Discrete Applied Mathematics 321 (2022) 90–108

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Rough analysis of computation trees

Mikhail Moshkov

Computer, Electrical and Mathematical Sciences and Engineering Division and Computational Bioscience Research Center, King Abdullah University of Science and Technology (KAUST), Thuwal 23955-6900, Saudi Arabia

ARTICLE INFO

Article history:
Received 4 January 2022
Received in revised form 7 June 2022
Accepted 17 June 2022
Available online xxxx

Keywords:
Structure
Computation tree
Complexity measure

ABSTRACT

This paper deals with computation trees over an arbitrary structure consisting of a set along with collections of functions and predicates that are defined on it. It is devoted to the comparative analysis of three parameters of problems with n input variables over this structure: the complexity of a problem description, the minimum complexity of a computation tree solving this problem deterministically, and the minimum complexity of a computation tree solving this problem nondeterministically. Rough classification of relationships among these parameters is considered and all possible seven types of these relations are enumerated. The changes of relation types with the growth of the number n of input variables are studied.

1. Introduction

Computation trees are well known models of algorithms. They are a natural generalization of decision trees: besides one-place operations of predicate type (attributes) which are used in decision trees, in computation trees many-place predicate and functional operations may be used. Just as in general algorithm theory, where both deterministic and nondeterministic algorithms are considered, it is expedient to study not only deterministic but also nondeterministic computation trees.

Linear decision trees and algebraic decision and computation trees were studied most intensively. Lower bounds on the complexity were obtained in [2,4,5] for linear decision trees, in [19–21] for algebraic decision trees, in [1,6] for algebraic computation trees, and in [7] for Pfaffian computation trees. Upper bounds on the complexity were obtained in [3,8,10] for linear decision trees and in [17] for quasilinear decision trees that includes linear decision trees and some kinds of algebraic decision trees. Nondeterministic linear decision and computation trees were studied in [9] and [11,17], respectively.

The complexity of deterministic decision trees over arbitrary infinite sets of k-valued attributes, $k \geq 2$, was studied in [16,17]. Relationships between deterministic and different kinds of nondeterministic decision trees over arbitrary infinite sets of k-valued attributes were investigated in [12,18].

In this paper, we study computation trees over an arbitrary structure $U = (A, F, P)$ consisting of a set A along with a collection of functions F and a collection of predicates P that are defined on it.

For each natural n, we describe a set $\mathcal{P}(U, n)$ of problems over U with n input variables. Each such problem with input variables $x_1, \ldots, x_n$ is given by a finite sequence of functional and predicate expressions over U. This sequence defines r functions $\alpha_1, \ldots, \alpha_r$ of the form $p(t_1, \ldots, t_m)$, where $p \in P$ and $t_1, \ldots, t_m$ are functions with variables from the set $\{x_1, \ldots, x_n\}$ obtained from functions contained in $F \cup \{x\}$ by the operation of substitution. The functions $\alpha_1, \ldots, \alpha_r$ divide the set A^n into areas in which these functions are constant. Each area is labeled with a finite nonempty set of solutions. For

E-mail address: mikhail.moshkov@kaust.edu.sa.

https://doi.org/10.1016/j.dam.2022.06.032

a given n-tuple $\bar{a} \in A^n$, we should find a solution from the set attached to the area to which $\bar{a}$ belongs. Various problems of combinatorial optimization, pattern recognition, computational geometry, etc., can be represented in such form.

We define a complexity measure ψ and, for each problem $z \in \mathcal{P}(U, n)$, we consider three parameters: $\psi_U^i(z)$ – the complexity of the problem z description, $\psi_U^d(z)$ – the minimum complexity of a computation tree that solves the problem z deterministically, and $\psi_U^a(z)$ – the minimum complexity of a computation tree that solves the problem z nondeterministically. The pair (U, ψ) is called a sm-pair ((structure, measure)-pair).

To study relationships among these parameters, for each $b, c \in \{i, d, a\}$, we consider two partial functions defined on the set of nonnegative integers:

$$\mathcal{U}_{U\psi n}^{bc}(m) = \max\{\psi_U^b(z) : z \in \mathcal{P}(U, n), \psi_U^c(z) \le m\},$$
$$\mathcal{L}_{U\psi n}^{bc}(m) = \min\{\psi_U^b(z) : z \in \mathcal{P}(U, n), \psi_U^c(z) \ge m\}.$$

If the value $\mathcal{U}_{U\psi n}^{bc}(m)$ is defined for some m, then it is the unimprovable upper bound on the value $\psi_U^b(z)$ for problems $z \in \mathcal{P}(U, n)$ such that the inequality $\psi_U^c(z) \le m$ holds. If the value $\mathcal{L}_{U\psi n}^{bc}(m)$ is defined for some m, then it is the unimprovable lower bound on the value $\psi_U^b(z)$ for problems $z \in \mathcal{P}(U, n)$ such that the inequality $\psi_U^c(z) \ge m$ holds.

The study of the functions $\mathcal{U}_{U\psi n}^{bc}$ and $\mathcal{L}_{U\psi n}^{bc}$ directly is, in general case, too complicated problem. Therefore, instead of the functions $\mathcal{U}_{U\psi n}^{bc}$ and $\mathcal{L}_{U\psi n}^{bc}$, we study their types $typ(\mathcal{U}_{U\psi n}^{bc})$ and $typ(\mathcal{L}_{U\psi n}^{bc})$ from the set $\{\alpha, \beta, \gamma, \delta, \varepsilon\}$. In particular, the type α means that the considered function $\mathcal{U}_{U\psi n}^{bc}$ has infinite domain and is bounded from above, types β, γ, and δ describe different variants of the growth of the unbounded function $\mathcal{U}_{U\psi n}^{bc}$ with an infinite domain, and the type ε means that the function $\mathcal{U}_{U\psi n}^{bc}$ has a finite domain.

All pairs $typ(\mathcal{L}_{U\psi n}^{bc})\, typ(\mathcal{U}_{U\psi n}^{bc})$, $b, c \in \{i, d, a\}$, form the n-type of the sm-pair (U, ψ) that is the table $typ(U, \psi, n)$ with three rows and three columns in which rows from top to bottom and columns from the left to the right are labeled with indices i, d, a, and the pair $typ(\mathcal{L}_{U\psi n}^{bc})\, typ(\mathcal{U}_{U\psi n}^{bc})$ is in the intersection of the row with index $b \in \{i, d, a\}$ and the column with index $c \in \{i, d, a\}$. We describe all possible seven n-types of sm-pairs. These results are similar to ones obtained for decision trees [12].

For the sm-pair (U, ψ), n-types can change with the increasing of n. To investigate this phenomenon, we study the infinite sequence

$$type(U, \psi) = typ(U, \psi, 1)typ(U, \psi, 2)\cdots$$

that is called the dynamic type of the sm-pair (U, ψ). In this paper, we describe all possible dynamic types of sm-pairs.

Some preliminary results in this direction were published without proofs in [13–15]. The publication of the final results with proofs was postponed for years: only in the present paper it was possible to finally resolve the issue of the structure of the set of all possible dynamic types of sm-pairs. This required considering of non-trivial constructions in Section 6. This section contains also examples of study of different sm-pairs.

The rest of the paper is organized as follows. In Sections 2 and 3, basic notions and main results are considered. Sections 4–7 are devoted to the proofs of auxiliary statements and main theorems. Section 8 contains some explanations of the results of this paper and Section 9 — short conclusions.

2. Basic notions

In this section, we consider the notions of structure, computation tree, problem, complexity measure, sm-pair ((structure,measure)-pair), type of function, n-type of sm-pair, and dynamic type of sm-pair.

2.1. Structures

Let $\omega = \{0, 1, 2, \ldots\}$ be the set of nonnegative integers, $E_2 = \{0, 1\}$, and $X = \{x_i : i \in \omega\}$ be the set of variables.

Definition 1. A structure is a triple $U = (A, F, P)$ such that A is a nonempty set (universe), F is a set of functions of the kind $f(x_1, \ldots, x_n)$, where $f : A^n \to A$ and $n \in \omega$ (if $n = 0$, then f is a constant), P is a nonempty set of predicates (relations) of the kind $p(x_1, \ldots, x_n)$, where $p : A^n \to E_2$ and $n \in \omega \setminus \{0\}$, and $F \cap P = \emptyset$.

Example 1. The structure considered in [1] contains the set of real numbers $\mathbb{R}$ as the universe, all constants from $\mathbb{R}$, $+$, -, $\times$, /, $\surd$ as functions, and $= 0$, > 0, ≥ 0 as predicates.

We denote by $[F]$ the set of all functions with variables from X obtained from functions contained in $F \cup \{x\}$ by the operation of substitution. We denote by $P[F]$ the set of all functions of the kind $p(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n \in [F]$ and p is a predicate from P with n variables.

An expression $x_j \leftarrow f(x_{l_1}, \ldots, x_{l_r})$, where f is a function from F with r variables, will be called a functional expression over U. An expression $p(x_{q_1}, \ldots, x_{q_k})$, where p is a predicate from P with $k \ge 1$ variables, will be called a predicate expression over U.

Consider a pair (Y, β), where Y is a finite nonempty subset of the set X and β is a finite sequence of functional and predicate expressions over U. We now correspond to each predicate expression from β a function from $P[F]$. Let $\beta = \beta_1, \ldots, \beta_m$ and s be the minimum number from ω such that all variables from Y and all variables from the expressions $\beta_1, \ldots, \beta_m$ are contained in the set $\{x_0, \ldots, x_s\}$. Let x_w be the variable from the set Y with the minimum index w.

For $i = 0, \ldots, m$, we define a sequence $t_i = t_{i0}, \ldots, t_{is}$ of functions from $[F]$ with variables from Y. Let $j \in \{0, \ldots, s\}$. If $x_j \in Y$, then $t_{0j} = x_j$. If $x_j \notin Y$, then $t_{0j} = x_w$. Let sequences $t_0, \ldots, t_i$, $0 \leq i < m$, be already defined. If β_{i+1} is a predicate expression, then $t_{i+1} = t_i$. If β_{i+1} is a functional expression $x_j \leftarrow f(x_{l_1}, \ldots, x_{l_n})$, then $t_{i+1} = t_{i0}, \ldots, t_{ij-1}, f(t_{il_1}, \ldots, t_{il_n}), t_{ij+1}, \ldots, t_{is}$.

Let there be exactly $r > 0$ predicate expressions $\beta_{c_1}, \ldots, \beta_{c_r}$ among $\beta_1, \ldots, \beta_m$, where $c_1 < \cdots < c_r$. For $i = 1, \ldots, r$, we associate with the expression β_{c_i} a function $\alpha_i \in P[F]$ with variables from Y. Let β_{c_i} be an expression $p(x_{q_1}, \ldots, x_{q_k})$. Then $\alpha_i = p(t_{c_i q_1}, \ldots, t_{c_i q_k})$. We denote by $\Pi(Y, \beta)$ the r-tuple $(\alpha_1, \ldots, \alpha_r)$.

Example 2. To illustrate the definitions, consider the simple structure $U_0 = (A_0, F_0, P_0)$, where $A_0 = \mathbb{R}$, $F_0 = \{f(x_1)\}$, $f(x_1) = x_1 - 1$, and $P_0 = \{p(x_1)\}$, $p(x_1) = 0$ if $x_1 \leq 0$ and $p(x_1) = 1$ if $x_1 > 0$.

For this structure, $[F_0] = \{x_i - j : i, j \in \omega\}$ and $P_0[F_0] = \{l_j(x_i) : i, j \in \omega\}$, where $l_j(x_i) = 0$ if $x_i \leq j$ and $l_j(x_i) = 1$ if $x_i > j$.

Let us consider a pair (Y_0, β^0), where $Y_0 = \{x_0\}$ and

$$\beta^0 = p(x_0), x_1 \leftarrow f(x_0), p(x_1).$$

In the sequence β^0, there are two predicate expressions. It is easy to show that $\Pi(Y_0, \beta^0) = (l_0(x_0), l_1(x_0))$.

2.2. Computation trees

A node in a finite directed tree is called the root, if it is the only node without entering edges. A tree, which has such a node, is called a finite directed tree with the root. The tree nodes without leaving edges are called terminal nodes. The tree nodes, which are neither the root nor terminal, will be called working nodes. A complete path in a finite directed tree with the root is any sequence $\xi = v_0, d_0, \ldots, v_m, d_m, v_{m+1}$ of nodes and edges of the tree such that v_0 is the root, v_{m+1} is a terminal node, and the edge d_i leaves the node v_i and enters the node v_{i+1} for $i = 0, \ldots, m$.

Definition 2. A computation tree over the structure $U = (A, F, P)$ is a pair $\Gamma = (Y, G)$, where Y is a finite nonempty subset of the set of variables X and G is a marked finite directed tree with the root, which has at least two nodes and satisfies the following conditions:

- The root and the edges leaving the root are not labeled.
- Each working node is a functional or a predicate node.
- Each functional node is labeled with a functional expression over U, and each edge leaving a functional node is not labeled.
- Each predicate node is labeled with a predicate expression over U, and each edge leaving a predicate node is labeled with a number from E_2.
- Each terminal node is labeled with a number from ω.

Definition 3. A computation tree is called deterministic if it satisfies the following conditions:

- There is exactly one edge leaving the root.
- Each functional node has exactly one edge leaving it.
- For each predicate node, edges leaving this node are labeled with pairwise different numbers from E_2.

The set of computation trees over the structure U will be denoted by $\mathcal{T}(U)$. Let $\Gamma = (Y, G)$ be a computation tree over U. Nodes, edges and paths in the tree G are called nodes, edges and paths in the computation tree Γ. The set Y is called the set of input variables for computation tree Γ. Let $Y = \{x_{l_1}, \ldots, x_{l_n}\}$ and $l_1 < \cdots < l_n$. Denote $\bar{x} = (x_{l_1}, \ldots, x_{l_n})$. We denote by $\Xi(\Gamma)$ the set of complete paths in Γ. Let $\xi = v_0, d_0, \ldots, v_m, d_m, v_{m+1}$ be a complete path in Γ. We denote by $\kappa(\xi)$ the number assigned to the node v_{m+1}. We now define a sequence $\beta(\xi)$ of functional and predicate expressions over U and a subset $\mathcal{A}(\xi)$ of the set A^n associated with ξ. If $m = 0$, then $\beta(\xi)$ is the empty sequence. Let $m > 0$, and let the expression β_j be assigned to the node v_j, $j = 1, \ldots, m$. Then $\beta(\xi) = \beta_1, \ldots, \beta_m$. If there are no predicate expressions in the sequence $\beta(\xi)$, then $\mathcal{A}(\xi) = A^n$. Let there are exactly $r > 0$ predicate expressions $\beta_{c_1}, \ldots, \beta_{c_r}$ among $\beta_1, \ldots, \beta_m$, where $c_1 < \cdots < c_r$. Let $\Pi(Y, \beta(\xi)) = (\alpha_1, \ldots, \alpha_r)$, and let δ_i be the number assigned to the edge d_{c_i}, $i = 1, \ldots, r$. Then $\mathcal{A}(\xi)$ is the set of solutions on A^n for the system of equations

$$\{\alpha_1(\bar{x}) = \delta_1, \ldots, \alpha_r(\bar{x}) = \delta_r\}.$$

Example 3 (*Continuation of Example 2*)**.** Let $Y_0 = \{x_0\}$. We now consider two computation trees $\Gamma_a = (Y_0, G_a)$ and $\Gamma_d = (Y_0, G_d)$ over the structure U_0 — see Fig. 1. The computation tree Γ_a is not deterministic, but the computation tree Γ_d is deterministic.

Let ξ be the complete path in the computation tree Γ_d, which is finished in the terminal node labeled with the number 2. Then $\beta(\xi) = x_1 \leftarrow f(x_0), p(x_1), p(x_0)$, $\Pi(Y_0, \beta(\xi)) = (l_1(x_0), l_0(x_0))$, and $A(\xi)$ is the set of solutions on $\mathbb{R}$ for the system of equations $\{l_1(x_0) = 0, l_0(x_0) = 1\}$, i.e., $A(\xi) = \{a \in \mathbb{R} : 0 < a \leq 1\}$.

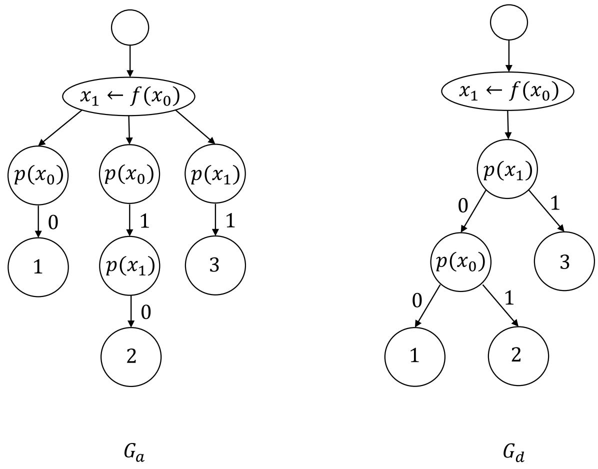

Fig. 1. Marked directed trees G_a and G_d from the computation trees $\Gamma_a = (Y_0, G_a)$ and $\Gamma_d = (Y_0, G_d)$.

