

Efficient Importance Sampling Algorithm Applied to the Performance Analysis of Wireless Communication Systems Estimation

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Abstract

When assessing the performance of wireless communication systems operating over fading channels, one often encounters the problem of computing expectations of some functional of sums of independent random variables (RVs). The outage probability (OP) at the output of Equal Gain Combining (EGC) and Maximum Ratio Combining (MRC) receivers is among the most important performance metrics that falls within this framework. In general, closed form expressions of expectations of functionals applied to sums of RVs are out of reach. A naive Monte Carlo (MC) simulation is of course an alternative approach. However, this method requires a large number of samples for rare event problems (small OP values for instance). Therefore, it is of paramount importance to use variance reduction techniques to develop fast and efficient estimation methods. In this work, we use importance sampling (IS), being known for its efficiency in requiring less computations for achieving the same accuracy requirement. In this line, we propose a state-dependent IS scheme based on a stochastic optimal control (SOC) formulation to calculate rare events quantities that could be written in a form of an expectation of some functional of sums of independent RVs. Our proposed algorithm is generic and can be applicable without any restriction on the univariate distributions of the different fading envelopes/gains or on the functional that is applied to the sum. We apply our approach to the Log-Normal distribution to compute the OP at the output of diversity receivers with and without co-channel interference. For each case, we show numerically that the proposed state-dependent IS algorithm compares favorably to most of the well-known estimators dealing with similar problems.

Keywords: Outage probability, Monte Carlo, rare event, importance sampling, stochastic optimal control.

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1 Introduction

In a probabilistic model, rare events are important events that happen infrequently with very small probabilities. The estimation of these probabilities has become a large area of research because of its many applications. Typical examples occur in the context of communication systems, where the rare event could be the event that the system fails to operate properly. For sophisticated networks such as ultra-reliable 5G or 6G systems, one can encounter the problem of estimating failure probabilities of the order of 10^{-9} [19].

Calculating rare events quantities that could be written in a form of an expectation of some functional of sums of independent random variables (RVs) is of paramount practical interest in many challenging applications in communication systems. For instance, the outage probability (OP), defined as the probability that the signal to noise ratio (SNR) falls below a given threshold at the output of equal gain combining (EGC) and maximum ratio combining (MRC) receivers, turns out to be equivalent to computing the cumulative distribution function (CDF) of sums of fading envelopes (EGC) or fading gains (MRC) [45]. Another relevant example is the computation of OP values in the presence of co-channel interference and additive white Gaussian noise (AWGN) for single-input-multiple-output (SIMO) and multiple-input-multiple-output (MIMO) wireless systems. For these settings and under some particular fading environments, the OP can be expressed as the expectation of a functional of sums of independent RVs.

Various works have proposed closed-form approximations of the OP under various wireless system configurations [12, 27, 33, 39, 42, 44, 46, 47, 49]. However, these different approximation methods are not generic. Moreover, their accuracy is not always guaranteed for all scenarios, as it can degrade for certain system's parameters. The Monte Carlo (MC) method can be used as a generic tool to cope with those problems. However, it is well-acknowledged that the estimation of rare event quantities with the naive MC sampler requires a prohibitively large number of simulation runs [38]. In order to improve the computational work of the naive MC method, variance reduction techniques have been used extensively. In this context, IS is among the most popular variance reduction technique that provides, when appropriately used, accurate estimates of rare event probabilities with a reduced number of simulation runs [38].

Variance reduction techniques have been widely discussed in literature, and a particular interest was devoted to propose algorithms for the efficient simulation of the right-tail of sums of RVs, i.e. the probability that the sum exceeds a sufficiently large threshold. In particular, for distributions with light right tails, i.e. decaying at an exponential rate or faster, it can be proved, under some regularity assumptions, that the popular exponential twisting IS approach [5] satisfies the logarithmic efficiency property, which is a useful metric used to assess the efficiency of an estimator. On the other hand, in the setting of heavy tailed distributions, which is for instance the case of Log-Normals and Weibulls with shape parameter strictly less than 1, the exponential twisting method is not applicable. Therefore, efficient algorithms have been developed for estimating tail probabilities involving heavy tailed RVs. In this context, [3] gave the first logarithmically efficient estimator for such probabilities using the conditional MC idea. The authors in [9] proposed an estimator with bounded relative error under distributions with regularly varying tails. The estimator in [9] was further extended to more general scenarios, see [7, 8, 26, 34, 35]. In addition to estimators based on conditional MC, various state-independent IS techniques have been proposed in [2, 36, 37, 40].

It was shown in [11] that state-independent change of measure for estimating certain rare events involving sums of heavy-tailed RVs cannot achieve logarithmic efficiency. Therefore, more

complex state-dependent IS algorithms have been proposed in the literature over the last few years to estimate probabilities for sums of heavy-tailed independent RVs. Of valuable interest are for instance the works developed in [21, 22, 23, 24, 29, 31]. The work in [23] developed an efficient state-dependent IS estimator with bounded relative error under distributions with regularly varying heavy tails. The estimator can be also adapted in light-tailed situations to provide strongly efficient algorithms. A related approach, based on the construction of Lyapunov inequalities has been also developed in [24] for the construction of strongly efficient estimators for large deviation probabilities of regularly varying random walks. These algorithms use parametric family of change of measure based on mixtures which are appropriately selected using Lyapunov bounds. Moreover, stochastic control and game theory has been used to build efficient state-dependent IS schemes for simulating rare events [29, 30, 32]. For instance, in the heavy tailed setting, the authors in [29] constructed dynamic IS estimators with nearly asymptotically optimal relative error for independent and identically distributed (i.i.d) non-negative regularly varying RVs. They considered a parametric family of change of measure whose parameters are determined via solving a deterministic, discrete time control problem. The closest work to our approach is in [30], where the authors proposed an approach based on connecting IS with stochastic optimal control (SOC). Note that the scope of the [30] is limited to the i.i.d case and to distributions with finite moment generating function. In our work, independence is the only assumption we make.

Few works have recently addressed the left-tail region, i.e. the probability that sums of non-negative RVs fall below a sufficiently small threshold [6, 13, 16, 17, 19, 20]. For instance, [6] considered the specific setting of the i.i.d sum of Log-Normal RVs. Its approach was based on the exponential twisting technique and was shown to be logarithmically efficient. The work of [17] proposed two unified hazard rate twisting based approaches that estimate the outage capacity values over generalized independent fading channels. The first estimator achieves the logarithmic efficiency for arbitrary fading models while the second one achieves the bounded relative error criterion for the majority of the well-known fading variates and the logarithmic efficiency for the Log-Normal case. Recently, [19] proposed an IS scheme based on sample rejection applied to the case of independent Rayleigh, correlated Rayleigh, and i.i.d Rice fading models. It was shown that the estimator satisfies the bounded relative error property.

In this paper, we propose a generic state-dependent IS approach to estimate rare event probabilities that could be written in the form of an expectation of some functional of sums of independent RVs. We adopt a SOC formulation to determine the optimal IS parameters, minimizing the variance or equivalently the second moment of the estimator, within a preselected class of measures. After formulating the SOC problem and describing the algorithm that will be used to derive the optimal controls, which are the optimal IS parameters, we apply our algorithm to two examples: the computation of the OP at the output of MRC and EGC receivers in a Log-Normal environment, and the computation of the OP in the presence of co-channel interferences and Gaussian noise in a Log-Normal environment as well. It is worth mentioning that our proposed algorithm is generic and not restricted to the Log-Normal environment. In fact, our algorithm can be applicable to compute the quantity of interest without any restriction on either the distribution of the univariate RVs in the sum or the expression of the functional applied to the sum. We show via some numerical simulations the outperformance of the proposed estimator, in terms of number of samples and computational work in order to meet a given prescribed tolerance, compared to the existing state of the art estimators dealing with similar problems.

The rest of the paper is organized as follows. In Section 2, we describe the problem setting

and present some challenging wireless communications applications that fall within the scope of applicability of our proposed approach. We introduce in the same section the concept of IS. Section 3 contains the main part of the work, where we explain our state-dependent IS scheme via a novel SOC formulation. We end the section by stating our algorithm. In Section 4, we apply our algorithm to two applications in wireless communications. We show that our algorithm compares favorably to some well-known estimators dealing with similar problems.

