Approximation of Hamilton-Jacobi equations with Caputo time-fractional derivative

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Abstract

In this paper, we investigate the numerical approximation of Hamilton-Jacobi equations with the Caputo time-fractional derivative. We introduce an explicit in time discretization of the Caputo derivative and a finite difference scheme for the approximation of the Hamiltonian. We show that the approximation scheme so obtained is stable under an appropriate CFL condition and converges to the unique viscosity solution of the Hamilton-Jacobi equation.

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1 Introduction

We define a class of finite difference schemes for the time-fractional Hamilton-Jacobi equation

$$\partial_t^\alpha u(t,x) + H(t,x,u(t,x),Du(t,x)) = 0 \qquad (t,x) \in Q_T := (0,T] \times \mathbb{T}^d, \tag{1.1}$$

where $\mathbb{T}^d$ is unit torus in $\mathbb{R}^d$. The symbol ∂_t^α, for $0 < \alpha \leq 1$, denotes the Caputo derivative

$$\partial_t^\alpha u(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial_s u(s,x)}{(t-s)^\alpha} ds$$

(note that ∂_t^α reduces to the standard time derivative ∂_t for $\alpha = 1$). Equation (1.1) is completed with the initial condition

$$u(x,0) = u_0(x), \qquad x \in \mathbb{T}^d. \tag{1.2}$$

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In recent times, there has been an increasing interest in the study of differential equations with time-fractional derivatives. Indeed, this kind of differential operators allows us to introduce new phenomena in differential models such as memory and trapping effects [11, 12, 15, 17]. Also, the numerical approximation of differential equations with fractional time-derivative has been extensively analyzed [3, 9, 10].

Since in general smooth solutions to Hamilton-Jacobi equations are not expected to exist, for equation (1.1) a theory of weak solutions, in viscosity sense, has been introduced, in [8, 13, 18]. Most of the results and techniques which hold in the classical case, i.e., for $\alpha = 1$, have been extended to the fractional case in order to prove the well-posedness of the Hamilton-Jacobi equation (1.1).

In the classical case, one of the most important properties of the viscosity solution theory is the stability with respect to the uniform convergence (see [2]). Starting with the seminal paper [5], this property has generated an enormous literature concerning the numerical approximation of Hamilton-Jacobi equations (see for example [6, 14, 16] and reference therein). Stability with respect to the uniform convergence is inherited by viscosity solutions of the Hamilton-Jacobi equation (1.1). Following [5], we define a general class of finite difference schemes for (1.1). We show that, under an appropriate CFL condition of the type $\Delta t^\alpha = O(\Delta x)$, these schemes are monotone, stable and consistent. Moreover, relying on an adaptation of the classical Barles-Souganidis convergence Theorem [4], we prove that the numerical solutions generated by these schemes converge to the unique viscosity solution of the limit problem. In order to verify the properties of the proposed schemes, we perform several numerical tests and, to analyze the order of the approximation error, we also compute exact solutions for some time-fractional Hamilton-Jacobi equations. We have only recently become aware that a similar problem was considered in [7].

The rest of the paper is organized as follows. In Section 2, we shortly review some basic properties of the theory of viscosity solution for (1.1). Section 3 is devoted to the description of a class of finite difference schemes and their properties. In Section 4, we prove a convergence result and in Section 5 we carry out some numerical tests.

2 Viscosity solutions for Hamilton-Jacobi equation with time-fractional derivative

In this section, we briefly review definitions and some results for the continuous problem (1.1) (we refer to [8, 13] for more details). For a function $f : [0, T] \to \mathbb{R}$ such that $f \in C^1((0, T]) \cap C([0, T])$ and $f' \in L^1((0, T))$, the Caputo time fractional derivative is defined by

$$\partial_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds, \tag{2.1}$$

for any $t \in (0, T]$. Using integration by parts and change of variables, (2.1) can be rewritten as

$$\partial_t^\alpha f(t) = J[f](t) + K_{(0,t)}[f](t), \tag{2.2}$$

where

$$J[f](t) := \frac{f(t) - f(0)}{t^\alpha \Gamma(1-\alpha)},$$
$$K_{(0,t)}[f](t) := \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{f(t) - f(t-\tau)}{\tau^{\alpha+1}} d\tau.$$