2.3. Problems

The set of nonempty finite subsets of the set ω will be denoted by $S(\omega)$.

Definition 4. A problem over the structure U is a tuple of the kind $z = (Y, \nu, \beta_1, \ldots, \beta_m)$, where Y is a nonempty finite subset of the set X, $m \in \omega \setminus \{0\}$, $\beta_1, \ldots, \beta_m$ are functional and predicate expressions over U, and there is $r \in \omega \setminus \{0\}$ such that $\nu : E_2^r \to S(\omega)$ and there are exactly r predicate expressions in the sequence $\beta_1, \ldots, \beta_m$.

The set Y is called the set of input variables for the problem z. We denote by $\beta(z)$ the sequence $\beta_1, \ldots, \beta_m$. Let $|Y| = n$ and $\Pi(Y, \beta(z)) = (\alpha_1, \ldots, \alpha_r)$. The problem z may be interpreted as a problem of searching for a number from the set $z(\bar{a}) = \nu(\alpha_1(\bar{a}), \ldots, \alpha_r(\bar{a}))$ for an arbitrary $\bar{a} \in A^n$. Different problems of pattern recognition, combinatorial optimization, and computational geometry can be represented in such form. We denote by $\mathcal{P}(U)$ the set of problems over the structure U.

Let $z = (Y_1, \nu, \beta_1, \ldots, \beta_m) \in \mathcal{P}(U)$ and $\Gamma = (Y_2, G) \in \mathcal{T}(U)$. Let $|Y_1| = n$.

Definition 5. We will say that the computation tree Γ solves the problem z nondeterministically if the following conditions hold:

- $Y_1 = Y_2$.
- $\bigcup_{\xi \in \Xi(\Gamma)} \mathcal{A}(\xi) = A^n$.
- For any $\bar{a} \in A^n$ and any $\xi \in \Xi(\Gamma)$ such that $\bar{a} \in \mathcal{A}(\xi)$, the relation $\kappa(\xi) \in z(\bar{a})$ holds.

Definition 6. We will say that the computation tree Γ solves the problem z deterministically if Γ is a deterministic computation tree, which solves z nondeterministically.

Example 4 (*Continuation of Examples 2 and 3*)**.** Let us consider a problem $z_0 = (Y_0, \nu_0, \beta^0)$ over the structure U_0, where $Y_0 = \{x_0\}$ and $\beta(z_0) = \beta^0$ is the sequence $p(x_0), x_1 \leftarrow f(x_0), p(x_1)$ considered in Example 2. We know that $\Pi(Y_0, \beta(z_0)) = (l_0(x_0), l_1(x_0))$. The map ν_0 is defined as follows: $\nu_0(0, 0) = \{1\}$, $\nu_0(1, 0) = \{2\}$, $\nu_0(1, 1) = \{1, 3\}$, and $\nu_0(0, 1) = \{4\}$.

We can reformulate the problem z_0 in the following way: for a given $a \in \mathbb{R}$, we should find a number from the set $z_0(a) = \nu_0(l_0(a), l_1(a))$. Fig. 2 represents a geometric interpretation of this problem. Two functions $l_0(x_0) = p(x_0)$ and $l_1(x_0) = p(f(x_0))$ divide the set of real numbers $\mathbb{R}$ into three areas that are labeled with the sets $\{1\}$, $\{2\}$ and $\{1, 3\}$, respectively. For a given $a \in \mathbb{R}$, we should recognize a number from the set attached to the area to which a belongs.

One can show that the computation tree $\Gamma_a = (Y_0, G_a)$ considered in Example 3 (see Fig. 1) solves the problem z_0 nondeterministically, and the computation tree $\Gamma_d = (Y_0, G_d)$ considered in the same example (see Fig. 1) solves the problem z_0 deterministically.

2.4. Complexity measures and sm-pairs

Let $U = (A, F, P)$ be a structure. Denote by $(F \cup P)^*$ the set of all finite words over the alphabet $F \cup P$, including the empty word λ.

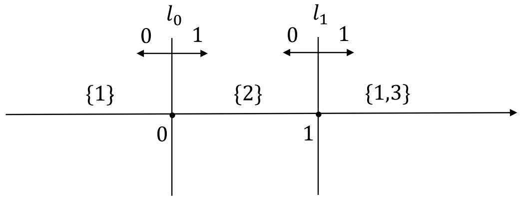

Fig. 2. Geometric interpretation of the problem z_0.

Definition 7. A complexity measure over the structure U is any map of the kind $\psi : (F \cup P)^* \to \omega$. The complexity measure ψ is called limited if it has the following properties:

- $\psi(\alpha_1\alpha_2) \le \psi(\alpha_1) + \psi(\alpha_2)$ for any $\alpha_1, \alpha_2 \in (F \cup P)^*$.
- $\psi(\alpha_1\alpha_2\alpha_3) \ge \psi(\alpha_1\alpha_3)$ for any $\alpha_1, \alpha_2, \alpha_3 \in (F \cup P)^*$.
- For any $\alpha \in (F \cup P)^*$, the inequality $\psi(\alpha) \ge |\alpha|$ holds, where $|\alpha|$ is the length of α.

We extend the complexity measure ψ onto the set of all finite sequences of functional and predicate expressions over U in the following way: $\psi(\beta) = \psi(\lambda)$ if β is the empty sequence. Let β be a nonempty sequence and $\beta = \beta_1, \dots, \beta_m$. Then $\psi(\beta) = \psi(\alpha)$, where $\alpha = b_1 \cdots b_m \in (F \cup P)^*$ and, for $i = 1, \dots, m$, if β_i is a predicate expression $p(x_{q_1}, \dots, x_{q_k})$, then $b_i = p$, and if β_i is a functional expression $x_j \leftarrow f(x_{l_1}, \dots, x_{l_r})$, then $b_i = f$.

We extend the complexity measure ψ onto the set $\mathcal{T}(U)$ of computation trees over U as follows: $\psi(\Gamma) = \max\{\psi(\beta(\xi)) : \xi \in \Xi(\Gamma)\}$ for any $\Gamma \in \mathcal{T}(U)$. The value $\psi(\Gamma)$ will be called the ψ-complexity of a computation tree Γ.

We now consider some examples of complexity measures. Let $w : (F \cup P) \to \omega \setminus \{0\}$. We define the function $\psi^w : (F \cup P)^* \to \omega$ in the following way: for any $\alpha \in (F \cup P)^*$, $\psi^w(\alpha) = 0$ if $\alpha = \lambda$, and $\psi^w(\alpha) = \sum_{i=1}^{m} w(b_i)$ if $\alpha = b_1 \cdots b_m$. The function ψ^w is a limited complexity measure over U and is called a weighted depth. If $w \equiv 1$, then the function ψ^w is called the depth.

Let ψ be a complexity measure over U and $z \in \mathcal{P}(U)$. The value $\psi_U^i(z) = \psi(\beta(z))$ is called the complexity of the problem z description. We denote by $\psi_U^d(z)$ the minimum ψ-complexity of a computation tree $\Gamma \in \mathcal{T}(U)$, which solves the problem z deterministically. We denote by $\psi_U^a(z)$ the minimum ψ-complexity of a computation tree $\Gamma \in \mathcal{T}(U)$, which solves the problem z nondeterministically.

Definition 8. A (structure,measure)-pair or, in short, a sm-pair is a pair (U, ψ) such that U is a structure and ψ is a complexity measure over U. If ψ is a limited complexity measure, then the pair (U, ψ) will be called a limited sm-pair.

Example 5 (*Continuation of Examples 2–4*)**.** We define a complexity measure ψ over the structure U_0 in the following way: for any word α over the alphabet $F_0 \cup P_0 = \{f, p\}$, $\psi(\alpha)$ is equal to the number of occurrences of the letter p in the word α. It is clear that the complexity measure ψ is not limited. It is easy to see that, for the computation trees Γ_a and Γ_d considered in Example 3, $\psi(\Gamma_a) = \psi(\Gamma_d) = 2$. One can show that, for the problem z_0 considered in Example 4, $\psi_{U_0}^i(z_0) = \psi_{U_0}^a(z_0) = \psi_{U_0}^d(z_0) = 2$. Later we will study the sm-pair (U_0, ψ).

2.5. n-Types and dynamic types of sm-pairs

Let (U, ψ) be a sm-pair and $n \in \omega \setminus \{0\}$. We denote by $\mathcal{P}(U, n)$ the set of problems from $\mathcal{P}(U)$ with n input variables.

We have the three parameters $\psi_U^i(z)$, $\psi_U^d(z)$, and $\psi_U^a(z)$ for any problem $z \in P(U, n)$, and we investigate the relationships between any two such parameters for problems from $\mathcal{P}(U, n)$. Let us consider, for example, the parameters $\psi_U^i(z)$ and $\psi_U^d(z)$. Let $m \in \omega$. We will study relations $\psi_U^i(z) \le m \Rightarrow \psi_U^d(z) \le u$ true for any $z \in \mathcal{P}(U, n)$. The minimum value of u is most interesting for us. This value (if exists) is equal to

$$\mathcal{U}_{U\psi n}^{di}(m) = \max\{\psi_U^d(z) : z \in \mathcal{P}(U, n), \psi_U^i(z) \le m\}.$$

We also study relations $\psi_U^i(z) \ge m \Rightarrow \psi_U^d(z) \ge l$. In this case, the maximum value of l is most interesting for us. This value (if exists) is equal to

$$\mathcal{L}_{U\psi n}^{di}(m) = \min\{\psi_U^d(z) : z \in \mathcal{P}(U, n), \psi_U^i(z) \ge m\}.$$

The two functions $\mathcal{U}_{U\psi n}^{di}$ and $\mathcal{L}_{U\psi n}^{di}$ describe how the behavior of the parameter $\psi_U^d(z)$ depends on the behavior of the parameter $\psi_U^i(z)$.

There are 18 similar functions for all ordered pairs of parameters $\psi_U^i(z)$, $\psi_U^d(z)$, and $\psi_U^a(z)$. These 18 functions well describe the relationships among the considered parameters. It would be very interesting to enumerate 18-tuples of these functions for all sm-pairs. But this is a very complicated problem.

In this paper, instead of functions we study types of functions. With any function, we associate its type from the set $\{\alpha, \beta, \gamma, \delta, \varepsilon\}$. For example, if a function has infinite domain of definition, and it is bounded from above, then its type is equal to α. Thus, we enumerate 18-tuples of types of functions. These tuples are represented as tables called the *n*-types of sm-pairs. We also consider infinite sequences of the kind 1-type of sm-pair 2-type of sm-pair ..., which are called dynamic types of sm-pairs and characterize changes of relationships among the considered parameters with the growth of the number of input variables.

We now give definitions of mentioned above notions. Let $b, c \in \{i, d, a\}$. We define partial functions $\mathcal{U}^{bc}_{U\psi n} : \omega \to \omega$ and $\mathcal{L}^{bc}_{U\psi n} : \omega \to \omega$ as follows:

$$\mathcal{U}^{bc}_{U\psi n}(m) = \max\{\psi^b_U(z) : z \in \mathcal{P}(U, n), \psi^c_U(z) \leq m\},$$

$$\mathcal{L}^{bc}_{U\psi n}(m) = \min\{\psi^b_U(z) : z \in \mathcal{P}(U, n), \psi^c_U(z) \geq m\}.$$

If the value $\mathcal{U}^{bc}_{U\psi n}(m)$ is defined, then it is the unimprovable upper bound on the value $\psi^b_U(z)$ for problems $z \in \mathcal{P}(U, n)$ such that the inequality $\psi^c_U(z) \leq m$ holds. If the value $\mathcal{L}^{bc}_{U\psi n}(m)$ is defined, then it is the unimprovable lower bound on the value $\psi^b_U(z)$ for problems $z \in P(U, n)$ such that the inequality $\psi^c_U(z) \geq m$ holds.

Let g be a partial function from ω to ω. We denote by $Dom(g)$ the domain of definition (domain for short) of g. Let $Dom^+(g) = \{n : n \in Dom(g), g(n) \geq n\}$ and $Dom^-(g) = \{n : n \in Dom(g), g(n) \leq n\}$.

Definition 9. The type of the function g is the value $typ(g) \in \{\alpha, \beta, \gamma, \delta, \varepsilon\}$ defined in the following way:

- If $Dom(g)$ is an infinite set and g is a bounded from above function, then $typ(g) = \alpha$.
- If $Dom(g)$ is an infinite set, $Dom^+(g)$ is a finite set, and g is an unbounded from above function, then $typ(g) = \beta$.
- If each of the sets $Dom^+(g)$ and $Dom^-(g)$ is an infinite set, then $typ(g) = \gamma$.
- If $Dom(g)$ is an infinite set and $Dom^-(g)$ is a finite set, then $typ(g) = \delta$.
- If $Dom(g)$ is a finite set, then $typ(g) = \varepsilon$.

To clarify the notion of a function type, consider in more detail the function $\mathcal{U}^{di}_{U\psi n}$. One can show that $\psi^d_U(z) \leq \psi^i_U(z)$ for any problem $z \in \mathcal{P}(U)$: based on the problem description, it is easy to construct a computation tree, which solves this problem deterministically and which complexity is equal to the complexity of the problem description. Therefore $typ(\mathcal{U}^{di}_{U\psi n}) \in \{\alpha, \beta, \gamma\}$. If $typ(\mathcal{U}^{di}_{U\psi n}) = \alpha$, then there is a positive constant p such that $\psi^d_U(z) \leq p$ for any problem $z \in \mathcal{P}(U, n)$. If $typ(\mathcal{U}^{di}_{U\psi n}) = \gamma$, then there are infinitely many numbers $m \in \omega$ for each of which there exists a problem $z \in \mathcal{P}(U, n)$ with $\psi^d_U(z) = \psi^i_U(z) = m$. The case $typ(\mathcal{U}^{di}_{U\psi n}) = \beta$ is the most interesting for us: the function $\mathcal{U}^{di}_{U\psi n}$ is not bounded from above and, for each problem with high enough complexity of description, there exists a computation tree, which solves this problem deterministically and which complexity is less than the complexity of the problem description.

Example 6 (*Continuation of Examples 2–5*)**.** Let $n \in \omega \setminus \{0\}$ and ψ be the complexity measure over the structure U_0 defined in Example 5. Denote $g(m) = \mathcal{U}^{di}_{U_0\psi n}(m)$. We will show that $typ(g) = \beta$. It is clear that $Dom(g) = \omega \setminus \{0\}$.

Let $m > 2n$. We now show that $g(m) < m$. Let $z \in \mathcal{P}(U_0, n)$ and $\psi^i_{U_0}(z) < m$. Then, as we already mentioned above, $\psi^d_{U_0}(z) \leq \psi^i_{U_0}(z) < m$. Let $\psi^i_{U_0}(z) = m$ and Y be the set of input variables of z. We now show that $\psi^d_{U_0}(z) < m$. It is clear that $\Pi(Y, \beta(z))$ is an m-tuple of functions of the form $l_j(x_i)$, where $x_i \in Y$ and $j \in \omega$. If in this tuple there are repeated functions, then, as it is easy to show, $\psi^d_{U_0}(z) < m$. Let all m functions in the tuple $\Pi(Y, \beta(z))$ are pairwise different. Since $m > 2n$, in the considered tuple, for some $x_i \in Y$, there are three functions $l_{j_1}(x_i)$, $l_{j_2}(x_i)$, $l_{j_3}(x_i)$ such that $j_1 < j_2 < j_3$. If $l_{j_2}(x_i) = 1$, then $l_{j_1}(x_i) = 1$. If $l_{j_2}(x_i) = 0$, then $l_{j_3}(x_i) = 0$. Therefore to find values of these three functions, it is enough to compute values of two of them. Using this fact, it is easy to show that $\psi^d_{U_0}(z) < m$. As a result, we obtain that $Dom^+(g)$ is a finite set.

We now show that g is unbounded from above function. Let us assume the contrary. In this case, there exists a constant t_1 such that $\psi^d_{U_0}(z) \leq t_1$ for any problem $z \in \mathcal{P}(U_0, n)$. From here it follows that there exists a constant t_2 such that, for any $z \in \mathcal{P}(U_0, n)$, there exists a computation tree over U_0, which solves the problem z deterministically and for which the number of terminal nodes is at most t_2. Evidently, it is impossible: for any $t \in \omega \setminus \{0\}$, we can find a problem $z \in \mathcal{P}(U_0, n)$ such that there exist tuples $\bar{a}_1, \ldots, \bar{a}_t \in \mathbb{R}^n$ for which the sets $z(\bar{a}_1), \ldots, z(\bar{a}_t)$ are pairwise disjoint. Any computation tree solving this problem deterministically should have at least t terminal nodes. Therefore the function g is unbounded from above. Thus, $typ(g) = typ(\mathcal{U}^{di}_{U_0\psi n}) = \beta$.

Definition 10. The *n*-type, $n \in \omega \setminus \{0\}$, of a sm-pair (U, ψ) is the table $typ(U, \psi, n)$ with three rows and three columns, in which rows from top to bottom and columns from the left to the right are labeled with indices i, d, a, and the pair $typ(\mathcal{L}^{bc}_{U\psi n})\ typ(\mathcal{U}^{bc}_{U\psi n})$ is in the intersection of the row with index $b \in \{i, d, a\}$ and the column with index $c \in \{i, d, a\}$.