2 Problem Setting

2.1 Objective and Applications

We consider $\mathbf{X} = (X_1, X_2, \dots, X_N)^t$ a random vector composed of independent positive components with PDFs $f_{X_1}(\cdot), f_{X_2}(\cdot), \dots, f_{X_N}(\cdot)$ and joint PDF $f(\mathbf{x}) = \prod_{n=1}^N f_{X_n}(x_n)$. Let $S_N = \sum_{n=1}^N X_n$ and $g: \mathbb{R}_+ \rightarrow \mathbb{R}$ a given function. Our aim is to develop, via a connection to a SOC formulation, a state-dependent IS algorithm to estimate rare event quantities that could be written in the following form

$$\alpha = \mathbb{E}[g(S_N)]. \quad (1)$$

Many wireless performance metrics can be written in the form of (1) such as the OP at the output of MRC or EGC diversity receivers. For example, the instantaneous SNR at the diversity receiver can be written as [48]

$$\gamma_{end} = \frac{E_s}{N_0 \sqrt{N^{1-p+q}}} \left(\sum_{i=1}^N R_i^p \right)^q, \quad (2)$$

where N is the number of diversity branches, E_s/N_0 is the SNR per symbol at the transmitter, and $R_i, i = 1, 2, \dots, N$, is the fading envelope. The parameter p and q are in $\{1, 2\}$ as follows

$$(p, q) = \begin{cases} (1, 2), & \text{EGC} \\ (2, 1), & \text{MRC} \end{cases} \quad (3)$$

The OP, defined as the probability that the SNR falls below a given threshold γ_{th} at the output of EGC and MRC, turns out to be equivalent to computing the CDF of sums of fading envelopes for EGC or fading gains for MRC [45] and hence can be expressed as in (1) with $g(x) = \mathbb{1}_{(x \leq a)}$, where $X_i = R_i^2$ and $a = \gamma_{th} N_0 / E_s$ (for MRC) or $X_i = R_i$ and $a = \sqrt{N_0 N \gamma_{th} / E_s}$ (for EGC), $i = 1, 2, \dots, N$.

Another performance metric, that can be expressed as in (1), is to evaluate the OP in the presence of co-channel interferences and noise. For single-input-single-output (SISO) systems, the OP is expressed as [18]

$$P_{out} = \mathbb{P}(\text{SINR} \leq \gamma_{th}) = \mathbb{P}\left(\frac{X_0}{\sum_{n=1}^N X_n + \eta} \leq \gamma_{th}\right), \quad (4)$$

where X_0 is the useful signal power, X_1, \dots, X_N are the received powers of the N interfering signals and η is the variance of the AWGN. We assume that X_0, \dots, X_N are independent. By conditioning on X_1, X_2, \dots, X_N and using the law of total expectation, we write (4) as

$$\mathbb{E}\left[F_{X_0}\left(\gamma_{th}\left(\sum_{n=1}^N X_n + \eta\right)\right)\right], \quad (5)$$

where $F_{X_0}(\cdot)$ is the CDF of the RV X_0 . This corresponds to the form in (1) with $g(x) = F_{X_0}(\gamma_{\text{th}}(x + \eta))$.

For SIMO systems, similar expressions as in (5) can be obtained for another type of diversity techniques which is selection combining. More precisely, when using the SNR-based approach, and under some assumptions, the expression of the SINR is expressed as in [43, eq.15], and hence the OP can be in the form of (5). Moreover, for MRC systems, when both the desired and interfering signals undergo Nakagami fading, the expression of the SINR can be simplified as in [1], so the OP can be expressed as in (5) as well. Additionally, the performance of MIMO MRC systems for Rayleigh-distributed with co-channel interference signals has been analyzed in [41], where it was proven that the SINR expression has the form of the SINR of SISO systems and thus, following the same strategy, the OP in this case can also be written as (5).

A further challenging problem involving sums of RVs is to consider discrete RVs. If we consider, like in [10], a detector array with N square shaped detectors of uniform area, we can express the probability of missed detection as

$$P_m = \mathbb{P} \left(\sum_{k=1}^N Z_k \alpha_k \leq \gamma_{\text{th}} \right), \quad (6)$$

where α_k are non-negative weights that capture the SNR content in the k^{th} detector of the array, and $\{Z_k\}_{k=1}^N$ are independent Poisson RVs. Thus, the problem yields to estimate a rare event probability in the form of (1) with X_1, \dots, X_N represent the weighted Poisson RVs and $g(x) = \mathbb{1}_{(x \leq \gamma_{\text{th}})}$.

2.2 Importance Sampling

The naive MC estimator of the quantity of interest in (1) is

$$\hat{\alpha}_{mc} = \frac{1}{M} \sum_{k=1}^M g \left(S_N^{(k)} \right), \quad (7)$$

where M is the number of simulation runs and $\{S_N^{(k)}\}_{k=1}^M$ represent independent realizations of the RV $S_N = \sum_{i=1}^N X_i$.

However, the naive MC method is computationally expensive, requiring substantial number of simulation runs to meet a given accuracy, when considering rare events probabilities. It is therefore necessary to use appropriate variance reduction techniques such as IS which can be used to overcome the failure of naive MC simulations and considerably reduce the computational work. The idea is to perform a change of measure under which the rare event is generated with higher probability than under the original distribution [38]. The IS technique consists on writing α as

$$\alpha = \mathbb{E}_{\tilde{f}} [\tilde{g}(\mathbf{X})], \quad (8)$$

where

$$\tilde{g}(\mathbf{x}) = g \left(\sum_{n=1}^N x_n \right) \frac{f(\mathbf{x})}{\tilde{f}(\mathbf{x})}, \quad (9)$$

and $\mathbb{E}_{\tilde{f}}[\cdot]$ is the expectation under which the vector \mathbf{X} has the joint PDF $\tilde{f}(\cdot)$. The IS estimator is then expressed as

$$\hat{\alpha}_{IS} = \frac{1}{M} \sum_{k=1}^M \tilde{g}(\mathbf{X}^{(k)}), \quad (10)$$

where $\{\mathbf{X}^{(k)}\}_{k=1}^M$ represent independent realizations of \mathbf{X} sampled according to $\tilde{f}(\cdot)$. For the case where $g(x) > 0, x \in \mathbb{R}_+$, it is well-known that the optimal change of measure minimizing the variance of the IS estimator is given by

$$f^*(\mathbf{x}) = \frac{f(\mathbf{x})g\left(\sum_{n=1}^N x_n\right)}{\alpha}, \quad \mathbf{x} \in \mathbb{R}_+^N. \quad (11)$$

This optimal change of measure yields zero variance, and thus is called the zero variance change of measure. However, it is not practical to use such change of measure as it assumes the knowledge of α , which is the unknown quantity.

3 IS via a SOC formulation

3.1 State-dependent IS approach

The idea we adopt is to link the problem of finding an efficient change of measure to a SOC problem. To be able to apply SOC to the current static problem, we embed it with the evolution of a Markov chain with the following dynamics

$$S_{n+1} = S_n + X_{n+1}, \quad n = 0, 1, \dots, N-1 \quad (12)$$

with $S_0 = 0$.

Instead of sampling X_{n+1} according to $f_{X_{n+1}}(\cdot)$, we perform a change of measure such that, given S_n , X_{n+1} is distributed according to $\tilde{f}_{X_{n+1}}(\cdot; \mu_{n+1}(S_n))$, where μ_{n+1} is a function of S_n . Using this idea, the new joint PDF can be written as

$$\tilde{f}(\mathbf{x}) = \prod_{n=1}^N \tilde{f}_{X_n}(x_n; \mu_n(s_{n-1})), \quad (13)$$

where $s_{n-1} = \sum_{i=1}^{n-1} x_i$.

