A Compact Super Resolution Phase Retrieval Method for Fourier Measurement

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Abstract—The method of super-resolution phase retrieval for Fourier phaseless measurement extends the signal model of phase retrieval from the discrete domain to a more realistic continuous domain. However, the existing three-stage solving framework contains unnecessary redundancy, so its core steps are loosely connected and have high computational complexity. Aiming at this, new algorithms are proposed for the following two key steps respectively, so as to achieve a compact super-resolution phase retrieval framework. First, for the super-resolution stage of the auto-correlation function, a non-redundant algorithm is proposed by merging items containing the same information. Then, a low complexity iterative algorithm is proposed for the support recovery by employing the relationship of elements we obtain from the previous step. Sufficient numerical simulations verify the effectiveness of the proposed algorithms and each step has better noise robustness compared with the existing method, but worse robustness when they combine. Therefore, we test the performance of different subalgorithm combination strategies, providing insight in meeting diverse requirements for noise robustness and computational complexity. Moreover, the effectiveness of the proposed method in a real system is verified by hardware experiment.

Index Terms—continuous domain, phaseless Fourier measurement, phase retrieval, super-resolution.

I. INTRODUCTION

In many measurement processes, it is usually more difficult to accurately measure the phase of signals than the amplitude. There are two main reasons for this fact: one is that the phase of some signals cannot be measured with existing technology [1]. The other is that in some measurement processes, the phase information is easy to be destroyed and lost [2]. Such a phaseless measurement process widely exists in applications such as mobile terminal measurement [3], near-field antenna measurement [4], [5], and electromagnetic inverse scattering [6]. In view of this problem, people began to realize that under certain conditions, the original signal can be reconstructed based on the amplitude-only information obtained from some linear measurements. This fundamentally provides a way to solve the estimation problem under a phaseless measurement, and the process of recovering the original signal from the amplitude-only measurement is called Phase Retrieval (PR), also known as solving quadratic equations [7]. Phase retrieval theory originated from crystallography, and has been widely used in X-ray imaging [8], [9], radar waveform optimization [10], optical image encryption [11], direction-of-arrival (DOA) estimation [12], [13], terahertz coded-aperture imaging [14] and more [15].

The research upsurge of phase retrieval is triggered by its successful application in imaging of micrometre-sized non-crystalline specimens [16]. After that, a plethora of algorithms and different types of phase retrieval problems have been widely exploited. The algorithm type expands from the initial alternating projection methods [17]–[19] to the convex optimization algorithms represented by PhaseLift [20]. These methods solve the problem of lack of convergence guarantee, but need to solve the problem in a space of higher dimension, so they suffer a higher computational complexity. Therefore, non-convex optimization algorithms represented by Wirtinger Flow (WF) [21] and its variants [22] are proposed. In terms of problem connotation, according to the different measurement process, phase retrieval is extended to random phase retrieval [23]–[25], Fourier phase retrieval [26]–[28] and convolution phase retrieval [29]. Among them, Fourier phase retrieval is the most widely used, which is also the problem type faced in this paper. However, it is noteworthy that all of the above algorithms are oriented to discrete signals, while the actual signals are often continuous. Such model mismatch often leads to inevitable approximate errors in the solution. Aiming at this problem, scholars have expanded the research field of phase retrieval. In [30], Prony’s method is first used to realize phase retrieval of a specific type of continuous signals. Subsequently, the concept of continuous domain phase retrieval is first proposed in [31]. Because the process of the continuous-domain phase retrieval has theoretically infinite resolution, it is also called super-resolution phase retrieval.

In this work, the background of [31] is considered, i.e., the super-resolution phase retrieval of the parameterized signal in continuous domain given the intensity of Fourier measurements. The existing solution frameworks deal with the time-domain sparse signal which can be completely determined by two kinds of parameters: support and amplitude. First, the support interval set and the amplitudes products set are obtained by super-resolution process of the auto-correlation function (ACF), and then the support and amplitudes are estimated successively. Although this framework is efficient and has sufficient theoretical guarantee, it still has some disadvantages. One of them is the redundancy of solution process, which makes the key steps of the two cores loosely connected. In addition, this directly leads to the high com-
computational complexity of the overall method. To solve this problem, we have proposed an algorithm that can eliminate the redundancy in super-resolution results of ACF [32]. However, this solution does not reduce the complexity of the whole process, limiting its availability in practice. In this paper, we propose a two-step support recovery algorithm based on non-redundancy positive support interval set for the support estimation process. By combining two algorithms, a compact super-resolution phase retrieval framework with low computational complexity is proposed. The effectiveness of the proposed framework is verified by both numerical simulations and hardware experiment. In addition, as opposed to lower complexity, the proposed subalgorithms combination has worse noise robustness compared to the existing method. To meet the different requirements for noise resistance and low complexity in practical applications, the corresponding combination strategy over the available subalgorithms is given.

We summarize the contributions of this paper as follows:

1) A super-resolution algorithm for ACF with redundancy removed is proposed. By using the symmetry of the results obtained from the super-resolution of ACF, we first combine the two complex exponentials containing the same information into one term by Euler’s formula. Then the cosine n-angle formula is utilized to transform the combined result into a form that can still be solved by Prony’s method. Therefore, the estimation of the non-redundant positive support interval is realized.

2) A two-step support recovery algorithm with low complexity is proposed. In the first step, the characteristics of the elements in the positive support interval are used to determine the range of support to be estimated. The second step is to use the greedy iterative algorithm to determine the optimal support estimation results within this range. It is worth noting that the key to this algorithm is the narrowing of the support search range in the first step. This makes the proposed algorithm reduce the complexity of the existing algorithm from $O(2^K)$ to $O(K^3)$ when the number of support is $K$.

3) A large number of numerical simulations are performed to examine the performance of the proposed algorithms. Our method can realize super-resolution phase retrieval of sparse signals and spline signals in continuous domain given the intensity of Fourier transform. Compared with the existing algorithm, the proposed method has better noise robustness at each step, but worse robustness when they combine. Therefore, we give a combination strategy of the subalgorithms when facing different requirements. In addition, the influence of scattering function energy overlap ratio on algorithm performance is also investigated. Finally, the effectiveness of the proposed algorithms is further verified in a real hardware system.

The rest of this paper is organized as follows: Section II states the problem and the existing super-resolution phase retrieval framework; Section III presents our research motivation; Super-resolution method of ACF with non-redundancy results is proposed in section IV; Section V gives the low complexity support recovery algorithm and briefly introduces the existing amplitudes recovery strategy; Section VI provides the complexity analysis; Numerical simulations are performed in Section VII while hardware experiment is implemented in Section VIII; Section IX concludes this paper.

**Notation:** Continuous time signals enclose the independent variables in parentheses $(\_)$, while discrete time signals are represented by square brackets $[\_]$. * is the convolution operation, $\lfloor \_ \rfloor$ represents the round down operation, $\max(\_)$ is to get the largest element. Obtaining the $\ell_2$-norm is represented by $\| \_\|_2$. $A \setminus B$ means to remove subset $B$ from set $A$.