Definition 11. The dynamic type of a sm-pair (U, ψ) is the infinite sequence $dtyp(U, \psi) = typ(U, \psi, 1)\ typ(U, \psi, 2) \cdots$.

3. Main results

The main goal of this paper is to describe the set of all dynamic types of sm-pairs and the set of all dynamic types of limited sm-pairs. The solution of this problem allows us to characterize roughly all possible with the growth of the number of input variables changes of relationships among the complexity of problem description, the minimum complexity of computation trees solving this problem deterministically, and the minimum complexity of computation trees solving this problem nondeterministically.

Define seven tables:

$T_1 =$

	i	d	a
i	$\varepsilon\alpha$	$\varepsilon\alpha$	$\varepsilon\alpha$
d	$\varepsilon\alpha$	$\varepsilon\alpha$	$\varepsilon\alpha$
a	$\varepsilon\alpha$	$\varepsilon\alpha$	$\varepsilon\alpha$

$T_2 =$

	i	d	a
i	$\gamma\gamma$	$\varepsilon\varepsilon$	$\varepsilon\varepsilon$
d	$\alpha\alpha$	$\varepsilon\alpha$	$\varepsilon\alpha$
a	$\alpha\alpha$	$\varepsilon\alpha$	$\varepsilon\alpha$

$T_3 =$

	i	d	a
i	$\gamma\gamma$	$\delta\varepsilon$	$\varepsilon\varepsilon$
d	$\alpha\beta$	$\gamma\gamma$	$\varepsilon\varepsilon$
a	$\alpha\alpha$	$\alpha\alpha$	$\varepsilon\alpha$

$T_4 =$

	i	d	a
i	$\gamma\gamma$	$\gamma\varepsilon$	$\varepsilon\varepsilon$
d	$\alpha\gamma$	$\gamma\gamma$	$\varepsilon\varepsilon$
a	$\alpha\alpha$	$\alpha\alpha$	$\varepsilon\alpha$

$T_5 =$

	i	d	a
i	$\gamma\gamma$	$\gamma\varepsilon$	$\gamma\varepsilon$
d	$\alpha\gamma$	$\gamma\gamma$	$\gamma\gamma$
a	$\alpha\gamma$	$\gamma\gamma$	$\gamma\gamma$

$T_6 =$

	i	d	a
i	$\gamma\gamma$	$\gamma\varepsilon$	$\gamma\varepsilon$
d	$\alpha\gamma$	$\gamma\gamma$	$\gamma\delta$
a	$\alpha\gamma$	$\beta\gamma$	$\gamma\gamma$

$T_7 =$

	i	d	a
i	$\gamma\gamma$	$\gamma\varepsilon$	$\gamma\varepsilon$
d	$\alpha\gamma$	$\gamma\gamma$	$\gamma\varepsilon$
a	$\alpha\gamma$	$\alpha\gamma$	$\gamma\gamma$

Let T be a table and $i \in \omega$. We denote by T^i the sequence $T \cdots T$, where T is repeated i times (if $i = 0$, then T^i is the empty sequence). We denote by T^∞ the infinite sequence $TTT\cdots$. Denote $\Delta = \{T_2^\infty, T_2^iT_3^\infty, T_2^iT_3^jT_4^\infty, T_2^iT_3^jT_4^kT_7^\infty, T_2^iT_5^\infty, T_2^iT_5^jT_6^\infty, T_2^iT_5^jT_6^kT_7^\infty : i, j, k \in \omega\}$.

Theorem 1. *For any sm-pair (U, ψ), the relation $dtyp(U, \psi) \in \{T_1^\infty\} \cup \Delta$ holds. For any sequence $\sigma \in \{T_1^\infty\} \cup \Delta$, there exists a sm-pair (U, ψ) such that $dtyp(U, \psi) = \sigma$.*

Theorem 2. *For any limited sm-pair (U, ψ), the relation $dtyp(U, \psi) \in \Delta$ holds. For any sequence $\sigma \in \Delta$, there exists a limited sm-pair (U, ψ) such that ψ is a weighted depth and $dtyp(U, \psi) = \sigma$.*

Example 7 (*Continuation of Examples 2–6*)**.** We know (see Example 6) that $typ(\mathcal{U}_{U_0\psi n}^{di}) = \beta$ for any $n \in \omega \setminus \{0\}$, where ψ is the complexity measure over the structure U_0 defined in Example 5. Using Theorem 1, we obtain that, for any $n \in \omega \setminus \{0\}$, $typ(U_0, \psi, n) = T_3$. Therefore $dtyp(U_0, \psi) = T_3^\infty$.

4. Possible upper *n*-types of sm-pairs

In this section, we will enumerate all possible upper n-types of sm-pairs.

Definition 12. The upper n-type, $n \in \omega \setminus \{0\}$, of a sm-pair (U, ψ) is the table $typ_u(U, \psi, n)$ with three rows and three columns in which rows from top to bottom and columns from the left to the right are labeled with indices i, d, a and the value $typ(\mathcal{U}_{U\psi n}^{bc})$ is in the intersection of the row with index $b \in \{i, d, a\}$ and the column with index $c \in \{i, d, a\}$.

We now define seven tables:

$t_1 =$

	i	d	a
i	α	α	α
d	α	α	α
a	α	α	α

$t_2 =$

	i	d	a
i	γ	ε	ε
d	α	α	α
a	α	α	α

$t_3 =$

	i	d	a
i	γ	ε	ε
d	β	γ	ε
a	α	α	α

$t_4 =$

	i	d	a
i	γ	ε	ε
d	γ	γ	ε
a	α	α	α

$t_5 =$

	i	d	a
i	γ	ε	ε
d	γ	γ	γ
a	γ	γ	γ

$t_6 =$

	i	d	a
i	γ	ε	ε
d	γ	γ	δ
a	γ	γ	γ

$$t_7 = \begin{array}{|c|ccc|} \hline & i & d & a \\ \hline i & \gamma & \varepsilon & \varepsilon \\ d & \gamma & \gamma & \varepsilon \\ a & \gamma & \gamma & \gamma \\ \hline \end{array}$$

In this section, we will prove the following two propositions.

Proposition 3. *For any sm-pair (U, ψ) and any $n \in \omega \setminus \{0\}$, the relation $typ_u(U, \psi, n) \in \{t_1, \dots, t_7\}$ holds.*

Proposition 4. *For any limited sm-pair (U, ψ) and any $n \in \omega \setminus \{0\}$, the relation $typ_u(U, \psi, n) \in \{t_2, \dots, t_7\}$ holds.*

First, we prove some auxiliary statements.

Lemma 5. *Let (U, ψ) be a sm-pair and $z \in \mathcal{P}(U)$. Then the inequalities $\psi_U^a(z) \le \psi_U^d(z) \le \psi_U^i(z)$ hold.*

Proof. Let $z = (Y, \nu, \beta_1, \dots, \beta_m)$. It is not difficult to construct a computation tree $\Gamma_0 \in \mathcal{T}(U)$, which solves the problem z deterministically and for which $\beta(\xi) = \beta(z) = \beta_1, \dots, \beta_m$ for any complete path ξ in the computation tree Γ_0. Evidently, $\psi(\Gamma_0) = \psi_U^i(z)$. Therefore $\psi_U^d(z) \le \psi_U^i(z)$. If a computation tree $\Gamma \in \mathcal{T}(U)$ solves the problem z deterministically, then the computation tree Γ solves the problem z nondeterministically. Therefore $\psi_U^a(z) \le \psi_U^d(z)$. □

Let (U, ψ) be a sm-pair, $n \in \omega \setminus \{0\}$, $m \in \omega$, and $b, c \in \{i, d, a\}$. The notation $\mathcal{U}_{U\psi n}^{bc}(m) = \infty$ means that the set $\{\psi_U^b(z) : z \in \mathcal{P}(U, n), \psi_U^c(z) \le m\}$ is infinite. Evidently, if $\mathcal{U}_{U\psi n}^{bc}(m) = \infty$, then $\mathcal{U}_{U\psi n}^{bc}(m+1) = \infty$. It is not difficult to prove the following statement.

Lemma 6. *Let (U, ψ) be a sm-pair, $n \in \omega \setminus \{0\}$, $m \in \omega$, and $b, c \in \{i, d, a\}$. Then*
(a) If there exists $m \in \omega$ such that $\mathcal{U}_{U\psi n}^{bc}(m) = \infty$, then $typ(\mathcal{U}_{U\psi n}^{bc}) = \varepsilon$, $Dom(\mathcal{U}_{U\psi n}^{bc}) = \emptyset$ if $m_0 = m_1$, and $Dom(\mathcal{U}_{U\psi n}^{bc}) = \{m : m \in \omega, m_0 \le m < m_1\}$ if $m_0 < m_1$, where $m_0 = \min\{\psi_U^c(z) : z \in \mathcal{P}(U, n)\}$ and $m_1 = \min\{m : m \in \omega, \mathcal{U}_{U\psi n}^{bc}(m) = \infty\}$.
(b) If there is no $m \in \omega$ such that $\mathcal{U}_{U\psi n}^{bc}(m) = \infty$, then $Dom(\mathcal{U}_{U\psi n}^{bc}) = \{m : m \in \omega, m \ge m_0\}$.

Let (U, ψ) be a sm-pair, $n \in \omega \setminus \{0\}$, and $b, c, e, f \in \{i, d, a\}$. The notation $\mathcal{U}_{U\psi n}^{bc} \lhd \mathcal{U}_{U\psi n}^{ef}$ means that, for any $m \in \omega$, the following conditions hold:

- If the value $\mathcal{U}_{U\psi n}^{bc}(m)$ is defined, then either $\mathcal{U}_{U\psi n}^{ef}(m) = \infty$ or the value $\mathcal{U}_{U\psi n}^{ef}(m)$ is defined and the inequality $\mathcal{U}_{U\psi n}^{bc}(m) \le \mathcal{U}_{U\psi n}^{ef}(m)$ holds.
- If $\mathcal{U}_{U\psi n}^{bc}(m) = \infty$, then $\mathcal{U}_{U\psi n}^{ef}(m) = \infty$.

We define a linear order $\preceq$ on the set $\{\alpha, \beta, \gamma, \delta, \varepsilon\}$ as follows: $\alpha \preceq \beta \preceq \gamma \preceq \delta \preceq \varepsilon$.

Lemma 7. *Let (U, ψ) be a sm-pair and $n \in \omega \setminus \{0\}$. Then $typ(\mathcal{U}_{U\psi n}^{bi}) \preceq typ(\mathcal{U}_{U\psi n}^{bd}) \preceq typ(\mathcal{U}_{U\psi n}^{ba})$ and $typ(\mathcal{U}_{U\psi n}^{ab}) \preceq typ(\mathcal{U}_{U\psi n}^{db}) \preceq typ(\mathcal{U}_{U\psi n}^{ib})$ for any $b \in \{i, d, a\}$.*

Proof. From the definition of the functions $\mathcal{U}_{U\psi n}^{bc}$, $b, c \in \{i, d, a\}$, and from Lemma 5 it follows that $\mathcal{U}_{U\psi n}^{bi} \lhd \mathcal{U}_{U\psi n}^{bd} \lhd \mathcal{U}_{U\psi n}^{ba}$ and $\mathcal{U}_{U\psi n}^{ab} \lhd \mathcal{U}_{U\psi n}^{db} \lhd \mathcal{U}_{U\psi n}^{ib}$ for any $b \in \{i, d, a\}$. Using these relations and Lemma 6 we obtain the statement of the lemma. □

Lemma 8. *Let (U, ψ) be a sm-pair, $n \in \omega \setminus \{0\}$, and $b, c \in \{i, d, a\}$. Then*
(a) $typ(\mathcal{U}_{U\psi n}^{bc}) = \alpha$ if and only if the function ψ_U^b is bounded from above on the set $\mathcal{P}(U, n)$.
(b) If the function ψ_U^b is unbounded from above on the set $\mathcal{P}(U, n)$, then $typ(\mathcal{U}_{U\psi n}^{bb}) = \gamma$.

Proof (*a*)**.** The first statement of the lemma is obvious. (b) Let the function ψ_U^b be unbounded from above on the set $\mathcal{P}(U, n)$. One can show that in this case the equality $\mathcal{U}_{U\psi n}^{bb}(m) = m$ holds for infinitely many $m \in \omega$. Therefore $typ(\mathcal{U}_{U\psi n}^{bb}) = \gamma$. □

Corollary 9. *Let (U, ψ) be a sm-pair, $n \in \omega \setminus \{0\}$, and $b \in \{i, d, a\}$. Then $typ(\mathcal{U}_{U\psi n}^{bb}) \in \{\alpha, \gamma\}$.*

Let $U = (A, F, P)$ be a structure and $n \in \omega \setminus \{0\}$. We denote by $P_n[F]$ the set of functions from $P[F]$ with variables from the set $\{x_1, \dots, x_n\}$. It is not difficult to prove the following statement.

Lemma 10. *Let (U, ψ) be a limited sm-pair, $U = (A, F, P)$, and $n \in \omega \setminus \{0\}$. Then the function ψ_U^d is bounded from above on the set $\mathcal{P}(U, n)$ if and only if the set $P_n[F]$ is a finite set.*

Lemma 11. *Let (U, ψ) be a sm-pair, $n \in \omega \setminus \{0\}$, and $typ(\mathcal{U}_{U\psi n}^{ii}) \ne \alpha$. Then $typ(\mathcal{U}_{U\psi n}^{id}) = typ(\mathcal{U}_{U\psi n}^{ia}) = \varepsilon$.*

Proof. By Lemma 8, the function ψ_U^i is unbounded from above on $\mathcal{P}(U,n)$. Let $r \in \omega$. Then there exists a problem $z = (Y, \nu, \beta_1, \dots, \beta_m) \in \mathcal{P}(U,n)$ such that $\psi_U^i(z) \geq r$. Let us consider the problem $z' = (Y, \nu', \beta_1, \dots, \beta_m)$, where $\nu' \equiv \{0\}$. It is clear that $\psi_U^i(z') \geq r$. Let $\Gamma = (Y, G)$ be a computation tree which consists of the root, the terminal node labeled with 0 and the edge leaving the root and entering the terminal node. One can show that the computation tree Γ solves the problem z' deterministically. Therefore $\psi_U^a(z') \leq \psi_U^d(z') \leq \psi(\Gamma) = \psi(\lambda)$. Taking into account that r is an arbitrary number from ω, we obtain $\mathcal{U}_{U\psi n}^{id}(\psi(\lambda)) = \infty$ and $\mathcal{U}_{U\psi n}^{ia}(\psi(\lambda)) = \infty$. By Lemma 6, $typ(\mathcal{U}_{U\psi n}^{id}) = typ(\mathcal{U}_{U\psi n}^{ia}) = \varepsilon$. □

Lemma 12. *Let (U, ψ) be a sm-pair and $n \in \omega \setminus \{0\}$. Then $typ(\mathcal{U}_{U\psi n}^{ai}) \in \{\alpha, \gamma\}$.*