The objective is to find the optimal controls $\mu_n : \mathbb{R}_+ \rightarrow \mathcal{A} \subset \mathbb{R}$, $n = 1, 2, \dots, N$, that minimizes the second moment of the IS estimator. Therefore, we define the cost function for $\mu_{n+1}, \dots, \mu_N \in \mathcal{A}^{N-n}$, $n = 0, \dots, N-1$, as

$$C_{n,s}(\mu_{n+1}, \dots, \mu_N) = \mathbb{E}_{\tilde{f}} \left[(g(S_N))^2 \prod_{i=n+1}^N \left(\frac{f_{X_i}(X_i)}{\tilde{f}_{X_i}(X_i; \mu_i(S_{i-1}))} \right)^2 \mid S_n = s \right], \quad (14)$$

where \mathcal{A} is the set of admissible Markov controls. We define also the value function as follows

$$u(n, s) = \inf_{\mu_{n+1}, \dots, \mu_N \in \mathcal{A}^{N-n}} C_{n,s}(\mu_{n+1}, \dots, \mu_N). \quad (15)$$

The above SOC formulation is flexible in the sense that the RVs are dependent. The same observation holds for the optimal change of measure (11). Therefore, if the family of PDFs $\tilde{f}_{X_n}(\cdot; \mu_n)$ is sufficiently large, we may expect the SOC formulation to deliver an estimator with performance close to that of the optimal estimator.

The question now is how to solve the minimization problem and to find the optimal controls μ_n , $n = 1, 2, \dots, N$. The idea is to solve it sequentially by going backward in time. In the following lemma, we state the dynamic programming equation solved by the value function u .

Proposition: For all $n \in \{0, 1, \dots, N-1\}$ and $s \geq 0$, we have

$$u(n, s) = \inf_{\mu \in A} \mathbb{E}_{\tilde{f}} \left[\left(\frac{f_{X_{n+1}}(X_{n+1})}{\tilde{f}_{X_{n+1}}(X_{n+1}; \mu)} \right)^2 u(n+1, S_{n+1}) \mid S_n = s \right], \quad (16)$$

and if the minimum is attained, we have

$$\mu_{n+1}(s) = \arg \min_{\mu \in A} \mathbb{E}_{\tilde{f}} \left[\left(\frac{f_{X_{n+1}}(X_{n+1})}{\tilde{f}_{X_{n+1}}(X_{n+1}; \mu)} \right)^2 u(n+1, S_{n+1}) \mid S_n = s \right], \quad (17)$$

with $u(N, x) = (g(x))^2$, $S_{n+1} = s + X_{n+1}$ and X_{n+1} is distributed according to $\tilde{f}_{X_{n+1}}(\cdot; \mu_{n+1}(s))$.

Proof: For simplicity, we assume that the optimal control is attained

$$u(n, s) = \min_{\mu_{n+1}, \dots, \mu_N \in A^{N-n}} C_{n,s}(\mu_{n+1}, \dots, \mu_N). \quad (18)$$

Step 1 Let $\mu_{n+1}^*, \dots, \mu_N^*$ be the optimal control minimizing (18). Then, we obtain

$$\begin{aligned} u(n, s) &= \mathbb{E}_{\tilde{f}} \left[(g(S_N))^2 \prod_{i=n+1}^N \left(\frac{f_{X_i}(X_i)}{\tilde{f}_{X_i}(X_i; \mu_i^*(S_{i-1}))} \right)^2 \mid S_n = s \right] \\ &= \mathbb{E}_{\tilde{f}} \left[\mathbb{E}_{\tilde{f}} \left[(g(S_N))^2 \prod_{i=n+2}^N \left(\frac{f_{X_i}(X_i)}{\tilde{f}_{X_i}(X_i; \mu_i^*(S_{i-1}))} \right)^2 \right. \right. \\ &\quad \left. \left. \times \left(\frac{f_{X_{n+1}}(X_{n+1})}{\tilde{f}_{X_{n+1}}(X_{n+1}; \mu_{n+1}^*(S_n))} \right)^2 \mid S_n = s, X_{n+1} \right] \mid S_n = s \right]. \end{aligned} \quad (19)$$

Knowing X_{n+1} and S_n , $\left(\frac{f_{X_{n+1}}(X_{n+1})}{\tilde{f}_{X_{n+1}}(X_{n+1}; \mu_{n+1}^*(S_n))} \right)^2$ will be deterministic. Thus, using the Markov property of S_n , we obtain

$$\begin{aligned} \mathbb{E}_{\tilde{f}} \left[(g(S_N))^2 \prod_{i=n+2}^N \left(\frac{f_{X_i}(X_i)}{\tilde{f}_{X_i}(X_i; \mu_i^*(S_{i-1}))} \right)^2 \mid S_n = s, X_{n+1} \right] &= C_{n+1, S_{n+1}}(\mu_{n+2}^*, \dots, \mu_N^*) \\ &\geq u(n+1, S_{n+1}). \end{aligned} \quad (20)$$

Hence, the following inequality holds

$$\begin{aligned} u(n, s) &\geq \mathbb{E}_{\tilde{f}} \left[\left(\frac{f_{X_{n+1}}(X_{n+1})}{\tilde{f}_{X_{n+1}}(X_{n+1}; \mu_{n+1}^*(s))} \right)^2 u(n+1, S_{n+1}) \mid S_n = s \right] \\ &\geq \min_{\mu \in A} \mathbb{E}_{\tilde{f}} \left[\left(\frac{f_{X_{n+1}}(X_{n+1})}{\tilde{f}_{X_{n+1}}(X_{n+1}; \mu)} \right)^2 u(n+1, S_{n+1}) \mid S_n = s \right]. \end{aligned} \quad (21)$$

Step 2 We choose the control μ_{n+1}^+ to be arbitrary and then, given the value of S_{n+1} , we choose the optimal controls $\mu_{n+2}^*, \dots, \mu_N^*$. Then, following lower bound holds

$$\begin{aligned}
u(n, s) &\leq \mathbb{E}_{\tilde{f}} \left[\left(\frac{f_{X_{n+1}}(X_{n+1})}{\tilde{f}_{X_{n+1}}(X_{n+1}; \mu_{n+1}^+(s))} \right)^2 (g(S_N))^2 \prod_{i=n+2}^N \left(\frac{f_{X_i}(X_i)}{\tilde{f}_{X_i}(X_i; \mu_i^*(S_{i-1}))} \right)^2 \mid S_n = s \right] \\
&\leq \mathbb{E}_{\tilde{f}} \left[\left(\frac{f_{X_{n+1}}(X_{n+1})}{\tilde{f}_{X_{n+1}}(X_{n+1}; \mu_{n+1}^+(s))} \right)^2 \right. \\
&\quad \times \mathbb{E}_{\tilde{f}} \left[(g(S_N))^2 \prod_{i=n+2}^N \left(\frac{f_{X_i}(X_i)}{\tilde{f}_{X_i}(X_i; \mu_i^*(S_{i-1}))} \right)^2 \mid S_n = s, X_{n+1} \right] \mid S_n = s \Big] \\
&= \mathbb{E}_{\tilde{f}} \left[\left(\frac{f_{X_{n+1}}(X_{n+1})}{\tilde{f}_{X_{n+1}}(X_{n+1}; \mu_{n+1}^+(s))} \right)^2 u(n+1, S_{n+1}) \mid S_n = s \right].
\end{aligned} \tag{22}$$

Taking the minimum over all controls $\mu_{n+1}^+(s)$ yields

$$u(n, s) \leq \min_{\mu \in A} \mathbb{E}_{\tilde{f}} \left[\left(\frac{f_{X_{n+1}}(X_{n+1})}{\tilde{f}_{X_{n+1}}(X_{n+1}; \mu)} \right)^2 u(n+1, S_{n+1}) \mid S_n = s \right], \tag{23}$$

and hence the proof is concluded using (22) and (23).

Remark: We can prove the proposition without the assumption that the minimum is attained. For that, we use a minimizing sequence μ_i of controls, satisfying

$$u(n, s) = \lim_{i \rightarrow \infty} C_{n,s}(\mu_i). \tag{24}$$

3.2 Hazard Rate Twisting family

Until now, we did not specify the family of PDFs $\tilde{f}_{X_n}(\cdot; \mu_n)$, $n = 1, \dots, N$. Our choice in this work is based on the well-known hazard rate twisting (HRT). The HRT technique was originally developed to deal with the right tail of sums of heavy tailed RVs [15, 37].

