A DUALITY APPROACH TO A PRICE FORMATION MFG MODEL

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Abstract. We study the connection between the Aubry-Mather theory and a mean-field game (MFG) price-formation model. We introduce a framework for Mather measures that is suited for constrained time-dependent problems in $\mathbb{R}$. Then, we propose a variational problem on a space of measures, from which we obtain a duality relation involving the MFG problem examined in [36].

1. Introduction

This paper studies the connection between Aubry-Mather theory and certain mean-field games (MFG) that model price formation. More precisely, we consider the MFG system

$$\begin{cases} -u_t(t,x) + H(x, \varpi(t) + u_x(t,x)) = 0 & (t,x) \in [0,T] \times \mathbb{R} \\ m_t(t,x) - \big(H_p(x,\varpi(t) + u_x(t,x))m(t,x)\big)_x = 0 & (t,x) \in [0,T] \times \mathbb{R} \\ -\int_{\mathbb{R}} H_p(x,\varpi(t)+u_x(t,x))m(t,x)dx = Q(t) & t \in [0,T] \end{cases}, \tag{1.1}$$

subject to initial-terminal conditions

$$\begin{cases} u(T,x) = u_T(x) \\ m(0,x) = m_0(x) \end{cases} \qquad x \in \mathbb{R}, \tag{1.2}$$

where Q, u_T, and m_0 are given functions, m_0 is a probability measure on $\mathbb{R}$, and the triplet (u, m, ϖ) is the unknown. Here, the state of a typical agent is the variable $x \in \mathbb{R}$ and represents the assets of that agent. The distribution of assets in the population of the agents at time t is encoded in the probability measure $m(\cdot, t)$. The agents change their assets by trading at a price $\varpi(t)$. The trading is subject to a balance condition encoded in the third equation in (1.1). This integral constraint that guarantees supply $Q(t)$ meets demand is represented by the term on the left-hand side of that condition.

As introduced in [36], u is the value function of an agent who trades a commodity with supply Q and price ϖ. The function u is characterized by the first equation in (1.1) and the terminal condition in (1.2). Each agent selects their trading rate in order to minimize a given cost functional (see (1.7) below). The optimal control selection is $-H_p(x,\varpi(t) + u_x(t,x))$. Under this optimal control, the density m describing the population of agents evolves according to the second equation in (1.1) and the initial condition in (1.2). The third equation in (1.1), which we refer to as the balance condition, is an integral constraint that guarantees supply meets demand.

Remark 1.1. The notion of solutions of (1.1) and (1.2) we consider is the following: $u \in C([0,T] \times \mathbb{R})$ solves the first equation in the viscosity sense, $m \in C([0,T], \mathcal{P}(\mathbb{R}))$ solves the second equation in the distributional sense, and $\varpi \in C([0,T])$.

The system (1.1) and (1.2) corresponds to the case $\epsilon = 0$ studied in [36]. Under Assumptions 4, 6 and 7 (see Section 2), the authors used a fixed-point argument and the vanishing viscosity method to prove the existence of a solution (u, m, ϖ), where u is Lipschitz and semiconcave in x, and differentiable m-almost everywhere, $m \in C([0,T], \mathcal{P}(\mathbb{R}))$ w.r.t. the 1-Wasserstein distance, and $\varpi \in W^{1,1}([0,T])$. Furthermore, under Assumption

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8, they obtained uniqueness of solutions, further differentiability of u in x for every x, and the boundedness of u_{xx} and m.

The connection between Hamilton-Jacobi equations and Aubtry-Mather theory is now well established; see, for example, [42], [19, 20, 21, 22], [17, 18, 5], or [46]. In particular, several generalizations of Aubry-Mather theory were developed to address problems like diffusions and study second-order Hamilton-Jacobi equations [30, 31]. In particular, duality methods, since the pioneering papers in [41] and [24] have been explored in multiple contexts, see for example [37]. Of great interest are the applications to the selection problem in the vanishing discount case, [32], [45] [35] and [34] and to the large time behavior of Hamilton-Jacobi equations [9], [38]. Recently applications of Aubry-Mather theory were developed for MFGs in [12] to study long-time behavior, and in [11], where the authors construct Mather measures to prove the existence of solutions for ergodic first-order MFG systems with state constraints.

The prototype MFG system corresponds to an optimal control problem for an agent who optimizes a cost function that depends on the aggregate behavior of other agents encoded in the population distribution m. In [36], the optimal control setting of the MFG system (1.1) and (1.2) corresponds to an agent interacting with the population through the price. At the same time, the balance condition between demand and supply is satisfied. This type of interaction arises in price formation models, where the commodity price being traded is an endogenous rather than an exogenous variable.

Price formation models were studied previously in [3] and [47] in the contex of revenue maximization by a producer. Earlier price models in the context of mean-field games include [8, 7, 43, 6] and [40]. Applications to electricity markets were examined in [44, 1, 23] and [14]. Price models with a market clearing condition were introduced in [36], [2], [33], [48] and [27]. The former work addresses a model for solar renewable energy certificate markets. Finally, [28] examines the effect of a major player.

The variational problem that we consider is a relaxed version of the Lagrangian formulation introduced in [36] to derive (1.1) and (1.2). We prove a duality formula (Theorem 1.3) between solutions of the MFG system and minimizers of a variational problem in the set of generalized Mather measures. For that, we begin by introducing the Legendre transform, L, of H; that is,

$$L(x,v)=\sup_{p\in\mathbb{R}}\left\{-pv-H(x,p)\right\}. \tag{1.3}$$

Our variational problem is

$$\inf_{\mu\in\mathcal{H}(m_0)}\int_0^T\int_{\mathbb{R}^2} L(x,v)+vu_T'(x)\ d\mu(t,x,v), \tag{1.4}$$

where $\mathcal{H}(m_0)$ is the set of admissible measures. These measures are Radon positive measures on $[0,T]\times\mathbb{R}^2$ that satisfy the following three conditions. First, the moment condition

$$\int_{[0,T]\times\mathbb{R}^2}(|x|^{\zeta_1}+|v|^{\zeta_2})d\mu(t,x,v)<\infty,$$

where ζ_1 and ζ_2 depend on the growth of the Hamiltonian in Assumption 2 and satisfy condition 3.1. Second, for some probability measure ν on $\mathbb{R}$, the Radon measure verifies

$$\int_{[0,T]\times\mathbb{R}^2}\varphi_t(t,x)+v\varphi_x(t,x)d\mu(t,x,v)=\int_{\mathbb{R}}\varphi(T,x)d\nu-\int_{\mathbb{R}}\varphi(0,x)dm_0 \tag{1.5}$$

for all suitable test functions φ. We refer to the previous as the holonomy condition, as it is motivated by the holonomy condition introduced in [42]. Lastly, the admissible measures satisfy the following balance condition

$$\int_{[0,T]\times\mathbb{R}^2}\eta(t)(v-Q(t))d\mu(t,x,v)=0$$

for all η continuous. If $u_T \in C^1(\mathbb{R})$ with u_T' bounded (see Assumption 3), the holonomy condition applied to $\varphi(t,x) = u_T(x)$ (see (3.3)) provides the identity

$$\int_{[0,T]\times\mathbb{R}^2} v u_T'(x) d\mu(t,x,v) = \int_{\mathbb{R}} u_T(x) d\nu - \int_{\mathbb{R}} u_T(x) dm_0.$$

Using the previous identity, the variational problem (1.4) is equivalent to

$$\inf_{\substack{\nu\in\mathcal{P}(\mathbb{R})\\ \mu\in\mathcal{H}(m_0,\nu)}} \int_0^T \int_{\mathbb{R}^2} L(x,v)\ d\mu(t,x,v) + \int_{\mathbb{R}} u_T(x) d\nu(x), \tag{1.6}$$

where $\mathcal{H}(m_0,\nu)$ is the set of measures that satisfy the moment condition, the holonomy condition for some probability measure ν on $\mathbb{R}$, and the balance condition. The difference between (1.6) and (1.4) is the term $-\int_{\mathbb{R}} u_T(x) dm_0$, which is independent of μ.

The motivation for this relaxed problem is as follows. In [36], each agent selects a control variable α aiming to solve

$$\inf_{\alpha\in\mathcal{A}} \int_0^T L(x(t),\alpha(t)) + \varpi(t)\alpha(t) dt + u_T(x(T)), \tag{1.7}$$

where $\dot{x}(t) = \alpha(t)$, and $\mathcal{A}$, the set of bounded measurable functions, is the set of admissible controls. The price ϖ is chosen so that the aggregate supply meets the demand. Here, following Mather's theory (see for example [32]), we introduced a relaxed version of problem (1.7). This relaxation is problem (1.6). The key idea is that each optimal trajectory $t \mapsto x^*(t)$ with optimal control $t \mapsto \alpha^*(t)$ solving (1.7) defines the measure $d\mu^*(t,x,v) = dt \times d_{\delta_{x^*(t)}} \times d_{\delta_{\alpha^*(t)}}$. This measure is supported on the path $t \mapsto (x^*(t), \alpha^*(t))$ and satisfies (1.5). Accordingly, the function in (1.7) becomes

$$\int_0^T \int_{\mathbb{R}^2} L(x,v) + \varpi v\ d\mu^*(t,x,v) + \int_{\mathbb{R}} u_T(x) d\nu^*(x)$$

where $d\nu^* = \delta_{x^*(T)}$; that is, the variational cost for the measure equals the variational cost for the optimal trajectory.

Our first result for the variational problem on measures (1.6) is a duality formula between minimizing measures and Hamilton-Jacobi equations that involves the following function. Let $h : \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ be

$$h(m_0,\nu) = \begin{cases} \inf\limits_{\mu\in\mathcal{H}(m_0,\nu)} \int_\Omega L(x,v) + v u_T'(x) d\mu(t,x,v), & \text{if } \mathcal{H}(m_0,\nu) \neq \emptyset, \\ +\infty & \text{if } \mathcal{H}(m_0,\nu) = \emptyset. \end{cases} \tag{1.8}$$

The main assumptions on L and u_T are stated in Section 2, after which, in Section 3, we develop a framework of Mather measures suitable for the MFG system (1.1) and (1.2). Finally, in that section, we prove the following theorem.

Theorem 1.2. *Let h be given by* (1.8) *and let ζ satisfy* (2.1). *Suppose Assumptions 1-4 hold. Assume that $\nu_T \in \mathcal{P}(\mathbb{R})$ is such that $\mathcal{H}(m_0,\nu_T) \neq \emptyset$. Then,*

$$h(m_0,\nu_T) = -\inf_{\varphi,\eta} \sup_{(t,x)} \Bigg(-\int_{\mathbb{R}} \varphi(0,x) dm_0(x) + \int_{\mathbb{R}} \varphi(T,x) d\nu_T(x) + T\big(-\varphi_t + Q\eta + H(x, \varphi_x + \eta + u_T')\big)\Bigg),$$

where $(t,x) \in [0,T]\times\mathbb{R}$, $\varphi \in \Lambda([0,T]\times\mathbb{R})$ and $\eta \in C([0,T])$.

The previous result is proved in Section 3 using Fenchel-Rockafellar's duality theorem.

Next, in Section 4, we establish additional results for the MFG system (1.1) and (1.2). In particular, in Proposition 4.2, we prove that ϖ solving (1.1) and (1.2) is Lipschitz continuous. This result was stated but not proved in [36]. Here, we give the full details of the proof.

Finally, in Section 5, we establish our main result, which is summarized in the following theorem.

Theorem 1.3. *Let* (u, m, ϖ) *solve* (1.1) *and* (1.2). *Suppose that Assumptions 1-8 hold. Then,*

$$\int_{\mathbb{R}} (u(0,x) - u_T(x))\, dm_0(x) - \int_0^T Q(t)\varpi(t)dt = \inf_{\mu \in \mathcal{H}(m_0)} \int_\Omega L(x,v) + v u_T'(x) d\mu(t,x,v).$$

In the previous theorem, the value of (1.4) is characterized by the solution of the MFG system (1.1) and (1.2). Although m does not appear explicitly on the right-hand side of the previous expression, it determines the balance condition for the MFG. Notice that for this minimization problem, u_T is fixed, whereas the terminal measure ν_T is varying (see Section 3).

2. Assumptions

Here, we present the main assumptions used in this paper. First, we consider the usual convexity assumption on the Hamiltonian, H, for which we require the strongest form of this property.

Assumption 1. *For all* $x \in \mathbb{R}$, *the map* $p \mapsto H(x,p)$ *is uniformly convex; that is, there exists a constant* $\kappa > 0$ *such that* $H_{pp}(x,p) \geqslant \kappa$ *for all* $(x,p) \in \mathbb{R}^2$.

The previous assumption guarantees not only convexity but also coercivity of H in the p variable (see [4], Corollary 11.17). Hence, the Legendre transform of H, given by (1.3), is well-defined, and it is convex and coercive in the second argument ([10], Theorem A.2.6).

The following four assumptions are used in Section 3 to establish duality results. The following growth conditions for H and the regularity for u_T and Q are required used when we apply Fenchel-Rockafellar's theorem.

Assumption 2. *There exists* $\gamma_1 \geqslant 1$, $\gamma_2 > 1$, *a positive constant* C, *and non-negative constants* C_1 *and* C_2 *such that, for all* $(x,p) \in \mathbb{R}^2$,

$$\begin{cases} -C_2|x|^{\gamma_1} + \dfrac{|p|^{\gamma_2}}{C\gamma_2} - C \leqslant H(x,p) \leqslant -C_1|x|^{\gamma_1} + \dfrac{|p|^{\gamma_2}C}{\gamma_2} + C, \\ |H_x(x,p)| \leqslant C(|p|^{\gamma_2} + 1), \\ |H_p(x,p)| \leqslant C(|p|^{\gamma_2 - 1} + 1). \end{cases}$$

Remark 2.1. Under Assumption 1, the Lagrangian, L, defined by (1.3), satisfies (see [10], Theorem A.2.6)

$$v = -H_p(x,p) \text{ if and only if } p = -L_v(x,v).$$

Furthermore, Assumption 2 implies a growth condition on L; that is,

$$C_1|x|^{\gamma_1} + \frac{|v|^{\gamma_2'}}{\gamma_2' C^{\gamma_2'/\gamma_2}} - C \leqslant L(x,v) \leqslant C_2|x|^{\gamma_1} + \frac{|v|^{\gamma_2'}C^{\gamma_2'/\gamma_2}}{\gamma_2'} + C, \tag{2.1}$$

where $1/\gamma_2 + 1/\gamma_2' = 1$. To see this, note that the first condition in Assumption 2 bounds the Legendre transform of H between the one of the functions

$$p \mapsto -C_1|x|^{\gamma_1} + \frac{|p|^{\gamma_2}C}{\gamma_2} + C \quad \text{and} \quad p \mapsto -C_2|x|^{\gamma_1} + \frac{|p|^{\gamma_2}}{C\gamma_2} - C.$$

Their transforms are the lower and upper bounds in (2.1), respectively.

Assumption 3. *The terminal cost satisfies* $u_T \in C^1(\mathbb{R})$, *and* $|u_T'| \leqslant C$ *for some* $C > 0$.

For the supply, we assume it is a smooth function of time.

Assumption 4. *The supply function,* Q, *is* $C^\infty([0,T])$.

The existence of generalized measures minimizing our variational problem (1.4) relies on the moment estimates that we impose for the initial distribution (see Proposition 5.2).

Assumption 5. *The initial density, m_0, is a probability measure in $\mathbb{R}$, and it has a finite absolute moment of order $\gamma > \gamma_1$; that is,*

$$\int_{\mathbb{R}} |x|^\gamma m_0(x)\mathrm{d}x < +\infty.$$

Following [36], we guarantee the solvability of (1.1) and (1.2) by considering, together with Assumption 4, the following conditions.

Assumption 6. *The Hamiltonian H is separable; that is,*

$$H(x,p) = \mathcal{H}(p) - V(x),$$

where $V \in C^2(\mathbb{R})$ is bounded from below and $|\mathcal{H}_{pp}|, |\mathcal{H}_{ppp}| \leqslant C$ for some constant $C > 0$.