**II. PROBLEM STATEMENT & SUPER RESOLUTION PHASE RETRIEVAL FRAMEWORK**

Super-resolution phase retrieval framework can be applied to parameterized signals: sparse signals and spline signals. Both of them have the same solving process, so this paper takes sparse signals as an example to state the problem.

**A. Problem Statement**

Consider the sparse signal with a compact structure as

$$x(t) = \sum_{k=0}^{K-1} c_k \phi(t - t_k) = \sum_{k=0}^{K-1} c_k \delta(t - t_k) * \phi(t),$$

(1)

where $\delta(t)$ is Dirac pulse, $\phi(t)$ is the scattering function, and $K$ is the number of support. To avoid a heavier notation, $c_k$ and the range of $\phi(t)$ are specified to belong to $\mathbb{R}$, and the results of this paper can be easily extended to $\mathbb{C}$. It can be seen that the sparse signal $x(t)$ can be completely determined by amplitudes $\{c_k\}_{k=0}^{K-1}$ and support $\{t_k\}_{k=0}^{K-1}$ on the premise that scattering function $\phi(t)$ is known. So we define

$$x^s(t) = \sum_{k=0}^{K-1} c_k \delta(t - t_k)$$

(2)

as the structure function. Then $x(t)$ can be regarded as $x^s(t)$ filtered by a known scattering function $\phi(t)$. The Fourier transform (FT) of signal $x(t)$ is defined as

$$\mathcal{F}x(\omega) := \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t} dt,$$

(3)

where $\omega$ is the frequency variable. In a phaseless measurement, only the intensity of the discrete samples of $\mathcal{F}x$ can be obtained,

$$y[n] := |\mathcal{F}x(n\Omega)|^2, \quad n = 0, 1, ..., N-1,$$

(4)

where $\Omega$ is the sampling frequency and $N$ is the number of samples.

In addition, we know from [33] that the ACF $a(t)$ of $x(t)$ can be determined by $|\mathcal{F}x(\omega)|^2$ through the inverse FT:

$$a(t) = x(t) * x(-t) = [x^s(t) * x^s(-t)] * \psi(t) = \mathcal{F}^{-1}[|\mathcal{F}x(\omega)|^2],$$

(5)

where $\mathcal{F}^{-1}$ is the inverse FT operator and $\psi(t)$ is the ACF of the scattering function $\phi(t)$. Therefore, the discrete samples of the FT of $a(t)$ is exactly equivalent to $y[n]$, i.e.,

$$A[n] = \mathcal{F}a(n\Omega) = |\mathcal{F}x(n\Omega)|^2 = y[n], \quad n = 0, 1, ..., N-1.$$  

(6)
Then the phase retrieval problem in continuous domain in this paper can be given.

Problem 1: Given the phaseless measurements $y[n]$, i.e., the discrete samples $A[n]$ of ACF, recover the support and amplitudes of parameterized continuous signal $x(t)$.

### B. Super Resolution Phase Retrieval Framework

The ACF $a(t)$ and $x(t)$ have the same structure, which makes the solution of Problem 1 possible.

$$a(t) = \left[ \sum_{k=0}^{K-1} \sum_{l=0}^{K-1} c_k c_l (t - (t_k - t_l)) \right] \ast \psi(t).$$  \hfill (7)

Thus, the phaseless measurement can be obtained as

$$A[n] = \mathcal{F}a(n\Omega) = |\mathcal{F}x(n\Omega)|^2$$
$$= \left[ \sum_{k=0}^{K-1} \sum_{l=0}^{K-1} c_k c_l e^{-j\Omega(t_k - t_l)} \right] \cdot |\Phi(n\Omega)|^2,$$  \hfill (8)

where $\Phi(n\Omega)$ is the discrete samples of $\mathcal{F}\phi(\omega)$.

The existing method divide the solution of Problem 1 into three stages. In the first stage, discrete samples $A[n]$ are used to recover the continuous ACF. When the scattering function is a Sinc function, near $n\Omega = 0$, (8) can be written as:

$$A[n] = \sum_{k=0}^{K-1} \sum_{l=0}^{K-1} c_k c_l e^{-j\Omega(t_k - t_l)}, \ n = 0, 1, \ldots, N - 1.$$  \hfill (9)

Recovering (7) from (9) can be solved by Prony's method [34]. Therefore, support interval $D$ and amplitudes products $A$ can be estimated, where

$$D := \{d_{kl}\}_{k,l=0}^{K-1} := \{t_k - t_l\}_{k,l=0}^{K-1},$$

$$A := \{a_{kl}\}_{k,l=0}^{K-1} := \{c_k c_l\}_{k,l=0}^{K-1}.$$  \hfill (10)

The next stage uses the iterative algorithm to estimate the support $T := \{t_{kl}\}_{k=0}^{K-1}$ from the interval set $D$. Finally, the third stage recovers $C := \{c_k\}_{k=0}^{N-1}$ from the amplitudes products $A$.

The measurement and recovery process of the super-resolution phase retrieval are summarized in Fig. 1.

### III. RESEARCH MOTIVATION

It can be seen that the process of super-resolution phase retrieval can be divided into two parts, the super-resolution of ACF and the estimation of support and amplitudes. The key to connecting the two parts is the support interval $D$ and the products of amplitudes $A$. It is evident from the structure of ACF as shown in (7), the parameters $k$ and $l$ have the same value range, so the elements in $D$ and $A$ must exist in pairs. If there is element $t_k - t_l$ in $D$ there must be another element $t_l - t_k$, and they are estimated as two unknown parameters. However, such two parameters, which are negatives of each other, contain exactly the same information. Likewise, their corresponding amplitudes products $c_k c_l$ and $c_l c_k$ are also redundant. Based on this symmetry, the support interval $D$ can be divided into the following three parts:

$$\begin{align*}
D^+ & := \{d_{kl}\}_{k,l=0}^{K-1} := \{t_k - t_l \ | \ t_k > t_l\}_{k,l=0}^{K-1}, \\
D^0 & := \{0\} := \{t_k - t_l \ | \ t_k = t_l\}_{k,l=0}^{K-1}, \\
D^- & := \{d_{kl}\}_{k,l=0}^{K-1} := \{t_k - t_l \ | \ t_k < t_l\}_{k,l=0}^{K-1},
\end{align*}$$

where $D^+$ and $D^-$ are redundant and either of these two subsets contains all the information about the support intervals. As a result, such redundant support intervals make it difficult to exploit the interrelationships between the elements. Therefore, the existing estimation supported iterative algorithm needs to traverse all elements in the support interval every time it updates the estimated support [31]. When there are $K$ elements in the support, the calculation complexity reaches $O(K^6)$, which makes the existing framework difficult to apply in practice. However, each element in the support interval is not isolated. Therefore, if the redundancy can be eliminated, the search scope can be narrowed by mining their relationship with each other.

Based on the above shortcomings of existing method and our insights, this paper proposes the following two algorithms:

1) A super-resolution algorithm for ACF with redundancy removed is proposed. It merges pairs of support intervals containing the same information into one term. The combined terms are then transformed into a form that can still be solved using Prony's method. In the end, only the positive support interval $D^+$ is output instead of $D$.