We define the hazard rate $\lambda_{X_i}(\cdot)$ associated to the RV X_i as

$$\lambda_{X_i}(x) = \frac{f_{X_i}(x)}{1 - F_{X_i}(x)}, \quad x > 0, \tag{25}$$

where $F_{X_i}(x) = \mathbb{P}(X_i \leq x)$ is the CDF of X_i , $i = 1, \dots, N$.

We define also the hazard function as

$$\Lambda_{X_i}(x) = -\log(1 - F_{X_i}(x)), \quad x > 0. \tag{26}$$

From (25) and (26), the PDF of X_i can be expressed as

$$f_{X_i}(x) = \lambda_{X_i}(x) \exp(-\Lambda_{X_i}(x)), \quad x > 0. \tag{27}$$

The HRT change of measure is obtained by twisting the hazard rate of each component X_i , $i = 1, \dots, N$ by a quantity $\mu_i < 1$ as follows

$$\begin{aligned}\tilde{f}_{X_i}(x; \mu_i) &= (1 - \mu_i)\lambda_{X_i}(x) \exp(-(1 - \mu_i)\Lambda_{X_i}(x)) \\ &= (1 - \mu_i)f_{X_i}(x) \exp(\mu_i\Lambda_{X_i}(x)), \quad x > 0.\end{aligned}\tag{28}$$

In order to efficiently handle the estimation of the right tail of the sum distribution, μ_i should satisfy $0 \leq \mu_i < 1$, $i = 1, \dots, N$. Consequently, the tail of the resulting distribution becomes much heavier to the right than the original one. However, this feature is not suitable for dealing with the left tail. Two approaches were proposed in [17] to adjust the HRT to handle the left tail region. The first is based on twisting the RVs $-X_1, \dots, -X_N$ instead of the original variates X_1, \dots, X_N . The second approach applies the HRT approach to X_1, \dots, X_N using a negative twisting parameter. Taking account of the appropriate twisting parameter, we will use the HRT change of measure given by (28) and the set A in this case is given by $A =]-\infty, 1[$.

3.3 Algorithm

Based on the result stated in the proposition, we propose a numerical algorithm to approximate the optimal controls μ_n , $n = 1, \dots, N$. We start by truncating the space \mathbb{R}_+ and work in the interval $[0, S]$, where S is a large number in \mathbb{R}_+ . There are particular cases that we will treat where S is naturally chosen. For instance, when estimating $\mathbb{P}(S_N \leq \gamma_{\text{th}})$, we know, due to the non-negativity of X_i , that $u(n, s) = 0$ for $s \geq \gamma_{\text{th}}$ and $n = 0, \dots, N$. In this case, S is chosen to be equal to γ_{th} . In the general case, S is chosen to be sufficiently large. Note that at each step of the backward algorithm, we use linear extrapolation to compute the value function for $s > S$.

Let us consider a mesh in the one dimensional s -space: $0 = s_0 < s_1, \dots < s_K = S$. The aim is to approximately compute $u(n, s_k)$ for all $n = 0, 1, \dots, N - 1$ and for all s_k , $k = 0, 1, \dots, K$. The algorithm is summarized as follows:

Step 1: For each s_k in the mesh, we solve

$$\begin{aligned}u(N - 1, s_k) &= \min_{\mu \in A} \mathbb{E}_{\tilde{f}} \left[\left(\frac{f_{X_N}(X_N)}{\tilde{f}_{X_N}(X_N; \mu)} \right)^2 (g(s_k + X_N))^2 \right] \\ &= \min_{\mu \in A} \int_0^{+\infty} \frac{(f_{X_N}(t))^2}{\tilde{f}_{X_N}(t; \mu)} (g(s_k + t))^2 dt,\end{aligned}\tag{29}$$

and

$$\mu_N(s_k) = \arg \min_{\mu \in A} \int_0^{+\infty} \frac{(f_{X_N}(t))^2}{\tilde{f}_{X_N}(t; \mu)} (g(s_k + t))^2 dt.\tag{30}$$

This step is not expensive since we need to compute a one dimensional integral for each point in the mesh and perform an optimization problem with respect to the parameter μ . As we will see later, in the case where the HRT family is used, the optimization problem turns out to be equivalent to finding the root of a non-linear equation.

Step 2: Having obtained $u(N-1, s_k)$ for all s_k in the grid, the next step corresponds to use again the result of the proposition to obtain an approximation of $u(N-2, s_k)$ and $\mu_{N-1}(s_k)$

$$u(N-2, s_k) = \min_{\mu \in A} \int_0^{+\infty} \frac{(f_{X_{N-1}}(t))^2}{\tilde{f}_{X_{N-1}}(t; \mu)} u(N-1, s_k + t) dt. \quad (31)$$

To be able to do this step, we need to know $u(N-1, s)$ for all s that are not necessarily in the grid. To overcome this issue, we proceed by interpolating between the points $u(N-1, s_k)$, $k = 0, 1, \dots, K$. As mentioned above, linear extrapolation is used for $s > S$ when needed.

Step 3: Having computed $\mu_n(s_k)$ for $n = 1, 2, \dots, N$ and for all s_k in the grid $k = 0, 1, 2, \dots, K$, the following step is to solve for μ_n , $n = 1, 2, \dots, N$, by going forward in time. More specifically, we start at $S_0 = 0$ and sample from $\tilde{f}_{X_1}(\cdot, \mu_1)$ to get S_1 . Note that $\mu_1(0)$ has been already computed in the resolution of the backward problem. We compute μ_2 as

$$\mu_2(\tilde{s}_1) = \arg \min_{\mu \in A} \int_0^{\infty} \frac{(f_{X_2}(t))^2}{\tilde{f}_{X_2}(t; \mu)} u(2, \tilde{s}_1 + t) dt. \quad (32)$$

Having computed μ_2 , we simulate S_2 as $S_2 = \tilde{s}_1 + X_2$, with X_2 sampled from $\tilde{f}_{X_2}(\cdot; \mu_2)$. We keep repeating this procedure until we get μ_N and then we sample X_N . It is worth mentioning that, in the case of smooth controls, the optimization problem (32) can be avoided by using instead an interpolation between the controls, obtained in the backward step, on the grid s_1, \dots, s_K .

Step 4: The forward problem is repeated M times. The proposed IS estimator is then given as

$$\hat{\alpha}_{\text{IS}} = \frac{1}{M} \sum_{k=1}^M g\left(S_N^{(k)}\right) \prod_{i=1}^N \frac{f_{X_i}\left(X_i^{(k)}\right)}{\tilde{f}_{X_i}\left(X_i^{(k)}, \mu_i\left(S_{i-1}^{(k)}\right)\right)}. \quad (33)$$

4 Numerical results

In this section, we present some selected numerical results in order to illustrate the performance of the proposed IS scheme. Within the wide scope of applicability of the proposed estimator, we focus on applying it to calculate the OP at the output of diversity receivers with and without co-channel interference. We consider the Log-Normal fading environment which is shown to exhibit a good fit to realistic propagation channels. We show that our proposed approach achieves a substantial reduction of the variance compared to other well-known IS algorithms.

In both applications, our objective is to efficiently estimate:

$$\alpha = \mathbb{E} \left[g \left(\sum_{i=1}^N X_i \right) \right], \quad (34)$$

where X_1, \dots, X_N are i.i.d Log-Normal RVs with parameters m and σ^2 . The PDF of X_i , $i = 1, \dots, N$, has the following expression

$$f_{X_i}(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - m)^2}{2\sigma^2}\right), \quad x > 0. \quad (35)$$

Recall that we used the HRT change of measure in (28) to build our estimator. Hence, we call our approach the HRT-SOC IS approach and the corresponding estimator will be denoted by $T_{\text{HRT-SOC}}$ which has the following expression

$$T_{\text{HRT-SOC}} = g(S_N) \prod_{i=1}^N \frac{e^{-\mu_i(S_{i-1})\Lambda_{X_i}(X_i)}}{(1 - \mu_i(S_{i-1}))}. \quad (36)$$

In this setting, each step of the backward algorithm can be expressed, for $k = 0, \dots, K$, as

$$u(n, s_k) = \min_{\mu \in]-\infty, 1[} \frac{1}{1 - \mu} \int_0^{+\infty} u(n+1, s_k + t) f_{X_{n+1}}(t) e^{-\mu \Lambda_{X_{n+1}}(t)} dt, \quad (37)$$

and the controls $\mu_n(s_k)$ are obtained by solving the following equation

$$1 - \mu_n(s_k) = \frac{\int_0^{+\infty} u(n-1, s_k + t) f_{X_{n-1}}(t) e^{-\mu_n(s_k) \Lambda_{X_{n-1}}(t)} dt}{\int_0^{+\infty} \Lambda_{X_{n-1}}(t) u(n-1, s_k + t) f_{X_{n-1}}(t) e^{-\mu_n(s_k) \Lambda_{X_{n-1}}(t)} dt}. \quad (38)$$

For the forward step, assuming that the control is smooth (which is motivated by numerical observations), we can compute the controls by interpolating between the points $\mu_n(s_k)$, $k = 0, \dots, K$.