Remark 2.2. Under the previous assumption, L, defined by (1.3), is separable as well; that is

$$L(x,v) = \mathcal{L}(v) + V(x),$$

where $\mathcal{L}$ is the Legendre transform of $\mathcal{H}$. Recalling that the Legendre transform is an involutive transformation, in case that $\mathcal{L}$ is uniformly convex, we have $\mathcal{L}_{vv} \geqslant \kappa'$ for some $\kappa' > 0$. Hence, ([10], Corollary A. 2.7)

$$\mathcal{H}_{pp} \leqslant \frac{1}{\kappa'}.$$

Furthermore, under Assumption 1, we obtain $\kappa < \mathcal{H}_{pp} \leqslant 1/\kappa'$. By abuse of notation, we set $\mathcal{H} = H$ and $\mathcal{L} = L$ when Assumption 6 holds.

Assumption 7. *The potential V, the terminal cost u_T, the initial density function m_0 are $C^2(\mathbb{R})$ functions and V, u_T are globally Lipschitz. Furthermore, there exists a constant $C > 0$ such that*

$$|V''| \leqslant C, \quad |u_T''| \leqslant C, \quad |m_0''| \leqslant C.$$

The following condition guarantees the uniqueness of solutions of (1.1) and (1.2).

Assumption 8. *The potential V and the terminal cost u_T are convex.*

Remark 2.3. Assume the Hamiltonian, H, satisfies Assumption 6, with a potential, V, satisfying Assumption 7. For Assumption 2 to hold, V has to satisfy $C \geqslant \mathrm{Lip}(V)$ and the growth condition

$$C_1|x|^{\gamma_1} - K \leqslant V(x) \leqslant C_2|x|^{\gamma_1} + K \tag{2.2}$$

for some $K > 0$, whereas $\mathcal{H}$ has to satisfy $|\mathcal{H}_p(p)| \leqslant C(|p|^{\gamma_2 - 1} + 1)$ and the growth condition

$$\frac{|p|^{\gamma_2}}{C\gamma_2} - C \leqslant H(p) \leqslant \frac{|p|^{\gamma_2}C}{\gamma_2} + C.$$

For instance, the Hamiltonian

$$H(x,p) = \left(1 + |p|^2\right)^{\frac{\gamma_2}{2}} - V(x)$$

satisfies all the assumptions above if V is a globally Lipschitz function that satisfies (2.2).

3. Duality results

This section considers generalized holonomic measures for time-dependent problems in $\mathbb{R}$ that are compatible with the integral constraint imposed by the balance condition. We use this formulation to prove Theorem 1.2 and for the proof of Theorem 1.3 in Section 5.

Fix $T > 0$. For $\gamma_1 \geqslant 1$ and $\gamma_2 > 1$ (see Assumption 2), let $\zeta = (\zeta_1, \zeta_2)$, where

$$0 < \zeta_1 \leqslant \gamma_1, \quad \text{and} \quad 1 < \zeta_2 < \gamma_2'. \tag{3.1}$$

Let $\Omega = [0,T] \times \mathbb{R} \times \mathbb{R}$. Let $\mathcal{R}(\Omega)$ be the set of signed Radon measures on Ω, $\mathcal{R}^+(\Omega)$ be the subset of non-negative elements of $\mathcal{R}(\Omega)$ ([25], page 212 and 222 or [15], Definition 1.9), and $\mathcal{P}(\mathbb{R})$ be the set of probability measures on $\mathbb{R}$. We define

$$\mathcal{H}_1 = \left\{ \mu \in \mathcal{R}^+(\Omega) : \int_\Omega (|x|^{\zeta_1} + |v|^{\zeta_2}) d\mu(t,x,v) < \infty \right\}. \tag{3.2}$$

This set is determined by the growth conditions for the Hamiltonian, as in Assumption 2. Next, let

$$\Lambda([0,T]\times\mathbb{R}) = \left\{ \varphi \in C^1([0,T]\times\mathbb{R}) : \ \varphi_t, \ \varphi_x \in L^\infty([0,T]\times\mathbb{R}) \right\}.$$

Notice that elements of $\Lambda([0,T]\times\mathbb{R})$ are globally Lipschitz continuous functions. This set corresponds to the set of test functions for the holonomy condition, which we define next. Given $m_0, \ \nu \in \mathcal{P}(\mathbb{R})$, let

$$\mathcal{H}_2(m_0,\nu) = \left\{ \mu \in \mathcal{R}^+(\Omega) : \begin{array}{l} \int_\Omega \varphi_t(t,x) + v\varphi_x(t,x) d\mu(t,x,v) \\ = \int_\mathbb{R} \varphi(T,x)d\nu - \int_\mathbb{R} \varphi(0,x)dm_0 \end{array} \quad \forall \varphi \in \Lambda([0,T]\times\mathbb{R}) \right\}. \tag{3.3}$$

As mentioned in the Introduction, we refer to the condition defining the set $\mathcal{H}_2(m_0,\nu)$ as the holonomy condition. For a given $\nu \in \mathcal{P}(\mathbb{R})$, the set $\mathcal{H}_2(m_0,\nu)$ may be empty. Nevertheless, as we show in Remark 3.2, there are probability measures satisfying $\mathcal{H}_2(m_0,\nu) \neq \emptyset$. In case m_0 satisfies a moment hypothesis (see Assumption 5), the identity that defines the holonomy condition is well-defined even if the terms are not finite.

Corresponding to the balance condition in (1.1), we set

$$\mathcal{H}_3 = \left\{ \mu \in \mathcal{R}^+(\Omega) : \int_\Omega \eta(t)(v - Q(t)) d\mu(t,x,v) = 0, \quad \forall \eta \in C([0,T]) \right\}. \tag{3.4}$$

Finally, we define

$$\mathcal{H}(m_0,\nu) := \mathcal{H}_1 \cap \mathcal{H}_2(m_0,\nu) \cap \mathcal{H}_3, \quad \text{and} \quad \mathcal{H}(m_0) = \bigcup_{\nu\in\mathcal{P}(\mathbb{R})} \mathcal{H}(m_0,\nu).$$

Remark 3.1. For any $\mu \in \mathcal{H}(m_0)$ there exists a unique $\nu \in \mathcal{P}(\mathbb{R})$ such that $\mu \in \mathcal{H}(m_0,\nu)$. To see this, let $\mu \in \mathcal{H}(m_0,\nu) \cap \mathcal{H}(m_0,\tilde{\nu})$. Let $\varphi \in C_c^1(\mathbb{R})$. Then, $\varphi \in \Lambda([0,T]\times\mathbb{R})$, and (3.3) holds for both ν and $\tilde{\nu}$, from which we obtain

$$\int_\mathbb{R} \varphi(x)d\nu(x) - \int_\mathbb{R} \varphi(x)dm_0(x) = \int_\mathbb{R} \varphi(x)d\tilde{\nu}(x) - \int_\mathbb{R} \varphi(x)dm_0(x);$$

that is, $\int_\mathbb{R} \varphi d\nu = \int_\mathbb{R} \varphi d\tilde{\nu}$ for any $\varphi \in C_c^1(\mathbb{R})$. Hence, $\nu = \tilde{\nu}$ ([25], Theorem 7.2). We denote the unique measure ν such that $\mu \in \mathcal{H}(m_0,\nu)$ as ν^μ.

Remark 3.2. If Assumptions 4 and 5 hold, $\mathcal{H}(m_0)$ is not empty. To see this, let $\overline{X}(t) = \int_0^t Q(s)ds$. Define $\nu \in \mathcal{P}(\mathbb{R})$ by

$$\int_\mathbb{R} f(x)d\nu(x) = \int_\mathbb{R} f(x + \overline{X}(T))dm_0(x) \tag{3.5}$$

for all $f \in C_c(\mathbb{R})$, and define $\mu \in \mathcal{R}^+(\Omega)$ by

$$\int_\Omega \psi(t,x,v)d\mu(t,x,v) = \int_0^T \int_\mathbb{R} \psi(t, x + \overline{X}(t), Q(t))dm_0(x)dt \tag{3.6}$$

for all $\psi \in C_c(\Omega)$. Next, we use the following cut-off function

$$\theta(x) = \begin{cases} 1 & x \in (-1,1) \\ \vartheta(x) & (-2,2)\setminus(-1,1) \\ 0 & x \in \mathbb{R}\setminus(-2,2), \end{cases}$$

where ϑ is chosen such that $\theta \in C^1(\mathbb{R})$, $0 \leqslant \theta \leqslant 1$, and $||\theta||_{C^1(\mathbb{R})} \leqslant c$. Let

$$h_n(x,v) = \theta\left(\frac{2x}{n}\right)\theta\left(\frac{2v}{n}\right)\left(|x|^{\zeta_1} + |v|^{\zeta_2}\right) \text{ and } g_n = \mathbb{1}_n(x)\mathbb{1}_n(v)\left(|x|^{\zeta_1} + |v|^{\zeta_2}\right),$$

where $\mathbb{1}_n$ is the characteristic function of the interval $[-n,n]$. g_n is a sequence of measurable functions that satisfy $0 \leqslant g_n(x,v) \leqslant |x|^{\zeta_1} + |v|^{\zeta_2}$ and $g_{\frac{n}{2}}(x,v) \to |x|^{\zeta_1} + |v|^{\zeta_2}$ pointwise for

$(x,y)\in\mathbb{R}^2$. Although the functions g_n are not continuous, they are Borel-measurable, and hence their integral w.r.t. μ is well-defined. Note that $g_{\frac{n}{2}}(x,v)\leqslant h_n(x,v)\leqslant g_n(x,v)$ and $h_n\in C_c(\Omega)$. Then,

$$\begin{aligned}\int_\Omega g_{\frac{n}{2}}(x,v)d\mu(t,x,v) &\leqslant \int_\Omega h_n(x,v)d\mu(t,x,v)\\ &= \int_0^T\int_{\mathbb{R}} h_n(x+\overline{X}(t),Q(t))dm_0(x)dt\\ &\leqslant \int_0^T\int_{\mathbb{R}} g_n(x+\overline{X}(t),Q(t))dm_0(x)dt\\ &\leqslant \int_0^T\int_{\mathbb{R}} |x+\overline{X}(t)|^{\zeta_1}+|Q(t)|^{\zeta_2}dm_0(x)dt\\ &\leqslant \int_0^T\int_{\mathbb{R}} 2^{\zeta_1-1}\left(|x|^{\zeta_1}+|\overline{X}(t)|^{\zeta_1}\right)+|Q(t)|^{\zeta_2}dm_0(x)dt\\ &= 2^{\zeta_1-1}\left(T\int_{\mathbb{R}}|x|^{\zeta_1}dm_0(x)+\|\overline{X}\|^{\zeta_1}_{L^{\zeta_1}([0,T])}\right)+\|Q\|^{\zeta_2}_{L^{\zeta_2}([0,T])}\\ &\leqslant 2^{\zeta_1-1}T\left(\int_{\mathbb{R}}|x|^{\zeta_1}dm_0(x)+T^{\zeta_1}\|Q\|^{\zeta_1}_\infty\right)+T\|Q\|^{\zeta_2}_\infty\\ &= C(\zeta_1,\zeta_2,T,m_0,Q),\end{aligned}$$

where $C(\zeta_1,\zeta_2,T,m_0,Q)$ is finite by Assumptions 4 and 5. Using the previous inequality and the Monotone Convergence Theorem, we conclude that μ satisfies (3.2). Therefore, for any $\varphi\in\Lambda([0,T]\times\mathbb{R})$, we have

$$\begin{aligned}\int_\Omega |\varphi_t(t,x)|+|v||\varphi_x(t,x)|d\mu(t,x,v)<\infty,\\ \int_{\mathbb{R}}|\varphi(T,x+\overline{X}(T))|dm_0(x)+\int_{\mathbb{R}}|\varphi(0,x)|dm_0(x)<\infty.\end{aligned}\tag{3.7}$$

Denote $\overline{M}=\max_{t\in[0,T]}|\overline{X}(t)|$ and let $\phi^n(t,x,v)=\varphi^n(t,x)\theta\left(\frac{v}{n}\right)$, where $\varphi^n(t,x)=\varphi(t,x)\theta\left(\frac{x}{n}\right)$. Because $\varphi\in\Lambda([0,T]\times\mathbb{R})$, from the definitions of ϕ^n, φ^n, and θ, we have

$$\begin{aligned}\max_{(t,x)\in[0,T]\times\mathbb{R}}|\varphi^n(t,x)| &= \max_{(t,x)\in[0,T]\times\mathbb{R}}\left|\varphi(t,x)\theta\left(\tfrac{x}{n}\right)\right|\leqslant Cn,\\ \max_{(t,x,v)\in\Omega}|\phi^n_t(t,x,v)|\leqslant\max_{(t,x)\in\Omega}|\varphi^n_t(t,x)| &= \max_{(t,x)\in[0,T]\times\mathbb{R}}\left|\varphi_t(t,x)\theta\left(\tfrac{x}{n}\right)\right|\leqslant C,\\ \|\phi^n_x\|_{C(\Omega)}\leqslant\max_{(t,x)\in[0,T]\times\mathbb{R}}|\varphi^n_x(t,x)| &= \max_{(t,x)\in[0,T]\times\mathbb{R}}\left|\varphi_x(t,x)\theta\left(\tfrac{x}{n}\right)+\tfrac{1}{n}\varphi(t,x)\theta'\left(\tfrac{x}{n}\right)\right|\leqslant C.\end{aligned}\tag{3.8}$$

Relying on these estimates from Assumption 5, we have

$$\begin{aligned}&\left|\int_0^T\int_{n<|x|<2n}\int_{n<|v|<2n}\phi^n_t+v\phi^n_x d\mu(t,x,v)\right|\\ &\leqslant\int_0^T\int_{n<|x|<2n}\int_{n<|v|<2n}C(|v|+1)d\mu(t,x,v)\\ &\leqslant\int_\Omega C(|v|+1)\left(\theta\left(\frac{2x}{n}-3\right)+\theta\left(\frac{2x}{n}+3\right)\right)\left(\theta\left(\frac{2v}{n}-3\right)+\theta\left(\frac{2v}{n}+3\right)\right)d\mu(t,x,v)\\ &\leqslant\int_0^T\int_{\mathbb{R}}2C(|Q(t)|+1)\left(\theta\left(\frac{2(x+\overline{X}(t))}{n}-3\right)+\theta\left(\frac{2(x+\overline{X}(t))}{n}+3\right)\right)dm_0(x)\\ &\leqslant TC\int_{\frac{n}{2}-\overline{M}<|x|<\frac{5}{2}n+\overline{M}}dm_0(x)=o(1).\end{aligned}\tag{3.9}$$