2) A low complexity iterative algorithm for support recovery is proposed. The proposed algorithm uses the relationship between the elements contained in $D^+$ to determine the range of estimated support $\hat{T}$. This dramatically reduces the number of traversals required.

### IV. ACF SUPER RESOLUTION WITH NON-REDUNDANT

In this section, a super-resolution algorithm for ACF with redundancy removed is proposed. The signal model for this algorithm can be sparse signal or spline signal, the former is the basis of the latter, so it is introduced first.

#### A. Sparse Signal

According to (3), The FT of $x(t)$ is

$$\mathcal{F}x(\omega) = \left( \sum_{k=0}^{K-1} c_k e^{-j\omega t_k} \right) \cdot \Phi(\omega),$$

where $\Phi(\omega)$ is the FT of $\phi(t)$. Thus the phaseless measurement in the continuous domain is

$$|\mathcal{F}x(\omega)|^2 = \left[ \sum_{k=0}^{K-1} c_k e^{-j\omega t_k} \right]^2 \cdot |\Phi(\omega)|^2$$

$$= \sum_{k=0}^{K-1} |c_k|^2 \cdot |\Phi(\omega)|^2$$

$$= \sum_{k=0}^{K-1} \sum_{l=0}^{K-1} c_k c_l e^{-j\omega(t_k - t_l)} \cdot |\Phi(\omega)|^2.$$
Fig. 1. The process of measurement and recovery of super-resolution phase retrieval. The samples of the intensity of the FT of the original signal \( x(t) \), which is obtained from the structure function \( x^s(t) \) filtered by the scattering function \( \phi(t) \), are the phaseless measurement \( A[n] \). This process can also be equivalent to: The ACF of \( x^s(t) \) is filtered by scattering function \( \psi(t) \) to get \( a(t) \), and finally the samples of its FT amplitudes are obtained. The recovery process of super-resolution phase retrieval: super-resolution of ACF is achieved by using phaseless measurement \( A[n] \), and then the support and amplitudes of \( x^s(t) \) are estimated. The blue line represents continuous signals and the red line represents discrete samples.

Since the scattering function \( \phi(t) \) must not be a meaningless constant function with a value of zero, we have,

\[
\frac{\|F_x(\omega)\|^2}{|\Phi(\omega)|^2} = \sum_{k=0}^{K-1} \sum_{l=0}^{K-1} c_k c_l e^{-j\omega(t_k-t_l)}. \tag{14}
\]

Set \( \gamma_{\omega(kl)} = e^{-j\omega(t_k-t_l)} \), then (14) can be rewritten as

\[
\frac{\|F_x(\omega)\|^2}{|\Phi(\omega)|^2} = \sum_{k=0}^{K-1} \sum_{l=0}^{K-1} c_k c_l \gamma_{\omega(kl)}. \tag{15}
\]

Since \( k \) and \( l \) have exactly the same range of values, \( c_k c_l \gamma_{\omega(kl)} \) and \( c_k c_l \gamma_{\omega(0(K-1))} \) must exist in pairs, such as \( c_0 c_{K-1} \gamma_{\omega(0(K-1))} \) and \( c_{K-1} c_0 \gamma_{\omega((K-1)0)} \). Using Euler’s formula, such a pair of elements can be combined into

\[
c_0 c_{K-1} \gamma_{\omega(0(K-1))} + c_{K-1} c_0 \gamma_{\omega((K-1)0)} = 2c_{K-1} c_0 \cos(\omega(t_{K-1}-t_0)). \tag{16}
\]

Therefore, we can rewrite (15) as

\[
\frac{\|F_x(\omega)\|^2}{|\Phi(\omega)|^2} = \|c\|^2 + 2|c_1 c_0 \cos(\omega(t_1-t_0)) + c_2 c_0 \cos(\omega(t_2-t_0)) + \ldots + c_{K-1} c_0 \cos(\omega(t_{K-1}-t_0))|. \tag{17}
\]

where \( \|c\|^2 \) is a known quantity. According to Parseval’s theorem, when the scattering function meets the two conditions that the scattering function has finite length and does not overlap at different support positions of the original signal \( x(t) \), we have

\[
\int_{-\infty}^{+\infty} |F_x(\omega)|^2 d\omega = c_0^2 \int_{-\infty}^{+\infty} |\phi(t)|^2 dt + c_1^2 \int_{-\infty}^{+\infty} |\phi(t)|^2 dt + \ldots + c_{K-1}^2 \int_{-\infty}^{+\infty} |\phi(t)|^2 dt. \tag{18}
\]

So the constant term in (17) is

\[
\|c\|^2 = \int_{-\infty}^{+\infty} |F_x(\omega)|^2 d\omega \int_{-\infty}^{+\infty} |\phi(t)|^2 dt. \tag{19}
\]

Defining

\[
f(\omega) := \frac{1}{2} \left( \frac{|F_x(\omega)|^2}{|\Phi(\omega)|^2} - \|c\|^2 \right), \tag{20}
\]

and according to (17), we get

\[
f(\omega) = \sum_{k=0}^{K-1} \sum_{l=0}^{K-1} c_k c_l \cos(\omega(t_k-t_l)), \quad k > l. \tag{21}
\]

To avoid complex expressions, we set \( \alpha_m = c_k c_l, \beta_m = t_k - t_l, k > l \), then (21) can be rewritten as

\[
f(\omega) = \sum_{m=0}^{M-1} \alpha_m \cos(\omega\beta_m), \tag{22}
\]

where \( M = K(K-1)/2 \). Considering the cosine property \( \cos(\omega\beta_m) = \cos(-\omega\beta_m) \), we can further set \( \beta_m > 0 \), i.e.

\[
\{\beta_m\}_{m=0}^{M-1} = \{t_k - t_l \mid t_k > t_l\}_{k,l=0}^{K-1} = D^+. \tag{23}
\]

Interestingly, this comes with an additional constraint \( t_0 < t_1 < \cdots < t_{K-1} \). Therefore, the support and its positive interval can be rewritten as

\[
\begin{align*}
T &= \{t_k \mid t_0 < t_1 < \cdots < t_{K-1}\}_{k=0}^{K-1}, \\
D^+ &= \{\beta_m\}_{m=0}^{M-1} = \{d_{kl} \mid k > l\}_{k,l=0}^{K-1} = \{t_k - t_l \mid k > l\}_{k,l=0}^{K-1}.
\end{align*} \tag{24}
\]

Note that from the Observation 3 in [31], \( D \) will not contain the same elements. Then \( D^+ \) also contains no repeating

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elements. In actual measurement, we can only get discrete samples of (22),
\[
f(n\Omega) = \sum_{m=0}^{M-1} \alpha_m \cos(n\Omega \beta_m), \quad n = 0, 1, \ldots, N-1. \tag{25}
\]
Such a discrete form can be expanded as a sum of exponents by using the cosine n-angle formula [35]
\[
\cos(n\theta) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \left( \frac{n-k}{k} \right) \left( \frac{n-1-k}{k-1} \right) (-1)^k 2^{-n-2k} \cos^{n-2k}(\theta) \tag{26}
\]
where \( \binom{n}{k} \) is binomial coefficient.