Proof. Using Lemma 7 and Corollary 9 we obtain $typ(\mathcal{U}_{U\psi n}^{ai}) \in \{\alpha, \beta, \gamma\}$. Assume that $typ(\mathcal{U}_{U\psi n}^{ai}) = \beta$. Then there exists $r \in \omega \setminus \{0\}$ such that $\mathcal{U}_{U\psi n}^{ai}(m) < m$ for any $m \in \omega$, $m > r$. We prove by induction on m that, for any problem $z \in \mathcal{P}(U,n)$, if $\psi_U^i(z) \leq m$, then $\psi_U^a(z) \leq r_0$, where $r_0 = \max\{r, \psi(\lambda)\}$. From Lemma 5 it follows that under the condition $m \leq r$ the considered statement holds. Let it hold for some $m \geq r$. We now show that this statement holds for $m+1$ too. Let $z \in \mathcal{P}(U,n)$ and $\psi_U^i(z) \leq m+1$. Since $m+1 > r$, $\psi_U^a(z) \leq m$. Let $\Gamma \in \mathcal{T}(U)$, $\psi(\Gamma) = \psi_U^a(z)$, and the computation tree Γ solves the problem z nondeterministically. Assume that in Γ there exists a complete path ξ such that in the sequence $\beta(\xi)$ there are no predicate expressions. In this case, a computation tree, which has the same set of input variables as z and consists of the root, the terminal node labeled with $\kappa(\xi)$, and the edge leaving the root and entering the terminal node, solves the problem z nondeterministically. Therefore $\psi_U^a(z) \leq \psi(\lambda) \leq r_0$. Assume now that, for each complete path ξ in the computation tree Γ, the sequence $\beta(\xi)$ contains a predicate expression. Let $\xi \in \Xi(\Gamma)$, $\xi = v_0, d_0, \dots, v_p, d_p, v_{p+1}$ and let the expression β_i be assigned to the node v_i, $i = 1, \dots, p$. Let there be exactly $t > 0$ predicate expressions $\beta_{c_1}, \dots, \beta_{c_t}$ among the expressions $\beta_1, \dots, \beta_p$, where $c_1 < \dots < c_t$. For $i = 1, \dots, t$, let δ_i be the number from the set E_2 assigned to the edge d_{c_i}. Let us consider a problem $z_\xi = (Y, \nu_\xi, \beta_1, \dots, \beta_p)$, where Y is the set of input variables for the problem z, $\nu_\xi(\delta_1, \dots, \delta_t) = \{\kappa(\xi)\}$ and $\nu_\xi(\bar\sigma) = \{\kappa(\xi)+1\}$ for any t-tuple $\bar\sigma \in E_2^t$ such that $\bar\sigma \neq (\delta_1, \dots, \delta_t)$. It is clear that $\psi_U^i(z_\xi) \leq m$. Using the inductive hypothesis we obtain that there exists a computation tree $\Gamma_\xi \in \mathcal{T}(U)$, which has the following properties: Γ_ξ solves the problem z_ξ nondeterministically and $\psi(\Gamma_\xi) \leq r_0$. Let $\mathcal{A}(\xi) \neq \emptyset$. We denote by $\tilde\Gamma_\xi$ a computation tree obtained from Γ_ξ by removal of all nodes and edges satisfying the following condition: there is no a complete path ξ' in Γ_ξ, which contains this node or edge and for which $\kappa(\xi') = \kappa(\xi)$. Let $\{\xi : \xi \in \Xi(\Gamma), \mathcal{A}(\xi) \neq \emptyset\} = \{\xi_1, \dots, \xi_q\}$. We identify the roots of computation trees $\tilde\Gamma_{\xi_1}, \dots, \tilde\Gamma_{\xi_q}$. Denote by Γ^* the obtained computation tree. It is not difficult to show that $\Gamma^* \in \mathcal{T}(U)$, $\psi(\Gamma^*) \leq r_0$ and the computation tree Γ^* solves the problem z nondeterministically. Thus, the considered statement holds. By Lemma 8, $typ(\mathcal{U}_{U\psi n}^{ai}) = \alpha$. The obtained contradiction shows that $typ(\mathcal{U}_{U\psi n}^{ai}) \in \{\alpha, \gamma\}$. □

Proof of Proposition 3. Let (U, ψ) be a sm-pair and $n \in \omega \setminus \{0\}$. By Corollary 9, $typ(\mathcal{U}_{U\psi n}^{ii}) \in \{\alpha, \gamma\}$. Using Corollary 9 and Lemma 7 we obtain $typ(\mathcal{U}_{U\psi n}^{di}) \in \{\alpha, \beta, \gamma\}$. From Lemma 12 it follows that $typ(\mathcal{U}_{U\psi n}^{ai}) \in \{\alpha, \gamma\}$.

(a) Let $typ(\mathcal{U}_{U\psi n}^{ii}) = \alpha$. Using Lemmas 7 and 8 we obtain $typ_u(U, \psi, n) = t_1$.

(b) Let $typ(\mathcal{U}_{U\psi n}^{ii}) = \gamma$ and $typ(\mathcal{U}_{U\psi n}^{di}) = \alpha$. Using Lemmas 7, 8, and 11 we obtain $typ_u(U, \psi, n) = t_2$.

(c) Let $typ(\mathcal{U}_{U\psi n}^{ii}) = \gamma$ and $typ(\mathcal{U}_{U\psi n}^{di}) = \beta$. Using Lemma 11 we obtain $typ(\mathcal{U}_{U\psi n}^{id}) = typ(\mathcal{U}_{U\psi n}^{ia}) = \varepsilon$. By Lemmas 7 and 12, $typ(\mathcal{U}_{U\psi n}^{ai}) = \alpha$. From this equality and from Lemma 8 it follows that $typ(\mathcal{U}_{U\psi n}^{ad}) = typ(\mathcal{U}_{U\psi n}^{aa}) = \alpha$. Using the equality $typ(\mathcal{U}_{U\psi n}^{di}) = \beta$, Lemma 7, and Corollary 9 we obtain $typ(\mathcal{U}_{U\psi n}^{dd}) = \gamma$. From the equalities $typ(\mathcal{U}_{U\psi n}^{dd}) = \gamma$, $typ(\mathcal{U}_{U\psi n}^{aa}) = \alpha$ and from Lemmas 6 and 8 it follows that $typ(\mathcal{U}_{U\psi n}^{da}) = \varepsilon$. Thus, $typ_u(U, \psi, n) = t_3$.

(d) Let $typ(\mathcal{U}_{U\psi n}^{ii}) = typ(\mathcal{U}_{U\psi n}^{di}) = \gamma$ and $typ(\mathcal{U}_{U\psi n}^{ai}) = \alpha$. Using Lemma 11 we obtain $typ(\mathcal{U}_{U\psi n}^{id}) = typ(\mathcal{U}_{U\psi n}^{ia}) = \varepsilon$. From Lemma 8 it follows that $typ(\mathcal{U}_{U\psi n}^{ad}) = typ(\mathcal{U}_{U\psi n}^{aa}) = \alpha$. Using Lemma 7 and Corollary 9 we obtain $typ(\mathcal{U}_{U\psi n}^{dd}) = \gamma$. Taking into account the equality $typ(\mathcal{U}_{U\psi n}^{aa}) = \alpha$ and Lemmas 6 and 8 we obtain $typ(\mathcal{U}_{U\psi n}^{da}) = \varepsilon$. Thus, $typ_u(U, \psi, n) = t_4$.

(e) Let $typ(\mathcal{U}_{U\psi n}^{ii}) = typ(\mathcal{U}_{U\psi n}^{di}) = typ(\mathcal{U}_{U\psi n}^{ai}) = \gamma$. Using Lemma 11 we obtain $typ(\mathcal{U}_{U\psi n}^{id}) = typ(\mathcal{U}_{U\psi n}^{ia}) = \varepsilon$. From Lemma 7 and Corollary 9 it follows that $typ(\mathcal{U}_{U\psi n}^{dd}) = typ(\mathcal{U}_{U\psi n}^{ad}) = typ(\mathcal{U}_{U\psi n}^{aa}) = \gamma$. By Lemma 7, $typ(\mathcal{U}_{U\psi n}^{da}) \in \{\gamma, \delta, \varepsilon\}$. Therefore $typ_u(U, \psi, n) \in \{t_5, t_6, t_7\}$. □

Proof of Proposition 4. Let (U, ψ) be a limited sm-pair, $U = (A, F, P)$, and $n \in \omega \setminus \{0\}$. Then, for any $\alpha \in (F \cup P)^*$, the inequality $\psi(\alpha) \geq |\alpha|$ holds. Therefore the function ψ_U^i is unbounded from above on the set $\mathcal{P}(U,n)$. By Lemma 8, $typ(\mathcal{U}_{U\psi n}^{ii}) \neq \alpha$. Therefore $typ_u(U, \psi, n) \neq t_1$. From this relation and Proposition 3 it follows that the statement of Proposition 4 holds. □

5. Possible upper dynamic types of sm-pairs

In this section, we begin the study of upper dynamic types of sm-pairs.

Definition 13. The upper dynamic type of a sm-pair (U, ψ) is the infinite sequence $dtyp_u(U, \psi) = typ_u(U, \psi, 1)typ_u(U, \psi, 2)\cdots$.

Denote $\Delta_u = \{t_2^\infty, t_2^i t_3^\infty, t_2^i t_3^j t_4^\infty, t_2^i t_3^j t_4^k t_7^\infty, t_2^i t_5^\infty, t_2^i t_5^j t_6^\infty, t_2^i t_5^j t_6^k t_7^\infty : i, j, k \in \omega\}$. In this section, we prove the following two propositions.

Proposition 13. *For any sm-pair (U, ψ), the relation $dtyp_u(U, \psi) \in \{t_1^\infty\} \cup \Delta_u$ holds.*

Proposition 14. *For any limited sm-pair (U, ψ), the relation $dtyp_u(U, \psi) \in \Delta_u$ holds.*

Let (U, ψ) be a sm-pair, $n \in \omega \setminus \{0\}$, and $b, c \in \{i, d, a\}$. The notation $\mathcal{U}^{bc}_{U\psi n} \lhd \mathcal{U}^{bc}_{U\psi n+1}$ means that, for any $m \in \omega$, the following conditions hold:

- If the value $\mathcal{U}^{bc}_{U\psi n}(m)$ is defined, then either $\mathcal{U}^{bc}_{U\psi n+1}(m) = \infty$ or the value $\mathcal{U}^{bc}_{U\psi n+1}(m)$ is defined and the inequality $\mathcal{U}^{bc}_{U\psi n}(m) \le \mathcal{U}^{bc}_{U\psi n+1}(m)$ holds.
- If $\mathcal{U}^{bc}_{U\psi n}(m) = \infty$, then $\mathcal{U}^{bc}_{U\psi n+1}(m) = \infty$.

Lemma 15. *Let (U, ψ) be a sm-pair, $n \in \omega \setminus \{0\}$, and $b, c \in \{i, d, a\}$. Then $typ(\mathcal{U}^{bc}_{U\psi n}) \preceq typ(\mathcal{U}^{bc}_{U\psi n+1})$.*

Proof. Let $U = (A, F, P)$. Let $z \in \mathcal{P}(U, n)$, $z = (Y, \nu, \beta_1, \dots, \beta_m)$, and s be the minimum number from ω such that all variables from Y and all variables from the expressions $\beta_1, \dots, \beta_m$ belong to the set $\{x_0, \dots, x_s\}$. Denote $r = s + 1$ and $\tilde{z} = (Y \cup \{x_r\}, \nu, \beta_1, \dots, \beta_m)$. We will prove that, for any $b \in \{i, d, a\}$, the equality $\psi^b_U(z) = \psi^b_U(\tilde{z})$ holds. It is clear that the considered equality holds if $b = i$. We now consider the case $b = a$.

One can show that there exists a computation tree $\Gamma_1 = (Y, G_1)$ over U satisfying the following conditions:

- Γ_1 solves the problem z nondeterministically.
- $\psi(\Gamma_1) = \psi^a_U(z)$.
- The variable x_r is not contained in the expressions assigned to nodes of Γ_1.

Denote $\Gamma_2 = (Y \cup \{x_r\}, G_1)$. Let $\bar{a}' = (a_1, \dots, a_n, a_{n+1}) \in A^{n+1}$, $\bar{a} = (a_1, \dots, a_n)$, ξ' be a complete path in Γ_2, and ξ be the complete path in Γ_1, which coincides with ξ'. One can show that $\tilde{z}(\bar{a}') = z(\bar{a})$ and $\bar{a}' \in \mathcal{A}(\xi')$ if and only if $\bar{a} \in \mathcal{A}(\xi)$. Using these relations it is not difficult to show that Γ_2 solves the problem $\tilde{z}$ nondeterministically. Therefore $\psi^a_U(\tilde{z}) \le \psi^a_U(z)$.

Let $\Gamma_3 = (Y \cup \{x_r\}, G_3)$ be a computation tree over U, which solves the problem $\tilde{z}$ nondeterministically and for which $\psi(\Gamma_3) = \psi^a_U(\tilde{z})$. Denote $\Gamma_4 = (Y, G_3)$. Let $\bar{a} = (a_1, \dots, a_n) \in A^n$, $\bar{a}' = (a_1, \dots, a_n, a_1)$, ξ be a complete path in Γ_4 and ξ' be the complete path in Γ_3, which coincides with ξ. One can show that $z(\bar{a}) = \tilde{z}(\bar{a}')$ and $\bar{a} \in \mathcal{A}(\xi)$ if and only if $\bar{a}' \in \mathcal{A}(\xi')$. Using these relations one can show that Γ_4 solves the problem z nondeterministically. Therefore $\psi^a_U(z) \le \psi^a_U(\tilde{z})$. Hence $\psi^a_U(z) = \psi^a_U(\tilde{z})$.

The case $b = d$ can be considered in a similar way. Thus, $\psi^b_U(z) = \psi^b_U(\tilde{z})$ for any problem $z \in \mathcal{P}(U, n)$ and for any $b \in \{i, d, a\}$. Taking into account that z is an arbitrary problem from $\mathcal{P}(U, n)$, one can show that $\mathcal{U}^{bc}_{U\psi n} \lhd \mathcal{U}^{bc}_{U\psi n+1}$. Using this relation and Lemma 6 we obtain $typ(\mathcal{U}^{bc}_{U\psi n}) \preceq typ(\mathcal{U}^{bc}_{U\psi n+1})$. □

We now define a partial order $\leqq$ on the set of tables $\{t_1, \dots, t_7\}$. For $b, c \in \{i, d, a\}$ and $i \in \{1, \dots, 7\}$, let t_i^{bc} be the letter from the set $\{\alpha, \beta, \gamma, \delta, \varepsilon\}$ that is in the table t_i in the intersection of the row with index b and the column with index c. Let $i, j \in \{1, \dots, 7\}$. Then $t_i \leqq t_j$ if and only if $t_i^{bc} \preceq t_j^{bc}$ for any $b, c \in \{i, d, a\}$. It is easy to check that the graph depicted in Fig. 3 is the Hasse diagram for the partially ordered set $(\{t_1, \dots, t_7\}, \leqq)$. Nodes of this diagram are tables $t_1, \dots, t_7$. An edge goes upward from t_i to t_j if $t_i \leqq t_j$ and there is no t_k, $t_k \notin \{t_i, t_j\}$ such that $t_i \leqq t_k \leqq t_j$.

Proof of Proposition 13. Let (U, ψ) be a sm-pair. Assume that there exists $n_0 \in \omega \setminus \{0\}$ for which $typ_u(U, \psi, n_0) = t_1$. Then, by Lemma 8, the function ψ^i_U is bounded from above on the set $\mathcal{P}(U, n_0)$. From here it follows that the function ψ^i_U is bounded from above on the set $\mathcal{P}(U, n)$ for any $n \in \omega \setminus \{0\}$. Using Lemmas 5 and 8 we obtain $dtyp_u(U, \psi) = t_1^\infty$.

Let for any $n \in \omega \setminus \{0\}$, the relation $typ_u(U, \psi, n) \neq t_1$ hold. In this case, by Proposition 3, $typ_u(U, \psi, n) \in \{t_2, \dots, t_7\}$ for any $n \in \omega \setminus \{0\}$. Using Lemma 15, we obtain that $typ_u(U, \psi, n) \leqq typ_u(U, \psi, n+1)$ for any $n \in \omega \setminus \{0\}$. Simple analysis of the Hasse diagram for the partially ordered set $(\{t_1, \dots, t_7\}, \leqq)$ depicted in Fig. 3 shows that the set Δ_u coincides with the set of infinite sequences $t_{i_1} t_{i_2} \cdots$ such that $t_{i_1}, t_{i_2}, \dots \in \{t_2, \dots, t_7\}$ and $t_{i_1} \leqq t_{i_2} \leqq \cdots$. Therefore $dtyp_u(U, \psi) \in \Delta_u$ □

Proof of Proposition 14. Let (U, ψ) be a limited sm-pair. Using Proposition 4 we obtain $dtyp_u(U, \psi) \neq t_1^\infty$. From this relation and Proposition 13 it follows that $dtyp_u(U, \psi) \in \Delta_u$. □

6. Realizable upper dynamic types of sm-pairs

In this section, we continue the study of upper dynamic types of sm-pairs and prove the following two propositions.

Proposition 16. *For any sequence $\tau \in \{t_1^\infty\} \cup \Delta_u$, there exists a sm-pair (U, ψ) such that $dtyp_u(U, \psi) = \tau$.*

Proposition 17. *For any sequence $\tau \in \Delta_u$, there exists a limited sm-pair (U, ψ) such that $dtyp_u(U, \psi) = \tau$ and ψ is a weighted depth.*

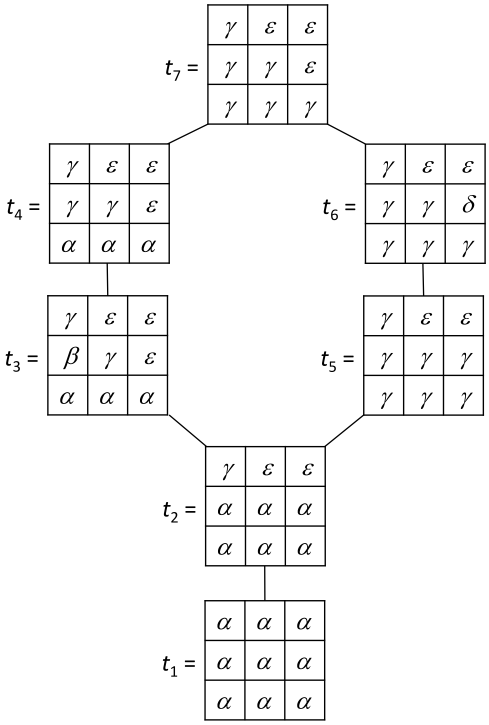

Fig. 3. Hasse diagram for the partially ordered set $(\{t_1, \dots, t_7\}, \leqq)$.

We now describe a construction that will be used in the proofs of the considered propositions.

For $i = 2, \dots, 7$, we define a sm-pair $\pi_i = (U_i, \psi_i)$, where $U_i = (A_i, F_i, P_i)$, $F_i = \emptyset$, P_i is a set of one-place predicates, and ψ_i is a weighted depth. It is clear that for the definition of the function ψ_i it is enough to define values of ψ_i on elements of the set P_i.