Furthermore, Assumption 5 with (3.8) implies that

$$\left|\left(\int_n^{2n-\overline{X}(T)}+\int_{-2n-\overline{X}(T)}^{-n}\right)\varphi^n(T,x+\overline{X}(T))dm_0(x)\right|$$

$$\leqslant \int_{n-\overline{M}<|x|<2n+\overline{M}} \left|\varphi^n(T, x+\overline{X}(T))\right| dm_0(x) \leqslant C \int_{n-\overline{M}<|x|<2n+\overline{M}} |x| dm_0(x) = o(1),$$

$$\left|\int_{n<|x|<2n} \varphi^n(0,x) dm_0(x)\right| \leqslant C \int_{n<|x|<2n} |x| dm_0(x) = o(1), \tag{3.10}$$

where $o(1) \to 0$ when $n \to \infty$. Note that $\phi^n \in C_c(\Omega)$ and $\operatorname{supp}(\phi^n) = [0,T] \times [-2n, 2n]^2$. Consequently, for all $n \geqslant n_0$, where n_0 satisfies $\frac{\|Q\|_\infty}{n_0} \leqslant 1$, by (3.6), we have

$$\begin{aligned}
&\int_0^T \int_{-2n}^{2n} \int_{-2n}^{2n} \phi_t^n(t,x,v) + v\phi_x^n(t,x,v) d\mu(t,x,v) \\
&= \int_0^T \int_{\mathbb{R}} \phi_t^n(t, x+\overline{X}(t), Q(t)) + Q(t)\phi_x^n(t, x+\overline{X}(t), Q(t)) dm_0(x) \mathrm{d}t \\
&= \int_0^T \int_{\mathbb{R}} \theta\left(\frac{Q(t)}{n}\right) \left(\varphi_t^n(t, x+\overline{X}(t)) + Q(t)\varphi^n(t, x+\overline{X}(t))\right) dm_0(x) dt \\
&= \int_{\mathbb{R}} \int_0^T \frac{d}{dt}\varphi^n(t, x+\overline{X}(t)) dm_0(x) dt \\
&= \int_{-2n-\overline{X}(T)}^{2n-\overline{X}(T)} \varphi^n(T, x+\overline{X}(T)) dm_0(x) - \int_{-2n}^{2n} \varphi^n(0,x) dm_0(x).
\end{aligned} \tag{3.11}$$

On the other hand, (3.9) and (3.10), yield

$$\begin{aligned}
&\int_0^T \int_{n<|x|<2n} \int_{n<|x|<2n} \phi_t^n(t,x) + v\phi_x^n(t,x) d\mu(t,x,v) \\
&- \left(\int_n^{2n-\overline{X}(T)} + \int_{-2n-\overline{X}(T)}^{-n}\right) \varphi^n(T, x+\overline{X}(T)) dm_0(x) + \int_{n<|x|<2n} \varphi^n(0,x) dm_0(x) = o(1).
\end{aligned}$$

Therefore, (3.11) with the definitions of ϕ^n, φ^n, θ implies

$$\begin{aligned}
&\int_0^T \int_{-n}^{n} \int_{-n}^{n} \varphi_t(t,x) + v\varphi_x(t,x) d\mu(t,x,v) \\
&\qquad = \int_{-n}^{n} \varphi(T, x+\overline{X}(T)) dm_0(x) - \int_{-n}^{n} \varphi(0,x) dm_0(x) + o(1).
\end{aligned} \tag{3.12}$$

With similar arguments, by using (3.5), we prove that

$$\begin{aligned}
&\int_{-n}^{n} \varphi(T, x+\overline{X}(T)) dm_0(x) - \int_{-n}^{n} \varphi(0,x) dm_0(x) \\
&\qquad = \int_{-n}^{n} \varphi(T,x) d\nu(x) - \int_{-n}^{n} \varphi(0,x) dm_0(x) + o(1).
\end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$\begin{aligned}
&\int_0^T \int_{-n}^{n} \int_{-n}^{n} \varphi_t(t,x) + v\varphi_x(t,x) d\mu(t,x,v) \\
&\qquad = \int_{-n}^{n} \varphi(T,x) d\nu(x) - \int_{-n}^{n} \varphi(0,x) dm_0(x) + o(1).
\end{aligned}$$

Letting $n \to \infty$ in the preceding identity and using (3.7), we conclude that μ satisfies (3.3).

Lastly, proceeding as before, we prove that μ verifies (3.4). Hence, $\mu \in \mathcal{H}(m_0, \nu)$ and, therefore, $\mathcal{H}(m_0) \neq \emptyset$.

The minimization in (1.4) is an infinite-dimensional optimization problem. To study the connection between solutions of (1.1) and the dual problem of (1.4), we compute the dual problem using Fenchel-Rockafellar's theorem ([49], Theorem 1.9.):

Theorem 3.3. *Let E be a normed vector space and let E^* be its topological dual space. Let f and g be convex functions on E with values in $\mathbb{R} \cup \{+\infty\}$. Denote by f^* and g^* the Legendre-Fenchel transforms of f and g, respectively, defined by*

$$f^*(x^*) = \sup_{x\in E} \left(\langle x^*, x\rangle - f(x)\right), \quad g^*(x^*) = \sup_{x\in E} \left(\langle x^*, x\rangle - g(x)\right).$$

Assume there exists $x_0 \in E$ such that $f(x_0),\ g(x_0) < +\infty$, and f is continuous at x_0. Then

$$\inf_{x\in E} f(x) + g(x) = \max_{y\in E^*} -f^*(-y) - g^*(y). \tag{3.14}$$

In the previous result, it is part of the theorem that the supremum in the right-hand side of (3.14) is a maximum.

Now, we introduce the definitions we need to apply Theorem 3.3. Recall that $\Omega = [0,T] \times \mathbb{R} \times \mathbb{R}$, and let $\zeta = (\zeta_1, \zeta_2)$ according to (3.1). Consider the normed vector space

$$C_\zeta(\Omega) := \Bigg\{ \phi \in C(\Omega) :\ \|\phi\|_\zeta := \sup_{\Omega} \left| \frac{\phi(t,x,v)}{1+|x|^{\zeta_1}+|v|^{\zeta_2}} \right| < \infty, \tag{3.15}$$
$$\lim_{|x|,|v|\to\infty} \frac{\phi(t,x,v)}{1+|x|^{\zeta_1}+|v|^{\zeta_2}} = 0 \text{ uniformly for } t\in[0,T] \Bigg\}.$$

Remark 3.4. Let ζ satisfy (3.1). The dual of $(C_\zeta(\Omega), \|\cdot\|_\zeta)$ is

$$\mathcal{U}^\zeta = \left\{ \mu \in \mathcal{R}(\Omega) :\ \int_\Omega (1+|x|^{\zeta_1}+|v|^{\zeta_2}) d|\mu|(t,x,v) < \infty \right\}.$$

To see this, let

$$C_0(\Omega) := \left\{ \psi \in C(\Omega) :\ \lim_{|x|,|v|\to\infty} \psi(t,x,v) = 0, \text{ uniformly for } t \in [0,T] \right\}.$$

From the Riesz Representation Theorem ([25], Theorem 7.17), we have that

$$C_0(\Omega)^* \text{ and } \mathcal{R}(\Omega) \text{ are isomorphic.} \tag{3.16}$$

Define $\Phi : C_0(\Omega) \to C_\zeta(\Omega)$ by $\Phi(\psi) = \phi := (1+|x|^{\zeta_1}+|v|^{\zeta_2})\psi$. Then Φ is a linear isometry since $\|\Phi(\psi)\|_\zeta = \|\psi\|_\infty$. Now, given $f \in C_\zeta(\Omega)^*$, define $F \in C_0(\Omega)^*$ by $F = f \circ \Phi$. Using (3.16), there exists $\tilde{\mu} \in \mathcal{R}(\Omega)$ such that

$$\langle F, \psi\rangle = \int_\Omega \psi(t,x,v)\ d\tilde{\mu}(t,x,v)$$

for all $\psi \in C_0(\Omega)$. Given $\phi \in C_\zeta(\Omega)$, let $\psi = \Phi^{-1}(\phi) = \frac{\phi}{1+|x|^{\zeta_1}+|v|^{\zeta_2}} \in C_0(\Omega)$. Then

$$\langle f, \phi\rangle = \langle F, \psi\rangle = \int_\Omega \phi(t,x,v)\ \frac{d\tilde{\mu}(t,x,v)}{1+|x|^{\zeta_1}+|v|^{\zeta_2}}.$$

Hence, because $(x,v) \mapsto \frac{1}{1+|x|^{\zeta_1}+|v|^{\zeta_2}}$ is continuous and bounded, the measure $d\mu(t,x,v) := \frac{1}{1+|x|^{\zeta_1}+|v|^{\zeta_2}} d\tilde{\mu}(t,x,v)$ is a Borel measure finite on compact sets. Therefore ([25], Theorem 7.8), μ is a Radon measure on Ω. Notice that any Hahn, and therefore, Jordan decomposition of $\tilde{\mu}$ ([25], Theorem 3.4) provides a corresponding decomposition for μ, from which we obtain that $d|\mu| = \frac{1}{1+|x|^{\zeta_1}+|v|^{\zeta_2}} d|\tilde{\mu}|$. Therefore,

$$\int_\Omega (1+|x|^{\zeta_1}+|v|^{\zeta_2}) d|\mu|(t,x,v) = \int_\Omega d|\tilde{\mu}|(t,x,v) < \infty.$$

On the other hand, any $\mu \in \mathcal{U}^\zeta$ defines a linear map on $C_\zeta(\Omega)$ by

$$\phi \mapsto \int_\Omega \phi(t,x,v) d|\mu|.$$

From the following inequality

$$\left| \int_\Omega (1+|x|^{\zeta_1}+|v|^{\zeta_2}) \frac{\phi(t,x,v)}{1+|x|^{\zeta_1}+|v|^{\zeta_2}} d|\mu| \right| \leqslant \|\phi\|_\zeta \int_\Omega (1+|x|^{\zeta_1}+|v|^{\zeta_2}) d|\mu|,$$

we see that this linear map is also bounded. Hence, we conclude that $C_\zeta(\Omega)^*$ and $\mathcal{U}^\zeta(\Omega)$ are isomorphic. It can be proved that they are isometrically isomorphic (see [25], Theorem 7.17).

Define (see Remark 3.4)

$$\mathcal{U}_1 = \left\{\mu \in \mathcal{U}^\zeta : \ \mu \geqslant 0, \ \int_\Omega d\mu = T\right\}. \tag{3.17}$$

Notice that $\mathcal{U}_1$ is the set of non-negative Radon measures that satisfy (3.2) and for which (3.3) holds for $\varphi(t,x) = t$. Now, we define an operator related to the left-hand side of (3.3). Take $v \in \mathbb{R}$. Define, $A^v : \ C^1([0,T] \times \mathbb{R}) \to C_\zeta(\Omega)$ by

$$\varphi \mapsto A^v\varphi = -\varphi_t - v\varphi_x.$$

Indeed, because $\varphi_t, \ \varphi_x \in C([0,T] \times \mathbb{R})$, and

$$\frac{|A^v\varphi|}{1+|x|^{\zeta_1}+|v|^{\zeta_2}} \leqslant \|\varphi\|_{C^1}\frac{1+|v|}{1+|x|^{\zeta_1}+|v|^{\zeta_2}} \leqslant \|\varphi\|_{C^1}\left(1+\sup_{v\in\mathbb{R}}\frac{|v|}{1+|v|^{\zeta_2}}\right) \leqslant C\|\varphi\|_{C^1},$$

we have $A^v\varphi \in C_\zeta(\Omega)$ and A^v is bounded. Therefore, A^v is a linear and bounded map. We use this map to define the following sets. Let $\mathcal{C} \subset C_\zeta(\Omega)$ be the closed subspace

$$\begin{aligned}\mathcal{C} = \mathrm{cl}_{\|\cdot\|_\zeta}\{\phi \in C_\zeta(\Omega) : \ \phi(t,x,v) = A^v\varphi(t,x) - (v-Q(t))\eta(t) \ \text{ for some} \\ \varphi \in \Lambda([0,T]\times\mathbb{R}), \ \eta \in C([0,T])\},\end{aligned} \tag{3.18}$$

where Q satisfies Assumption 4, and $\mathrm{cl}_{\|\cdot\|_\zeta}$ denotes the closure with respect to $\|\cdot\|_\zeta$. Notice that $\mathcal{C}$ is convex because A^v is linear.

Given a linear and bounded operator $B : C([0,T]\times\mathbb{R}) \to \mathbb{R}$, let

$$\mathcal{U}_2(B) = \mathrm{cl}_{\mathrm{weak}}\left\{\mu \in \mathcal{U}^\zeta : \ \int_\Omega A^v\varphi d\mu(t,x,v) = B\varphi, \ \forall\varphi \in \Lambda([0,T]\times\mathbb{R})\right\}, \tag{3.19}$$

where $\mathrm{cl}_{\mathrm{weak}}$ denotes the closure with respect to weak convergence of measures ([16], Definition 1.31.). The choice of the operator B determines whether $\mathcal{U}_2(B) \neq \emptyset$. For instance, given $\nu_T \in \mathcal{P}(\mathbb{R})$, for the operators

$$B\varphi = \int_\mathbb{R}\varphi(0,x)dm_0(x) - \int_\mathbb{R}\varphi(T,x)d\nu_T(x), \tag{3.20}$$

and A^v as before, (3.19) corresponds to (3.3), and Remark 3.2 shows that $\mathcal{U}_2(B) \neq \emptyset$. Analogously, (see (3.4)) we define

$$\mathcal{U}_3 = \mathrm{cl}_{\mathrm{weak}}\left\{\mu \in \mathcal{U}^\zeta : \ \int_\Omega \eta(t)(v-Q(t)) \ d\mu(t,x,v) = 0, \ \forall\eta \in C([0,T])\right\}. \tag{3.21}$$

Remark 3.5. Let B as in (3.20). If $m_0, \nu \in \mathcal{P}(\mathbb{R})$ are such that $\mathcal{H}(m_0,\nu) \neq \emptyset$, then

$$\mathcal{H}(m_0,\nu) = \mathcal{U}_1 \cap \mathcal{U}_2(B) \cap \mathcal{U}_3.$$

To see this, notice that (3.17), (3.19), and (3.21) imply $\mathcal{U}_1 \cap \mathcal{U}_2(B) \cap \mathcal{U}_3 \subset \mathcal{H}(m_0,\nu)$. For the opposite inclusion, let $\mu \in \mathcal{H}(m_0,\nu)$ and let $A = \left\{(t,x,v) \in \Omega : \ 1 < |x|^{\zeta_1} + |v|^{\zeta_2}\right\}$. We have that $|\mu| = \mu$ because $\mu \geqslant 0$. Writing $\int_\Omega d\mu = \int_A d\mu + \int_{A^c} d\mu$, where A^c denotes the complement of the set A, we see that

$$\int_A d\mu \leqslant \int_A |x|^{\zeta_1} + |v|^{\zeta_2} d\mu \leqslant \int_\Omega |x|^{\zeta_1} + |v|^{\zeta_2} d\mu < \infty,$$

and $\int_{A^c} d\mu$ is finite because A^c is compact and μ is a Radon measure. Hence, $\mu \in \mathcal{U}^\zeta$. Moreover, since μ satisfies (3.3) and (3.4), we have that $\mu \in \mathcal{U}_2(B) \cap \mathcal{U}_3$. Using $\varphi(t,x) = t$ in (3.3), we obtain that $\mu \in \mathcal{U}_1$. Therefore $\mu \in \mathcal{U}_1 \cap \mathcal{U}_2(B) \cap \mathcal{U}_3$.

Now, we introduce the functionals we will use in the context of the Fenchel-Rockafellar theorem. Define $f : C_\zeta(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$f(\phi) = T \sup_{(t,x,v)\in\Omega} \left(\phi(t,x,v) - L(x,v) - vu_T'(x)\right). \tag{3.22}$$

Since f is the supremum of affine functions, f is convex. The following result proves continuity for this map.

