with respect to the final output $C_0$, for each iteration.

![Fig. 13: Convergence plot for all the three simulation. The plot shows the relative difference of the output pulley $C_i$ with respect to the ideal reference $C_{ref}$ and with respect to the final output $C_0$, for each iteration.](image1)

Let $\{P_1, P_2, \ldots, P_n\}$ be the vertices for the convex set $C$ in $\mathbb{R}^2$ with $n \in \mathbb{N}$ vertices. Let $\triangle_1, \triangle_2, \ldots, \triangle_n$ be the set of triangles which define the convex hull $C$. Since the set $C$ is convex, at each time one (and only one) vertex is pulling on the timing belt. Also, only one vertex controls the belt’s movement, until the next vertex takes its place. As a consequence, we are interested in finding the set of transition angles $\{w_1, w_2, \ldots, w_n\}$, with the corresponding initial and final values for $m$: $\{m_{1,1}, m_{2,1}, \ldots, m_{n,1}\}$ and $\{m_{1,1}, m_{2,1}, \ldots, m_{n,1}\}$, i.e., $m \in [m_{1,1}, m_{1,1}]$ while the vertex $P_i$ is controlling the belt’s movement.

In Fig. 14 we observe a picture of the instant moment of alignment for the triangle’s vertices $P_{i-1}$ and $P_i$, with the belt-bearing contact point $Q$. At the alignment position, the triangle $\Delta_i$ stops controlling the belt’s movement, and the triangle $\Delta_{i-1}$ starts controlling the belt. In the same way, point $P_i$, which was pulling from the belt, is substituted by the point $P_{i-1}$. The value $r_y$ is the bearing’s radius, and $r_x$ is the distance between the bearing and pulley rotation points.

Fig. 14: Alignment position for the $i$-th triangle from the convex hull.

![Fig. 14: Alignment position for the $i$-th triangle from the convex hull.](image2)

In Subsection VII-B we provide the trigonometric equalities used to find the values of all variables during the transition time. From the position in Fig. 14 vertex $P_i$ is pulling from the timing belt, and we want to calculate the change in the belt’s position due to a slight rotation of the pulley.

Since $\frac{dm}{dw} = 1$ and $\frac{du}{db} = 1$, we calculate $\frac{du}{dm} = \frac{db}{dm} = \frac{du}{db} \cdot \frac{dm}{dw}$. Then, for the triangle $\Delta_i$, we derive

$$b(m) = \sqrt{\frac{r_{i-1}^2 + r_{xy}^2 - 2r_{i-1}r_{xy} \cos(m)}} \quad (9)$$

to find

$$\frac{db}{dm}(m) = \frac{1}{2} \frac{2r_{i-1}r_{xy} \sin(m)}{\sqrt{\frac{r_{i-1}^2 + r_{xy}^2 - 2r_{i-1}r_{xy} \cos(m)} \quad (10)}}$$

In the following construction, we will only consider the clockwise rotation for the $x$-axis timing pulley.

Fig. 15: Pictures before (left) and after (right) the transition time.

![Fig. 15: Pictures before (left) and after (right) the transition time.](image3)

$$\text{Final output}$$

The field of self-replicating, self-repair machines is up-
where \( m \in [m_i^-, m_i^+] \). Recall that
\[
\sum_{i=1}^{n} \mu \left( \left[ m_i^-, m_i^+ \right] \right) = 2 \pi, \text{ where } \mu(\cdot) \text{ is the Lebesgue measure, so we can construct an analytic expression of }
\frac{du}{dw}(u) \text{ for the complete pulley’s revolution by concatenating the value of (10) for each segment } [m_i^-, m_i^+] \text{ with } i \in \{1, \ldots, n\}.
\]

**B. Trigonometric Identities**

We provide the trigonometric identities utilized during Subsection VII-A to calculate the instant derivative in the computational model. We refer to Fig. 14 to identify the variables. \( r_x, r_y, r_i, \) and \( a_i \) are well defined, and are considered data provided by the block CH. We assume alignment between \( Q, P_{i-1}, \) and \( P_i \). The following deduction reflects what is implemented on Git [25]:

1) Calculate fixed parameters: \( r_{xy} = \sqrt{r_x^2 + r_y^2}, \alpha = \arctan \left( \frac{r_y}{r_x} \right), \beta = \pi - \alpha - \pi/2. \)

2) Calculate the angles for triangle \( \Delta_1 : \gamma_i = \arccos \left( \frac{r_{i-1}^2 + r_i^2 - a_i^2}{2r_{i-1}r_i} \right), \psi_i = \arccos \left( \frac{r_{i-1}^2 + r_i^2 - a_i^2}{2r_{i-1}r_i} \right), \) and \( \omega_i = \pi - \gamma_i - \psi_i. \)

3) Find value of \( b : n = \arcsin \left( \frac{r_{i-1}}{r_{xy}} \sin (\pi - \psi_i) \right), m = \psi_i - n, \) and \( \beta = r_{i-1} \sin (\pi - \psi_i) \).

4) Find transition angles: \( m_i^+ = m_i^- + \Delta m_i, \) \( \Delta m_i = m_i^- + \gamma_i - m_{i+1}^- \) for \( i \in \{1, \ldots, n - 1\}, \) and \( \Delta m_n = m_n^- + \gamma_n - m_1^-. \)

**C. Theoretical Limits**

In Subsection IV-C we prove that, when the reference \( C_{ref}^{A,1} \) is the ideal timing pulley, it is a globally stable point. In addition, utilizing Assumption 1 we get that, for any initial shape \( C_0^{A,1} \), the ideal circular reference \( C_{ref}^{A,1} \) is a globally stable point for the iteration, which guarantees self-repair for any imperfection or degradation in the pulley. Finally, we observe that the sequence always converges to the ideal reference \( C_{ref}^{A,1} \) for all positive values of \( r \), which implies that any circular pulley that the printer is designed for is a globally stable point.

**REFERENCES**