Define the sm-pair π_2 as follows: $A_2 = \{0\}$, $P_2 = \{q_1\}$, $q_1(0) = 0$, and $\psi_2(q_1) = 1$.

Define the sm-pair π_3 as follows: $A_3 = \omega$, $P_3 = \{l_i : i \in \omega\}$,

$$l_i(j) = \begin{cases} 0, & j \leq i, \\ 1, & j > i, \end{cases}$$

and $\psi_3(l_i) = 1$ for any $i, j \in \omega$, i.e., $\psi_3 = h$.

Define the sm-pair π_4 as follows: $A_4 = \omega$, P_4 is the set of mappings of the kind $f : \omega \to \{0, 1\}$, and $\psi_4(f) = 1$ for any $f \in P_4$, i.e., $\psi_4 = h$.

Define the sm-pair π_5 as follows: $A_5 = \omega$, $P_5 = \{q_i : i \in \omega\}$,

$$q_i(j) = \begin{cases} 0, & j \neq i, \\ 1, & j = i \end{cases}$$

for any $i, j \in \omega$, $\psi_5(q_0) = 1$, and $\psi_5(q_i) = i \sum_{t=0}^{i-1} \psi_5(q_t)$ for $i \geq 1$.

Define the sm-pair π_6 as follows: $A_6 = \omega$, $P_6 = \{q_{2i}, q_{2i+1}, p_{2i} : i \in \omega \setminus \{0\}\}$, $q_{2i}(j) = 1$ if and only if $j = 2i$, $q_{2i+1}(j) = 1$ if and only if $j = 2i + 1$, $p_{2i}(j) = 1$ if and only if $j = 2i$ or $j = 2i + 1$, and $\psi_6(q_{2i}) = \psi_6(q_{2i+1}) = \psi_6(p_{2i}) = i$ for any $i \in \omega \setminus \{0\}$ and $j \in \omega$.

Define the sm-pair π_7 as follows: $A_7 = \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, $P_7 = \{l_i : i \in \omega\} \cup \{q_{-k} : k \in \omega \setminus \{0\}\}$,

$$l_i(j) = \begin{cases} 0, & j \leq i, \\ 1, & j > i, \end{cases} \quad q_{-k}(j) = \begin{cases} 0, & j \neq -k, \\ 1, & j = -k, \end{cases}$$

$\psi_7(l_i) = 1$, and $\psi_7(q_{-k}) = k$ for any $i \in \omega$, $k \in \omega \setminus \{0\}$, and $j \in \mathbb{Z}$.

Let $r \in \{2, \dots, 7\}$ and $n \in \omega \setminus \{0\}$. Define a sm-pair $\pi_r^{(n)} = (U_r^{(n)}, \psi_r^{(n)})$, where $U_r^{(n)} = (A_r^{(n)}, F_r^{(n)}, P_r^{(n)})$ and $\psi_r^{(n)}$ is a weighted depth. Let $K = \{k_i : i \in \omega\}$ be a set such that $k_i \neq k_j$ if $i \neq j$ and $K \cap \mathbb{Z} = \emptyset$. For any $c \in A_r \cup K$, denote

$c^{(n)} = (c, n)$. Then $A_r^{(n)} = \{a^{(n)} : a \in A_r\} \cup \{k_0^{(n)}, k_1^{(n)}, \ldots, k_{n-1}^{(n)}\}$, $F_r^{(n)} = \emptyset$, $P_r^{(n)} = \{g^{(n)} : g \in P_r\}$, where

$$g^{(n)}(x_1, \ldots, x_n) = \begin{cases} g(a), & x_1 = k_1^{(n)}, \ldots, x_{n-1} = k_{n-1}^{(n)}, \\ & x_n = (a, n) \in A_r^{(n)}, \\ 0, & \text{otherwise}, \end{cases} \quad (1)$$

and $\psi_r^{(n)}(g^{(n)}) = \psi_r(g)$ for any $g \in P_r$.

Let $i_1, \ldots, i_m \in \{2, \ldots, 7\}$, $n_1, \ldots, n_m \in \omega \setminus \{0\}$, and $n_1 < \cdots < n_m$. Define the sm-pair $\pi_{i_1}^{(n_1)} \oplus \cdots \oplus \pi_{i_m}^{(n_m)}$ as follows: $\pi_{i_1}^{(n_1)} \oplus \cdots \oplus \pi_{i_m}^{(n_m)} = (U, \psi)$, $U = (A, F, P)$, $A = A_{i_1}^{(n_1)} \cup \cdots \cup A_{i_m}^{(n_m)}$, $F = \emptyset$, $P = P_{i_1}^{(n_1)} \cup \cdots \cup P_{i_m}^{(n_m)}$, and ψ is a weighted depth. For any $j \in \{1, \ldots, m\}$ and $g^{(n_j)} \in P_{i_j}^{(n_j)}$, $g^{(n_j)}$ is equal to 0 on tuples that do not belong to the set $(A_{i_j}^{(n_j)})^{n_j}$, and is defined by (1) with $n = n_j$ and $r = i_j$ on tuples from $(A_{i_j}^{(n_j)})^{n_j}$, and $\psi(g^{(n_j)}) = \psi_{i_j}(g^{(n_j)})$.

Let $\tau \in \Delta_u$ and $\tau = t_{v_1}^{w_1} \cdots t_{v_m}^{w_m}$, where $v_1, \ldots, v_m$ are pairwise different numbers from $\{2, \ldots, 7\}$, $w_m = \infty$ and if $m \geq 2$, then $w_j \in \omega \setminus \{0\}$ for $j = 1, \ldots, m-1$. Define the sm-pair (U_τ, ψ_τ) as follows:

$$(U_\tau, \psi_\tau) = \pi_{v_1}^{(1)} \oplus \pi_{v_2}^{(w_1+1)} \oplus \cdots \oplus \pi_{v_m}^{(w_1+\cdots+w_{m-1}+1)}.$$

In particular, if $m = 1$, then $(U_\tau, \psi_\tau) = \pi_{v_1}^{(1)}$.

Let $\tau \in \Delta_u$, $(U_\tau, \psi_\tau) = \pi_{v_1}^{(n_1)} \oplus \cdots \oplus \pi_{v_m}^{(n_m)}$, $n \in \omega \setminus \{0\}$, $r \in \{1, \ldots, m\}$, $n \geq n_r$ and if $r < m$, then $n < n_{r+1}$. Let $U_\tau = (A, F, P)$. It is clear that all functions from $P[F]$ are of the kind $g(x_{i_1}, \ldots, x_{i_s})$, where $g \in P$. Since $n < n_{r+1}$, any function from $P[F]$ that depends on n variables and does not equal identically to 0 is of the kind $g(x_{i_1}, \ldots, x_{i_{n_j}})$, where $j \leq r$, $g \in P_{v_j}^{(n_j)}$ and $x_{i_1}, \ldots, x_{i_{n_j}}$ are pairwise different variables. The tuple $(x_{i_1}, \ldots, x_{i_{n_j}})$ will be called the sort of the considered function. Let Y be a set of variables and $|Y| = n$. Then a function depending on n_j pairwise different variables from Y may have exactly $(n)_{n_j} = n(n-1)\cdots(n-n_j+1)$ different sorts.

We now prove some statements about properties of sm-pairs (U_τ, ψ_τ), where $\tau \in \Delta_u$.

Lemma 18. *Let $\tau \in \Delta_u$, $(U_\tau, \psi_\tau) = \pi_{v_1}^{(n_1)} \oplus \cdots \oplus \pi_{v_m}^{(n_m)}$, $r \in \{1, \ldots, m\}$, $v_r = 2$, $n \in \omega \setminus \{0\}$, $n \geq n_r$ and if $r < m$, then $n < n_{r+1}$. Then $typ_u(U_\tau, \psi_\tau, n) = t_2$.*

Proof. Since $v_r = 2$, $r = 1$. Denote $(U, \psi) = (U_\tau, \psi_\tau)$. It is clear that the function ψ_U^i is unbounded from above on the set $\mathcal{P}(U, n)$. Using Lemma 8 we obtain that $typ(\mathcal{U}_{U\psi n}^{ii}) = \gamma$. Let $U = (A, F, P)$. Taking into account that in the case $m \geq 2$ the inequality $n < n_2$ holds, one can show that all functions from $P[F]$ depending on n variables are equal identically to 0. Therefore the function ψ_U^d is bounded from above on the set $\mathcal{P}(U, n)$. From here and from Lemma 8 it follows that $typ(\mathcal{U}_{U\psi n}^{di}) = \alpha$. Using Proposition 3 we obtain $typ_u(U_\tau, \psi_\tau, n) = t_2$. □

Lemma 19. *Let $\tau \in \Delta_u$, $(U_\tau, \psi_\tau) = \pi_{v_1}^{(n_1)} \oplus \cdots \oplus \pi_{v_m}^{(n_m)}$, $r \in \{1, \ldots, m\}$, $v_r = 3$, $n \in \omega \setminus \{0\}$, $n \geq n_r$ and if $r < m$, then $n < n_{r+1}$. Then $typ_u(U_\tau, \psi_\tau, n) = t_3$.*

Proof. One can show that $r = 1$ or $r = 2$. In the latter case, $v_1 = 2$. Denote $(U, \psi) = (U_\tau, \psi_\tau)$. Let $U = (A, F, P)$. Since $n < n_{r+1}$, all functions from $P[F]$, which depend on n variables and do not equal identically to 0 are of the kind $l_i^{(n_r)}(x_{j_1}, \ldots, x_{j_{n_r}})$, where $x_{j_1}, \ldots, x_{j_{n_r}}$ are pairwise different variables.

We now prove that $typ(\mathcal{U}_{U\psi n}^{di}) = \beta$. Using Corollary 9 and Lemma 7 we obtain $typ(\mathcal{U}_{U\psi n}^{di}) \in \{\alpha, \beta, \gamma\}$. Show that $typ(\mathcal{U}_{U\psi n}^{di}) \neq \gamma$. Let $m \in \omega \setminus \{0\}$, $z = (Y, \nu, \beta_1, \ldots, \beta_t) \in \mathcal{P}(U, n)$, and $t \leq m$. One can transform the problem z into a problem $z' = (Y, \nu', \alpha_1, \ldots, \alpha_{t'})$ such that $z(\bar{a}) = z'(\bar{a})$ for any $\bar{a} \in A^n$, $t' \leq t$ and, for $s = 1, \ldots, t'$, the expression α_s is an expression of the kind $l_i^{(n_r)}(x_{j_1}, \ldots, x_{j_{n_r}})$, where $x_{j_1}, \ldots, x_{j_{n_r}}$ are pairwise different variables from Y. There are exactly $(n)_{n_r}$ different sorts of such functions. Using an approach similar to the binary search algorithm it is not difficult to show that, for any c functions of the same sort, there exists a computation tree over U, which computes values of the considered c functions and which depth is at most $1 + \log_2 c$. Using this fact it is not difficult to show that $h_U^d(z') \leq (n)_{n_r}(1 + \log_2 t')$ and $h_U^d(z) \leq (n)_{n_r}(1 + \log_2 t)$. Taking into account that z is an arbitrary problem from $\mathcal{P}(U, n)$ such that $\psi_U^i(z) \leq m$, we obtain $\mathcal{U}_{U\psi n}^{di}(m) \leq (n)_{n_r}(1 + \log_2 m)$. Therefore $\mathcal{U}_{U\psi n}^{di}(m) < m$ for large enough m and $typ(\mathcal{U}_{U\psi n}^{di}) \neq \gamma$.

We now prove that $typ(\mathcal{U}_{U\psi n}^{di}) \neq \alpha$. Assume the contrary. Using Lemma 8 we obtain that there exists a number $p \in \omega$ such that $\psi_U^d(z) \leq p$ for any problem $z \in \mathcal{P}(U, n)$. Therefore there exists a number $q \in \omega \setminus \{0\}$ that satisfies the following condition: for any problem $z \in \mathcal{P}(U, n)$, there exists a computation tree $\Gamma \in \mathcal{T}(U)$, which solves the problem z deterministically and has at most q terminal nodes. Let us consider a problem

$$z' = (Y, \nu, l_1^{(n_r)}(x_1, \ldots, x_{n_r}), \ldots, l_q^{(n_r)}(x_1, \ldots, x_{n_r}))$$

from $\mathcal{P}(U, n)$, where $Y = \{x_1, \ldots, x_n\}$, $\nu : E_2^q \to \mathcal{S}(\omega)$ and $\nu(\bar{\delta}_1) \cap \nu(\bar{\delta}_2) = \emptyset$ for any $\bar{\delta}_1, \bar{\delta}_2 \in E_2^q$ such that $\bar{\delta}_1 \neq \bar{\delta}_2$. It is easy to check that there are $q+1$ tuples $\bar{a}_1, \ldots, \bar{a}_{q+1} \in A^n$, such that $z(\bar{a}_i) \cap z(\bar{a}_j) = \emptyset$ for any $i, j \in \{1, \ldots, q+1\}$, $i \neq j$. Let Γ be an arbitrary computation tree over U, which solves the problem z' deterministically. It is not difficult to show that the tree Γ should have at least $q+1$ terminal nodes. We obtain a contradiction. Therefore $typ(\mathcal{U}^{di}_{U\psi n}) = \beta$. From this equality and Proposition 3 it follows that $typ_u(U_\tau, \psi_\tau, n) = t_3$. $\square$

Lemma 20. *Let $\tau \in \Delta_u$, $(U_\tau, \psi_\tau) = \pi^{(n_1)}_{v_1} \oplus \cdots \oplus \pi^{(n_m)}_{v_m}$, $r \in \{1, \ldots, m\}$, $v_r = 4$, $n \in \omega \setminus \{0\}$, $n \geq n_r$ and if $r < m$, then $n < n_{r+1}$. Then $typ_u(U_\tau, \psi_\tau, n) = t_4$.*

Proof. One can show that if $r > 1$, then $\{v_1, \ldots, v_{r-1}\} \subseteq \{2, 3\}$. Denote $(U, \psi) = (U_\tau, \psi_\tau)$. Let $U = (A, F, P)$.

We show that $typ(\mathcal{U}^{ai}_{U\psi n}) = \alpha$. Let Y be a set of n variables. A system of equations

$$\{g_1 = \delta_1, \ldots, g_m = \delta_m\}, \tag{2}$$

where $g_1, \ldots, g_m$ are functions from P depending on variables from Y and $\delta_1, \ldots, \delta_m \in E_2$, is called a system of equations over Y. We show that, for any consistent on A^n system of equations over Y, there exists a system of equations over Y, which has the same set of solutions from A^n and contains at most $m_0 = 2\sum_{j=1}^{r}(n)_{n_j}$ equations. Consider a consistent on A^n system of equations (2). Remove from this system all equations $g_i = \delta_i$ such that $g \equiv 0$. Denote the obtained system by S. This system can contain functions of $(n)_{n_r}$ sorts from $P^{(n_r)}_4$ and if $v_t = 3$ for some $t \in \{1, \ldots, r-1\}$, then this system can contain functions of $(n)_{n_t}$ sorts from $P^{(n_t)}_3$ too.

Divide the system S into at most $(n)_{n_r} + (n)_{n_t}$ subsystems each of which contains equations composed from functions of the same sort and from the same set $P^{(n_r)}_4$ or $P^{(n_t)}_3$. Let Σ be one of such subsystems. Let Σ consist of equations with the left parts from $P^{(n_r)}_4$. One can show that there exists a function $g \in P^{(n_r)}_4$ depending on variables from Y and a number $\delta \in E_2$ such that the set of solutions on A^n of the equation system Σ coincides with the set of solutions on A^n of the equation system $\Sigma' = \{g = \delta\}$. Let now Σ consist of equations with the left parts from $P^{(n_t)}_3$. One can show that there exists a subsystem Σ' of the system Σ, which has the same set of solutions as the system Σ and contains at most two equations. Replace in the system S each subsystem Σ with the corresponding subsystem Σ' and denote the obtained system by S'. It is clear that S' has the same set of solutions as S and contains at most m_0 equations.

We now show that, for any problem $z = (Y, \nu, \beta_1, \ldots, \beta_m) \in \mathcal{P}(U, n)$, the inequality $\psi^a_U(z) \leq m_0$ holds. It is clear that there exists a problem $z' = (Y, \nu, \alpha_1, \ldots, \alpha_m) \in \mathcal{P}(U, n)$ such that $\alpha_1, \ldots, \alpha_m$ are expressions depending on variables from Y and $z(\bar{a}) = z'(\bar{a})$ for any $\bar{a} \in A^n$. For any $\bar{\delta} = (\delta_1, \ldots, \delta_m) \in E^m_2$, denote by $S(\bar{\delta})$ the system of equations

$$\{\alpha_1 = \delta_1, \ldots, \alpha_m = \delta_m\}.$$

Let the system $S(\bar{\delta})$ be consistent on the set A^n. Then there exists a system of equations

$$\{\gamma_1 = \sigma_1, \ldots, \gamma_t = \sigma_t\}$$

over Y such that $t \leq m_0$ and the set of solutions of this system coincides with the set of solutions of the system $S(\bar{\delta})$.