Before giving a general result, we first present several heuristic steps. Define \( \{\varepsilon[n]\}_{n=0}^{N-1} \) as a sequence of exponentially weighted sum forms,
\[
\varepsilon[n] := \sum_{m=0}^{M-1} \alpha_m \left[ \cos(\Omega \beta_m) \right]^n. \tag{27}
\]
The first two terms of \( \varepsilon[n] \) can be derived directly from \( f(n\Omega) \),
\[
\begin{align*}
\varepsilon[0] &= \sum_{m=0}^{M-1} \alpha_m \left[ \cos(\Omega \beta_m) \right]^0 = f(0\Omega), \\
\varepsilon[1] &= \sum_{m=0}^{M-1} \alpha_m \left[ \cos(\Omega \beta_m) \right]^1 = f(1\Omega).
\end{align*}
\]
The natural next step is how we get the value of \( \varepsilon[2] \),
\[
\varepsilon[2] = \sum_{m=0}^{M-1} \alpha_m \left[ \cos(\Omega \beta_m) \right]^2. \tag{29}
\]
Using (25) and (26), we have
\[
f(2\Omega) = \sum_{m=0}^{M-1} \alpha_m \cos(2\Omega \beta_m)
= \sum_{m=0}^{M-1} \alpha_m \left[ 2 \cos(\Omega \beta_m)^2 - 1 \right]
= 2 \cdot \sum_{m=0}^{M-1} \alpha_m \cos(\Omega \beta_m)^2 - \sum_{m=0}^{M-1} \alpha_m,
\]
So we get
\[
\varepsilon[2] = \frac{1}{2} (f(2\Omega) + \varepsilon[0]). \tag{31}
\]
And also according to (25) and (26),
\[
f(3\Omega) = \sum_{m=0}^{M-1} \alpha_m \cos(3\Omega \beta_m)
= \sum_{m=0}^{M-1} \alpha_m \left[ 4 \cos(\Omega \beta_m)^3 - 3 \cos(\Omega \beta_m)^2 \right]
= 4 \cdot \sum_{m=0}^{M-1} \alpha_m \cos(\Omega \beta_m)^3 - 3 \cdot \sum_{m=0}^{M-1} \alpha_m \cos(\Omega \beta_m).
\]
So it is easy to get
\[
\varepsilon[3] = \frac{1}{4} (f(3\Omega) + 3\varepsilon[1]). \tag{33}
\]
At this point, a general expression can be given that
\[
\varepsilon[n] = 2^{1-N} (f(n\Omega) - G_n), \tag{34}
\]
where
\[
G_n := \sum_{m=0}^{M-1} \left( \left( \frac{n-k}{n} \right) \left( \frac{n-1-k}{k-1} \right) (-1)^k 2^{-n-2k} \cos^{n-2k}[n - 2k]. \tag{35}
\]
Since (34) is a recursive expression, the sequence \( \{\varepsilon[n]\}_{n=0}^{N-1} \) can be completely determined after \( \varepsilon[0] \) and \( \varepsilon[1] \) are determined.

The above process can be summarized as obtaining sequence \( \{\varepsilon[n]\}_{n=0}^{N-1} \) from the phaseless measurement \( f(n\Omega) \) of sparse signal \( x(t) \). So the problem becomes how to use \( \{\varepsilon[n]\}_{n=0}^{N-1} \) to achieve the super-resolution of the ACF, i.e. to calculate parameters \( \{\alpha_m\}_{m=0}^{M-1} \) and \( \{\beta_m\}_{m=0}^{M-1} \). Because \( \varepsilon[n] \) has the form of an exponentially weighted sum as shown in (27), the Prony’s method can be used to estimate \( \{\cos(\Omega \beta_m)\}_{m=0}^{M-1} \) [34], [36], [37]. Then, as long as the sampling interval satisfies
\[
\Omega < \frac{\pi}{\max(\beta_m)}, \tag{36}
\]
the inverse cosine function can be used to uniquely determine \( \{\beta_m\}_{m=0}^{M-1} \). In addition, the \( \{\alpha_m\}_{m=0}^{M-1} \) can be obtained by injecting \( \{\cos(\Omega \beta_m)\}_{m=0}^{M-1} \) in (27) and solving a linear system of equations.

**B. Spline Signal**

A spline signal \( \tilde{z}(t) \) is a signal whose derivative is a Dirac pulse train, so its \( r \)th derivative is given by
\[
\frac{d^r}{dt^r} \tilde{z}(t) = \sum_{k=0}^{K-1+r} c_k \delta(t - t_k). \tag{37}
\]
Similar to (1), scattering function \( \phi(t) \) is used to filter (37), then we have
\[
\frac{d^r}{dt^r} \tilde{z}(t) := \frac{d^r}{dt^r} (\tilde{z}(t) * \phi(t))
= \sum_{k=0}^{K-1+r} c_k \delta(t - t_k) * \phi(t). \tag{38}
\]
Applying the FT to (38), we now obtain
\[
(j\omega)^r F \tilde{z}(\omega) = \left( \sum_{k=0}^{K-1+r} c_k e^{-j\omega t_k} \right) \Phi(\omega), \tag{39}
\]
and thus
\[
\omega^{2r} |F \tilde{z}(\omega)|^2 = \left( \sum_{k=0}^{K-1+r} \sum_{l=0}^{K-1+r} c_k c_l e^{-j\omega(t_k-t_l)} \right) |\Phi(\omega)|^2. \tag{40}
\]
It is interesting that (40) and (13) have exactly the same form except for the scale factor \( \omega^{2r} \). Since both \( \omega \) and \( r \) are known parameters, it is very easy to extend the super-resolution of sparse signal ACF to spline signal.
V. LOW COMPLEXITY SUPPORT RECOVERY

In this section, a low complexity support recovery algorithm is proposed. In addition, the existing amplitudes estimation strategies are briefly introduced for completeness.

A. Ambiguity Analysis in Support Recovery

Support recovery is the next stage of super-resolution phase retrieval, which can be described as: given the positive support interval set $D^+$, recover the support $T$ of the original signal.

Before formally introducing the support recovery algorithm, we make some remarks. First, the trivial ambiguity is started following.

Remark 1: Even if the labels are known, only the supports with irremovable unknown shifts and reflections can be recovered from the positive support interval set.

To prove that, $\tilde{T} := -T + \delta t = \{-t_k + \delta t\}_{k=0}^{K-1}$ obtained by shifting and reflecting $T$ is given. Then the corresponding positive support interval set $\tilde{D}^+$ can be obtained as

$$\tilde{D}^+ = \{-t_k + \delta t\} - \{-t + \delta t\} \bigcap \tilde{D} = \{t_l - t_k | t_l > t_k\}_{K-1}^{0} = \tilde{D}^+.$$ (41)

Second, non-trivial ambiguities usually exist except shifts and reflections, but unique solutions can be obtained under certain condition.

Remark 2: If the support $t_k$ is randomly and independently drawn from a sufficiently smooth distribution, then the solution of support recovery is unique [38].