The advantage of rewriting the Caputo derivative in the form (2.2) is explained in [1, 8, 18]. We denote by $USC(\overline{Q_T})$ (resp., $LSC(\overline{Q_T})$) the class of the upper semi-continuous (resp., lower semi-continuous) functions in $\overline{Q_T}$. The class of the test functions for the problem (1.1) is given by

$$\mathcal{C} := \{\varphi \in C^1((0,T] \times \mathbb{T}^d) \cup C([0,T] \times \mathbb{T}^d) \,|\, \partial_t \varphi(\cdot, x) \in L^1(0,t) \text{ for every } x \in \mathbb{T}^d\}.$$

Definition 2.1. A function $u \in USC(\overline{Q_T})$ (resp. $LSC(\overline{Q_T})$) is said a viscosity subsolution (resp. supersolution) of (1.1)–(1.2) if

- for any $\varphi \in \mathcal{C}$ and for any $(\hat{t}, \hat{x}) \in (0,T] \times \mathbb{T}^d$ such that

$$\max_{(0,T] \times \mathbb{T}^d} (u - \varphi) = (u - \varphi)(\hat{t}, \hat{x}) \quad (\text{resp. } \min_{(0,T] \times \mathbb{T}^d})$$

  then

$$J[u](\hat{t}, \hat{x}) + K_{(0,t)}[\phi](\hat{t}, \hat{x}) + H(\hat{t}, \hat{x}, u(\hat{t}, \hat{x}), D\varphi(\hat{t}, \hat{x})) \le 0 \quad (\text{resp. } \ge 0);$$

- $u(0,x) \le u_0(x)$ (resp. $u(0,x) \ge u_0(x)$) for any $x \in \mathbb{T}^d$.

If a function $u : \overline{Q_T} \to \mathbb{R}$ is both a viscosity sub- and supersolution, then u is said a viscosity solution of (1.1)-(1.2).

For other equivalent definitions of viscosity solutions for (1.1), we refer to [8]. We consider the following assumptions on the Hamiltonian H and on the initial datum u_0.

(H1) $H : \overline{Q_T} \times \mathbb{R} \times \mathbb{R}^d$ is continuous;

(H2) there exists a modulus $\omega : [0, \infty) \to [0, \infty)$ such that

$$|H(t,x,r,p) - H(t,y,r,p)| \le \omega(|x-y|(1+|p|))$$

for all (t,x,r,p), $(t,y,r,p) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$;

(H3) $r \mapsto H(t,x,r,p)$ is nondecreasing for all $(t,x,p) \in Q_T \times \mathbb{R}^d$;

(H4) $u_0 : \mathbb{T}^d \to \mathbb{R}$ is a continuous function.

The first result is a comparison principle for (1.1).

Theorem 2.2. Assume (H1)-(H3). Let $u \in USC(\overline{Q_T})$ and $v \in LSC(\overline{Q_T})$ be a subsolution and a supersolution of (1.1), respectively. If $u(x,0) \le v(x,0)$ for $x \in \mathbb{T}^d$, then $u \le v$ on Q_T.

The proof of the previous result is based on an adaptation of the classical doubling of variables method in viscosity solution theory.
We also recall an existence result for viscosity solutions of (1.1). For a locally bounded function $u : \overline{Q_T} \to \mathbb{R}$, u^* and u_* denote respectively the upper and lower semi-continuous envelope, defined for $(t,x) \in \overline{Q_T}$ by

$$u^*(t,x) = \lim_{\delta\to 0^+} \sup\{u(s,y)\,|\,(s,y) \in B((t,x),\delta) \cap \overline{Q_T}\},$$

and by $u_*(x) = -(-u)^*$.

Theorem 2.3. Assume (H1). Let $u^- \in USC(\overline{Q_T})$ and $u^+ \in LSC(\overline{Q_T})$ be a subsolution and a supersolution of (1.1) such that $(u^-)_* > -\infty$ and $(u^+)^* < +\infty$ on $\overline{Q_T}$. If $u^- \le u^+$, then there exists a solution u of (1.1) that satisfies $u^- \le u \le u^+$ in $\overline{Q_T}$.

By Theorems 2.2 and 2.3, it follows an existence and uniqueness result for the solution of (1.1)-(1.2).

Corollary 2.4. Assume (H1)-(H4). Then there exists a unique continuous viscosity solution of (1.1)-(1.2).

Existence and uniqueness results for the problem of (1.1)-(1.2) in a bounded domain with boundary conditions in viscosity sense are discussed in [13].