Denote by $\Gamma(\bar{\delta})$ the computation tree $(Y, G(\bar{\delta}))$, where $G(\bar{\delta})$ is a tree with the root consisting of unique complete path $v_0, d_0, \ldots, v_t, d_t, v_{t+1}$ in which, for $i = 1, \ldots, t$, the node v_i is labeled with the expression γ_i and the edge d_i is labeled with the number σ_i, and the node v_{i+1} is labeled with the minimum number from the set $\nu(\bar{\delta})$.

Identify the roots of the trees $G(\bar{\delta})$, where $\bar{\delta} \in E^m_2$ and the system of equations $S(\bar{\delta})$ is consistent. Denote the obtained tree by G. By Γ we denote the pair (Y, G). It is not difficult to prove that Γ is a computation tree over U, which solves the problem z nondeterministically and for which $\psi(\Gamma) \leq m_0$. Thus, $\psi^a_U(z) \leq m_0$ for any problem $z \in \mathcal{P}(U, n)$. From here and from Lemma 8 it follows that $typ(\mathcal{U}^{ai}_{U\psi n}) = \alpha$.

We now show that $typ(\mathcal{U}^{di}_{U\psi n}) = \gamma$. Using Lemmas 6 and 7, and Corollary 9 we obtain $Dom(\mathcal{U}^{di}_{U\psi n}) = \omega \setminus \{0\}$. Let $m \in \omega \setminus \{0\}$. One can show that there exist functions $f_1, \ldots, f_m \in P^{(n_r)}_{v_r}$ such that, for any $\delta_1, \ldots, \delta_m \in E_2$, the system of equations

$$\{f_1(\bar{x}) = \delta_1, \ldots, f_m(\bar{x}) = \delta_m\},$$

where $\bar{x} = (x_1, \ldots, x_{n_r})$, is consistent on A^n.

Consider the problem $z = (Y, \nu, f_1(\bar{x}), \ldots, f_m(\bar{x}))$, where $Y = \{x_1, \ldots, x_n\}$, $\nu : E^m_2 \to \mathcal{S}(\omega)$ and $\nu(\bar{\delta}_1) \cap \nu(\bar{\delta}_2) = \emptyset$ for any $\bar{\delta}_1, \bar{\delta}_2 \in E^m_2$ such that $\bar{\delta}_1 \neq \bar{\delta}_2$. It is clear that $\psi^i_U(z) = m$. Let Γ be a computation tree over U, which solves the problem z deterministically and for which $\psi(\Gamma) = \psi^d_U(z)$. Evidently, the computation tree Γ must have at least 2^m terminal nodes. Therefore $h(\Gamma) \geq m$ and $\psi^d_U(z) \geq m$. Thus, $\mathcal{U}^{di}_{U\psi n}(m) \geq m$. By Lemma 5, $\mathcal{U}^{di}_{U\psi n}(m) = m$. Taking into account that m is an arbitrary number from $\omega \setminus \{0\}$, we obtain $typ(\mathcal{U}^{di}_{U\psi n}) = \gamma$. From this equality, from equality $typ(\mathcal{U}^{ai}_{U\psi n}) = \alpha$, and from Proposition 3 it follows that $typ_u(U, \psi, n) = t_4$. $\square$

Lemma 21. *Let $\tau \in \Delta_u$, $(U_\tau, \psi_\tau) = \pi^{(n_1)}_{v_1} \oplus \cdots \oplus \pi^{(n_m)}_{v_m}$, $r \in \{1, \ldots, m\}$, $v_r = 5$, $n \in \omega \setminus \{0\}$, $n \geq n_r$ and if $r < m$, then $n < n_{r+1}$. Then $typ_u(U_\tau, \psi_\tau, n) = t_5$.*

Proof. One can show that $r = 1$ or $r = 2$. In the latter case, $v_1 = 2$. Denote $(U, \psi) = (U_\tau, \psi_\tau)$. Let $U = (A, F, P)$. We now show that $typ(\mathcal{U}^{da}_{U\psi n}) = \gamma$. For an arbitrary $i \in \omega$, denote $c_i = \psi(q^{(n_r)}_i)$. Remind that $c_0 = 1$ and $c_i = i\sum_{j=0}^{i-1} c_j + 2$ for $i \geq 1$. Consider a problem $z_i = (Y, \nu, q^{(n_r)}_i(x_1, \ldots, x_{n_r}))$, where $Y = \{x_1, \ldots, x_n\}$, $\nu(0) = \{0\}$ and $\nu(1) = \{1\}$. It is clear that $\psi_U^i(z_i) = c_i$. We now show that $\psi_U^a(z_i) = \psi_U^d(z_i) = c_i$. One can prove that there exists a computation tree Γ over U, which solves the problem z_i nondeterministically and satisfies the following conditions: $\psi(\Gamma) = \psi_U^a(z_i)$ and all expressions attached to nodes of Γ depend on variables from Y only. We now show that the expression $q_i^{(n_r)}(x_1, \ldots, x_{n_r})$ is attached to a node of Γ. Assume the contrary. Consider two n-tuples from A^n: $\bar{\alpha} = (k_0^{(n_r)}, \ldots, k_0^{(n_r)})$ and $\bar{\beta} = (k_1^{(n_r)}, \ldots, k_{n_r-1}^{(n_r)}, i^{(n_r)}, k_0^{(n_r)}, \ldots, k_0^{(n_r)})$. Note that all functions from P depending on variables from Y take the value 0 on the tuple $\bar{\alpha}$, and all functions from P depending on variables from Y with the exception of $q_i^{(n_r)}(x_1, \ldots, x_{n_r})$ take the value 0 on the tuple $\bar{\beta}$. It is clear that $z_i(\bar{\alpha}) = \{0\}$ and $z_i(\bar{\beta}) = \{1\}$. Let ξ be a complete path in Γ such that $\bar{\alpha} \in \mathcal{A}(\xi)$. Evidently, all edges of this path with the exception of the first one are labeled with the number 0. Therefore $\bar{\beta} \in \mathcal{A}(\xi)$, but this is impossible since $z_i(\bar{\alpha}) \cap z_i(\bar{\beta}) = \emptyset$. Hence the expression $q_i^{(n_r)}(x_1, \ldots, x_{n_r})$ is attached to at least one node of Γ. Taking into account that $\psi_U^i(z_i) = c_i$ and using Lemma 5 we obtain that $\psi_U^a(z_i) = \psi_U^d(z_i) = c_i$. Hence if the value $\mathcal{U}_{U\psi n}^{da}(c_i)$ is definite, then it satisfies the inequality $\mathcal{U}_{U\psi n}^{da}(c_i) \geq c_i$.

Let $i \in \omega \setminus \{0\}$ and $i \geq (n)_{n_r}$. We now prove that the value $\mathcal{U}_{U\psi n}^{da}(c_i - 1)$ is definite and $\mathcal{U}_{U\psi n}^{da}(c_i - 1) < c_i - 1$. One can show that the set $\{z : z \in \mathcal{P}(U, n), \psi_U^a(z) \leq c_i - 1\}$ is not empty. Let $z = (Y, \nu, \beta_1, \ldots, \beta_t)$ be an arbitrary problem from this set. One can prove that there exists a computation tree Γ over U, which solves the problem z nondeterministically and satisfies the following conditions: $\psi(\Gamma) \leq c_i - 1$ and all expressions attached to nodes of Γ depend on variables from Y. Denote by $\Phi(\Gamma)$ the set of all functions that are attached to nodes of Γ and are not identically equal to 0. One can show that there exists a computation tree Γ', which solves the problem z deterministically by sequential computing values of all functions from $\Phi(\Gamma)$. Since all functions from $\Phi(\Gamma)$ are not identically equal to 0, there are of the kind $q_j^{(n_r)}(x_{s_1}, \ldots, x_{s_{n_r}})$, where $x_{s_1}, \ldots, x_{s_{n_r}}$ are pairwise different variables from Y. Since $\psi(\Gamma) \leq c_i - 1, j \leq i - 1$. It is clear that $\psi(\Gamma') = \sum_{f \in \Phi(\Gamma)} \psi(f) \leq (n)_{n_r} \sum_{j=0}^{i-1} c_j$. Since $i \geq (n)_{n_r}$, $\psi(\Gamma') \leq i \sum_{j=0}^{i-1} c_j = c_i - 2 < c_i - 1$. Hence $\psi_U^d(z) < c_i - 1$. Taking into account that z is an arbitrary problem from $\mathcal{P}(U, n)$ such that $\psi_U^a(z) \leq c_i - 1$, we obtain that the value $\mathcal{U}_{U\psi n}^{da}(c_i - 1)$ is definite and satisfies the inequality $\mathcal{U}_{U\psi n}^{da}(c_i - 1) < c_i - 1$. Therefore $Dom(\mathcal{U}_{U\psi n}^{da})$ is an infinite set.

By Lemma 6, $Dom(\mathcal{U}_{U\psi n}^{da}) = \omega$. Since, for any $i \in \omega$, the inequality $\mathcal{U}_{U\psi n}^{da}(c_i) \geq c_i$ holds, $Dom^+(\mathcal{U}_{U\psi n}^{da})$ is an infinite set. Since, for any $i \in \omega$, $i \geq (n)_{n_r}$, the inequality $\mathcal{U}_{U\psi n}^{da}(c_i - 1) < c_i - 1$ holds, the set $Dom^-(\mathcal{U}_{U\psi n}^{da})$ is infinite. Therefore $typ(\mathcal{U}_{U\psi n}^{da}) = \gamma$. Using Proposition 3 we obtain that $typ_u(U, \psi, n) = t_5$. □

Lemma 22. *Let $\tau \in \Delta_u$, $(U_\tau, \psi_\tau) = \pi_{s_1}^{(n_1)} \oplus \cdots \oplus \pi_{s_m}^{(n_m)}$, $r \in \{1, \ldots, m\}$, $s_r = 6$, $n \in \omega \setminus \{0\}$, $n \geq n_r$ and if $r < m$, then $n < n_{r+1}$. Then $typ_u(U_\tau, \psi_\tau, n) = t_6$.*

Proof. One can show that if $r > 1$, then $\{s_1, \ldots, s_{r-1}\} \subseteq \{2, 5\}$. Denote $(U, \psi) = (U_\tau, \psi_\tau)$. Let $U = (A, F, P)$.

Show that, for any $m \in \omega$, the value $\mathcal{U}_{U\psi n}^{da}(m)$ is definite. One can prove that the set $\{z : z \in \mathcal{P}(U, n), \psi_U^a(z) \leq m\}$ is not empty. Let $z = (Y, \nu, \beta_1, \ldots, \beta_t)$ be an arbitrary problem from this set. Let Φ_Y be the set of all functions from $P[F]$ with variables from Y, which are not identically equal to 0 on A^n. One can show that the set Φ_Y contains only functions of the kind $q_{2i}^{(n_r)}(x_{j_1}, \ldots, x_{j_{n_r}})$, $q_{2i+1}^{(n_r)}(x_{j_1}, \ldots, x_{j_{n_r}})$, $p_{2i}^{(n_r)}(x_{j_1}, \ldots, x_{j_{n_r}})$, where $x_{j_1}, \ldots, x_{j_{n_r}}$ are pairwise different variables from Y, and if $s_t = 5$ for some $t \in \{1, \ldots, r-1\}$, then the set Φ_Y contains also functions of the kind $q_i^{(n_t)}(x_{j_1}, \ldots, x_{j_{n_t}})$, where $x_{j_1}, \ldots, x_{j_{n_t}}$ are pairwise different variables from Y. Denote $\Phi_Y(m) = \{f : f \in \Phi_Y, \psi(f) \leq m\}$. One can show that $\Phi_Y(m)$ is a finite set and $\psi_U^d(z) \leq \sum_{f \in \Phi_Y(m)} \psi(f)$. Taking into account that z is an arbitrary problem from $\mathcal{P}(U, n)$ such that $\psi_U^a(z) \leq m$, we obtain that the value $\mathcal{U}_{U\psi n}^{da}(m)$ is definite.

We now show that, for any $m \in \omega \setminus \{0\}$, the inequality $\mathcal{U}_{U\psi n}^{da}(m) > m$ holds. Consider the problem $z_m = (Y, \nu, q_{2m}^{(n_r)}(\bar{x}), q_{2m+1}^{(n_r)}(\bar{x}), p_{2m}^{(n_r)}(\bar{x}))$, where $Y = \{x_1, \ldots, x_n\}$, $\bar{x} = (x_1, \ldots, x_{n_r})$, $\nu((1, 0, 1)) = \{1\}$, $\nu((0, 1, 1)) = \{2\}$, and $\nu(\bar{\delta}) = \{0\}$ for any 3-tuple $\bar{\delta} \in E_2^3 \setminus \{(1, 0, 1), (0, 1, 1)\}$. Consider the computation tree Γ_0 containing exactly three complete paths ξ_0, ξ_1, and ξ_2, where $\xi_i = v_0, d_{0i}, v_{1i}, d_{1i}, v_{2i}$ for $i = 0, 1, 2$. For $i = 0, 1, 2$, the node v_0 and the edge d_{0i} are not labeled, and the node v_{2i} is labeled with the number i. The node v_{10} is labeled with the expression $p_{2m}^{(n_r)}(\bar{x})$ and the edge d_{10} is labeled with the number 0. The node v_{11} is labeled with the expression $q_{2m}^{(n_r)}(\bar{x})$ and the edge d_{11} is labeled with the number 1. The node v_{12} is labeled with the expression $q_{2m+1}^{(n_r)}(\bar{x})$ and the edge d_{12} is labeled with the number 1. One can show that Γ_0 solves the problem z_m nondeterministically and $\psi(\Gamma_0) = m$. Therefore $\psi_U^a(z_m) \leq m$.

Show that $\psi_U^d(z_m) \geq 2m$. One can prove that there exists a computation tree Γ over U, which solves the problem z_m deterministically and satisfies the following conditions: $\psi(\Gamma) = \psi_U^d(z_m)$ and all functions attached to nodes of Γ depend on variables from the set Y only. Define three n-tuples $\bar{\alpha}_0$, $\bar{\alpha}_1$, and $\bar{\alpha}_2$ from A^n: $\bar{\alpha}_0 = (k_0^{(n_r)}, \ldots, k_0^{(n_r)})$, $\bar{\alpha}_1 = (k_1^{(n_r)}, \ldots, k_{n_r-1}^{(n_r)}, (2m)^{(n_r)}, k_0^{(n_r)}, \ldots, k_0^{(n_r)})$, and $\bar{\alpha}_2 = (k_1^{(n_r)}, \ldots, k_{n_r-1}^{(n_r)}, (2m+1)^{(n_r)}, k_0^{(n_r)}, \ldots, k_0^{(n_r)})$. It is clear that $z_m(\bar{\alpha}_0) = \{0\}$, $z_m(\bar{\alpha}_1) = \{1\}$, and $z_m(\bar{\alpha}_2) = \{2\}$. For $i = 0, 1, 2$, denote by φ_i the set of all functions from Φ_Y, which take value 1 on the tuple $\bar{\alpha}_i$. One can show that $\varphi_0 = \emptyset$, $\varphi_1 = \{q_{2m}^{(n_r)}(\bar{x}), p_{2m}^{(n_r)}(\bar{x})\}$, and $\varphi_2 = \{q_{2m+1}^{(n_r)}(\bar{x}), p_{2m}^{(n_r)}(\bar{x})\}$.