Lemma 3.6. *Let ζ satisfy* (3.1). *Under Assumptions 1-3, the map f is continuous on $C_\zeta(\Omega)$.*

Proof. Let $(\phi_n)_{n\in\mathbb{N}}, \phi \in C_\zeta(\Omega)$ be such that $\lim_n \|\phi_n - \phi\|_\zeta = 0$. The first condition in (3.15) and the convergence of ϕ_n guarantees the existence of $C > 0$ such that $\|\phi_n\|_\zeta, \|\phi\|_\zeta \leqslant C$ for all n; that is,

$$|\phi_n(t,x,v)|, |\phi(t,x,v)| \leqslant C(1 + |x|^{\zeta_1} + |v|^{\zeta_2}) \quad \text{for all } (t,x,v) \in \Omega,\ n \in \mathbb{N}.$$

Let $\alpha > C$. By Assumption 2, using (2.1) (see Remark 2.1), we have

$$C_1|x|^{\gamma_1} + \frac{|v|^{\gamma_2'}}{\gamma_2' C^{\gamma_2'/\gamma_2}} - C \leqslant L(x,v), \quad \text{for all} \quad (x,v) \in \mathbb{R}^2.$$

Adding the term $vu_T'(x)$ to both sides of the previous inequality, we get

$$\frac{1}{\gamma_2' C^{\gamma_2'/\gamma_2}} \left(\frac{\gamma_2' C^{\gamma_2'/\gamma_2} C_1 |x|^{\gamma_1} + |v|^{\gamma_2'} + vu_T'(x) - \gamma_2' C^{\gamma_2'/\gamma_2} C}{1 + |x|^{\zeta_1} + |v|^{\zeta_2}} \right) \leqslant \frac{L(x,v) + vu_T'(x)}{1 + |x|^{\zeta_1} + |v|^{\zeta_2}}$$

for all $(x,v) \in \mathbb{R}^2$. By Assumption 3, u_T' is bounded. Hence, according to (3.1), the left-hand side of the previous expression goes to $+\infty$ when $|x|, |v| \to +\infty$. Hence, we can find $r > 0$ such that $|x|, |v| \geqslant r$ implies

$$-\frac{L(x,v) + vu_T'(x)}{1 + |x|^{\zeta_1} + |v|^{\zeta_2}} \leqslant -\alpha.$$

Let $(x,v) \in ([-r,r]^2)^c$, where A^c denotes the complement of the set A, and let $t \in [0,T]$. Using the previous bound, we have

$$\begin{aligned} \phi_n(t,x,v) - L(x,v) - vu_T'(x) &\leqslant \phi_n(t,x,v) - \alpha(1 + |x|^{\zeta_1} + |v|^{\zeta_2}) \\ &\leqslant (C - \alpha)(1 + |x|^{\zeta_1} + |v|^{\zeta_2}) \\ &< 0, \end{aligned}$$

for $n \in \mathbb{N}$. Hence,

$$f(\phi_n) = T \sup_{(t,x,v)\in[0,T]\times[-r,r]^2} \left(\phi_n(t,x,v) - L(x,v) - vu_T'(x)\right),$$

and the same holds for ϕ. Because the convergence on $C_\zeta(\Omega)$ implies uniform convergence on $[0,T] \times [-r,r]^2$, we obtain

$$f(\phi_n) \to f(\phi). \qquad \square$$

Proposition 3.7. *Let ζ satisfy* (3.1). *Suppose that Assumptions 1-3 hold. Let $\mu \in \mathcal{U}^\zeta$. If $\mu \ngeq 0$ then $f^*(\mu) = +\infty$.*

Proof. Let $\mu \in \mathcal{U}^\zeta$ be such that $\mu \ngeq 0$. Regarding μ as a linear map, by Remark 3.4, there exists $\phi \in C_\zeta(\Omega)$ such that $0 \leqslant \phi$ and $\int_\Omega \phi(t,x,v) d\mu < 0$. Let $\phi_n = -n\phi$, for $n \in \mathbb{N}$. Thus, the sequence $(\phi_n)_{n\in\mathbb{N}}$ in $C_\zeta(\Omega)$ satisfies

$$\phi_n \leqslant 0 \quad \text{and} \quad \int_\Omega \phi_n d\mu \to +\infty. \tag{3.23}$$

Let $\tilde{\phi}_n = \phi_n + vu_T'$, for $n \in \mathbb{N}$. By Assumption 3, we have $vu_T' \in C_\zeta(\Omega)$. Therefore, $\tilde{\phi}_n \in C_\zeta(\Omega)$ for $n \in \mathbb{N}$. Moreover, $\int_\Omega \tilde{\phi}_n d\mu \to +\infty$ as well. From (3.22) we get

$$f(\tilde{\phi}_n) = T \sup_{(t,x,v)\in\Omega} (\phi_n - L).$$

By Assumption 2, using (2.1) (see Remark 2.1) and the first condition in (3.23), we get

$$\phi_n(t,x,v) - L(x,v) \leqslant -C_1|x|^{\gamma_1} - \frac{|v|^{\gamma_2'}}{\gamma_2' C^{\gamma_2'/\gamma_2}} + C \leqslant C.$$

Thus, $f(\tilde{\phi}_n) \leqslant TC$. Hence, we conclude that

$$+\infty = \lim_n \int_\Omega \tilde{\phi}_n d\mu - TC \leqslant \lim_n \int_\Omega \tilde{\phi}_n d\mu - f(\tilde{\phi}_n) \leqslant f^*(\mu). \qquad \square$$

Proposition 3.8. *Let ζ satisfy* (3.1). *Suppose that Assumptions 1-3 hold. Let $\mu \in \mathcal{U}^\zeta$. If $\mu \geqslant 0$, then*

$$f^*(\mu) \geqslant \int_\Omega L + vu_T' \, d\mu + \sup_{\psi \in C_\zeta(\Omega)} \left(\int_\Omega \psi d\mu - T \sup_\Omega \psi \right).$$

Proof. From (2.1), we can add a constant C to L and assume that $0 \leqslant L$. Using Remark 2.1, let L_n be a sequence in $C_\zeta(\Omega)$ such that $0 \leqslant L_n \leqslant L_{n+1} \leqslant L$ and $L_n \to L$ pointwise. Fix $n \in \mathbb{N}$, $\phi \in C_\zeta(\Omega)$ and let $\psi = \phi - vu_T' - L_n \in C_\zeta(\Omega)$. Then

$$\begin{aligned}\int_\Omega \phi d\mu - f(\phi) &= \int_\Omega (\psi + L_n + vu_T') \, d\mu - T \sup_\Omega (\psi + L_n - L) \\ &\geqslant \int_\Omega (\psi + L_n + vu_T') \, d\mu - T \sup_\Omega \psi.\end{aligned}$$

By the Monotone Convergence Theorem, we have $\int_\Omega L_n d\mu \to \int_\Omega L d\mu$. Therefore,

$$f^*(\mu) = \sup_{\phi \in C_\zeta(\Omega)} \int_\Omega \phi d\mu - f(\phi) \geqslant \int_\Omega L + vu_T' \, d\mu + \sup_{\psi \in C_\zeta(\Omega)} \left(\int_\Omega \psi \, d\mu - T \sup_\Omega \psi \right). \qquad \square$$

Proposition 3.9. *Let ζ satisfy* (3.1). *Suppose that Assumptions 1-3 hold. Let f be as in* (3.22) *and let f^* be its Legendre transform; that is,*

$$f^*(\mu) = \sup_{\phi \in C_\zeta(\Omega)} \left(\int_\Omega \phi \, d\mu - f(\phi) \right).$$

Then,

$$f^*(\mu) = \begin{cases} \int_\Omega L + vu_T' \, d\mu & \mu \in \mathcal{U}_1 \\ +\infty & \textit{otherwise} \end{cases}.$$

Proof. By Proposition 3.7, if $\mu \ngeqslant 0$ then $f^*(\mu) = +\infty$. Let $\mu \geqslant 0$. If $\mu \notin \mathcal{U}_1$, by definition, $\int_\Omega d\mu \neq T$ (see (3.17)). Define $\phi_\alpha = \alpha + vu_T' - C \in C_\zeta(\Omega)$, where $\alpha \in \mathbb{R}$ and C is given by Assumption 2. Then, by (2.1), we obtain

$$f(\phi_\alpha) = T \sup_\Omega (\alpha - C - L) \leqslant T\alpha.$$

Adding $\alpha \int_\Omega d\mu$ and rearranging the previous expression, we get

$$\alpha \int_\Omega d\mu - T\alpha \leqslant \alpha \int_\Omega d\mu - f(\phi_\alpha),$$

which implies that

$$\left(\int_\Omega d\mu - T \right) \sup_{\alpha \in \mathbb{R}} \alpha \leqslant \sup_{\alpha \in \mathbb{R}} \int_\Omega \alpha d\mu - f(\phi_\alpha) \leqslant f^*(\mu).$$

From the preceding inequality, we conclude that $f^*(\mu) = +\infty$. On the other hand, if $\mu \in \mathcal{U}_1$, by definition, $\int_\Omega d\mu = T$. For any $\phi \in C_\zeta(\Omega)$, we have

$$\int_\Omega \phi - L - vu_T' \, d\mu \leqslant \int_\Omega \sup_\Omega (\phi - L - vu_T') \, d\mu = T \sup_\Omega (\phi - L - vu_T') = f(\phi).$$

Rearranging the previous inequality, we obtain

$$\int_\Omega \phi d\mu - f(\phi) \leqslant \int_\Omega L + vu_T' \, d\mu,$$

and we conclude that $f^*(\mu) \leqslant \int_\Omega L + vu_T' \, d\mu$. Finally, we take $\psi \equiv 0$ in Proposition 3.8 to obtain $f^*(\mu) \geqslant \int_\Omega L + vu_T' \, d\mu$. The result follows. $\square$

Now, we define the second functional we use in the Fenchel-Rockafellar theorem. Recall the definition of $\mathcal{C}$ in (3.18). Fix (see (3.19) and (3.21))

$$\overline{\mu} \in \mathcal{U}_2(B) \cap \mathcal{U}_3. \tag{3.24}$$

Define $g : C_\zeta(\Omega) \to \mathbb{R} \cup \{+\infty\}$ as

$$g(\phi) = \begin{cases} -\int_\Omega \phi d\overline{\mu}, & \phi \in \mathcal{C} \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.25}$$

Proposition 3.10. *Let ζ satisfy* (3.1)*. Suppose that Assumption 4 holds. Assume that $B : C([0,T] \times \mathbb{R}) \to \mathbb{R}$ is a linear and bounded operator such that $\mathcal{U}_2(B) \cap \mathcal{U}_3 \neq \emptyset$. Then*

$$g^*(\mu) = \begin{cases} 0, & -\mu \in \mathcal{U}_2(B) \cap \mathcal{U}_3 \\ +\infty, & \textit{otherwise.} \end{cases}$$

Proof. Let $\mu \in \mathcal{U}^\zeta$ and define $\hat{\mu} = \mu + \overline{\mu} \in \mathcal{R}(\Omega)$.

Assume that $-\mu \in \mathcal{U}_2(B) \cap \mathcal{U}_3$. Then, $\hat{\mu}$ satisfies

$$\int_\Omega A^v \varphi - (v - Q(t))\eta(t) \; d\hat{\mu} = 0$$

for all $\varphi \in \Lambda([0,T] \times \mathbb{R})$ and $\eta \in C([0,T])$. Because $\hat{\mu}$ defines a linear and bounded functional on $C_\zeta(\Omega)$ (see Section 3.4), the continuity under $\| \cdot \|_\zeta$ guarantees that $\int_\Omega \phi \; d\hat{\mu} = 0$ for all $\phi \in \mathcal{C}$; that is, $\int_\Omega \phi \; d\mu = -\int_\Omega \phi \; d\overline{\mu} = g(\phi)$ for all $\phi \in \mathcal{C}$. Hence,

$$g^*(\mu) = \sup_{\phi \in C_\zeta(\Omega)} \left(\int_\Omega \phi \; d\mu - g(\phi) \right) = \sup_{\phi \in \mathcal{C}} \left(\int_\Omega \phi \; d\mu - g(\phi) \right) = 0.$$

Now, assume that $-\mu \notin \mathcal{U}_2(B) \cap \mathcal{U}_3$. Then, either $-\mu \notin \mathcal{U}_2(B)$ or $-\mu \notin \mathcal{U}_3$. In the first alternative, there exists $\varphi \in \Lambda([0,T] \times \mathbb{R})$ such that

$$-\int_\Omega A^v \varphi d\mu(x,q,s) \neq B\varphi,$$

we have

$$\int_\Omega A^v \varphi d\hat{\mu} = \int_\Omega A^v \varphi d\mu + \int_\Omega A^v \varphi d\overline{\mu} \neq 0.$$

Define $\hat{\phi} = A^v \varphi$. Then $\hat{\phi} \in \mathcal{C}$ and satisfies $\int_\Omega \hat{\phi} \; d\hat{\mu} \neq 0$, and using (3.25), we obtain

$$\sup_{\phi \in C_\zeta(\Omega)} \left(\int_\Omega \phi \; d\mu - g(\phi) \right) = \sup_{\phi \in \mathcal{C}} \left(\int_\Omega \phi \; d\mu - g(\phi) \right) \geqslant \int_\Omega \hat{\phi} \; d\mu - g(\hat{\phi}) = \int_\Omega \hat{\phi} \; d\hat{\mu}.$$

Let $\alpha_n = n \; \mathrm{sgn}\left(\int_\Omega \hat{\phi} \; d\hat{\mu} \right)$, where $\mathrm{sgn}(\cdot)$ denotes the sign function, and $\hat{\phi}_n = \alpha_n A^v \varphi$, for $n \in \mathbb{N}$. Because $\alpha_n \varphi$ is a sequence in $\Lambda([0,T] \times \mathbb{R})$, $\hat{\phi}_n$ is a sequence in $\mathcal{C}$. Furthermore, the previous inequality implies

$$g^*(\mu) \geqslant n \; \mathrm{sgn} \left(\int_\Omega \hat{\phi} \; d\hat{\mu} \right) \int_\Omega \hat{\phi} \; d\hat{\mu}$$

for all $n \in \mathbb{N}$. Hence $g^*(\mu) = +\infty$.

In the second alternative, there exists $\eta \in C([0,T])$ such that

$$\int_\Omega \eta(t)(v - Q(t)) \; d\mu \neq 0,$$

we have

$$\int_\Omega \eta(v - Q)\varphi d\hat{\mu} = \int_\Omega \eta(v - Q)\varphi d\mu \neq 0.$$

Define $\hat{\phi} = -(v - Q)\eta$. Then $\hat{\phi} \in \mathcal{C}$ and satisfies $\int_\Omega \hat{\phi} \; d\hat{\mu} \neq 0$. Proceeding as before, we obtain $g^*(\mu) = +\infty$. □

Theorem 3.11. *Let ζ satisfy* (3.1). *Suppose that Assumptions 1-4 hold. Assume that $B : C([0,T]\times\mathbb{R}) \to \mathbb{R}$ is a linear and bounded operator such that $\mathcal{U}_2(B)\cap\mathcal{U}_3 \neq \emptyset$. Then*

$$\inf_{\varphi,\eta}\left(T\sup_{(t,x)}\left(-\varphi_t + Q\eta + H(x,\varphi_x+\eta+u_T')\right) - B\varphi\right) = \max_{\mu}\left(-\int_\Omega L + vu_T'\, d\mu\right),$$

where the supremum is taken over $(t,x)\in[0,T]\times\mathbb{R}$, the infimum is taken over $\varphi\in\Lambda([0,T]\times\mathbb{R})$, $\eta\in C([0,T])$, and the maximum is taken on $\mu\in\mathcal{U}_1\cap\mathcal{U}_2(B)\cap\mathcal{U}_3$.