Although the interference of non-trivial ambiguity can be excluded, the positive support interval $D^+$ still corresponds to an infinite number of feasible solutions according toRemark 1. Therefore, we narrow the range of solutions by merging the trivial ambiguities to facilitate the subsequent performance evaluation. This is reasonable because the trivial ambiguity is a common phenomenon in phase retrieval problem and cannot be removed without other information [1], wherein many works about ambiguity elimination based on prior information or special measurement structure have been reported [28], [39], [40]. As this is not the focus of this study, we ignore the trivial ambiguities and consider it a successful reconstruction when reaching any trivial ambiguous estimation [32], [41]–[43].

Remark 3: Assuming

$$T = \{t_k | t_0 = 0, t_0 < t_1 < \ldots < t_{K-1}\}_{K-1}^{0},$$ (42)

then support recovery contains two possible solutions.

To show that, given

$$T' = \{t' | t'_0 \neq 0, t'_0 < t'_1 < \ldots < t'_{K-1}\}_{K-1}^{0},$$ (43)

to be a feasible solution to $D^+$. According to Remark 1, two ambiguity solutions $T_1$ and $T_2$ with $t_0 = 0$ can be obtained. Where $T_1$ is obtained by shifting $t'_0$:

$$T_1 = T' - t'_0 = \{t_0 = t'_0 - t'_0 = 0, \ldots, t_k = t'_k - t'_0, \ldots, t_{K-1} = t'_{K-1} - t'_0 | t_0 < t_1 < \cdots < t_{K-1}\}_{K-1}^{0}. $$ (44)

In addition, by reflection and shift $t'_{K-1}$, we have

$$T_2 = -T' + t'_{K-1} = \{t_0 = -t'_{K-1} + t'_{K-1} = 0, \ldots, t_k = -t'_k + t'_{K-1}, \ldots, t_{K-1} = -t'_{0} + t'_{K-1} | t_0 < t_1 < \cdots < t_{K-1}\}_{K-1}^{0}. $$ (45)

Therefore, we regard either of these two ambiguous solutions as a successful estimate.

B. Support Recovery

The proposed iterative algorithm for support recovery described below can be divided into two steps. First, we determine the range $S$ of the support to be estimated, and then the optimal estimated support $\tilde{T}$ is selected from $S$ by iterations. Before introducing step 1, some properties of positive support interval $D^+$ are given.

Property 1: For any element of $D^+$, we have

$$d_{kl} = d_{k(l-1)} + d_{k(l-2)} + \cdots + d_{l(l+1)},$$ (46)

This can be proved by the following simple expansion,

$$d_{k(l-1)} + d_{k(l-2)} + \cdots + d_{l(l+1)} = (t_k - t_{l-1}) + (t_{l-1} - t_{l-2}) + \cdots + (t_{l+1} - t_l) $$ (47)

According to Property 1, the elements in $D^+$ can be arranged in the following table:

$$
\begin{array}{ccccccc}
  d_{(K-1)0} & d_{(K-1)1} & \vdots & \vdots & \vdots \\
  d_{(K-2)0} & d_{(K-2)1} & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  d_{10} & d_{11} & \cdots & d_{(K-1)(K-3)} & d_{(K-1)(K-2)} \\
  d_{0} & d_{1} & \cdots & d_{(K-1)(K-3)} & d_{(K-1)(K-2)} \\
\end{array}
$$

There are $K-1$ rows in this table, with labels $1, 2, \ldots, K-1$ from bottom to top. The $i$-th row element is $\{d_{(i+1)i}\}_{i=0}^{K-1}$, which are the sum of the adjacent $i$ elements in the first row. Therefore, we define the elements $\{d_{(i+1)i}\}_{i=0}^{K-2}$ in the first row as the basic elements.

By mining Property 1, the order of the size of some elements in $D^+$ can be determined.

Property 2: Given positive support interval $D^+$, we have \(\max(D^+) = d_{K-1}0\) and

$$d_{k\ell} > d_{e(k-i)\ell}, \; i = 1, 2, \ldots, k - l - 1.$$ (49)

To show it, Property 1 can be used to expand $d_{k\ell}$, $d_{e(K-1)0}$ into a sum of the basic elements,

$$d_{k\ell} = d_{k(k-1)} + d_{k(k-2)} + \cdots + d_{\ell(l+1)};$$

$$d_{e(K-1)0} = d_{e(K-1)(K-2)} + \cdots + d_{e(k-1)k} + d_{k(k-1)} + \cdots + d_{e(l)l};$$

$$d_{k\ell} = d_{k(k-1)} + \cdots + d_{e(k-i-1)k} + \cdots + d_{e(l)l};$$

$$d_{e(k-1)0} = d_{e(k-1)(k-2)} + \cdots + d_{e(l)l} + \cdots + d_{e(l)l}.$$ (50)
In noisy case, the elements in $\mathcal{P}$ are the $K-2$ pairs whose sum is closest to $\max \left( \hat{D}^+ \right)$. According to Property 3, we have

$$\mathcal{P} = \{ \hat{d}_{10}, \hat{d}_{(K-1)1}, \hat{d}_{20}, \hat{d}_{(K-1)2}, \ldots, \hat{d}_{(K-2)0}, \hat{d}_{(K-1)(K-2)} \}. \tag{56}$$

If $\hat{t}_0 = 0$ is reasonably set, we define

$$\hat{T}^+ := \{ \hat{d}_{10}, \hat{d}_{20}, \ldots, \hat{d}_{(K-1)0} \} \subset \hat{T} = \{ 0, \hat{T}^+ \}, \tag{57}$$

where $\hat{d}_{10} = \hat{t}_1 - \hat{t}_0 = \hat{t}_2 - \hat{t}_1 = \hat{t}_0 = \hat{t}_{(K-1)}$. Define $\mathcal{S} := \{ \mathcal{P}, \max (\hat{D}^+) \}$, and then from (54) (56) (57) we get

$$\hat{T}^+ \subset \mathcal{S}. \tag{58}$$

**Step 2:** The estimated support $\hat{T}^+$ is obtained by selecting the optimal subset from set $\mathcal{S}$ by iterations. Initialization is performed before iterations begin. According to (54) (57) (58), when $t_0 = 0$, $\max (\hat{D}^+) \in \hat{T}^+$, so this is our first initial value. The corresponding positive support interval sets of the two ambiguity solutions $\hat{T}_1 = \hat{T} - \hat{t}_0$ and $\hat{T}_2 = \hat{T} + \hat{t}_{K-1}$ in Remark 3 are defined as $\hat{D}_1^+$ and $\hat{D}_2^+$ respectively, then

$$\left\{ \begin{array}{l}
\hat{d}_{mn} = (\hat{t}_m - \hat{t}_0) - (\hat{t}_n - \hat{t}_0) = \hat{t}_m - \hat{t}_n, m,n \in \mathcal{D}_1^+; \\
\hat{d}_{(K-1-n)(K-1-m)} = (-\hat{t}_m + \hat{t}_{K-1}) - (-\hat{t}_n + \hat{t}_{K-1}) = \hat{t}_m - \hat{t}_n, m,n \in \mathcal{D}_2^+;
\end{array} \right. \tag{59}$$

where $m > n$. This means that $\mathcal{D}_1^+$ and $\mathcal{D}_2^+$ contain exactly the same elements in them, but their labels are symmetric. In addition, according to Property 2, the elements in $\mathcal{P}$ satisfy that

$$\left\{ \begin{array}{l}
\hat{d}_{(K-2)0} \geq \hat{d}_{(K-3)0} \geq \cdots \geq \hat{d}_{10} \\
\hat{d}_{(K-1)1} \geq \hat{d}_{(K-1)2} \geq \cdots \geq \hat{d}_{(K-1)(K-2)}.
\end{array} \right. \tag{60}$$