The relative error will serve as a measure of efficiency of the estimators. The relative error of the naive MC estimator and the proposed IS estimator are defined respectively through the use of the Central Limit Theorem (CLT) [5] as

$$\epsilon_{\text{MC}} = C \frac{\sqrt{\alpha(1-\alpha)}}{\sqrt{M\alpha}}, \quad \epsilon_{\text{HRT-SOC}} = C \frac{\sqrt{\text{Var}[T_{\text{HRT-SOC}}]}}{\sqrt{M\alpha}}, \quad (39)$$

where C is the confidence constant equal to 1.96 for 95 % confidence interval.

We compare our estimator, defined in (36), to other existing estimators when calculating the OP at the output of diversity receivers with and without co-channel interference. For instance, the use of the Log-Normal setting with the HRT technique will allow us to compare our estimator with the approach in [14] which used the HRT without SOC; i.e, the control is constant, independent of the state and time. We denote this method as HRT. As we will see later in the numerical experiments, the HRT-SOC technique reduces the variance substantially compared to other approaches. On the other hand, it requires additional time, which we call backward cost, to determine the optimal controls.

Let M_{HRT} and $M_{\text{HRT-SOC}}$ be the number of required simulation runs for respectively the HRT estimator T_{HRT} and our proposed estimator $T_{\text{HRT-SOC}}$ to ensure a relative error equal to TOL. The total cost of the HRT-SOC and the HRT approaches have the following expressions

$$W_{\text{HRT-SOC}} = \underbrace{N \times K \times T_b}_{\text{Backward cost}} + \underbrace{M_{\text{HRT-SOC}} \times T_f}_{\text{Forward cost}}, \quad (40)$$

$$W_{\text{HRT}} = \underbrace{M_{\text{HRT}} \times T_f}_{\text{Forward cost}}, \quad (41)$$

where T_b is the time required in the backward algorithm to calculate a single control and T_f the cost per sample in the forward step (it is approximately the same for both approaches), As it will

be observed in Figure 2 and Figure 4, the amount of variance reduction compared to the HRT technique increases as the quantity of interest becomes rarer. Thus, we have $M_{\text{HRT}} \gg M_{\text{HRT-SOC}}$ especially for rare regions. As a consequence, we expect that in the regime of rare events and for a fixed N , the backward time can be neglected compared to the forward cost of the HRT, which will be seen in Figure 3.

In the case where the backward time dominates the forward time of the HRT, we propose an improved version of the HRT-SOC estimator, which we will call the aggregate method (HRT-SOC-AG) that aims to reduce the backward cost without considerably affecting the amount of variance reduction.

4.1 Aggregate method

The idea is to divide the sum S_N into B blocks and compute the controls for each block rather than for each $X_i, i = 1, \dots, N$. By doing so, we reduce the backward cost from $N \times K \times T_b$ to $B \times K \times T_b$. We call this method aggregate method. In other words, if we choose B blocks, such that $B \leq N$, we consider the following dynamics,

$$S_{n_m+b_{m+1}} = S_{n_m} + \sum_{i=n_m+1}^{n_m+b_{m+1}} X_i, \quad m = 0, 1, \dots, B-1, \quad (42)$$

where $n_m = \sum_{j=1}^m b_j$, and $b_m, m = 1, 2, \dots, B$, are chosen such that $n_B = \sum_{j=1}^B b_j = N$. The idea that we adopt is to have the same control $\mu_m(S_{n_{m-1}})$ for each X_i from $i = n_{m-1} + 1$ to $i = n_m$. Thus, we define the B new controls μ_1^X, \dots, μ_B^X such that

$$\mu_i = \mu_m^X \text{ for } n_{m-1} < i \leq n_m, \quad i = 1, \dots, N, \quad m = 1, \dots, B. \quad (43)$$

It is important to mention that, with this proposed approach, we decrease the cost of the backward step with the price of increasing the variance.

In order to determine μ_1^X, \dots, μ_B^X , we use the dynamics proposed in (42) instead of the initial dynamics (12) to define a reformulated dynamic programming equation. In other words, we use the same steps as those followed in the proof of the proposition, but instead of conditioning on X_{n+1} , we condition on $X_{n_m+1}, \dots, X_{n_m+b_m}$. By using the same control μ_{m+1} for each $X_i, i = n_m + 1, \dots, n_m + b_m$, as explained in (43), we obtain

$$u(m, s_k) = \min_{\mu \in]-\infty, 1[} \int_{[0, +\infty[^{b_m}} \frac{e^{-\mu \sum_{j=n_m+1}^{n_m+b_m} \Lambda_{X_j}(t_j)}}{(1-\mu)^{b_m}} \prod_{j=n_m+1}^{n_m+b_m} f_{X_j}(t_j) \times u(m+1, s_k + t_{n_m+1} + \dots + t_{n_m+b_m}) dt_{n_m+1} \dots dt_{n_m+b_m}. \quad (44)$$

Instead of solving the above equation, we propose to minimize an approximate upper bound of it. This will become clearer in the next two subsections.

4.2 OP at the output EGC and MRC receivers in a Log-Normal environment without co-channel interference

The computation of the OP at the output of receivers with EGC or MRC diversity schemes is equivalent to evaluating the CDF of the sum of fading envelopes for EGC or channel gains for

MRC. Therefore, our interest in the first application goes to the estimation of the left-tail region of the form of

$$\mathbb{P} \left(\sum_{i=1}^N X_i \leq \gamma_{\text{th}} \right). \quad (45)$$

We compare our approach to the HRT technique, see [14] and to the exponential twisting estimator, see [6]. We also use the improved version to ameliorate the results. When applying the aggregate method, instead of solving (44), we propose to minimize an approximate upper bound of it. More precisely, for the i.i.d Log-Normal case, one can show that

$$\sum_{j=n_m+1}^{n_m+b_m} \Lambda_{X_j}(t_j) \leq \Lambda_X \left(\sum_{j=n_m+1}^{n_m+b_m} t_j \right), \quad t_j > 0. \quad (46)$$

holds asymptotically, i.e. when the sum $\sum_{j=n_m+1}^{n_m+b_m} t_j$ is sufficiently small, where X has the same distribution as $X_j, j = n_m + 1, \dots, n_m + b_m$. This result can be proven using the asymptotic result of the tail of a Normal distribution in [4]. Using the inequality (46), the twisting parameters μ_{m+1}^X are then selected as the argmin of the following approximated upper bound

$$u(m, s_k) \lesssim \min_{\mu \in]-\infty, 1[} \int_{[0, S-s_k]} \frac{e^{-\mu \Lambda_X(y)}}{(1-\mu)^{b_m}} f_{\sum_{j=n_m+1}^{n_m+b_m} X_j}(y) u(n_m + b_m, s_k + y) dy, \quad (47)$$

where $f_{\sum_{j=n_m+1}^{n_m+b_m} X_j}(\cdot)$ is the PDF of $\sum_{j=n_m+1}^{n_m+b_m} X_j$. Given that the PDF of sums of i.i.d Log-Normal RVs is unknown, we suggest to approximate it by a univariate Log-Normal PDF $f_{Y_{m+1}}(\cdot)$, whose parameters are computed using moment matching, see [28]. Finally, we get

$$u(m, s_k) \approx \min_{\mu \in]-\infty, 1[} \int_{[0, S-s_k]} \frac{e^{-\mu \Lambda_X(y)}}{(1-\mu)^{b_m}} f_{Y_{m+1}}(y) u(n_m + b_m, s_k + y) dy, \quad m = 0, \dots, B-1. \quad (48)$$

and μ_1^X, \dots, μ_B^X are obtained as

$$\mu_{m+1}^X(s_k) \approx \arg \min_{\mu \in]-\infty, 1[} \int_{[0, S-s_k]} \frac{e^{-\mu \Lambda_X(y)}}{(1-\mu)^{b_m}} f_{Y_{m+1}}(y) u(n_m + b_m, s_k + y) dy, \quad m = 0, \dots, B-1. \quad (49)$$

We plot in Figure 2 the number of samples, needed by the different approaches, to ensure $\text{TOL} = 5\%$ as a function of γ_{th} . The used range of γ_{th} ensures a range of probabilities between 2×10^{-12} and 6×10^{-6} . For the aggregate method, we choose a constant parameter b , i.e. $b_m = 2$ for all $m = 1, \dots, B$ with $B = \frac{N}{2}$.