Consider the complete path ξ_0 in Γ such that $\bar{\alpha}_0 \in \mathcal{A}(\xi_0)$. It is clear that $\bar{\alpha}_1 \notin \mathcal{A}(\xi_0)$ and $\bar{\alpha}_2 \notin \mathcal{A}(\xi_0)$. Therefore the expression $p_{2m}^{(n_r)}(\bar{x})$ or both expressions $q_{2m}^{(n_r)}(\bar{x})$ and $q_{2m+1}^{(n_r)}(\bar{x})$ are among expressions attached to the nodes of ξ_0. If nodes of ξ_0 are labeled with at least two expressions from the set $B = \{q_{2m}^{(n_r)}(\bar{x}), q_{2m+1}^{(n_r)}(\bar{x}), p_{2m}^{(n_r)}(\bar{x})\}$, then $\psi(\Gamma) \geq 2m$. Let only one expression from the set B be attached to the nodes of ξ_0. Then this is $p_{2m}^{(n_r)}(\bar{x})$. Let the considered expression be attached to the node v of the path ξ_0. Consider the complete path ξ_1 in Γ such that $\bar{\alpha}_1 \in \mathcal{A}(\xi_1)$. Since Γ is a deterministic computation tree, the path ξ_1 contains the node v. Suppose that no one node of the path ξ_1 is labeled with an expression from the set $\{q_{2m}^{(n_r)}(\bar{x}), q_{2m+1}^{(n_r)}(\bar{x})\}$. Then $\bar{\alpha}_2 \in \mathcal{A}(\xi_1)$ but this is impossible. Hence nodes of ξ_1 are labeled with at least two expressions from the set B. Therefore $\psi(\Gamma) \geq 2m$ and $\psi_U^d(z_m) \geq 2m$. Taking into account that $\psi_U^a(z_m) \leq m$, we obtain $\mathcal{U}_{U\psi n}^{da}(m) \geq 2m$ for any $m \in \omega \setminus \{0\}$. Thus, $typ(\mathcal{U}_{U\psi n}^{da}) = \delta$. Using Proposition 3 we obtain $typ_u(U, \psi, n) = t_6$. □

Lemma 23. *Let $\tau \in \Delta_u$, $(U_\tau, \psi_\tau) = \pi_{s_1}^{(n_1)} \oplus \cdots \oplus \pi_{s_m}^{(n_m)}$, $s_m = 7$, $n \in \omega \setminus \{0\}$, and $n \geq n_m$. Then $typ_u(U_\tau, \psi_\tau, n) = t_7$.*

Proof. Denote $(U, \psi) = (U_\tau, \psi_\tau)$. Let $U = (A, F, P)$. Show that $typ(\mathcal{U}_{U\psi n}^{da}) = \varepsilon$. Let $t \in \omega \setminus \{0\}$. Consider the problem $z_t = (Y, \nu, l_1^{(n_m)}(\bar{x}), \ldots, l_t^{(n_m)}(\bar{x}))$ from $\mathcal{P}(U, n)$, where $Y = \{x_1, \ldots, x_n\}$, $\bar{x} = (x_1, \ldots, x_{n_m})$, $\nu : E_2^t \to \mathcal{S}(\omega)$, and $\nu(\bar{\delta}_1) \cap \nu(\bar{\delta}_2) = \emptyset$ for any $\bar{\delta}_1, \bar{\delta}_2 \in E_2^t$ such that $\bar{\delta}_1 \neq \bar{\delta}_2$. It is not difficult to show that, for any consistent on A^n system of equations

$$\{l_1^{(n_m)}(\bar{x}) = \delta_1, \ldots, l_t^{(n_m)}(\bar{x}) = \delta_t\},$$

where $\delta_1, \ldots, \delta_t \in E_2$, there exists a subsystem, which has the same set of solutions and which contains at most two equations. Using this fact it is not difficult to show that $\psi_U^a(z_t) \leq 2$.

Let us prove that there is no $c \in \omega$ such that $\psi_U^d(z_t) \leq c$ for any $t \in \omega \setminus \{0\}$. Assume the contrary. Then there exists a number $w \in \omega \setminus \{0\}$ satisfying the following condition: for any $t \in \omega \setminus \{0\}$, there exists a computation tree Γ over U, which solves the problem z_t deterministically and has at most w terminal nodes. Let us consider the problem z_w. Let Γ be an arbitrary computation tree over U, which solves the problem z_w deterministically. It is not difficult to show that Γ must have at least $w + 1$ terminal nodes. We obtain a contradiction. Thus, $\psi_U^a(z_t) \leq 2$ for any $t \in \omega \setminus \{0\}$ and there is no $c \in \omega$ such that $\psi_U^d(z_t) \leq c$ for any $t \in \omega \setminus \{0\}$. Therefore $\mathcal{U}_{U\psi n}^{da}(2) = \infty$. Using Lemma 6 we obtain $typ(\mathcal{U}_{U\psi n}^{da}) = \varepsilon$.

We now prove that the function ψ_U^a is unbounded from above on the set $\mathcal{P}(U, n)$. Let $i \in \omega \setminus \{0\}$. Consider the problem $\eta_i = (Y, \nu, q_{-i}^{(n_m)}(\bar{x}))$ over U, where $Y = \{x_1, \ldots, x_n\}$, $\bar{x} = (x_1, \ldots, x_{n_m})$, $\nu((0)) = \{0\}$, and $\nu((1)) = \{1\}$. Show that $\psi_U^a(\eta_i) \geq i$. One can prove that there exists a computation tree Γ over U, which solves the problem η_i nondeterministically and satisfies the following conditions: $\psi(\Gamma) = \psi_U^a(\eta_i)$, and all expressions attached to nodes of Γ depend on variables from Y only. Show that the expression $q_{-i}^{(n_m)}(\bar{x})$ is attached to a node of Γ. Assume the contrary. Consider two n-tuples from A^n: $\bar{\alpha} = (k_0^{(n_m)}, \ldots, k_0^{(n_m)})$ and $\bar{\beta} = (k_1^{(n_m)}, \ldots, k_{n_m-1}^{(n_m)}, (-i)^{(n_m)}, k_0^{(n_m)}, \ldots, k_0^{(n_m)})$. It is clear that $\eta_i(\bar{\alpha}) = \{0\}$ and $\eta_i(\bar{\beta}) = \{1\}$. One can show that all functions from P depending on variables from Y take the value 0 on the tuple $\bar{\alpha}$, and all functions from P, depending on variables from Y, with the exception of $q_{-i}^{(n_m)}(\bar{x})$, take the value 0 on the tuple $\bar{\beta}$. Let ξ be a complete path in Γ such that $\bar{\alpha} \in \mathcal{A}(\xi)$. Evidently, all edges of this path with the exception of the first one are labeled with 0. Therefore $\bar{\beta} \in \mathcal{A}(\xi)$ but this is impossible since $\eta_i(\bar{\alpha}) \cap \eta_i(\bar{\beta}) = \emptyset$. Hence the expression $q_{-i}^{(n_m)}(\bar{x})$ is attached to a node of Γ. Therefore $\psi(\Gamma) \geq i$ and $\psi_U^a(\eta_i) \geq i$. Thus, the function ψ_U^a is unbounded from above on the set $\mathcal{P}(U, n)$. Using Lemma 8 we obtain $typ(\mathcal{U}_{U\psi n}^{aa}) = \gamma$. From this equality, the equality $typ(\mathcal{U}_{U\psi n}^{da}) = \varepsilon$, and from Proposition 3 it follows that $typ_u(U, \psi, n) = t_7$. □

Lemma 24. *Let (U, ψ) be a sm-pair such that $\psi \equiv 0$. Then $dtyp_u(U, \psi) = t_1^\infty$.*

Proof. Let $n \in \omega \setminus \{0\}$. It is clear that the function ψ_U^i is bounded from above on the set $\mathcal{P}(U, n)$. Using Lemma 8 we obtain $typ(\mathcal{U}_{U\psi n}^{ii}) = \alpha$. From this equality and from Proposition 3 it follows that $typ_u(U, \psi, n) = t_1$. □

Proof of Proposition 16. Let $\tau \in \{t_1^\infty\} \cup \Delta_u$. Assume that $\tau = t_1^\infty$. Consider an arbitrary sm-pair (U, ψ) such that $\psi \equiv 0$. By Lemma 24, $dtyp_u(U, \psi) = t_1^\infty$. Assume now that $\tau \in \Delta_u$. Consider the sm-pair (U_τ, ψ_τ). Using Lemmas 18–23 we obtain $dtyp_u(U_\tau, \psi_\tau) = \tau$. □

Proof of Proposition 17. Let $\tau \in \Delta_u$. Consider the sm-pair (U_τ, ψ_τ). By construction, ψ_τ is a weighted depth. Therefore (U_τ, ψ_τ) is a limited sm-pair. From Lemmas 18–23 it follows that $dtyp_u(U_\tau, \psi_\tau) = \tau$. □

7. Proofs of Theorems 1 and 2

Let us define a function $\rho : \{\alpha, \beta, \gamma, \delta, \varepsilon\} \to \{\alpha, \beta, \gamma, \delta, \varepsilon\}$ as follows: $\rho(\alpha) = \varepsilon$, $\rho(\beta) = \delta$, $\rho(\gamma) = \gamma$, $\rho(\delta) = \beta$, and $\rho(\varepsilon) = \alpha$. The following statement (Proposition 5 from [12]) allows us to analyze the relationships between upper types and types of sm-pairs.

Proposition 25. *Let B be a nonempty set, $f : B \to \omega$, $g : B \to \omega$, $\mathcal{U}^{fg}(n) = \max\{f(b) : b \in B, g(b) \leq n\}$, and $\mathcal{L}^{gf}(n) = \min\{g(b) : b \in B, f(b) \geq n\}$ for any $n \in \omega$. Then $typ(\mathcal{L}^{gf}) = \rho(typ(\mathcal{U}^{fg}))$.*

Using Proposition 25 we obtain the following statement.

Proposition 26. *Let (U, ψ) be a sm-pair, $n \in \omega \setminus \{0\}$, and $typ_u(U, \psi, n) = t_i$ for some $i \in \{1, \ldots, 7\}$. Then $typ(U, \psi, n) = T_i$.*

Proof of Theorem 1. The statement of the theorem follows from Propositions 13, 16, and 26. □

Proof of Theorem 2. The statement of the theorem follows from Propositions 14, 17, and 26. □

8. Explanations of results

Rather unusual formulations of the results (in particular, the use of the types of functions) require additional explanations.

Let (U, ψ) be a sm-pair, and $n \in \omega \setminus \{0\}$. We will start by considering the upper n-type $typ_u(U, \psi, n)$ of the sm-pair (U, ψ). After that, discuss the lower n-type $typ_l(U, \psi, n)$ of the sm-pair (U, ψ) and the n-type $typ(U, \psi, n)$ of the sm-pair (U, ψ). We end by looking at the dynamic type $dtyp(U, \psi)$ of the sm-pair (U, ψ).

8.1. Upper n-types of sm-pairs

For any $b, c \in \{i, d, a\}$, the matrix $typ_u(U, \psi, n)$ (the upper n-type of the sm-pair (U, ψ)) contains the value $typ(\mathcal{U}^{bc}_{U\psi n}) \in \{\alpha, \beta, \gamma, \delta, \varepsilon\}$ in the intersection of the row b and the column c, where

$$\mathcal{U}^{bc}_{U\psi n}(m) = \max\{\psi^b_U(z) : z \in \mathcal{P}(U, n), \psi^c_U(z) \le m\}$$

for any $m \in \omega$. If the value $\mathcal{U}^{bc}_{U\psi n}(m)$ is defined for some $m \in \omega$, then it is the unimprovable upper bound on the value $\psi^b_U(z)$ for problems $z \in \mathcal{P}(U, n)$ such that the inequality $\psi^c_U(z) \le m$ holds. From Propositions 13, 14, 16, and 17 it follows that $\{t_1, \ldots, t_7\}$ is the set of all possible upper n-types of sm-pairs and $\{t_2, \ldots, t_7\}$ is the set of all possible upper n-types of limited sm-pairs.

Let us remind that $Dom(\mathcal{U}^{bc}_{U\psi n})$ is the domain of $\mathcal{U}^{bc}_{U\psi n}$, $Dom^+(\mathcal{U}^{bc}_{U\psi n}) = \{m : m \in Dom(\mathcal{U}^{bc}_{U\psi n}), \mathcal{U}^{bc}_{U\psi n}(m) \ge m\}$, and $Dom^-(\mathcal{U}^{bc}_{U\psi n}) = \{m : m \in Dom(\mathcal{U}^{bc}_{U\psi n}), \mathcal{U}^{bc}_{U\psi n}(m) \le m\}$.

By Lemma 6, the set $Dom(\mathcal{U}^{bc}_{U\psi n})$ is finite if and only if there exists $m \in \omega$ such that $\mathcal{U}^{bc}_{U\psi n}(m) = \infty$, i.e., the set $\{\psi^b_U(z) : z \in \mathcal{P}(U, n), \psi^c_U(z) \le m\}$ is infinite. In this case, $Dom(\mathcal{U}^{bc}_{U\psi n}) = \emptyset$ if $m_0 = m_1$ and $Dom(\mathcal{U}^{bc}_{U\psi n}) = \{m : m \in \omega, m_0 \le m < m_1\}$ if $m_0 < m_1$, where $m_0 = \min\{\psi^c_U(z) : z \in \mathcal{P}(U, n)\}$ and $m_1 = \min\{m : m \in \omega, \mathcal{U}^{bc}_{U\psi n}(m) = \infty\}$. If the set $Dom(\mathcal{U}^{bc}_{U\psi n})$ is infinite, then $Dom(\mathcal{U}^{bc}_{U\psi n}) = \{m : m \in \omega, m \ge m_0\}$.

Let us remind that, by Lemma 5, $\psi^a_U(z) \le \psi^d_U(z) \le \psi^i_U(z)$ for any problem $z \in \mathcal{P}(U, n)$.

The equality $typ(\mathcal{U}^{bc}_{U\psi n}) = \alpha$ means that the set $Dom(\mathcal{U}^{bc}_{U\psi n})$ is infinite and the function $\mathcal{U}^{bc}_{U\psi n}$ is bounded from above. This equality can hold for any pair bc, $b, c \in \{i, d, a\}$. By Lemma 8, $typ(\mathcal{U}^{bc}_{U\psi n}) = \alpha$ if and only if the function ψ^b_U is bounded from above on the set $\mathcal{P}(U, n)$. Later we will often omit words "from above on the set $\mathcal{P}(U, n)$" and write that the function ψ^b_U is bounded or that the function ψ^b_U is unbounded. The function ψ^i_U can be bounded only for sm-pairs that are not limited. The function ψ^d_U can be bounded for sm-pairs that are limited, but this case is in some sense degenerate — see Lemma 10.

The equality $typ(\mathcal{U}^{bc}_{U\psi n}) = \beta$ means that $Dom(\mathcal{U}^{bc}_{U\psi n})$ is an infinite set, $Dom^+(\mathcal{U}^{bc}_{U\psi n})$ is a finite set, and $\mathcal{U}^{bc}_{U\psi n}$ is an unbounded from above function. This equality can hold only if $bc = di$. One can show that $typ(\mathcal{U}^{di}_{U\psi n}) = \beta$ if and only if the function ψ^d_U is unbounded and there exists $p \in \omega$ such that $\psi^d_U(z) < \psi^i_U(z)$ for any problem $z \in \mathcal{P}(U, n)$ such that $\psi^i_U(z) \ge p$.

The equality $typ(\mathcal{U}^{bc}_{U\psi n}) = \gamma$ means that each of the sets $Dom^+(\mathcal{U}^{bc}_{U\psi n})$ and $Dom^-(\mathcal{U}^{bc}_{U\psi n})$ is an infinite set. This equality can hold for any pair bc, $b, c \in \{i, d, a\}$, with the exception of id and ia. One can show that, for $bc \in \{di, ai, ad\}$, $typ(\mathcal{U}^{bc}_{U\psi n}) = \gamma$ if and only if, for any $q \in \omega$, there exists a problem $z \in \mathcal{P}(U, n)$ such that $\psi^b_U(z) = \psi^c_U(z) \ge q$. If $bc \in \{ii, dd, aa\}$, then $typ(\mathcal{U}^{bc}_{U\psi n}) = \gamma$ if and only if the function ψ^b_U is unbounded — see Lemma 8. One can show that $typ(\mathcal{U}^{da}_{U\psi n}) = \gamma$ if and only if the set $Dom(\mathcal{U}^{da}_{U\psi n})$ is infinite, the function ψ^d_U is unbounded and, for any $q \in \omega$, there exists a number $m \in \omega$, $m \ge q$, such that, for any problem $z \in \mathcal{P}(U, n)$ with $\psi^a_U(z) \le m$, the inequality $\psi^d_U(z) \le m$ holds.

The equality $typ(\mathcal{U}^{bc}_{U\psi n}) = \delta$ means that $Dom(\mathcal{U}^{bc}_{U\psi n})$ is an infinite set and $Dom^-(\mathcal{U}^{bc}_{U\psi n})$ is a finite set. This equality can hold only if $bc = da$. One can show that $typ(\mathcal{U}^{da}_{U\psi n}) = \delta$ if and only if $Dom(\mathcal{U}^{da}_{U\psi n})$ is an infinite set and there exists $p \in \omega$ such that, for any $m \in \omega$, $m \ge p$, there exists a problem $z \in \mathcal{P}(U, n)$ such that $\psi^a_U(z) \le m$ and $\psi^d_U(z) > m$.

The equality $typ(\mathcal{U}^{bc}_{U\psi n}) = \varepsilon$ means that $Dom(\mathcal{U}^{bc}_{U\psi n})$ is a finite set. This equality can hold only if $bc \in \{id, ia, da\}$. By Lemma 6, $typ(\mathcal{U}^{bc}_{U\psi n}) = \varepsilon$ if and only if there exists $m \in \omega$ such that $\mathcal{U}^{bc}_{U\psi n}(m) = \infty$, i.e., the set $\{\psi^b_U(z) : z \in \mathcal{P}(U, n), \psi^c_U(z) \le m\}$ is infinite. The case $typ(\mathcal{U}^{bc}_{U\psi n}) = \varepsilon$ is not very informative: for large enough m, there are problems $z \in \mathcal{P}(U, n)$ for which $\psi^c_U(z) \le m$ but we cannot derive any upper bound on the value $\psi^b_U(z)$ for these problems.

Let $typ_u(U, \psi, n) = t \in \{t_1, \ldots, t_7\}$. We now consider possible behavior of the rows in the table t, which is closely related to the boundedness of the functions ψ^i_U, ψ^d_U, and ψ^a_U.

The row i in the matrix t is equal to (α, α, α) if the function ψ^i_U is bounded (if $t = t_1$) and is equal to $(\gamma, \varepsilon, \varepsilon)$ if the function ψ^i_U is unbounded (if $t \in \{t_2, \ldots, t_7\}$).