Proof. Recall that f is convex, and by Lemma 3.6, f is continuous on $C_\zeta(\Omega)$. By definition, g is convex. Therefore, to use Theorem 3.3, we need to find $\phi\in C_\zeta(\Omega)$ such that $f(\phi), g(\phi) < +\infty$. Take $\varphi(t,x) = Ct - u_T(x)$, where C is given by Assumption 2. Then $\phi = A^v\varphi = -C + vu_T' \in \mathcal{C}$. By Assumption 2 and (2.1), we have

$$f(\phi) \leqslant 0.$$

From the definition of g (see (3.25)),

$$g(\phi) = -B\varphi,$$

and by Assumption 3, $B\varphi$ is finite. Hence, relying on the duality relation between $C_\zeta(\Omega)$ and $\mathcal{U}^\zeta$ (see Remark 3.4), we apply Theorem 3.3 to get

$$\inf_{\phi\in C_\zeta(\Omega)}(f(\phi)+g(\phi)) = \max_{\mu\in\mathcal{U}^\zeta}(-f^*(-\mu) - g^*(\mu)) = \max_{\mu\in\mathcal{U}^\zeta}(-f^*(\mu) - g^*(-\mu)).$$

From Proposition 3.9 and Proposition 3.10, it follows that

$$\max_{\mu\in\mathcal{U}^\zeta}(-f^*(\mu) - g^*(-\mu)) = \max_{\mu\in\mathcal{U}_1\cap\mathcal{U}_2(B)\cap\mathcal{U}_3}\left(-\int_\Omega L + vu_T'\, d\mu\right).$$

By (3.25),

$$\inf_{\phi\in C_\zeta(\Omega)}(f(\phi)+g(\phi)) = \inf_{\phi\in\mathcal{C}}\left(T\sup_\Omega(\phi - L - vu_T') - \int_\Omega \phi d\overline{\mu}\right),$$

and using the definition of $\mathcal{C}$ in (3.18), the selection of $\overline{\mu}$ in (3.24), and the definition of the Legendre transform (1.3), we obtain

$$\begin{aligned}
&\inf_{\phi\in\mathcal{C}}\left(T\sup_\Omega(\phi - L - vu_T') - \int_\Omega \phi d\overline{\mu}\right)\\
&= \inf_{\substack{\varphi\in\Lambda([0,T]\times\mathbb{R})\\ \eta\in C([0,T])}}\left(T\sup_\Omega(A^v\varphi - (v-Q)\eta - L - vu_T') - \int_\Omega A^v\varphi - (v-Q)\eta\, d\overline{\mu}\right)\\
&= \inf_{\substack{\varphi\in\Lambda([0,T]\times\mathbb{R})\\ \eta\in C([0,T])}}\left(T\sup_\Omega(-\varphi_t + Q\eta - v(\varphi_x+\eta+u_T') - L) - B\varphi\right)\\
&= \inf_{\substack{\varphi\in\Lambda([0,T]\times\mathbb{R})\\ \eta\in C([0,T])}}\left(T\sup_{(t,x)\in[0,T]\times\mathbb{R}}\left(-\varphi_t + Q\eta + \sup_{v\in\mathbb{R}}(-v(\varphi_x+\eta+u_T') - L)\right) - B\varphi\right)\\
&= \inf_{\substack{\varphi\in\Lambda([0,T]\times\mathbb{R})\\ \eta\in C([0,T])}}\left(T\sup_{(t,x)\in[0,T]\times\mathbb{R}}\left(-\varphi_t + Q\eta + H(x,\varphi_x+\eta+u_T')\right) - B\varphi\right).
\end{aligned}$$

The result follows. □

Now, we use the duality result from Theorem 3.11 to prove Theorem 1.2.

Proof of Theorem 1.2. Let B be given by (3.20) and consider the set $\mathcal{U}_2(B)$ according to (3.19). By assumption, $\mathcal{H}(m_0,\nu_T)\neq\emptyset$, from which (1.8) and Remark 3.5 imply

$$h(m_0,\nu_T) = \inf_{\mu\in\mathcal{H}(m_0,\nu_T)}\int_\Omega L + vu_T' d\mu = \inf_{\mu\in\mathcal{U}_1\cap\mathcal{U}_2(B)\cap\mathcal{U}_3}\int_\Omega L + vu_T'\, d\mu.$$

In particular, $\mathcal{U}_2(B)\cap\mathcal{U}_3\neq\emptyset$. The conclusion follows by invoking Theorem 3.11 and the previous equality. □

4. Preliminary results on MFG

Here, we consider approximations of Lipschitz continuous solutions of the Hamilton-Jacobi equation in (1.1). We provide a commutation lemma, which states that the approximated solutions are sub-solutions of an approximate Hamilton-Jacobi equation. Then, we improve the result in [36], where the authors proved that ϖ solving (1.1) and (1.2) satisfies $\varpi \in W^{1,1}([0,T])$. A better result can be established as ϖ is Lipschitz continuous, as we prove here. This result, in turn, enables the use of the commutation lemma.

4.1. **A commutation lemma.** The commutation lemmas presented in [38] and [46] are applied to a Hamilton-Jacobi equation where the state variable is constrained to the d-dimensional torus; that is, periodic boundary conditions. Here, we present a version of this lemma that is valid for the non-periodic case and takes into account the dependence of the Hamilton-Jacobi equation on the price variable.

We start by introducing smooth approximations to the solutions of (1.1). Let $\rho, \theta \in C_c^\infty(\mathbb{R};[0,\infty))$ be symmetric standard mollifiers, i.e.

$$\operatorname{supp}\rho, \operatorname{supp}\theta \subset [-1,1],\ \rho(t)=\rho(-t),\ \theta(x)=\theta(-x),\ \text{and } \|\rho\|_{L^1(\mathbb{R})}=\|\theta\|_{L^1(\mathbb{R})}=1.$$

For $0<\alpha<T$, set $\rho^\alpha(t):=\alpha^{-1}\rho(\alpha^{-1}t)$, $t\in\mathbb{R}$ and $\theta^\alpha(x):=\alpha^{-1}\theta(\alpha^{-1}x)$, $x\in\mathbb{R}$. Then, we have that $\|\rho^\alpha\|_{L^1(\mathbb{R})}=\|\theta^\alpha\|_{L^1(\mathbb{R})}=1$, and

$$\int_0^\infty \rho^\alpha(s)\int_{\mathbb{R}}\theta^\alpha(y)|y|\ dyds,\ \int_0^\infty \rho^\alpha(s)\int_{\mathbb{R}}\theta^\alpha(y)s\ dyds \leqslant \alpha. \tag{4.1}$$

For $w\in C([\alpha,T]\times\mathbb{R})$, define $w^\alpha\in C^\infty([\alpha,T]\times\mathbb{R})$ as

$$w^\alpha(t,x)=\int_0^\infty \rho^\alpha(s)\int_{\mathbb{R}}\theta^\alpha(y)w(t-s,x-y)dyds,\quad (t,x)\in[\alpha,T]\times\mathbb{R}. \tag{4.2}$$

Lemma 4.1. *Suppose that Assumptions 1 and 2 hold. Let (w,m,ϖ) solve* (1.1) *(see Remark 1.1). Assume further that w is Lipschitz in x and ϖ is Lipschitz. Let w^α be defined as in* (4.2). *Then, there exists $C'>0$ depending on ϖ, H and the Lipschitz constants of w and ϖ such that*

$$-w_t^\alpha + H(x,\varpi + w_x^\alpha)\leqslant C'\alpha,\quad \textit{for all}\quad (t,x)\in[\alpha,T]\times\mathbb{R}. \tag{4.3}$$

Proof. To obtain the desired inequality, we write the left-hand side of (4.3) as a convolution between $\rho^\alpha\theta^\alpha$ and the left-hand side of the first equation in (1.1). Thus, for the first term, we have

$$-w_t^\alpha(t,x)=\int_0^\infty\rho^\alpha(s)\int_{\mathbb{R}}\theta^\alpha(y)(-w_t(t-s,x-y))dyds. \tag{4.4}$$

For the second term, by Jensen's inequality ([29], Theorem 204), we have

$$\begin{aligned} H(x,\varpi(t)+w_x^\alpha(t,x)) &= H\left(x,\int_0^\infty\rho^\alpha(s)\int_{\mathbb{R}}\theta^\alpha(y)(\varpi(t)+w_x(t-s,x-y))dyds\right)\\ &\leqslant \int_0^\infty\rho^\alpha(s)\int_{\mathbb{R}}\theta^\alpha(y)H(x,\varpi(t)+w_x(t-s,x-y))dyds. \end{aligned}\tag{4.5}$$

Let $t\in[\alpha,T]$, $s\in[0,\alpha]$, $x,y\in\mathbb{R}$, and

$$q(t,x;s,y):=H\big(x,\varpi(t)+w_x(t-s,x-y)\big)-H\big(x-y,\varpi(t-s)+w_x(t-s,x-y)\big).$$

Using Assumption 2 and the Lipschitz continuity of w and ϖ, we get

$$\begin{aligned} |q(t,x;s,y)| &\leqslant \big|H\big(x,\varpi(t)+w_x(t-s,x-y)\big)-H(x,\varpi(t-s)+w_x(t-s,x-y))\big|\\ &\quad+\big|H\big(x,\varpi(t-s)+w_x(t-s,x-y)\big)-H(x-y,\varpi(t-s)+w_x(t-s,x-y))\big|\\ &\leqslant C|\varpi(t)-\varpi(t-s)|\left(|\varpi(t-s)+w_x(t-s,x-y)|^{\gamma_2-1}+1\right)\\ &\quad+C|y|\big(|\varpi(t-s)+w_x(t-s,x-y)|^{\gamma_2}+1\big)\\ &\leqslant C's\left(|\varpi(t-s)+w_x(t-s,x-y)|^{\gamma_2-1}+1\right)\\ &\quad+C|y|\big(|\varpi(t-s)+w_x(t-s,x-y)|^{\gamma_2}+1\big) \end{aligned}$$

$$\leqslant C'(s+|y|),$$

where C' depends on ϖ, γ_2, and the Lipschitz constants of w and ϖ. From the previous inequality and (4.1), we obtain

$$\int_0^\infty \rho^\alpha(s)\int_{\mathbb{R}}\theta^\alpha(y)q(t,x;s,y)dyds \leqslant \int_0^\infty \rho^\alpha(s)\int_{\mathbb{R}}\theta^\alpha(y)C'(s+|y|)dyds \leqslant C'\alpha. \tag{4.6}$$

Then, from (4.5) and (4.6), we have

$$\begin{aligned}&H(x,\varpi(t)+w_x^\alpha(t,x))\\ &\leqslant \int_0^\infty \rho^\alpha(s)\int_{\mathbb{R}}\theta^\alpha(y)H(x-y,\varpi(t-s)+w_x(t-s,x-y))dyds + C'\alpha.\end{aligned}$$

Using the preceding inequality and (4.4), we get

$$\begin{aligned}&-w_t^\alpha + H(x,\varpi+w_x^\alpha)\\ &\leqslant \int_0^\infty \rho^\alpha(s)\int_{\mathbb{R}}\theta^\alpha(y)\big(-w_t(t-s,x-y)+H(x-y,\varpi(t-s)+w_x(t-s,x-y))\big)dyds\\ &\quad + C'\alpha,\end{aligned}$$

which implies (4.3). □

4.2. **Lipschitz continuity of the price.** We begin by recalling the following techniques and results from [36] if Assumptions 4, 6, and 7 hold. Firstly, to prove the existence of a solution (u,m,ϖ) of (1.1) and (1.2), the authors used the vanishing viscosity method, which relies on the following regularized version of (1.1)

$$\begin{cases}-u_t(t,x)+H(x,\varpi(t)+u_x(t,x)) = \epsilon u_{xx}\\ m_t(t,x)-\big(H_p(x,\varpi(t)+u_x(t,x))m(t,x)\big)_x = \epsilon m_{xx}(t,x) & (t,x)\in[0,T]\times\mathbb{R},\\ -\int_{\mathbb{R}} H_p(x,\varpi(t)+u_x(t,x))m(t,x)dx = Q(t)\end{cases} \tag{4.7}$$

subject to (1.2), where $\epsilon>0$. Secondly, the proof of existence of a solution $(u^\epsilon,m^\epsilon,\varpi^\epsilon)$ of (4.7) and (1.2) uses a fixed-point argument. This argument shows that $(u^\epsilon,m^\epsilon,\varpi^\epsilon)$ satisfies

$$\dot{\varpi}^\epsilon = \frac{-\dot{Q}-\int_{\mathbb{R}} H_{pp}(x,\varpi^\epsilon+u_x^\epsilon)H_x(x,\varpi^\epsilon+u_x^\epsilon)m^\epsilon + \epsilon H_{ppp}(x,\varpi^\epsilon+u_x^\epsilon)(u_{xx}^\epsilon)^2 m^\epsilon\,dx}{\int_{\mathbb{R}} H_{pp}(x,\varpi^\epsilon+u_x^\epsilon)m^\epsilon\,dx}, \tag{4.8}$$

and $\varpi^\epsilon(0)$ is determined by

$$\int_{\mathbb{R}} H_p(x,\varpi^\epsilon(0)+u^\epsilon(0,x))m_0(x)\,dx = -Q(0).$$

Using (4.8), we can deduce the Lipschitz continuity of ϖ, where (u,m,ϖ) solves (1.1) and (1.2), as we show next.

Proposition 4.2. *Suppose that Assumptions 1, 4, 6 and 7 hold. Then, there exists a solution (u,m,ϖ) of (1.1) and (1.2) such that ϖ is Lipschitz continuous.*

Proof. The existence of a solution (u,m,ϖ) of (1.1) and (1.2) is guaranteed by Theorem 1 in [36]. We aim to prove that ϖ, obtained in [36], is Lipschitz. To obtain this solution, the authors considered, for $\epsilon>0$, solutions $(u^\epsilon,m^\epsilon,\varpi^\epsilon)$ of (4.7) and (1.2) that satisfy (4.8). Extracting a sub-sequence if necessary, it is guaranteed that $\varpi^\epsilon\to\varpi$ uniformly. To prove that ϖ is Lipschitz, we consider the right-hand side of (4.8). By Assumption 1, we have

$$t\mapsto \frac{1}{\int_{\mathbb{R}} H_{pp}(x,\varpi^\epsilon+u_x^\epsilon)m^\epsilon\,dx} \leqslant \frac{1}{\kappa} \quad \text{for all} \quad t\in[0,T]. \tag{4.9}$$

By Assumptions 6 and 7, $|H_x|=|V'|\leqslant \operatorname{Lip}(V)$, where $\operatorname{Lip}(V)$ denotes the Lipschitz constant of V. Hence, Assumption 6 implies that

$$t\mapsto \int_{\mathbb{R}} H_{pp}(x,\varpi^\epsilon+u_x^\epsilon)H_x(s,\varpi^\epsilon+u_x^\epsilon)m^\epsilon\,dx \leqslant \frac{\operatorname{Lip}(V)}{\kappa'} \quad \text{for all} \quad t\in[0,T]. \tag{4.10}$$

By Assumption 6 and Assumption 1, we have

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}|H_{ppp}(x,\varpi^\epsilon+u_x^\epsilon)|(u_{xx}^\epsilon)^2m^\epsilon dxdt &\leqslant C\int_0^T\int_{\mathbb{R}}(u_{xx}^\epsilon)^2m^\epsilon dxdt\\ &\leqslant \frac{C}{\kappa}\int_0^T\int_{\mathbb{R}}H_{pp}(x,\varpi^\epsilon+u_x^\epsilon)(u_{xx}^\epsilon)^2m^\epsilon dxdt. \quad (4.11)\end{aligned}$$

Assumptions 6, 7 and Proposition 5 in [36] guarantee that the term

$$\int_0^T\int_{\mathbb{R}}H_{pp}(x,\varpi^\epsilon+u_x^\epsilon)(u_{xx}^\epsilon)^2m^\epsilon \mathrm{d}x\mathrm{d}t \quad (4.12)$$

has an upper bound that is independent of ϵ. Hence, using Assumption 4, (4.9), (4.10) and (4.11), we can write (4.8) as

$$\dot{\varpi}^\epsilon=\vartheta_\infty^\epsilon+\epsilon\vartheta_1^\epsilon,$$

where

$$\vartheta_\infty^\epsilon=\frac{-\dot{Q}-\int_{\mathbb{R}}H_{pp}(x,\varpi^\epsilon+u_x^\epsilon)H_x(x,\varpi^\epsilon+u_x^\epsilon)m^\epsilon dx}{\int_{\mathbb{R}}H_{pp}(x,\varpi^\epsilon+u_x^\epsilon)m^\epsilon\ dx}\in L^\infty([0,T]),$$

$$\vartheta_1^\epsilon=\frac{-\int_{\mathbb{R}}H_{ppp}(x,\varpi^\epsilon+u_x^\epsilon)(u_{xx}^\epsilon)^2m^\epsilon\ dx}{\int_{\mathbb{R}}H_{pp}(x,\varpi^\epsilon+u_x^\epsilon)m^\epsilon\ dx}\in L^1([0,T]),$$

and they satisfy

$$\|\vartheta_1^\epsilon\|_{L^1([0,T])}\leqslant C' \quad \text{and} \quad \|\vartheta_\infty^\epsilon\|_{L^\infty([0,T])}\leqslant C'$$

for $\epsilon\to 0$, where C' is independent of ϵ. Hence, ([26], Proposition 1.202) passing to a sub-sequence, there exists $\mu\in\mathcal{R}([0,T])$ such that ϑ_1^ϵ converges in the weak-$\star$ topology to μ; that is,

$$\int_0^T\vartheta_1^\epsilon\eta\ dt\to\int_0^T\eta\ d\mu \quad \text{for all} \quad \eta\in C([0,T]). \quad (4.13)$$

Passing to a further sub-sequence if necessary, ([26], Proposition 2.46) there exists $\vartheta_\infty\in L^\infty([0,T])$ such that $\vartheta_\infty^\epsilon$ converges in the weak-$\star$ topology to ϑ_∞; that is,

$$\int_0^T\vartheta_\infty^\epsilon\eta\ dt\to\int_0^T\vartheta_\infty\eta\ dt \quad \text{for all} \quad \eta\in L^1([0,T]). \quad (4.14)$$

Let $\eta\in C_c^1((0,T))$. By uniform convergence, we have that

$$\int_0^T(\vartheta_\infty^\epsilon+\epsilon\vartheta_1^\epsilon)\,\eta\ dt=\int_0^T\dot{\varpi}^\epsilon\eta\ dt=-\int_0^T\varpi^\epsilon\dot{\eta}\ dt\to-\int_0^T\varpi\dot{\eta}\ dt,$$

and by (4.13) and (4.14), we have that

$$\int_0^T(\vartheta_\infty^\epsilon+\epsilon\vartheta_1^\epsilon)\,\eta\ dt\to\int_0^T\vartheta_\infty\eta\ dt.$$

Hence, $\dot{\varpi}=\vartheta_\infty$ in the sense of distributions. Thus, $\varpi\in W^{1,\infty}([0,T])$, which is equivalent to ([15], Theorem 4.5) ϖ being Lipschitz continuous in $[0,T]$. □

5. Proof of Theorem 1.3

Here, we use the results from Sections 3 and 4 to prove Theorem 1.3. We divide the proof into two lemmas, Lemma 5.1 and Lemma 5.6.