Our choice of the parameter $K = 20$ is motivated by Figure 1, where we plot the variance as a function of K . We notice that the larger K is, the smaller the variance will be. Note that the backward step will be costly when K is large. Besides, variance reduction for $K > 20$ is minimal compared to the increased cost of solving the backward problem.

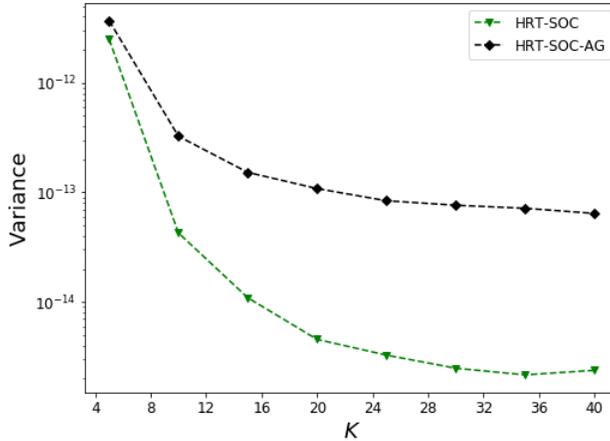


Figure 1: The variance as a function of K with parameters : $N = 10$, $m = 0$ dB, $\sigma = 3$ dB, TOL = 0.05, $b = 2$.

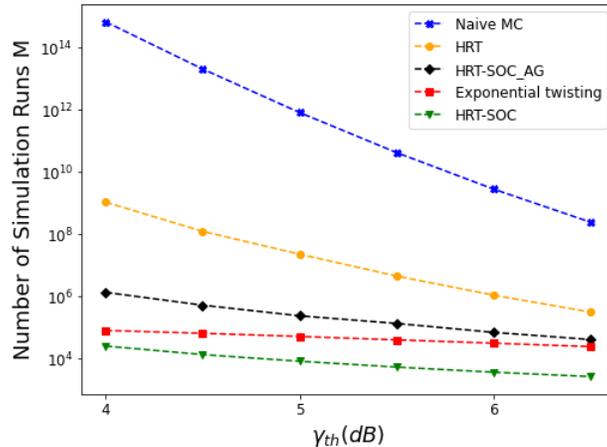


Figure 2: Number of required simulation runs for 5% relative error with parameters : $N = 10$, $K = 20$, $m = 0$ dB, $\sigma = 3$ dB, TOL = 0.05, $b = 2$.

We see from Figure 2 that the number of samples required by naive MC simulations increases at a faster rate as we decrease the threshold. We also observe that the HRT-SOC approach requires the smallest number of simulation runs. Particularly, it saves a considerable number of samples compared to the HRT approach. For example, the number of simulations is reduced by about 41775 times for a small threshold (4 dB) which corresponds to an OP value of 2×10^{-12} . On the other hand, the HRT-SOC-AG requires an additional number of samples, compared to the HRT-SOC approach, to reach 5% relative error, which means that the variance has increased as expected. However, we still get better variance reduction compared to the HRT technique.

We further our analysis by studying the computational work of each method. In Figure 3, we

plot the total time needed by the exponential twisting, the HRT, the HRT-SOC and the HRT-SOC-AG techniques, to ensure 5% relative error, as a function of the threshold. We also plot the time required by the HRT-SOC and HRT-SOC-AG techniques to show the time required for the backward step compared to that required for the forward step.

We notice that our proposed estimator is the best in terms of computational time for small thresholds (corresponds to an OP less than 3.6×10^{-8}). As the event becomes rarer, the time gap between our approach and the other IS techniques increases significantly. Additionally, Figure 2 and Figure 3 show that the HRT approach requires a huge number of samples in order to estimate an OP of the order of 2×10^{-12} with a good accuracy. However, for an OP greater than 3.6×10^{-8} , our approach is expensive compared to the other approaches. This is due to the additional computational time required to perform the backward step for each threshold which exceeds the time required for the rest of the techniques when the number of samples is not sufficiently large. Nevertheless, this was ameliorated when we used the improved version. In fact, the HRT-SOC-AG reduces the CPU time by about 1.7 times compared to the HRT-SOC approach for $\gamma_{\text{th}} \geq 5 \text{ dB}$. This means that with this choice of b , the efficiency of the aggregate method in terms of time reduction exceeds the loss in terms of variance. It is important to note that this choice of $b_m, m = 1, \dots, B$ is not optimal and despite this, it gives better results than the HRT-SOC approach.

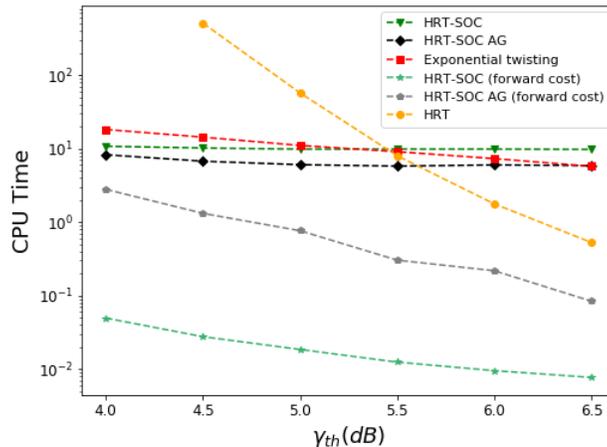


Figure 3: CPU time required for 5% relative error with parameters : $N = 10$, $K = 20$, $m = 0 \text{ dB}$, $\sigma = 3 \text{ dB}$, $\text{TOL} = 0.05$, $b = 2$.

Another possible experiment is to study the efficiency as a function of the number of antennas N for a fixed threshold and investigate the number of simulation runs required for each method as well as the computational time. The results are shown respectively in Figure 4 and Figure 5. The range of the OP is between 10^{-5} and 2.5×10^{-12} when using a range of N between 9 and 13 antennas and a fixed threshold $\gamma_{\text{th}} = 6 \text{ dB}$. For the aggregate method, we use $b_m = 2$, $m = 1, \dots, \frac{N}{2}$ for N even, and $b_m = 2, m = 1, \dots, \frac{N-3}{2}$, $b_{\frac{N-1}{2}} = 3$ for N odd.

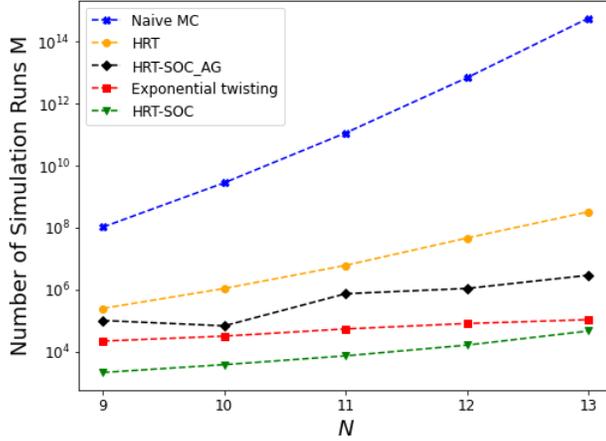


Figure 4: Number of required simulation runs for 5% relative error with parameters : $K = 20$, $\gamma_{th} = 6 \text{ dB}$, $m = 0 \text{ dB}$, $\sigma = 3 \text{ dB}$, $TOL = 0.05$.