Therefore, $\max (\mathcal{P}) = \hat{d}_{(K-2)0}$ or $\max (\mathcal{P}) = \hat{d}_{(K-1)1}$, and their corresponding supports are trivial ambiguities to each other. Set

$$p_{\max 1} := \max (\mathcal{P}) = \hat{d}_{(K-2)0} = \hat{t}_2 \in \hat{T}^+, \tag{61}$$

which gives us second initial value. It is worth noting that, according to (56) (61),

$$p_{\max 0} = \hat{d}_{(K-1)(K-2)} \notin \hat{T}^+. \tag{62}$$

In summary, we can obtain the initialization of the estimated support and search scope as

$$\left\{ \begin{array}{l}
\hat{T}_2^+ = \left\{ p_{\max 1}, \max (\hat{D}^+) \right\}; \\
\mathcal{S}_2 = \mathcal{S} \backslash \left\{ \max (\hat{D}^+) \right\}, p_{\max 0}, p_{\max 1}, \end{array} \right. \tag{63}$$

Next, the remaining $K - 3$ elements in $\hat{T}^+$ are determined by iterations, and the iteration index is defined as $j = 2, \ldots, K - 2$. Then, in the $j$-th iteration, element $p$ in $\mathcal{S}$ is selected as $\hat{t}_{j+1}$, so that the corresponding positive support interval corresponding to $\hat{T}^+_{j+1} = \hat{T}^+_j \cup \hat{t}_{j+1}$ is a subset of $\hat{D}^+$. 
The proposed method needs to estimate into a matrix completion problem, for which a simple and products set, the estimation of amplitudes can be transformed repeating elements in the amplitudes set and the amplitudes of support as Algorithm 1. The algorithm finally returns the estimated result $\hat{p}$ where $\hat{p} = \arg \min_{p \in \mathbb{C}^N} \| p \|_2$. However, the existing methods in [31] need to estimate $K^2 - K + 1$ parameters in this process. This shows that our method reduces the number of parameters to be estimated by nearly half.

The computational complexity of the proposed algorithms is mainly in two processes. The first is to calculate $\{ \epsilon[i,j]\}_{n=0}^N$ recursively in ACF super-resolution. According to (34) and (35), it takes $\frac{N^2}{2}$ additions and $\frac{N^2}{2} + 1$ multiplications to compute $\varepsilon[i]$. So the complexity required to obtain $\{ \varepsilon[i,j]\}_{n=0}^N$ is $\mathcal{O}(N^2)$. In addition, according to the requirements of Prony’s method [34],

$$N = 2M + 1 = K(K - 1) + 1$$

in our model. Thus, the computational complexity of the recursive process is $\mathcal{O}(K^4)$. The other is the proposed Algorithm 1 for estimating support. Because there are $K(K - 1)/2$ elements in $\hat{D}^+$, it takes $\mathcal{O}(K^5)$ calculations to get the maximum when updating it to $\hat{D}^+ \setminus \max(\hat{D}^+)$. In determining the supporting range $\hat{S}$, the sum of any two elements in $\hat{D}^+$ should be calculated first to obtain the set $\hat{Q}$. It takes $\frac{K^2 - K + 1}{2}$ additions, and the computational complexity is $\mathcal{O}(K^5)$. Then the calculation of complexity $2\mathcal{O}(K^4)$ is used to take the absolute value of the difference between all elements in $\hat{Q}$ and $\max(\hat{D}^+)$. Finally, the set $\hat{P}$ can be obtained by minimizing $K - 2$ times of total complexity

$$\mathcal{O}(K^4)$$

and then the set $\hat{S}$ can be obtained. So far, the computational complexity of Step 1 of Algorithm 1 is

$$\mathcal{O}(K^2) + \mathcal{O}(K^4) + 20(K^4) + \mathcal{O}(K^5) = \mathcal{O}(K^5).$$

In the support estimation, because $\hat{P}$ has $2(K - 2)$ elements, the computational complexity of obtaining $p_{\text{max}1}$ is $\mathcal{O}(K)$. Before calculating the computational complexity of iterations, the number of elements in each set to be traversed is given. $\hat{D}^+$ has $O(K^2)$ elements, $\hat{T}^+$ has $O(K)$ elements, and $\hat{S}$ has $O(K)$ elements. In each loop, we traverse each element in the estimated support $\hat{T}^+$ and obtain the difference between it and all elements in $\hat{S}$. This requires

$$\mathcal{O}(K) \cdot \mathcal{O}(K) = \mathcal{O}(K^2)$$

computational complexity. In addition, for each of these differences, $\mathcal{O}(K^2)$ comparisons are required to obtain the closest element in $\hat{D}^+$. Since $K - 3$ cycles are required, the computational complexity of Step 2 of Algorithm 1 is

$$(K - 3) \mathcal{O}(K^2) \cdot \mathcal{O}(K^2) = \mathcal{O}(K^5).$$

### VI. Complexity Analysis

Table I Complexity Comparison

<table>
<thead>
<tr>
<th>Number of parameters to be estimated</th>
<th>Computational complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K^2 - K + 1$</td>
<td>$\mathcal{O}(K^6)$</td>
</tr>
<tr>
<td>$K(K - 1)/2$</td>
<td>$\mathcal{O}(K^5)$</td>
</tr>
</tbody>
</table>

In the super-resolution of ACF, according to (22), the proposed method needs to estimate $M = K(K - 1)/2$ intermediate parameters. However, the existing methods in [31]
VII. NUMERICAL SIMULATIONS

A. Verification of Effectiveness

Firstly, the simulations are carried out to verify the effectiveness of the proposed algorithms. We choose two different types of structure function models, and each model sets two different support numbers $K$. The whole simulation process and the results of each part are shown in Fig. 3.

Inset (a) shows the structure functions of the original signals. From top to bottom, there are four functions: Dirac pulse train of $K = 3$, Dirac pulse train of $K = 4$, non-uniform spline of $K = 3$ and non-uniform spline signal $K = 4$. The selection of such structure functions covers the signal models applicable to the super-resolution method of ACF proposed in this paper. Then the structure functions are convolved with four different scattering functions (as shown in inset (b)) to achieve filtering. It can be seen that the scattering functions can be of arbitrary form as long as they satisfy the simple constraints of limited length and non-overlapping. Inset (c) shows the time-domain form of the filtered signals, i.e. the original signals. The phaseless measurement shown in inset (d) is obtained by applying FT to the original signals and keeping only the intensity. The red line represents the discrete samples used for support estimation by Prony’s method, whose number is the minimum value required, i.e. $2 \lceil K (K - 1) + 1 \rceil + 1$.