Lemma 5.1. *Let $m_0\in\mathcal{P}(\mathbb{R})$. Suppose that Assumptions 1-8 hold. Let (u,m,ϖ) solve (1.1) and (1.2). Then,*

$$\int_{\mathbb{R}}(u(0,x)-u_T(x))\,dm_0(x)-\int_0^T\varpi(t)Q(t)dt\leqslant\inf_{\mu\in\mathcal{H}(m_0)}\int_\Omega L(x,v)+vu_T'(x)d\mu(t,x,v).$$

Proof. By Assumptions 4, 6, 7, and 8, Theorem 1 in [36] guarantees the existence of a unique (u, m, ϖ) solving (1.1) and (1.2). Because u is continuous (see Remark 1.1), let w^α be the function given by (4.2); that is,

$$w^\alpha(t,x) = \int_0^\infty \rho^\alpha(s) \int_{\mathbb{R}} \theta^\alpha(y) u(t-s, x-y) dy ds, \quad (t,x) \in [\alpha, T] \times \mathbb{R}. \tag{5.1}$$

For $(t,x) \in [0,T] \times \mathbb{R}$, set

$$u^\alpha(t,x) = w^\alpha\left(\tfrac{T-\alpha}{T}t + \alpha, x\right) - u_T(x),$$

which is $C^1([0,T] \times \mathbb{R})$ due to Assumption 3 and (5.1). By Assumptions 6 and 7, the map $x \mapsto u(t,x)$ is Lipschitz for $0 \leqslant t \leqslant T$ ([36], Proposition 1), and the Lipschitz constant depends on T and the estimates for V and u_T. Hence, u_x is bounded independently of t. Therefore, $u^\alpha_x \in L^\infty([0,T] \times \mathbb{R})$ because

$$\begin{aligned} u^\alpha_x(t,x) &= w^\alpha_x\left(\tfrac{T-\alpha}{T}t + \alpha, x\right) - u_T'(x) \\ &= \int_0^\infty \rho^\alpha(s) \int_{\mathbb{R}} \theta^\alpha(y) u_x\left(\tfrac{T-\alpha}{T}t + \alpha - s, x - y\right) dy ds - u_T'(x). \end{aligned}$$

Furthermore, recalling that u is a viscosity solution to the first equation in (4.7) with $\varepsilon = 0$, we have that the first equation in (4.7) with $\varepsilon = 0$ holds a.e. $(t,x) \in (0,T) \times \mathbb{R}$. Using this and the facts that $u_x \in L^\infty((0,T) \times \mathbb{R})$ and $\varpi \in W^{1,\infty}([0,T])$, we deduce that $u_t \in L^\infty_{loc}((0,T) \times \mathbb{R})$. Thus, $u^\alpha_t \in L^\infty((0,T) \times \mathbb{R})$ because

$$\begin{aligned} u^\alpha_t(t,x) &= \tfrac{T-\alpha}{T} w^\alpha_t\left(\tfrac{T-\alpha}{T}t + \alpha, x\right) \\ &= \tfrac{T-\alpha}{T} \int_0^\infty \rho^\alpha(s) \int_{\mathbb{R}} \theta^\alpha(y) u_t\left(\tfrac{T-\alpha}{T}t + \alpha - s, x - y\right) dy ds. \end{aligned}$$

Hence, $u^\alpha \in \Lambda([0,T] \times \mathbb{R})$. Now, take $\mu \in \mathcal{H}(m_0)$ (see Remark 3.2). By (3.3), we have

$$\int_\Omega u^\alpha_t(t,x) + v u^\alpha_x(t,x) d\mu(t,x,v) = \int_{\mathbb{R}} u^\alpha(T,x) d\nu^\mu(x) - \int_{\mathbb{R}} u^\alpha(0,x) dm_0(x). \tag{5.2}$$

By Assumption 1 and (1.3), using $p = \varpi\left(\tfrac{T-\alpha}{T}t + \alpha\right) + u^\alpha_x(t,x) + u_T'(x)$, it follows that

$$\begin{aligned} -v u^\alpha_x(t,x) \leqslant{}& L(x,v) + v u_T'(x) + v\varpi\left(\tfrac{T-\alpha}{T}t + \alpha\right) \\ &+ H\big(x, \varpi\left(\tfrac{T-\alpha}{T}t + \alpha\right) + u^\alpha_x(t,x) + u_T'(x)\big). \end{aligned} \tag{5.3}$$

Moreover, by the Lipschitz continuity of u in x and Proposition 4.2, we apply Lemma 4.1 to w defined by (5.1) to get

$$-w^\alpha_t\left(\tfrac{T-\alpha}{T}t + \alpha, x\right) + H\left(x, \varpi\left(\tfrac{T-\alpha}{T}t + \alpha\right) + w^\alpha_x\left(\tfrac{T-\alpha}{T}t + \alpha, x\right)\right) \leqslant C'\alpha,$$

for $(t,x) \in [0,T] \times \mathbb{R}$; that is,

$$-\tfrac{T}{T-\alpha} u^\alpha_t + H\left(x, \varpi\left(\tfrac{T-\alpha}{T}t + \alpha\right) + u^\alpha_x(t,x) + u_T'(x)\right) \leqslant C'\alpha, \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}. \tag{5.4}$$

Therefore, by (5.2), (5.3), (5.4), and using $\varphi = t$ in (3.2), we have

$$\begin{aligned} &\int_{\mathbb{R}} u^\alpha(0,x) dm_0(x) - \int_{\mathbb{R}} u^\alpha(T,x) d\nu^\mu(x) \\ &= \int_\Omega -u^\alpha_t(t,x) - v u^\alpha_x(t,x) d\mu(t,x,v) \\ &\leqslant \int_\Omega -u^\alpha_t d\mu(t,x,v) + \int_\Omega L(x,v) + v u_T' d\mu(t,x,v) \\ &\quad + \int_\Omega v\varpi\left(\tfrac{T-\alpha}{T}t + \alpha\right) + H\left(x, \varpi\left(\tfrac{T-\alpha}{T}t + \alpha\right) + u^\alpha_x + u_T'\right) d\mu(t,x,v) \\ &\leqslant \int_\Omega L(x,v) + v u_T' d\mu(t,x,v) + \int_\Omega v\varpi\left(\tfrac{T-\alpha}{T}t + \alpha\right) d\mu(t,x,v) + (T-\alpha)C'\alpha \\ &\quad + \int_\Omega H\left(x, \varpi\left(\tfrac{T-\alpha}{T}t + \alpha\right) + u^\alpha_x + u_T'\right) - \tfrac{T-\alpha}{T} H\left(x, \varpi\left(\tfrac{T-\alpha}{T}t + \alpha\right) + u^\alpha_x + u_T'\right) d\mu(t,x,v). \end{aligned}$$

Taking $\alpha \to 0$ in the previous inequality and using (3.4) with $\eta = \varpi$, we obtain

$$\int_{\mathbb{R}} (u(0,x) - u_T(x))\, dm_0(x) \leqslant \int_\Omega L(x,v) + v u_T' d\mu(t,x,v) + \int_\Omega Q(t)\varpi(t) d\mu(t,x,v). \quad (5.5)$$

Finally, taking $\varphi(t,x) = \int_0^t Q(s)\varpi(s)ds$ in (3.3), we have

$$\int_\Omega Q(t)\varpi(t)d\mu(t,x,v) = \int_0^T Q(t)\varpi(t)dt.$$

Hence, (5.5) becomes

$$\int_{\mathbb{R}} (u(0,x) - u_T(x))\, dm_0(x) - \int_0^T Q(t)\varpi(t)dt \leqslant \int_\Omega L(x,v) + v u_T' d\mu(t,x,v).$$

Since $\mu \in \mathcal{H}(m_0)$ is arbitrary, the preceding inequality completes the proof. □

For the second part of the proof of Theorem 1.3, we rely on (4.7), the regularized version of (1.1), subject to (1.2). We recall that, if Assumptions 4, 6 and 7 hold, ([36], Theorem 1) there exists a solution $(u^\epsilon, m^\epsilon, \varpi^\epsilon)$ of (4.7) and (1.2), where u^ϵ is a viscosity solution of the first equation, Lipschitz and semiconcave in x, and differentiable m^ϵ-almost everywhere, $m^\epsilon \in C([0,T], \mathcal{P}(\mathbb{R}))$ w.r.t. the 1-Wasserstein distance, and $\varpi^\epsilon \in W^{1,1}([0,T])$ is continuous. Moreover, if $\epsilon > 0$ or $\epsilon = 0$ and Assumption 8 holds, this solution is unique. Using the previous results for the solution of (4.7) when $\epsilon > 0$, we take $\epsilon \to 0$ to exhibit a measure $\mu \in \mathcal{H}(m_0)$ for which the inequality

$$\int_{\mathbb{R}} u(0,x) - u_T(x) dm_0(x) - \int_0^T \varpi(t)Q(t)dt \geqslant \int_0^T \int_{\mathbb{R}^2} L(x,v) + v u_T'(x) d\mu(t,x,v)$$

holds. We begin by establishing the following moment estimate for the probability measures m^ϵ when $\epsilon > 0$.

Proposition 5.2. *Suppose Assumptions 1, 2, 4, 6-8 hold. Assume further that m_0 satisfies Assumption 5. Then, for $0 < \epsilon < 1$, the solution $(u^\epsilon, m^\epsilon, \varpi^\epsilon)$ of (4.7)-(1.2) satisfies*

$$\int_{\mathbb{R}} |x|^\gamma m^\epsilon(t,x)dx < C \quad \textit{for almost every } t \in [0,T],$$

where the constant C is independent of ϵ.

Proof. By Assumptions 6, 7 and 8, there exist a unique solution $(u^\epsilon, m^\epsilon, \varpi^\epsilon)$ of (4.7) and (1.2) ([36], Theorem 1). Then, by Assumptions 1, 2, 4, 6 and 7, and the bounds on ϖ^ϵ and u_x^ϵ (see Proposition 4.2 and [36], Propositions 1 and 6), we have, for $0 < \epsilon < 1$,

$$\begin{aligned}
|H_p(\varpi^\epsilon(t) + u_x^\epsilon(t,x))| &\leqslant C\left(|\varpi^\epsilon(t) + u_x^\epsilon(t,x)|^{\gamma_2 - 1} + 1\right) \\
&\leqslant C\left(C'(\gamma_2)\left(\|\varpi^\epsilon\|_\infty^{\gamma_2-1} + \mathrm{Lip}(u^\epsilon)^{\gamma_2-1}\right) + 1\right) \\
&\leqslant C\left(C'(\gamma_2)\left(\epsilon^{\gamma_2-1}C' + \mathrm{Lip}(u^\epsilon)^{\gamma_2-1}\right) + 1\right) \\
&= C_1\epsilon^{\gamma_2-1} + C_2 \\
&\leqslant \tilde{C}, \qquad (5.6)
\end{aligned}$$

where $C'(\gamma_2) = \max\{2^{\gamma_2-2}, 1\}$ and $\mathrm{Lip}(u^\epsilon)$, and therefore C_1, C_2, and $\tilde{C}$ are independent of ϖ^ϵ and ϵ. Furthermore, u^ϵ defines the optimal feedback in a stochastic optimal control problem, for which the optimal trajectory satisfies

$$\mathrm{dx}_t = -H_p(\mathrm{x}_t, \varpi^\epsilon(t) + u_x^\epsilon(t,\mathrm{x}_t))\mathrm{d}t + \sqrt{2\epsilon}\mathrm{d}W_t, \quad (5.7)$$

where W_t is a one-dimensional Brownian motion (see [36]). Using Assumptions 2 and 6, the vector field

$$(t,x) \mapsto H_p(x, \varpi^\epsilon(t) + u_x^\epsilon(t,x)) = H_p(\varpi^\epsilon(t) + u_x^\epsilon(t,x))$$

is bounded and uniformly Lipschitz. Hence, $m(t,\cdot) = \mathcal{L}(\mathrm{x}_t)$, where $\mathcal{L}(\mathrm{x})$ denotes the law of the random variable x, is a weak solution of the second equation in (4.7) ([13], Lemma

4.2.3), and by Assumption 8, this weak solution is unique. Hence $m^\epsilon(t,\cdot)=\mathcal{L}(\mathrm{x}_t)$. Writing (5.7) as

$$\mathrm{x}_t = x + \int_0^t -H_p(\mathrm{x}_t, \varpi^\epsilon(t) + u_x^\epsilon(t,\mathrm{x}_t))\mathrm{d}t + \int_0^t \sqrt{2\epsilon}\mathrm{d}W_t,$$

where $x\in\mathbb{R}$, and using (5.6), we have, for $0<\epsilon<1$,

$$\begin{aligned}|\mathrm{x}_t|^\gamma &\leqslant 2^{\gamma-1}\left(|x|^\gamma + 2^{\gamma-1}\left(T^\gamma\tilde{C}^\gamma + \sqrt{2\epsilon}^\gamma|W_t|^\gamma\right)\right)\\ &\leqslant 2^{\gamma-1}|x|^\gamma + C_1' + C_2'|W_t|^\gamma,\end{aligned} \tag{5.8}$$

where C_1' and C_2' are independent of ϖ^ϵ and ϵ. Because W_t is normally distributed w.r.t. the measure $m^\epsilon(t,x)dx$ in $\mathbb{R}$, we have

$$\mathbb{E}[|W_t|^\gamma] = \int_{\mathbb{R}} |W_t|^\gamma m^\epsilon(t,x)dx = \frac{(2t)^{\gamma/2}}{\pi}\Gamma(\tfrac{\gamma+1}{2}),$$

where $\Gamma(\cdot)$ denotes the Gamma function. Integrating (5.8) w.r.t. $m^\epsilon(t,x)dx$, using the previous formula, and recalling the initial condition for m^ϵ in (1.2), we obtain that m^ϵ satisfies

$$\int_{\mathbb{R}} |x|^\gamma m^\epsilon(t,x)dx \leqslant 2^{\gamma-1}\int_{\mathbb{R}}|x|^\gamma m_0(x)dx + C_1' + C_2'\frac{(2T)^{\gamma/2}}{\pi}\Gamma(\tfrac{\zeta_1+1}{2}).$$