The row a in the matrix t is equal to (α, α, α) if the function ψ^a_U is bounded (if $t \in \{t_1, t_2, t_3, t_4\}$) and is equal to (γ, γ, γ) if the function ψ^a_U is unbounded (if $t \in \{t_5, t_6, t_7\}$).

The behavior of the row d in the matrix t is more complicated. If the function ψ_U^d is bounded (if $t \in \{t_1, t_2\}$), then this row is equal to (α, α, α). If the function ψ_U^d is unbounded and the function ψ_U^a is bounded (if $t \in \{t_3, t_4\}$), then the row d is equal to (x, γ, ε), where $x \in \{\beta, \gamma\}$. If each of the functions ψ_U^i, ψ_U^d, and ψ_U^a is unbounded (if $t \in \{t_5, t_6, t_7\}$), then the row d is equal to (γ, γ, x), where $x \in \{\gamma, \delta, \varepsilon\}$.

8.2. Lower n-types of sm-pairs

In this section, we consider lower n-types of sm-pairs.

Definition 14. The lower n-type, $n \in \omega \setminus \{0\}$, of a sm-pair (U, ψ) is the table $typ_l(U, \psi, n)$ with three rows and three columns in which rows from top to bottom and columns from the left to the right are labeled with indices i, d, a and the value $typ(\mathcal{L}_{U\psi n}^{bc})$ is in the intersection of the row with index $b \in \{i, d, a\}$ and the column with index $c \in \{i, d, a\}$.

For any $b, c \in \{i, d, a\}$ and $m \in \omega$,

$$\mathcal{L}_{U\psi n}^{bc}(m) = \min\{\psi_U^b(z) : z \in \mathcal{P}(U, n), \psi_U^c(z) \geq m\}.$$

If the value $\mathcal{L}_{U\psi n}^{bc}(m)$ is defined for some $m \in \omega$, then it is the unimprovable lower bound on the value $\psi_U^b(z)$ for problems $z \in \mathcal{P}(U, n)$ such that the inequality $\psi_U^c(z) \geq m$ holds.

One can show that the set $Dom(\mathcal{L}_{U\psi n}^{bc})$ is finite if and only if the function ψ_U^c is bounded. In this case, $Dom(\mathcal{L}_{U\psi n}^{bc}) = \{0, \ldots, M\}$, where $M = \max\{\psi_U^c(z) : z \in \mathcal{P}(U, n)\}$. If the function ψ_U^c is unbounded, then $Dom(\mathcal{L}_{U\psi n}^{bc}) = \omega$.

Define seven tables:

$l_1 =$

	i	d	a
i	ε	ε	ε
d	ε	ε	ε
a	ε	ε	ε

$l_2 =$

	i	d	a
i	γ	ε	ε
d	α	ε	ε
a	α	ε	ε

$l_3 =$

	i	d	a
i	γ	δ	ε
d	α	γ	ε
a	α	α	ε

$l_4 =$

	i	d	a
i	γ	γ	ε
d	α	γ	ε
a	α	α	ε

$l_5 =$

	i	d	a
i	γ	γ	γ
d	α	γ	γ
a	α	γ	γ

$l_6 =$

	i	d	a
i	γ	γ	γ
d	α	γ	γ
a	α	β	γ

$l_7 =$

	i	d	a
i	γ	γ	γ
d	α	γ	γ
a	α	α	γ

Using Proposition 25 we obtain that $\{l_1, \ldots, l_7\}$ is the set of all possible lower n-types of sm-pairs and $\{l_2, \ldots, l_7\}$ is the set of all possible lower n-types of limited sm-pairs. Moreover, $typ_l(U, \psi, n) = l_i$, $i \in \{1, \ldots, 7\}$, if and only if $typ_u(U, \psi, n) = t_i$.

From Proposition 25 it follows that, for any pair bc, $b, c \in \{i, d, a\}$,

$$typ(\mathcal{L}_{U\psi n}^{bc}) = \rho(typ(\mathcal{U}_{U\psi n}^{cb})),$$

where $\rho(\alpha) = \varepsilon$, $\rho(\beta) = \delta$, $\rho(\gamma) = \gamma$, $\rho(\delta) = \beta$, and $\rho(\varepsilon) = \alpha$. Using this equality and criteria of the behavior of the value $typ(\mathcal{U}_{U\psi n}^{cb})$ described in the previous section we can obtain the criteria of the behavior of the value $typ(\mathcal{L}_{U\psi n}^{bc})$. As examples, we consider the criteria for equalities $typ(\mathcal{L}_{U\psi n}^{bc}) = \alpha$ and $typ(\mathcal{L}_{U\psi n}^{bc}) = \varepsilon$.

The equality $typ(\mathcal{L}_{U\psi n}^{bc}) = \alpha$ means that the set $Dom(\mathcal{L}_{U\psi n}^{bc})$ is infinite and the function $\mathcal{L}_{U\psi n}^{bc}$ is bounded from above. This equality can hold if $bc \in \{di, ai, ad\}$. One can show that $typ(\mathcal{L}_{U\psi n}^{bc}) = \alpha$ if and only if there exists $p \in \omega$ such that $\mathcal{U}_{U\psi n}^{cb}(p) = \infty$, i.e., the set $\{\psi_U^c(z) : z \in \mathcal{P}(U, n), \psi_U^b(z) \leq p\}$ is infinite. In this case, $\mathcal{L}_{U\psi n}^{bc}(m) \leq p$ for any $m \in \omega$.

The equality $typ(\mathcal{L}_{U\psi n}^{bc}) = \varepsilon$ means that $Dom(\mathcal{L}_{U\psi n}^{bc})$ is a finite set. This equality can hold for any pair bc, $b, c \in \{i, d, a\}$. One can show that $typ(\mathcal{L}_{U\psi n}^{bc}) = \varepsilon$ if and only if the function ψ_U^c is bounded. The case $typ(\mathcal{L}_{U\psi n}^{bc}) = \varepsilon$ is not as bad as the case $typ(\mathcal{U}_{U\psi n}^{bc}) = \varepsilon$. In the former case, the value $\mathcal{L}_{U\psi n}^{bc}(m)$ is undefined only if there are no problems $z \in \mathcal{P}(U, n)$ such that $\psi_U^c(z) \geq m$. In the latter case, if the value $\mathcal{U}_{U\psi n}^{bc}(m)$ is undefined, then there exist problems $z \in \mathcal{P}(U, n)$ with $\psi_U^c(z) \leq m$.

Let $typ_l(U, \psi, n) = l \in \{l_1, \ldots, l_7\}$. We now consider possible behavior of the columns in the table l.

The column i in the matrix l is equal to $(\varepsilon, \varepsilon, \varepsilon)^T$ if the function ψ_U^i is bounded (if $l = l_1$) and is equal to $(\gamma, \alpha, \alpha)^T$ if the function ψ_U^i is unbounded (if $l \in \{l_2, \ldots, l_7\}$).

The column a in the matrix l is equal to $(\varepsilon, \varepsilon, \varepsilon)^T$ if the function ψ_U^a is bounded (if $l \in \{l_1, l_2, l_3, l_4\}$) and is equal to $(\gamma, \gamma, \gamma)^T$ if the function ψ_U^a is unbounded (if $l \in \{l_5, l_6, l_7\}$).

The behavior of the column d in the matrix l is more complicated. If the function ψ_U^d is bounded (if $l \in \{l_1, l_2\}$), then this column is equal to $(\varepsilon, \varepsilon, \varepsilon)^T$. If the function ψ_U^d is unbounded and the function ψ_U^a is bounded (if $l \in \{l_3, l_4\}$), then the column d is equal to $(x, \gamma, \alpha)^T$, where $x \in \{\delta, \gamma\}$. If each of the functions ψ_U^i, ψ_U^d, and ψ_U^a is unbounded (if $l \in \{l_5, l_6, l_7\}$), then the column d is equal to $(\gamma, \gamma, x)^T$, where $x \in \{\gamma, \beta, \alpha\}$.

8.3. n-Types of sm-pairs

From Theorems 1 and 2 it follows that $\{T_1, \ldots, T_7\}$ is the set of all possible n-types of sm-pairs and $\{T_2, \ldots, T_7\}$ is the set of all possible n-types of limited sm-pairs.

For $i \in \{1, \ldots, n\}$ and $b, c \in \{i, d, a\}$, we denote by t_i^{bc} the value in the intersection of the row b and the column c in the matrix t_i and by l_i^{bc} the value in the intersection of the row b and the column c in the matrix l_i. Then the matrix T_i has the pair $l_i^{bc} t_i^{bc}$ in the intersection of the row b and the column c.

In the table

	i	d	a
i	$\gamma\gamma, \varepsilon\alpha$	$\gamma\varepsilon, \delta\varepsilon, \varepsilon\alpha, \varepsilon\varepsilon$	$\gamma\varepsilon, \varepsilon\alpha, \varepsilon\varepsilon$
d	$\alpha\alpha, \alpha\beta, \alpha\gamma, \varepsilon\alpha$	$\gamma\gamma, \varepsilon\alpha$	$\gamma\gamma, \gamma\delta, \gamma\varepsilon, \varepsilon\alpha, \varepsilon\varepsilon$
a	$\alpha\alpha, \alpha\gamma, \varepsilon\alpha$	$\alpha\alpha, \alpha\gamma, \beta\gamma, \gamma\gamma, \varepsilon\alpha$	$\gamma\gamma, \varepsilon\alpha$

in the intersection of the row with index $b \in \{i, d, a\}$ and the column with index $c \in \{i, d, a\}$, we have all possible pairs that appear in tables $T_1, \ldots, T_7$ in the intersection of the row and the column with the same indices. Note that out of 25 different pairs pq, $p, q \in \{\alpha, \beta, \gamma, \delta, \varepsilon\}$, only ten pairs $\alpha\alpha$, $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$, $\gamma\gamma$, $\gamma\delta$, $\gamma\varepsilon$, $\delta\varepsilon$, $\varepsilon\alpha$, $\varepsilon\varepsilon$ are present in tables $T_1, \ldots, T_7$.

The situation, when $typ(\mathcal{L}_{U\psi n}^{bc})typ(\mathcal{U}_{U\psi n}^{bc}) \in \{\alpha\alpha, \beta\gamma, \gamma\gamma, \gamma\delta\}$ is good enough: the difference between the lower and upper bounds $\mathcal{L}_{U\psi n}^{bc}$ and $\mathcal{U}_{U\psi n}^{bc}$ is reasonable (they have infinite domains and are either both bounded from above or both unbounded from above). For the rest of the cases, the situation is worse. For pairs $\alpha\beta$ and $\alpha\gamma$, there is a too big gap between lower and upper bounds $\mathcal{L}_{U\psi n}^{bc}$ and $\mathcal{U}_{U\psi n}^{bc}$: the lover bound is bounded from above and the upper bound is unbounded from above. For pairs $\gamma\varepsilon$, $\delta\varepsilon$, $\varepsilon\alpha$, and $\varepsilon\varepsilon$, at least one of the bounds has finite domain.

8.4. Dynamic types for sm-pairs

We defined the linear order $\preceq$ on the set $\{\alpha, \beta, \gamma, \delta, \varepsilon\}$ as follows: $\alpha \preceq \beta \preceq \gamma \preceq \delta \preceq \varepsilon$. Based on this order, we defined the partial order $\leqq$ on the set of tables $\{t_1, \ldots, t_7\}$. Let $i, j \in \{1, \ldots, 7\}$. Then $t_i \leqq t_j$ if and only if $t_i^{bc} \preceq t_j^{bc}$ for any $b, c \in \{i, d, a\}$. The graph depicted in Fig. 3 is the Hasse diagram for the partially ordered set $(\{t_1, \ldots, t_7\}, \leqq)$. Nodes of this diagram are tables $t_1, \ldots, t_7$. An edge goes upward from t_i to t_j if $t_i \leqq t_j$ and there is no t_k, $t_k \notin \{t_i, t_j\}$, such that $t_i \leqq t_k \leqq t_j$. We now define a partial order $\leqq$ on the set of tables $\{T_1, \ldots, T_7\}$: for any $i, j \in \{1, \ldots, 7\}$, $T_i \leqq T_j$ if and only if $t_i \leqq t_j$.

From Theorems 1 and 2 it follows that, for limited sm-pairs, the set of all possible dynamic types coincides with the set Δ of infinite sequences $T_{i_1}T_{i_2}\cdots$ such that $T_{i_1}, T_{i_2}, \ldots \in \{T_2, \ldots, T_7\}$ and $T_{i_1} \leqq T_{i_2} \leqq \cdots$. For arbitrary sm-pairs the set of all possible dynamic types coincides with the set $\Delta \cup T_1^\infty$.

9. Conclusions

In this paper, we studied computation trees over arbitrary structures. We described the set of all possible dynamic types for (i) arbitrary sm-pairs ((structure, measure)-pairs), (ii) for limited sm-pairs, and, in fact, (iii) for limited sm-pairs with a weighted depth as the complexity measure. The question about the set of all possible dynamic types for sm-pairs that have the depth as the complexity measure is open. In the future, we are planning to consider both this issue and various issues related to computational trees over structures with finite collections of predicates and functions.

Acknowledgments

Research reported in this publication was supported by King Abdullah University of Science and Technology (KAUST), Saudi Arabia. The author is grateful to the anonymous reviewer for useful comments and suggestions.

References

[1] M. Ben-Or, Lower bounds for algebraic computation trees (Preliminary Report), in: The 15th Annual ACM Symposium on Theory of Computing, 1983, pp. 80–86.
[2] A. Björner, L. Lovász, A.C. Yao, Linear decision trees: Volume estimates and topological bounds, in: The 24th Annual ACM Symposium on Theory of Computing, 1992, pp. 170–177.
[3] D.P. Dobkin, R.J. Lipton, Multidimensional searching problems, SIAM J. Comput. 5 (2) (1976) 181–186.
[4] D.P. Dobkin, R.J. Lipton, A lower bound of the $(1/2)n^2$ on linear search programs for the knapsack problem, J. Comput. System Sci. 16 (3) (1978) 413–417.
[5] D.P. Dobkin, R.J. Lipton, On the complexity of computations under varying sets of primitives, J. Comput. System Sci. 18 (1) (1979) 86–91.
[6] A. Gabrielov, N.N. Vorobjov, On topological lower bounds for algebraic computation trees, Found. Comput. Math. 17 (1) (2017) 61–72.
[7] D. Grigoriev, N.N. Vorobjov, Complexity lower bounds for computation trees with elementary transcendental function gates, Theoret. Comput. Sci. 157 (2) (1996) 185–214.
[8] F.M. auf der Heide, A polynomial linear search algorithm for the n-dimensional knapsack problem, J. ACM 31 (3) (1984) 668–676.
[9] F.M. auf der Heide, Nondeterministic versus probabilistic linear search algorithms, in: The 26th Annual Symposium on Foundations of Computer Science, 1985, pp. 65–73.

[10] M. Moshkov, On conditional tests, Sov. Phys. Dokl. 27 (1982) 528–530.
[11] M. Moshkov, On the relations between the depth of deterministic and nondeterministic acyclic programs in basis $\{x+y, x-y, 1; sign(x)\}$ (in Russian), in: Mathematical Problems in Computational Theory, in: Banach Center Publications, vol. 21, PWN, Polish Scientific Publishers, 1988, pp. 523–529.
[12] M. Moshkov, Comparative analysis of deterministic and nondeterministic decision tree complexity. Global approach, Fundam. Informaticae 25 (2) (1996) 201–214.
[13] M. Moshkov, Comparative analysis of time complexity of deterministic and nondeterministic tree-programs (in Russian), in: Actual Problems of Modern Mathematics, Vol. 3, Novosibirsk, NII MIOO NGU, 1997, pp. 117–124.
[14] M. Moshkov, Rough analysis of tree-programs, in: The Fifth European Congress on Intelligent Techniques and Soft Computing, Vol. 1, 1997, pp. 231–235.
[15] M. Moshkov, Rough analysis of tree-program time complexity, in: The 7th International Conference Information Processing and Management of Uncertainty in Knowledge-Based Systems, Vol. 2, 1998, pp. 1376–1380.
[16] M. Moshkov, Classification of infinite information systems depending on complexity of decision trees and decision rule systems, Fundam. Informaticae 54 (4) (2003) 345–368.
[17] M. Moshkov, Time complexity of decision trees, in: J.F. Peters, A. Skowron (Eds.), Trans. Rough Sets III, in: Lecture Notes in Computer Science, vol. 3400, Springer, 2005, pp. 244–459.
[18] M. Moshkov, Comparative Analysis of Deterministic and Nondeterministic Decision Trees, in: Intelligent Systems Reference Library, vol. 179, Springer, 2020.
[19] J.M. Steele, A.C. Yao, Lower bounds for algebraic decision trees, J. Algorithms 3 (1) (1982) 1–8.
[20] A.C. Yao, Algebraic decision trees and Euler characteristics, Theoret. Comput. Sci. 141 (1&2) (1995) 133–150.
[21] A.C. Yao, Decision tree complexity and Betti numbers, J. Comput. System Sci. 55 (1) (1997) 36–43.