Finally, by combining the proposed super-resolution method of ACF, the proposed support estimation algorithm and the existing amplitudes recovery method, the parameters of the original structure function are estimated. It can be seen from the reconstructed results comparison inset (e) that perfect recovery can be achieved in the absence of noise. It should be noted that the reconstruction result under $K = 4$ Dirac pulse train is a reflection of the original signal, which is one of the trivial ambiguities stated in Remark 1. As we mentioned in subsection V-A, such a result is regarded as a successful

So far, it can be obtained that the computational complexity of the framework proposed in this article is

$$\mathcal{O}(K^4) + \mathcal{O}(K^5) + \mathcal{O}(K^5) = \mathcal{O}(K^5).$$

This is an order of magnitude lower than the $\mathcal{O}(K^6)$ calculations required by the existing framework in [31].

The complexity comparison between the proposed method and the existing method is summarized in Table I.
Because the proposed method estimates the positive support interval set corresponding to the original structure function. The reconstruction error is the mean of the $\ell_2$-norm error of 10000 Monte Carlo experiments.

reconstruction. The above simulations prove the correctness and effectiveness of the proposed algorithms.

B. Robustness of Noise & Computational Complexity

When algorithms are used in real systems, noise is inevitable. Therefore, on the basis of verifying the correctness and effectiveness, we also test the noise robustness of the proposed two algorithms respectively.

First, the numerical simulations of noise robustness evaluation of ACF super-resolution method are presented. Dirac pulse train with support number $K$ of 3 and 5 is selected as the structure function, and its support and amplitudes are randomly selected from 0 to 1 and 1 to 10 respectively. Sinc function is used as scattering function to filter the structure function to obtain the original signal. It should be noted that this scattering function is chosen because the existing method can only be applied to the case of Sinc scattering function filtering. Because Sinc function is infinitely long, it does not meet the scattering function requirement of the proposed ACF super-resolution algorithm. However, when the bandwidth of Sinc function is large enough, most of the energy is contained in several main lobes of the time-domain waveform, while other side lobes can be ignored, so it can be regarded as a signal of finite length. After FT of the original signal, discrete samples of its intensity are obtained, and zero mean additive random White Gaussian noise with SNR from $-20$ dB to 30 dB is added to simulate the phaseless measurement under the noise condition. Finally, the existing and proposed ACF super-resolution methods are used to estimate the support interval of the original structure function respectively, and the estimation result is defined as $\tilde{D}$. It is worth noting that the number of discrete samples used for Prony’s method in the two algorithms is the minimum value required, i.e. $2[K(K-1)+1]+1$. Because the proposed method estimates the positive support interval $D^+$, its completion should be $\tilde{D} = \{-D^+, 0, D^+\}$.

The quantitative metric of reconstruction error we use is $\ell_2$-norm error:

$$\text{error} = \|\text{sort}(D) - \text{sort}(\tilde{D})\|_2,$$  \hfill (72)

where $D$ is the interval set corresponding to the original random support, and $\text{sort} (\cdot)$ is the elements of the set sorted in ascending order. According to the above setting, 10000 Monte Carlo experiments are performed at each noise intensity and the average value is taken to obtain the final reconstruction error. The results of numerical simulations are shown in Fig. 4. It can be seen that the reconstruction error decreases with the increase of SNR. In addition, compared with existing super-resolution algorithms for ACF, the proposed method has better noise robustness under different noise and support numbers. This is because the Prony’s method used in the two methods has better robustness when the number of estimated parameters is smaller.

Next, the noise robustness of support estimation algorithms is assessed by numerical simulations. We randomly select $K$ values from 0 to 1 to form the support $T$ to be restored, and set $K$ as 3 and 5 respectively. Then the support interval set $D$ corresponding to $T$ is obtained, and zero mean additive White Gaussian noise with SNR of $-20$ dB to 30 dB is added to it. Finally, the estimation result $\hat{T}$ of the original support is obtained by using the existing and proposed support recovery algorithms, in which the proposed algorithm uses the positive support interval with noise. Before the quantitative metric of reconstruction error is given, the trivial ambiguity solutions with zero of the original support $T$ are considered. According to the Remark 1, $K$ trivial ambiguity solutions containing 0 can be obtained through shift. In addition, another $K$ ambiguity solutions could be obtained through the reflection according to the Remark 2. So we define the set of $2K$ trivial
ambiguity solutions with zero as \( \{ \hat{T}_j \}^{2K}_{j=1} \). Then the \( \ell_2 \)-norm error of the reconstruction result is

\[
\text{error} = \min \left( \left\| \text{sort} \left( \hat{T} \right) - \text{sort} \left( \{ \hat{T}_j \}^{2K}_{j=1} \right) \right\|_2 \right).
\]

(73)

Under the above conditions, the average value of errors after 10000 Monte Carlo experiments is carried out for each SNR is the final reconstruction error. The results of the numerical simulations are presented in fig. 5. It can be seen that the reconstruction errors of all algorithms decrease with the increase of SNR. In the case of different support numbers, the proposed support recovery algorithm has better noise robustness than the existing method. This is mainly because the characteristics of elements in positive support interval are less affected by noise. This means that, in the range of experimental noise intensity, the preliminary search range determination step unique to the proposed algorithm has better noise robustness than the common iterative solution process.

In addition, we count the running time in the above noise robustness simulations, and giving the computation time comparison of the proposed algorithm and the existing method in each stage in Table II. For the data in Table II, the following notes are given. The results are obtained in seconds (s) using Matlab2021a on a PC with 16 GB RAM and 3.2 GHz Intel Core i7 processor. The values in Table II represent the difference \( t_{\text{diff}} \) between the average running time of the proposed algorithm \( t_{\text{pro}} \) and the existing algorithm \( t_{\text{exi}} \) at each stage under all SNR conditions (a total of 26,000 times) in the noise robustness simulations, i.e., \( t_{\text{diff}} = t_{\text{pro}} - t_{\text{exi}} \).

By analyzing the results in Table II, we can draw the following conclusions. In the first stage, the proposed super-resolution algorithm of ACF needs longer running time than the existing method, i.e., it has higher computational complexity. For the support estimation algorithm in the second stage, the proposed algorithm has lower computational complexity. Combining the two stages, the proposed algorithm has lower computational complexity than the existing method. The above conclusions are consistent with the theoretical analysis in VI.