By Assumption 5, the right-hand side of the previous inequality is bounded independently of ϵ, for $0<\epsilon<1$, as stated. ☐

Let $t\in[0,T]$. Define $\beta_t^\epsilon\in\mathcal{P}(\mathbb{R}^2)$ by

$$\int_{\mathbb{R}^2}\psi(x,p)d\beta_t^\epsilon(x,p) = \int_{\mathbb{R}}\psi(x,\varpi^\epsilon(t)+u_x^\epsilon(t,x))m^\epsilon(t,x)\mathrm{d}x \quad \text{for all } \psi\in C_{\bar{\zeta}}(\mathbb{R}^2),$$

where $C_\zeta(\mathbb{R}^2)=\{\phi\in C(\mathbb{R}^2) : \lim_{|x|,|v|\to\infty}\frac{\phi(x,v)}{1+|x|^{\zeta_1}+|v|^{\zeta_2}}=0\}$ and $\bar{\zeta}=(\gamma_1,\gamma_2')$. Note that the well definiteness of the measure β_t^ϵ is ensured by Proposition 5.2. Relying on the definition of β_t^ϵ, we define $\mu_t^\epsilon\in\mathcal{P}(\mathbb{R}^2)$ by

$$\int_{\mathbb{R}^2}\psi(x,-L_v(x,v))d\mu_t^\epsilon(x,v) = \int_{\mathbb{R}^2}\psi(x,p)d\beta_t^\epsilon(x,p) \quad \text{for all } \psi\in C_{\bar{\zeta}}(\mathbb{R}^2).$$

If Assumption 1 holds, the relation $v=-H_p(x,p)$ if and only if $p=-L_v(x,v)$ (see Remark 2.1), implies

$$\int_{\mathbb{R}^2}\psi(x,-H_p(x,p))d\beta_t^\epsilon(x,p) = \int_{\mathbb{R}^2}\psi(x,v)d\mu_t^\epsilon(x,v).$$

Finally, we define $\beta^\epsilon,\mu^\epsilon\in\mathcal{U}^{\bar{\zeta}}\cap\mathcal{R}^+(\Omega)$ by

$$\int_\Omega f(t,x,v)d\beta^\epsilon(t,x,v) = \int_0^T\int_{\mathbb{R}^2} f(t,x,v)d\beta_t^\epsilon(x,v)\mathrm{d}t,$$

and

$$\int_\Omega f(t,x,v)d\mu^\epsilon(t,x,v) = \int_0^T\int_{\mathbb{R}^2} f(t,x,v)d\mu_t^\epsilon(x,v)\mathrm{d}t, \tag{5.9}$$

for all $f\in C_{\bar{\zeta}}(\Omega)$ (see Remark 3.4). Under Assumptions 1, 2, 4, 6, 7 and 8, the non-negative and finite Radon measures μ^ϵ defined by (5.9) have a weak limit in $\mathcal{U}^{\bar{\zeta}}$ as $\epsilon\to 0$.

We show the existence of a weak limit of the Radon measures μ^ϵ defined by (5.9).

Proposition 5.3. *Suppose Assumptions 1, 2, 4-8 hold. Then, there exists $\mu\in\mathcal{U}^{\bar{\zeta}}\cap\mathcal{R}^+(\Omega)$, where $\bar{\zeta}=(\gamma_1,\gamma_2')$, such that, up to a sub-sequence, the sequence of Radon measures μ^ϵ defined by* (5.9) *weakly converge to μ; that is, for all $f\in C_{\bar{\zeta}}(\Omega)$*

$$\int_\Omega f(t,x,v)d\mu^\epsilon(t,x,v) \to \int_\Omega f(t,x,v)d\mu(t,x,v). \tag{5.10}$$

Proof. By (5.6) and Proposition 5.2, we have

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}}\left(1+|x|^\gamma+|H_p(x,\varpi^\epsilon(t)+u_x^\epsilon(t,x))|^{\gamma_2'+1}\right)m^\epsilon(t,x)\mathrm{d}x\mathrm{d}t\\&\leqslant\int_0^T\int_{\mathbb{R}}\left(1+|x|^\gamma+\tilde{C}^{\gamma_2'+1}\right)m^\epsilon(t,x)\mathrm{d}x\mathrm{d}t\\&\leqslant C(1+\tilde{C}^{\gamma_2'+1}).\end{aligned}$$

Using the previous inequality, an argument similar to that in Remark 3.2 shows that the probability measures μ^ϵ defined by (5.9) satisfy

$$\begin{aligned}&\int_\Omega\left(1+|x|^\gamma+|v|^{\gamma_2'+1}\right)d\mu^\epsilon(t,x,v)\\&=\int_0^T\int_{\mathbb{R}^2}\left(1+|x|^\gamma+|v|^{\gamma_2'+1}\right)d\mu_t^\epsilon(x,v)\mathrm{d}t\\&\leqslant C(1+\tilde{C}^{\gamma_2'+1}),\end{aligned}\tag{5.11}$$

where C and $\tilde{C}$ are independent of ϵ. Hence, $\mu^\epsilon\in\mathcal{U}^{\bar{\zeta}}\cap\mathcal{R}^+(\Omega)$, with $\bar{\zeta}=(\gamma_1,\gamma_2')$. Furthermore, (5.11) implies that the measure $\nu^\epsilon=\left(1+|x|^{\gamma_1}+|v|^{\gamma_2'}\right)\mu^\epsilon(t,x,v)$ belongs to $\mathcal{R}^+(\Omega)$ and

$$\int_\Omega\left(t^{\alpha_0}+|x|^{\alpha_0}+|v|^{\alpha_0}\right)d\nu^\epsilon(t,x,v)<C,$$

where $0<\alpha_0<\min\{(\gamma-\gamma_1),\frac{1}{\gamma}(\gamma_2'+1)(\gamma-\gamma_1),\frac{\gamma}{\gamma_2'+1},1\}$. Therefore, as $\epsilon\to0$, the sequence ν^ϵ is tight ([39], Proposition 2.23). Hence, by Prohorov's Theorem ([39], Theorem 2.29), there exists $\nu\in\mathcal{R}^+(\Omega)$ such that, up to a sub-sequence, which we still denote by ν^ϵ, ν^ϵ weakly converges to ν; that is,

$$\int_\Omega\psi(t,x,v)d\nu^\epsilon(t,x,v)\to\int_\Omega\psi(t,x,v)d\nu(t,x,v)\quad\text{for all}\quad\psi\in C_b(\Omega).\tag{5.12}$$

Now, taking $\mu=\frac{1}{1+|x|^{\gamma_1}+|v|^{\gamma_2'}}\nu$, we notice that $\mu\in\mathcal{U}^{\bar{\zeta}}\cap\mathcal{R}^+(\Omega)$. Moreover, recalling the definition of ν^ϵ from (5.12), we deduce (5.10). □

Next, we show that the weak limit provided by Proposition 5.3 belongs to $\mathcal{H}(m_0)$.

Proposition 5.4. *Suppose Assumptions 1, 2, 4-8 hold. Let $\mu\in\mathcal{R}(\Omega)$ be such that, up to a sub-sequence, the Radon measures μ^ϵ defined by* (5.9) *weakly converge to μ. Then, $\mu\in\mathcal{H}(m_0)$.*

Proof. The existence of μ is given by Proposition 5.3. By (5.11), we have that $\mu\in\mathcal{H}_1$. Let (u,m,ϖ) be the solution of (1.1) and (1.2) ([36], Theorem 1) . Let $\varphi\in C_c^1([0,T]\times\mathbb{R})$. Because m is a weak solution of the second equation in (1.1), we have

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}}\big(\varphi_t(t,x)-H_p(x,\varpi+u_x)\varphi_x(t,x)\big)m(t,x)dx\\&=\int_{\mathbb{R}}\varphi(T,x)m(T,x)\mathrm{d}x-\int_{\mathbb{R}}\varphi(0,x)m_0(x)\mathrm{d}x,\end{aligned}$$

and by (5.9)

$$\begin{aligned}\int_\Omega\varphi_t(t,x)+v\varphi_x(t,x)d\mu^\epsilon(t,x,v)&=\int_0^T\int_{\mathbb{R}^2}\varphi_t(t,x)+v\varphi_x(t,x)\ d\mu_t^\epsilon(x,v)dt\\&=\int_0^T\int_{\mathbb{R}^2}\varphi_t(t,x)-H_p(x,p)\varphi_x(t,x)\ d\beta_t^\epsilon(x,p)dt\\&=\int_0^T\int_{\mathbb{R}}\varphi_t(t,x)-H_p(x,\varpi^\epsilon+u_x^\epsilon)\varphi_x(t,x)\ m^\epsilon(t,x)dxdt.\end{aligned}$$

Now, taking into account (5.11) and arguing as in Remark 3.2, we deduce that the previous two identities also hold for any $\varphi \in \Lambda([0,T]\times\mathbb{R})$. Hence, $\mu \in \mathcal{H}_2(m_0,\nu)$ for $\nu = m(T,\cdot) \in \mathcal{P}(\mathbb{R})$. Finally, the third equation in (4.7) gives

$$\int_\Omega \eta(t)(v - Q(t))d\mu^\epsilon(t,x,v) = \int_0^T\int_\mathbb{R} \eta(t)(-H_p(x,\varpi^\epsilon + u_x^\epsilon) - Q(t))m^\epsilon(t,x)dxdt = 0,$$

for all $\eta \in C([0,T])$, which implies that $\mu \in \mathcal{H}_3$. Therefore, $\mu \in \mathcal{H}(m_0)$ as stated. □

Next, we prove the following technical lemma.

Lemma 5.5. *Let $x_n > y_n$, $y_n \to +\infty$ and $\lim_{n\to\infty}\frac{x_n}{y_n} = 1$. Suppose that $\phi \in L^1(\mathbb{R})$ and $\int_\mathbb{R}|x|^\sigma\phi\mathrm{d}x < \infty$ for some $\sigma > 0$. Then,*

$$\lim_{n\to\infty}\frac{1}{2x_n}\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y}\phi(x)\mathrm{d}x\mathrm{d}y = \int_\mathbb{R}\phi(x)\mathrm{d}x. \tag{5.13}$$

Proof. After exchanging the order of the integrals on the left-hand side in (5.13), we have

$$\begin{aligned}\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y}\phi(x)\mathrm{d}x\mathrm{d}y &= \int_{-2y_n}^{0}\phi(x)\int_{-y_n}^{x+y_n}\mathrm{d}y\mathrm{d}x\\ &+ \int_0^{x_n-y_n}\phi(x)\int_{-y_n}^{x_n}\mathrm{d}y\mathrm{d}x + \int_0^{x_n-y_n}\phi(x)\int_{x+y_n}^{x_n}\mathrm{d}y\mathrm{d}x + \int_{x_n-y_n}^{2x_n}\phi(x)\int_{-x_n+x}^{x_n}\mathrm{d}y\mathrm{d}x\\ &= 2x_n\int_{-2y_n}^{2x_n}\phi(x)\mathrm{d}x + 2(y_n-x_n)\int_{-2y_n}^{0}\phi(x)\mathrm{d}x + \int_{-2y_n}^{0}\phi(x)x\mathrm{d}x - \int_0^{2x_n}\phi(x)x\mathrm{d}x.\end{aligned}$$

Dividing the proceeding equation by $2x_n$ and letting $n\to\infty$, we deduce (5.13). □

Now, relying on the previous results, we complete the second part of the proof of Theorem 1.3. This is the content of the following Lemma.

Lemma 5.6. *Suppose Assumptions 1-8 hold. Let (u,m,ϖ) solve (1.1) and (1.2). Then*

$$\int_\mathbb{R}(u(0,x)-u_T(x))\,dm_0(x) - \int_0^T\varpi(t)Q(t)dt \geqslant \inf_{\mu\in\mathcal{H}(m_0)}\int_0^T\int_{\mathbb{R}^2}L(x,v)+vu_T'(x)d\mu(t,x,v).$$

Proof. By Assumption 1 and (1.3), the following identity holds

$$L(x,v) = H_p(x,-L_v(x,v))(-L_v(x,v)) - H(x,-L_v(x,v)). \tag{5.14}$$

Let $t\in[0,T]$. By Remark 2.1 and (5.6), we have

$$\int_\mathbb{R}L(x,-H_p(x,\varpi^\epsilon+u_x^\epsilon))m^\epsilon(t,x)dx \leqslant \int_\mathbb{R}(C_2|x|^{\gamma_1}+C)\,m^\epsilon(t,x)dx,$$

where C is independent of x and ϵ. From the previous inequality, Assumption 5 and Proposition 5.2, and an argument similar to that in Remark 3.2, we get that the integral $\int_{\mathbb{R}^2}L(x,v)d\mu_t^\epsilon(x,v)$ exists and is finite. Hence, we integrate both sides of (5.14) w.r.t. μ_t^ϵ, and we use the definition of β_t^ϵ to obtain

$$\begin{aligned}\int_{\mathbb{R}^2}L(x,v)d\mu_t^\epsilon(x,v) &= \int_{\mathbb{R}^2}H_p(x,-L_v(x,v))(-L_v(x,v)) - H(x,-L_v(x,v))d\mu_t^\epsilon(x,v)\\ &= \int_{\mathbb{R}^2}H_p(x,p)p - H(x,p)d\beta_t^\epsilon(x,p)\\ &= \int_\mathbb{R}\big(H_p(x,\varpi^\epsilon+u_x^\epsilon)(\varpi^\epsilon+u_x^\epsilon) - H(x,\varpi^\epsilon+u_x^\epsilon)\big)m^\epsilon dx.\end{aligned}\tag{5.15}$$

Let $a\in[-\frac{1}{2},0]$ be such that $|m^\varepsilon(t,a)|<\infty$. By Proposition 5.2 and Assumption 5, we have that $\int_\mathbb{R}|x|m^\epsilon dx<\infty$. Rewriting the first momentum of m^ε

$$\int_\mathbb{R}|x|m^\epsilon\mathrm{d}x = \int_{-a}^0|x|m^\epsilon\mathrm{d}x + \sum_{n=0}^\infty\int_n^{n+1}xm^\epsilon\mathrm{d}x + \int_{-n-a-1}^{-n-a}|x|m^\epsilon dx < \infty,$$

we deduce that there exists N_0 such that for all $N \geqslant N_0$

$$\int_N^{N+1} xm^\epsilon dx + \int_{-N-a-1}^{-N-a} |x| m^\epsilon \mathrm{d}x = \int_N^{N+1} xm^\epsilon dx + \int_{-N-1}^{-N} |x-a| m^\epsilon(t,x-a)\mathrm{d}x \leqslant \frac{C}{N}.$$

The previous estimates with Chebyshev's inequality imply

$$\left|\{x \in [N,N+1]: \quad xm^\epsilon > \frac{1}{\sqrt{N}}\}\right| \leqslant \sqrt{N}\int_N^{N+1} xm^\epsilon dx \leqslant \frac{C}{\sqrt{N}},$$

$$\left|\{x \in [-N-1,-N]: \quad |x-a| m^\epsilon(t,x-a) > \frac{1}{\sqrt{N}}\}\right| \leqslant \sqrt{N}\int_{-N-a-1}^{-N-a} |x| m^\epsilon dx \leqslant \frac{C}{\sqrt{N}}.$$

Because $a \in [-\frac{1}{2}, 0]$, there exists a sequence $\{x_n\}$ such that

$$\begin{aligned} &x_n \geqslant 0, \quad \lim_{n\to\infty} x_n = +\infty, \\ &\lim_{n\to\infty} x_n m^\epsilon(t,2x_n) = \lim_{n\to\infty} x_n m^\epsilon(t,-2x_n-2a) = 0. \end{aligned} \tag{5.16}$$

Let $y_n = x_n + a$.