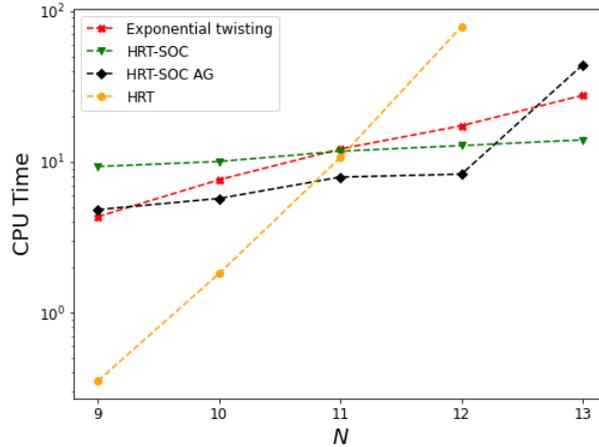


Figure 5: CPU time required for 5% relative error with parameters : $K = 20$, $\gamma_{th} = 6 \text{ dB}$, $m = 0 \text{ dB}$, $\sigma = 3 \text{ dB}$, $TOL = 0.05$.

We note from Figure 4 that the HRT-SOC approach is more efficient requiring less simulation runs than the HRT and the exponential twisting approaches. For $N = 13$, our method requires 7455 times less simulation runs than the HRT technique to meet the same accuracy requirement. In addition, we observe that the amount of variance reduction for the HRT-SOC-AG technique depends on whether N is odd or even. Moreover, the HRT-SOC-AG method requires larger number of simulation runs compared to the HRT-SOC technique to reach a fixed precision TOL but it is more efficient in terms of CPU time for $N \leq 12$. It is worth mentioning that when the event becomes rarer (for small γ_{th} and large N), the improved approach, with fixed choice of b , becomes less efficient in terms of CPU time than the HRT-SOC approach. Indeed, in these cases, the number of samples is large enough that the backward time is neglected. Thus, it is more efficient to reduce

the variance rather than to reduce the cost of the backward step. These results allow us to conclude that the choice of b_m , $m = 1, \dots, B$, is very important and should be adaptively chosen to give better results. More precisely, for a fixed parameters γ_{th} , TOL and N , the following optimization problem should be solved

$$\min_{b, M, K} B \times K \times T_b + M_{\text{HRT-SOC-AG}}(b) \times T_f, \quad (50)$$

such that

$$C^2 \frac{\text{Var} [T_{\text{HRT-SOC-AG}}(b)]}{M_{\text{HRT-SOC-AG}}(b)\alpha^2} \leq \text{TOL}^2.$$

We observe from the above optimization problem that an optimal choice of b_m in the case of a very rare event is $b_m = 1$, $m = 1, \dots, B$ with $B = N$. On the other hand, when the event becomes less rare, an optimal choice of B is to take a single block; i.e. $b_1 = N$. By doing this, the HRT-SOC-AG technique reduces to the HRT technique since in this case the controls are state-independent. Future work can be devoted to solve the previous optimization problem. By using optimal values of b_m , we expect the HRT-SOC-AG estimator to achieve better performances.

4.3 OP in the presence of co-channel interference in a Log-Normal environment for SISO systems

We consider a SISO system and recall that the OP in the presence of co-channel interference and noise is expressed as

$$P_{out} = \mathbb{E} \left[F_{X_0} \left(\gamma_{\text{th}} \left(\sum_{n=1}^N X_n + \eta \right) \right) \right],$$

where X_1, \dots, X_N are the interfering power signal, assumed to be i.i.d Log-Normal RVs with parameters m and σ^2 .

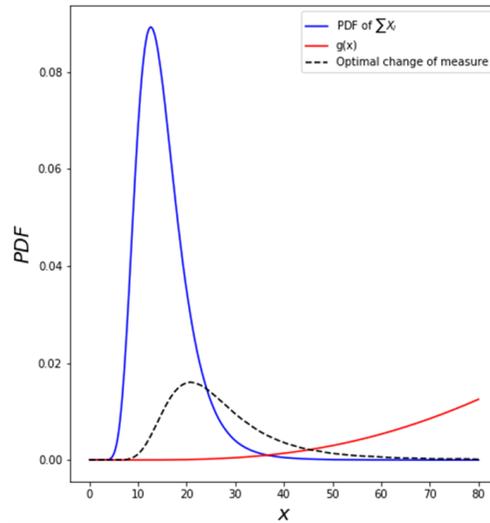


Figure 6: Motivation of using IS with $N = 10$, $m_0 = 10 \text{ dB}$, $\sigma_0 = 4 \text{ dB}$, $m = 0 \text{ dB}$, $\sigma = 4 \text{ dB}$, $\gamma_{\text{th}} = -18 \text{ dB}$, $\eta = -10 \text{ dB}$.

Let us denote the PDF of $\sum_{i=1}^N X_i$ by $f_{\sum_{i=1}^N X_i}(\cdot)$. In order to motivate the need for IS to efficiently estimate P_{out} , we plot in Figure 6 the quantities $f_{\sum_{i=1}^N X_i}$, g , and the optimal IS PDF which is proportional to $g f_{\sum_{i=1}^N X_i}$. Note that the product $g f_{\sum_{i=1}^N X_i}$ in Figure 6 is not normalized, i.e., it is an unnormalized PDF.

We see clearly that sampling from the original PDF of $\sum_{i=1}^N X_i$ is not efficient, i.e. when sampling from the original PDF, most of the samples will fall in the region where g takes almost zero values. Hence, the computation of P_{out} behaves like a rare event problem and can be therefore tackled using our proposed HRT-SOC technique.

The comparison is made with respect to the estimator of [18] which is based on a covariance matrix scaling (CS) technique. It transforms the problem of evaluating the OP to that of computing the probability that a sum of correlated Log-Normal RVs exceeds a certain threshold. We also compare our approach to the exponentially tilted (ET) estimator of [25].

We will also use the HRT-SOC-AG method, proposed in the previous subsection, to further improve the computational work of the HRT-SOC technique. Recall that the reformulated dynamic programming equation is

$$u(m, s_k) = \min_{\mu \in]-\infty, 1[} \int_{[0, +\infty[^{b_m}} \frac{e^{-\mu \sum_{j=n_m+1}^{n_m+b_m} \Lambda_{X_j}(t_j)}}{(1-\mu)^{b_m}} \times \prod_{j=n_m+1}^{n_m+b_m} f_{X_j}(t_j) u\left(m+1, s_k + \sum_{j=n_m+1}^{n_m+b_m} t_j\right) dt_{n_m+1} \cdots dt_{n_m+b_m}. \quad (51)$$

Next, using the following inequality, proven in [37], which is particularly satisfied in the case of i.i.d Log-Normal RVs and holds for $\sum_{j=n_m+1}^{n_m+b_m} t_j$ large enough,

$$\sum_{j=n_m+1}^{n_m+b_m} \Lambda_{X_j}(t_j) \geq \Lambda_X\left(\sum_{j=n_m+1}^{n_m+b_m} t_j\right) - \epsilon, \quad t_j > 0, \text{ for all } \epsilon > 0, \quad (52)$$

we can write

$$u(m, s_k) \approx \min_{\mu \in]-\infty, 1[} \int_{[0, +\infty[} \frac{e^{-\mu \Lambda_X(y)}}{(1-\mu)^{b_m}} f_{Y_{m+1}}(y) u(n_m + b_m, s_k + y) dy, \quad m = 0, \dots, B-1. \quad (53)$$