3 A class of finite difference schemes

In this section we describe a finite difference scheme for the approximation of (1.1). For simplicity of notations, we assume that the Hamiltonian H depends only on the state and gradient variables, i.e. $H = H(x,p)$, and that the dimension d is equal to 2. The extension for general H and d will be clear from this special case.
Let $\mathbb{T}_h^2$ be a uniform grid on the torus with step h, (this supposes that $1/h$ is an integer), and denote by $x_{i,j}$ a generic point in $\mathbb{T}_h^2$ (an anisotropic mesh with steps h_1 and h_2 is possible too and we have taken $h_1 = h_2$ only for simplicity). The value $U_{i,j}^n$ denotes the numerical approximation of the function u at $(x_{i,j}, t_n) = (ih, jh, n\Delta t)$, $i,j \in \mathbb{Z}$, $n = 0,\dots,N$ (assuming that $N = T/\Delta t$ is an integer). We also denote by U^n the grid function taking the value $U_{i,j}^n$ at $x_{i,j} \in \mathbb{T}_h^2$.
We start by describing the numerical approximation of the Caputo time-fractional derivative ∂_t^α introduced in [9]. The numerical derivative is obtained by approximating the time-derivative inside the fractional integral in (2.1) via finite difference and writing in compact form the expression so obtained. We approximate $\partial_t^\alpha u(x_{i,j}, t_{n+1})$ by

$$\begin{aligned} D_{\Delta t}^\alpha U_{i,j}^{n+1} &= \frac{1}{\Gamma(1-\alpha)} \sum_{m=0}^{n} \int_{t_m}^{t_{m+1}} \frac{U_{i,j}^{m+1} - U_{i,j}^m}{\Delta t} \frac{1}{(t_{n+1}-s)^\alpha} ds \\ &= \frac{1}{\Gamma(1-\alpha)(1-\alpha)} \sum_{m=0}^{n} \frac{U_{i,j}^{m+1} - U_{i,j}^m}{\Delta t} \left(-\frac{1}{(t_{n+1}-t_{m+1})^{\alpha-1}} + \frac{1}{(t_{n+1}-t_m)^{\alpha-1}} \right) \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{m=0}^{n} \frac{(n+1-m)^{1-\alpha} - (n-m)^{1-\alpha}}{\Delta t^\alpha} (U_{i,j}^{m+1} - U_{i,j}^m), \end{aligned} \tag{3.1}$$

since $t_n - t_m = (n-m)\Delta t$. Defined

$$\rho_\alpha = \Gamma(2-\alpha)\Delta t^\alpha, \tag{3.2}$$

we obtain by (3.1)

$$\begin{aligned}
\rho_\alpha D^\alpha_{\Delta t} U^{n+1}_{i,j} &= \sum_{m=0}^{n} \Big((n+1-m)^{1-\alpha} - (n-m)^{1-\alpha}\Big)(U^{m+1}_{i,j} - U^m_{i,j}) \\
&= -\Big((n+1)^{1-\alpha} - n^{1-\alpha}\Big)U^0_{i,j} \\
&\quad - \sum_{m=1}^{n} \Big(2(n+1-m)^{1-\alpha} - (n+2-m)^{1-\alpha} - (n-m)^{1-\alpha}\Big)U^m_{i,j} + U^{n+1}_{i,j} \\
&= U^{n+1}_{i,j} - \sum_{m=0}^{n} c^{n+1}_m U^m_{i,j},
\end{aligned}$$

where

$$\begin{aligned}
c^{n+1}_0 &= (n+1)^{1-\alpha} - n^{1-\alpha} \\
c^{n+1}_m &= 2(n+1-m)^{1-\alpha} - (n+2-m)^{1-\alpha} - (n-m)^{1-\alpha}
\end{aligned}$$

for $1 \le m \le n$. Thus, the approximation of the Caputo time-derivative is given by

$$D^\alpha_{\Delta t} U^n_{i,j} = \frac{1}{\rho_\alpha}\left(U^{n+1}_k - \sum_{m=0}^{n} c^{n+1}_m U^j_k\right). \tag{3.3}$$

Remark 3.1. Denoted by $r^{n+1}_{\Delta t}$ the truncation error, in [9] it is proved that

$$r^{n+1}_{\Delta t} \le c_u \Delta t^{2-\alpha}$$

where c_u is a constant depending on the second order time-derivative of u. Hence the temporal accuracy of the scheme is of order $2-\alpha$.

In the following we summarize some properties of the coefficients c_m in (3.3)

Lemma 3.2. (i) $c^{n+1}_m > 0$ for $0 \le m \le n$.

(ii) $c^{n+2}_0 - c^{n+1}_0 = -c^{n+2}_1$.

(iii) $c^{n+2}_{m+1} = c^{n+1}_m$ for $1 \le m \le n$.

(iv) $\sum_{m=0}^{n} c^{n+1}_m = 1$.