By Assumption 5 follows that there exists $\sigma > 0$ such that $\int_{\mathbb{R}} |x|^{\gamma_1+\sigma} m_0 \mathrm{d}x < \infty$. Then, relying on Proposition 5.2 and using Lemma 5.5, we rewrite (5.15)

$$\begin{aligned} &\int_{\mathbb{R}^2} L(x,v) d\mu_t^\epsilon(x,v) = \lim_{n\to\infty} \frac{1}{2x_n}\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y} \big(H_p(x,\varpi^\epsilon+u_x^\epsilon)(\varpi^\epsilon+u_x^\epsilon) \\ &- H(x,\varpi^\epsilon+u_x^\epsilon)\big) m^\epsilon \mathrm{d}x\mathrm{d}y = \lim_{n\to\infty}\frac{1}{2x_n}\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y} H_p(x,\varpi^\epsilon+u_x^\epsilon) m^\epsilon \big(u^\epsilon-u_T\big)_x \\ &+ H_p(x,\varpi^\epsilon+u_x^\epsilon)\big(\varpi^\epsilon+u_T'\big)m^\epsilon - H(x,\varpi^\epsilon+u_x^\epsilon)m^\epsilon \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{5.17}$$

Because H is separable, $u^\epsilon(t,\cdot), u_T \in \mathrm{Lip}(\mathbb{R})$, from Proposition 5.2, we deduce

$$\lim_{n\to\infty}\frac{1}{2x_n}\left|\int_{-y_n}^{x_n} H_p(x,\varpi^\epsilon+u_x^\epsilon) m^\epsilon\big(u^\epsilon-u_T\big)\Big|_{-y_n+y}^{x_n+y} \mathrm{d}y\right| \leqslant \lim_{n\to\infty}\frac{C}{x_n}\int_{\mathbb{R}} |x| m^\epsilon \mathrm{d}x = 0.$$

Integrating by parts the first term on the right-hand side in (5.17), using the preceding equality, and the definition of β_t^ϵ, (5.17) becomes

$$\begin{aligned} &\int_{\mathbb{R}^2} L(x,v) d\mu_t^\epsilon(x,v) \\ &= \lim_{n\to\infty}\frac{1}{2x_n}\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y} -\big(H_p(x,\varpi^\epsilon+u_x^\epsilon)m^\epsilon\big)_x\big(u^\epsilon-u_T\big) - H(x,\varpi^\epsilon+u_x^\epsilon)m^\epsilon \mathrm{d}x\mathrm{d}y \\ &+ \lim_{n\to\infty}\frac{1}{2x_n}\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y}\int_{\mathbb{R}} H_p(x,p)\big(\varpi^\epsilon+u_T'\big) d\beta_t^\epsilon(x,p)\mathrm{d}y. \end{aligned} \tag{5.18}$$

Next, we prove well definiteness of several integrals. Note that Assumption 5 and the definition of μ_t^ϵ, yield

$$\begin{aligned} \left|\int_{\mathbb{R}^2} v|x|^\sigma\big(\varpi^\epsilon+u_T'\big) d\mu_t^\epsilon(x,v)\right| &= \left|\int_{\mathbb{R}^2} H_p(x,p)|x|^\sigma\big(\varpi^\epsilon+u_T'\big) d\beta_t^\epsilon(x,p)\right| \\ &= \left|\int_{\mathbb{R}} H_p(x,\varpi^\epsilon+u_x^\epsilon)|x|^\sigma\big(\varpi^\epsilon+u_T'\big) m^\epsilon \mathrm{d}x\right| \leqslant C, \end{aligned}$$

for $\gamma_1 < \gamma_1 + \sigma < \gamma$. Relying on the preceding estimate and considering Lemma 5.5, we obtain

$$-\lim_{n\to\infty}\frac{1}{2x_n}\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y}\int_{\mathbb{R}} v\big(\varpi^\epsilon+u_T'\big) d\mu_t^\epsilon(x,v) = -\int_{\mathbb{R}^2} v\big(\varpi^\epsilon+u_T'\big) d\mu_t^\epsilon(x,v). \tag{5.19}$$

By the second-order energy estimate in (4.12) and using Young's inequality, we have

$$\int_0^T\int_{\mathbb{R}} |u_{xx}^\epsilon||x|^{\frac{\gamma}{2}} m^\epsilon \mathrm{d}x\mathrm{d}t \leqslant \int_0^T\int_{\mathbb{R}} (u_{xx}^\epsilon)^2 m^\epsilon + |x|^\gamma m^\epsilon \mathrm{d}x\mathrm{d}t \leqslant C.$$

Hence, Lemma 5.5 implies that

$$\int_0^T\int_{\mathbb{R}}|u^\epsilon_{xx}|m^\epsilon \mathrm{d}x\mathrm{d}t=\lim_{n\to\infty}\frac{1}{2x_n}\int_0^T\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y}|u^\epsilon_{xx}|m^\epsilon \mathrm{d}x\mathrm{d}t\leqslant C.$$

Using the previous estimate and taking into account that $u^\epsilon\in \mathrm{Lip}(\mathbb{R})$ for all $t\in[0,T]$, we get

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{2x_n}\left|\int_0^T\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y}m^\epsilon_x u^\epsilon_x \mathrm{d}x\mathrm{d}y\mathrm{d}t\right| &\leqslant \lim_{n\to\infty}\frac{1}{2x_n}\left|\int_0^T\int_{-y_n}^{x_n}m^\epsilon(t,x_n+y)u^\epsilon_x(t,x_n+y)\right.\\ \left.-m^\epsilon(t,-y_n+y)u^\epsilon_x(t,-y_n+y)\mathrm{d}y\mathrm{d}t\right| &+\lim_{n\to\infty}\frac{1}{2x_n}\int_0^T\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y}|u^\epsilon_{xx}|m^\epsilon\mathrm{d}x\mathrm{d}t\\ \leqslant \lim_{n\to\infty}\frac{1}{x_n}\int_0^T\int_{\mathbb{R}}m^\epsilon|u^\epsilon_x|\mathrm{d}y\mathrm{d}t+\int_0^T\int_{\mathbb{R}}|u^\epsilon_{xx}|m^\epsilon\mathrm{d}y\mathrm{d}t\leqslant C.\end{aligned} \tag{5.20}$$

Because $|m^\varepsilon(t,a)|<\infty$ from (5.16), we have

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{2x_n}\left|\int_{-y_n}^{x_n}m^\epsilon_x u^\epsilon\Big|_{-y_n+y}^{x_n+y}\mathrm{d}y\right|\\ &\leqslant \lim_{n\to\infty}\frac{1}{2x_n}\left|\int_{-y_n}^{x_n}m^\epsilon_x(t,x_n+y)u^\epsilon(t,x_n+y)\mathrm{d}y-\int_{-y_n}^{x_n}m^\epsilon_x(t,-y_n+y)u^\epsilon(t,-y_n+y)\mathrm{d}y\right|\\ &=\lim_{n\to\infty}\frac{1}{2x_n}\left|\int_{x_n-y_n}^{2x_n}m^\epsilon_x(t,y)u^\epsilon(t,y)\mathrm{d}y-\int_{-2y_n}^{x_n-y_n}m^\epsilon_x(t,y)u^\epsilon(t,y)\mathrm{d}y\right|\\ &\leqslant \lim_{n\to\infty}\frac{1}{2x_n}\left(m^\epsilon(t,2x_n)|u^\epsilon(t,2x_n)|+m^\epsilon(t,-2y_n)|u^\epsilon(t,-2y_n)|+2m^\epsilon(t,a)|u^\epsilon(t,a)|\right.\\ &\left.+2\int_{\mathbb{R}}m^\epsilon|u^\epsilon_x|\mathrm{d}x\right)=0.\end{aligned} \tag{5.21}$$

Furthermore, (5.20) and (5.21), yield

$$\begin{aligned}\left|\lim_{n\to\infty}\frac{1}{2x_n}\int_0^T\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y}m^\epsilon_{xx}u^\epsilon\mathrm{d}x\mathrm{d}y\mathrm{d}t\right| &\leqslant \left|\lim_{n\to\infty}\frac{1}{2x_n}\int_0^T\int_{-y_n}^{x_n}m^\epsilon_x u^\epsilon\Big|_{-y_n+y}^{x_n+y}\mathrm{d}y\mathrm{d}t\right|\\ &+\left|\lim_{n\to\infty}\frac{1}{2x_n}\int_0^T\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y}m^\epsilon_x u^\epsilon_x\mathrm{d}x\mathrm{d}y\mathrm{d}t\right|\leqslant C.\end{aligned} \tag{5.22}$$

Note that (5.20), (5.21) and (5.22) also hold for u_T.

Because $\varpi^\epsilon\in C[0,T]$, then $H(x,\varpi^\epsilon+u^\epsilon_x)\in L^\infty_{loc}([0,T]\times\mathbb{R})$, which with the regularity of heat equation implies that $u_x\in C([0,T]\times\mathbb{R})$ and $u_{xx}\in L^p_{loc}([0,T]\times\mathbb{R})$ for every $p\in[1,\infty)$. Therefore, the second and the first equation in (4.7) imply

$$\begin{aligned}-\big(H_p(x,\varpi^\epsilon+u^\epsilon_x)m^\epsilon\big)_x\big(u^\epsilon-u_T\big)&=\left(\epsilon m^\epsilon_{xx}-m^\epsilon_t\right)\big(u^\epsilon-u_T\big),\\ -H(x,\varpi^\epsilon+u^\epsilon_x)m^\epsilon&=-\left(\epsilon u^\epsilon_{xx}+u^\epsilon_t\right)m^\epsilon.\end{aligned}$$

Relying on (5.22) and using the preceding identities and the identities in (5.19) after integrating on $[0,T]$ the equation in (5.18), we obtain

$$\begin{aligned}
&\int_\Omega L(x,v)d\mu_t^\epsilon(x,v)\mathrm{d}t = -\int_0^T\int_{\mathbb{R}^2} v\big(\varpi^\epsilon + u_T'\big)d\mu_t^\epsilon(x,v)\mathrm{d}t\\
&+\lim_{n\to\infty}\frac{1}{2x_n}\int_0^T\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y} -m_t^\epsilon\big(u^\epsilon - u_T\big) + \epsilon m_{xx}^\epsilon\big(u^\epsilon - u_T\big) - u_t^\epsilon m^\epsilon - \epsilon u_{xx}^\epsilon m^\epsilon \mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&=\lim_{n\to\infty}\frac{1}{2x_n}\int_0^T\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y} -\big(m^\epsilon\big(u^\epsilon - u_T\big)\big)_t + \epsilon m_{xx}^\epsilon\big(u^\epsilon - u_T\big) - \epsilon u_{xx}^\epsilon m^\epsilon \mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&-\int_\Omega v\big(\varpi^\epsilon + u_T'\big)d\mu_t^\epsilon(x,v)\mathrm{d}t.
\end{aligned} \tag{5.23}$$

Taking into account (5.21) and integrating by parts, we have

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{2x_n}\left|\int_0^T\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y} m_{xx}^\epsilon u^\epsilon - u_{xx}^\epsilon m^\epsilon \mathrm{d}x\mathrm{d}y\mathrm{d}t\right|\\
&=\lim_{n\to\infty}\frac{1}{2x_n}\left|\int_0^T\int_{-y_n}^{x_n} m_x^\epsilon u^\epsilon - u_x^\epsilon m^\epsilon\Big|_{-y_n+y}^{x_n+y}\mathrm{d}y\mathrm{d}t\right|\\
&\leqslant \lim_{n\to\infty}\frac{1}{2x_n}\left(\left|\int_0^T\int_{-y_n}^{x_n} m_x^\epsilon u^\epsilon\Big|_{-y_n+y}^{x_n+y}\mathrm{d}y\mathrm{d}t\right| + C\int_0^T\int_{\mathbb{R}} m^\epsilon \mathrm{d}x\mathrm{d}t\right)\\
&=0.
\end{aligned} \tag{5.24}$$

Thus, recalling the definition of μ^ϵ, (5.9), and by using (5.22), (5.24) in (5.23), we get

$$\begin{aligned}
\int_\Omega L(x,v)d\mu^\epsilon(t,x,v) &= \lim_{n\to\infty}\frac{1}{2x_n}\int_0^T\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y} -\left(m^\epsilon\left(u^\epsilon - u_T\right)\right)_t - \epsilon m_{xx}^\epsilon u_T \mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\quad - \int_\Omega v\big(\varpi^\epsilon + u_T'\big)d\mu^\epsilon(t,x,v).
\end{aligned}$$

After rearranging the terms in the previous equation, we obtain

$$\begin{aligned}
&\int_\Omega L(x,v) + vu_T' d\mu^\epsilon(t,x,v) = \lim_{n\to\infty}\frac{1}{2x_n}\int_0^T\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y}\big(u^\epsilon(0,x) - u_T(x)\big)m_0\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&-\epsilon\lim_{n\to\infty}\frac{1}{2x_n}\int_0^T\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y} m_{xx}^\epsilon u_T\mathrm{d}x\mathrm{d}y\mathrm{d}t - \int_\Omega v\varpi^\epsilon d\mu^\epsilon(t,x,v).
\end{aligned} \tag{5.25}$$

Now, we pass to the limit in (5.25) as follows. By Assumptions 1, 4, 6 and 7, Theorem 1 in [36] guarantees the existence of a sequence such that $u^\epsilon \to u$ and $\varpi^\epsilon \to \varpi$ uniformly, where, for ϖ, u solves the first equation in (1.1) in the viscosity sense. Furthermore, by Proposition 4.2, $\varpi \in W^{1,\infty}([0,T])$. Remark 1 implies that $L(x,v) + vu_T'(x)$ belongs to $C_{\bar\zeta}(\Omega)$, therefore extracting a further sub-sequence out of the previous sequence, Proposition 5.3 gives the existence of a weak limit $\mu \in \mathcal{U}^{\bar\zeta} \cap \mathcal{R}(\Omega)$ for μ^ϵ and (5.10) holds for $L(x,v) + vu_T'(x)$. Using these, by letting $\epsilon \to 0$ in (5.25), we obtain

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^2} L(x,v) + vu_T'(x)d\mu(t,x,v) &= \lim_{n\to\infty}\frac{1}{2x_n}\int_0^T\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y}(u(0,x) - u_T(x))m_0\mathrm{d}x\mathrm{d}y\\
&\quad - \int_\Omega v\varpi(t)d\mu(t,x,v).
\end{aligned} \tag{5.26}$$

Furthermore, by Proposition 4.2, $\varpi \in W^{1,\infty}([0,T])$, and by Proposition 5.4 $\mu \in \mathcal{H}(m_0)$. In particular, $\mu \in \mathcal{H}_3$. Therefore,

$$\int_0^T\int_{\mathbb{R}^2} v\varpi(t)d\mu(t,x,v) = \int_0^T Q(t)\varpi(t)\mathrm{d}t.$$

By Assumption 5 and Lemma 5.5, we deduce that

$$\lim_{n\to\infty}\frac{1}{2x_n}\int_0^T\int_{-y_n}^{x_n}\int_{-y_n+y}^{x_n+y}(u(0,x)-u_T(x))m_0\mathrm{d}x\mathrm{d}y=\int_0^T\int_{\mathbb{R}}u(0,x)-u_T(x)dm_0(x).$$

Therefore, from (5.26), we obtain

$$\int_\Omega L(x,v)+vu_T'(x)d\mu(t,x,v)=\int_{\mathbb{R}}u(0,x)-u_T(x)dm_0(x)-\int_0^T Q(t)\varpi(t)dt,$$

which completes the proof. □

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