The large value of $\sum_{j=n_m+1}^{n_m+b_m} t_j$ is motivated from Figure 6, where it shows that the change of measure tends to increase the value of the sum as we go to the regime of rare events. We study the efficiency of the four IS schemes in terms of the number of samples needed to ensure a fixed accuracy requirement. To this end, we plot in Figure 7 the number of samples needed to ensure $TOL = 5\%$ as a function of γ_{th} . This Figure reveals that, the HRT-SOC approach saves a substantial number of samples compared to the other approaches. For instance, it can be seen that the CS technique requires approximately 2000 times as many simulations as needed by the HRT-SOC scheme. Interestingly, we observe also that the aggregate method did not affect the amount of variance reduction. We further proceed investigating the gain in terms of the required computational time. We present in Figure 8 the total CPU time needed by the four techniques to achieve the fixed accuracy TOL. We note that the HRT-SOC approach requires less CPU time than the ET approach for all the

range of considered thresholds. In particular, when $\gamma_{\text{th}} = -30 \text{ dB}$, it is 13 times more efficient than the ET scheme. Compared to the CS approach, the HRT-SOC technique is more efficient when $\gamma_{\text{th}} < -25 \text{ dB}$, which corresponds to an OP less than 3×10^{-8} . Observe also that, the required computational time for the HRT-SOC technique is almost the same in the considered range of threshold, while the CS and the ET approaches require much more time as the threshold decreases. Moreover, the HRT-SOC-AG technique requires less time than the HRT-SOC technique using $b = 2$ to estimate the quantity of interest α . Therefore, our improved approach widens the region over which our proposed approach outperforms the CS approach.

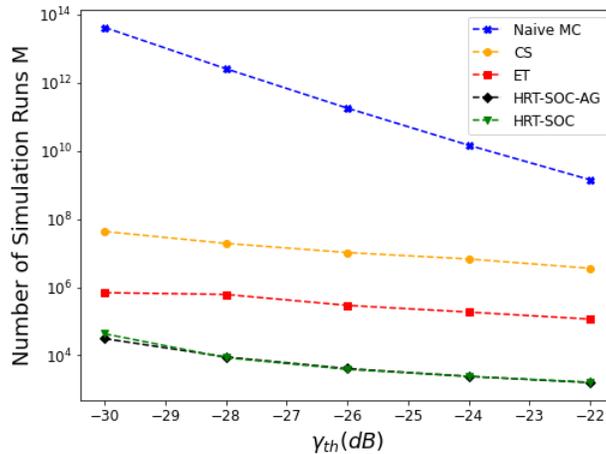


Figure 7: Number of required simulation runs with parameters : $N = 10$, $K = 20$, $S = 40$, $\text{TOL} = 0.05$, $\eta = -10 \text{ dB}$, $m_0 = 10 \text{ dB}$, $\sigma_0 = 4 \text{ dB}$, $m = 0 \text{ dB}$, $\sigma = 4 \text{ dB}$.

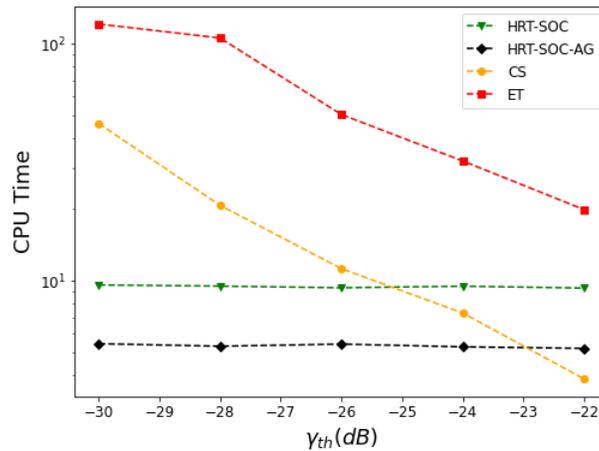


Figure 8: CPU time required for 5% relative error with parameters : $N = 10$, $K = 20$, $S = 40$, $\text{TOL} = 0.05$, $\eta = -10 \text{ dB}$, $m_0 = 10 \text{ dB}$, $\sigma_0 = 4 \text{ dB}$, $m = 0 \text{ dB}$, $\sigma = 4 \text{ dB}$.

In a last experiment, we aim to study the impact of varying the accuracy TOL on our approaches as well as on the other IS approaches. To this end, we respectively plot in Figure 9 and Figure 10 the number of simulation runs and the CPU time needed when varying TOL for a fixed γ_{th} and N . With this choice, the OP is approximately equal 10^{-7} .

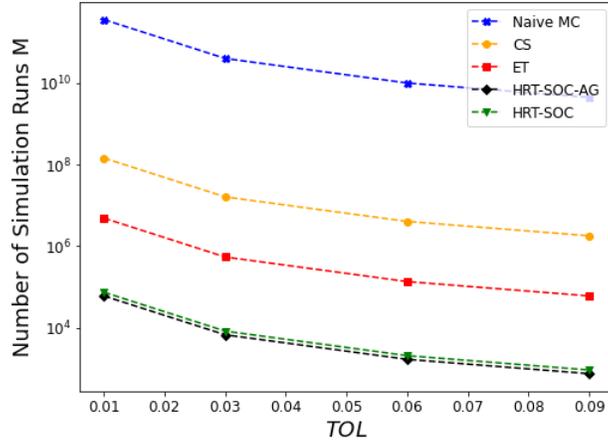


Figure 9: Number of required simulation runs with parameters : $N = 10$, $K = 20$, $S = 40$, $\gamma_{\text{th}} = -24 \text{ dB}$, $\eta = -10 \text{ dB}$, $m_0 = 10 \text{ dB}$, $\sigma_0 = 4 \text{ dB}$, $m = 0 \text{ dB}$, $\sigma = 4 \text{ dB}$.

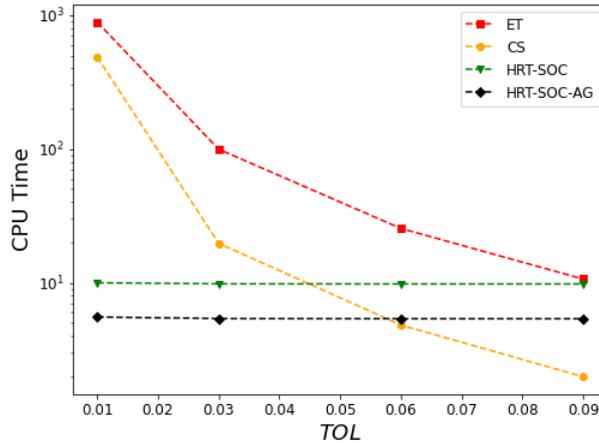


Figure 10: CPU time required for 5% relative error with parameters : $N = 10$, $K = 20$, $S = 40$, $\gamma_{\text{th}} = -24 \text{ dB}$, $\eta = -10 \text{ dB}$, $m_0 = 10 \text{ dB}$, $\sigma_0 = 4 \text{ dB}$, $m = 0 \text{ dB}$, $\sigma = 4 \text{ dB}$.

Figure 9 confirms again the high gains of our proposed methods compared to all other IS approaches. As a matter of fact, our approaches are 2000 times (respectively 65 times) more efficient than the CS (respectively the ET) approaches for all values of TOL. Furthermore, we observe from Figure 10 that the required time of our methods compared to the other algorithms remains unchanged for the considered range of TOL. Moreover, similarly to the previous conclusions,

the computational time needed by our proposed algorithm is less than that needed by the the ET algorithm for all TOL. Additionally, the outperformances of our method compared to the CS approach are more important for a small values of TOL. Finally, the HRT-SOC-AG method increases the threshold, below which our proposed method performs better than the CS approach, from 0.045 to 0.058.

5 Conclusions

We developed a generic state dependent IS algorithm in order to efficiently estimate rare events quantities that could be written in a form of an expectation of some functional of sums of independent RVs. These problems find their applications in the performance analysis of wireless communications systems operating over fading channels. Within a pre-selected class of change of measures, the optimal IS parameters are determined via a connection to a SOC formulation. Our numerical experiments verified the ability of the proposed approach to accurately and efficiently estimate the quantity of interest in the rare event regime. It was shown numerically that our proposed approach yields a substantial amount of variance reduction compared to other well-known estimators. Additionally, our estimator requires less CPU time than the other proposed approaches in the rare regions. We also proposed an aggregate method to further improve the efficiency in terms of computational time.

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