<table>
<thead>
<tr>
<th>( K = 3 )</th>
<th>Stage1</th>
<th>Stage2</th>
<th>Entirety</th>
</tr>
</thead>
<tbody>
<tr>
<td>( +1.9 \times 10^{-5} )</td>
<td>( -2.5904 \times 10^{-5} )</td>
<td>( -2.3604 \times 10^{-5} )</td>
<td></td>
</tr>
<tr>
<td>( K = 5 )</td>
<td>( +2.2 \times 10^{-5} )</td>
<td>( -1.2038 \times 10^{-4} )</td>
<td>( -1.2038 \times 10^{-4} )</td>
</tr>
</tbody>
</table>

### C. Algorithms Combination Strategy

In this subsection the noise robustness of the whole algorithm is evaluated by numerical simulations, and the algorithms combination strategy under different requirements is given. According to the introduction of the algorithms, the existing and proposed algorithms can be used in combination. Because in the whole super-resolution phase retrieval process, ACF super-resolution is the first stage, and support recovery is the second stage. We use \( \text{exi1} \) and \( \text{pro1} \) to indicate the existing and proposed super-resolution methods of ACF respectively, and use \( \text{exi2} \) and \( \text{pro2} \) to indicate the existing and proposed support recovery algorithms respectively. Thus, the following three strategies of algorithms combination, \( \text{exi1} - \text{exi2} \), \( \text{pro1} - \text{pro2} \), and \( \text{pro1} - \text{exi2} \), can be obtained. It should be noted that when combination \( \text{pro1} - \text{exi2} \) is used, positive support interval \( D^+ \) estimated by \( \text{pro1} \) should be completed into interval set \( D = \{-D^+, 0, D^+\} \). In addition, due to the reconstruction error in the case of noise, the existing super-resolution methods of ACF cannot guarantee the symmetry of the elements in the estimated support interval set. The positive support interval cannot be selected for the proposed support estimation algorithm. Therefore, the combination strategy of \( \text{exi1} - \text{pro2} \) cannot be realized.

The structure function is set as Dirac pulse train with support number \( K = 6 \). Its support and amplitudes are randomly selected on a scale from 0 to 1 and 1 to 10, respectively. Sinc function is used as scattering function to filter the structure function to obtain the original signal. Discrete samples of FT intensity of the original signal are obtained, and zero mean additive white Gaussian noise with SNR ranging from -20dB to 30dB is added. The number of discrete samples used for Prony’s method in the two methods is the minimum value required. The super-resolution phase retrieval of the original signal is achieved by using the three combination strategies of algorithms to obtain the estimated support \( \hat{T} \). The quantitative metric of reconstruction error is the same as (73). After 10000 Monte Carlo experiments, the average error is taken as the final estimation error. The simulation results are shown in Fig. 6.

It can be seen that the reconstruction errors of all combination strategies decrease with the increase of SNR. The algorithm combination \( \text{pro1} - \text{pro2} \) has the worst noise robustness, but it has the lowest algorithm complexity and intermediate parameter redundancy according to the analysis in section VI. The algorithm combination \( \text{exi1} - \text{exi2} \) has higher computational complexity than \( \text{pro1} - \text{pro2} \), but it...
The hardware system consists of two parts: analog circuit and digital signal processing. In analog signal processing, the structure function $x^a(t)$ needs to be filtered to achieve the signal cannot be expanded into the form on the right-hand side of the equation in (18). This causes a deviation between the $|c|^2$ obtained by (19) and its true value, thus reducing the performance of the proposed algorithm. Therefore, according to (74), the energy overlap ratio of the scattering functions can be defined as
\[
\gamma = \frac{\int_{-\infty}^{\infty} |F_x(\omega)|^2 d\omega - E_{\text{no}}}{E_{\text{no}}}. \tag{75}
\]

In the simulation, we select two original signals $x_1(t)$ and $x_2(t)$ with the same support and scattering function, which take the Dirac pulse train with $K = 2$ as the structure function. The shape of the scattering function is shown in Fig. 7(a), and its duration is 0.2 s. Set the support of the two original signals as $\{0, 0.2\}$, and then move the support 0.2 toward 0 to achieve energy overlap of the scattering functions, as shown in Fig. 7(a) and Fig. 7(b). For each original signal, the proposed algorithms are used to obtain the estimation $\hat{F}$ of the support under different energy overlap ratios. Finally, the $L_2$-error in (73) is used as the quantitative metric of reconstruction errors, and its change versus the energy overlap ratio is plotted in Fig. 7(c). The simulation results show that the reconstruction error increases with the increase of scattering function energy overlap ratio under two different signal forms. The reason is that with the increase of energy overlap ratio, the difference between the $|c|^2$ obtained by (19) and its true value is greater, which leads to worse performance of the proposed algorithms. Moreover, when the original signal is $x_2(t)$, due to the different amplitudes corresponding to the two supports, the energy overlap ratio of the scattering functions cannot reach 1 even if the supports overlap completely, but this does not affect the above conclusion. When the scattering function is selected, the simulation result can be used to limit the energy overlap rate of the scattering functions according to the requirement of reconstruction error.

### VIII. Hardware Experiment

In addition to numerical simulations, in this section, a hardware experimental platform is built to verify the effectiveness of the proposed methods in the actual system. The hardware system consists of two parts: analog circuit and digital signal processing. In analog signal processing, the structure function $x^a(t)$ needs to be filtered to achieve the...
The processing time is the time to perform an operation using Intel Core i7 processor. It can be seen that the proposed method has lower running time, which is consistent with the theoretical analysis in section VI and the simulation results in subsection VII-B. Some key signals in the hardware experiment are plotted in Fig. 9. It can be seen that the methods proposed in this paper can effectively realize super-resolution phase retrieval of the original signal in the actual system, when it is filtered by the scattering function of arbitrary shape.

**IX. CONCLUSION**

Aiming at the redundancy of solving process in existing super-resolution phase retrieval frameworks and the problem of ultra-high complexity caused by this, we improve the two core steps of super-resolution phase retrieval respectively, and propose a compact solution framework. First, the redundancy of the estimated results is eliminated by merging the symmetric terms containing the same information in the super-resolution process of the ACF. Then, by mining the characteristics of elements in the positive support interval set,
Fig. 9. Several key signals in hardware experiments: (a) scattering function of arbitrary shape; (b) The original signal obtained by convolving the structure function with the scattering function; (c) Comparison of reconstruction results, where the solid blue line is the original structure function and the dotted red line is the recovery signal.

<table>
<thead>
<tr>
<th>Structure function</th>
<th>Support interval (s)</th>
<th>Support (s)</th>
<th>Amplitudes products (V)</th>
<th>Amplitudes (V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^s(t) = \sum_{k=0}^{K-1} \delta(t - t_k)$, $K = 3, 0 \leq t \leq 1$ (s)</td>
<td>true value</td>
<td>estimation</td>
<td>true value</td>
<td>estimation</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3047</td>
<td>0.2</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>0.25</td>
<td>0.2527</td>
<td>0.5</td>
<td>0.3047</td>
<td>0.5047</td>
</tr>
<tr>
<td>0.55</td>
<td>0.5499</td>
<td>0.75</td>
<td>0.5499</td>
<td>0.7499</td>
</tr>
</tbody>
</table>

$^1$ The proposed algorithm and the existing algorithm obtain the same support estimation results. The processing time of the proposed algorithm is $1.2631 \times 10^{-3}$ s, the processing time of the existing method is $3.7414 \times 10^{-5}$ s, and the time difference between them is $2.4783 \times 10^{-5}$ s.

the traversal scope required for iterations is reduced, and the computational complexity of the support estimation algorithm is reduced. The effectiveness of the proposed method is verified by sufficient numerical simulations, and each step has better noise robustness compared with existing methods. Because the overall robustness of the proposed algorithm is worse than that of the existing method, this paper also gives the combination strategy of algorithms when facing different requirements. Finally, we examine the effectiveness of the algorithm in the real system. In addition, it is worth mentioning that how to make the super-resolution phase retrieval method suitable for more signal types and how to improve its noise robustness are worthy of further study in the future.

REFERENCES