Proof. (i) When $m = 0$, it is clear that $c^{n+1}_0 > 0$. Consider the case where $1 \le m \le n$. Because of the strong concavity of the function $x^{1-\alpha}$ for $x \ge 0$, by Jensen's inequality, we have

$$\frac{(n+2-m)^{1-\alpha} + (n-m)^{1-\alpha}}{2} < (n+1-m)^{1-\alpha}.$$

Thus, it follows that $c^{n+1}_m > 0$.

(ii) By definition,

$$c_0^{n+2} - c_0^{n+1} = (n+2)^{1-\alpha} - 2(n+1)^{1-\alpha} + n^{1-\alpha} = -c_1^{n+2}.$$

(iii) By definition,

$$\begin{aligned} c_{m+1}^{n+2} &= 2((n+2)-(m+1))^{1-\alpha} - ((n+3)-(m+1))^{1-\alpha} - ((n+1)-(m+1))^{1-\alpha} \\ &= 2(n+1-m)^{1-\alpha} - (n+2-m)^{1-\alpha} - (n-m)^{1-\alpha} = c_m^{n+1}. \end{aligned}$$

(iv) We have

$$\begin{aligned} \sum_{m=0}^{n} c_m^{n+1} &= (n+1)^{1-\alpha} - n^{1-\alpha} + \sum_{m=1}^{n} 2(n+1-m)^{1-\alpha} - (n+2-m)^{1-\alpha} - (n-m)^{1-\alpha} \\ &= (n+1)^{1-\alpha} - n^{1-\alpha} + \sum_{m=1}^{n} 2(n+1-m)^{1-\alpha} - \sum_{m=0}^{n-1}(n+1-m)^{1-\alpha} - \sum_{m=2}^{n+1}(n+1-m)^{1-\alpha} \\ &= (n+1)^{1-\alpha} - n^{1-\alpha} + 2n^{1-\alpha} + 2 - (n+1)^{1-\alpha} - n^{1-\alpha} - 1 = 1. \end{aligned}$$

□

For the approximation of the Hamiltonian in (1.1) we follow the approach in [5]. We introduce the finite difference operators

$$(D_1^+U)_{i,j} = \frac{U_{i+1,j} - U_{i,j}}{h} \quad \text{and} \quad (D_2^+U)_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{h}, \tag{3.4}$$

and define

$$[D_hU]_{i,j} = \left((D_1^+U)_{i,j}, (D_1^+U)_{i-1,j}, (D_2^+U)_{i,j}, (D_2^+U)_{i,j-1}\right)^T. \tag{3.5}$$

In order to approximate the Hamiltonian H in equation (1.1), we consider a numerical Hamiltonian $g : \mathbb{T}^2 \times \mathbb{R}^4 \to \mathbb{R}$, $(x, q_1, q_2, q_3, q_4) \mapsto g(x, q_1, q_2, q_3, q_4)$ satisfying the following conditions:

(G1) g is non increasing with respect to its second and fourth arguments, and nondecreasing with respect to its third and fifth arguments.

(G2) g is consistent with the Hamiltonian H, i.e.

$$g(x, q_1, q_1, q_2, q_2) = H(x, q), \quad \forall x \in \mathbb{T}^2, \forall q = (q_1, q_2) \in \mathbb{R}^2.$$

(G3) g is locally Lipschitz continuous.

Hence, recalling the approximation (3.3) of the Caputo time derivative, we consider the explicit finite difference scheme

$$\frac{1}{\rho_\alpha}\left(U_{i,j}^{n+1} - \sum_{m=0}^{n} c_m^{n+1} U_{i,j}^m\right) + S(x_{i,j}, h, U_{i,j}^n, [U^n]_{i,j}) = 0, \tag{3.6}$$

for $i,j=1,\dots,1/h$, $n=0,\dots,N-1$, where ρ_α is defined as in (3.2) and

$$S(x_{i,j},h,U^n_{i,j},[U^n]_{i,j})=g(x_{i,j},(D_1^+U^n)_{i,j},(D_1^+U^n)_{i-1,j},(D_2^+U^n)_{i,j},(D_2^+U^n)_{i,j-1}). \quad (3.7)$$

The scheme is completed with the initial condition

$$U^0_{i,j}=u_0(x_{i,j}). \quad (3.8)$$

Note that U^{n+1} depends on all the past history U^m, $m=0,\dots,n$ of the solution.
For $\alpha=1$, the scheme (3.6) reduces to the standard finite difference approximation

$$\frac{U^{n+1}_{i,j}-U^n_{i,j}}{\Delta t}+S(x_{i,j},h,U^n_{i,j},[U^n]_{i,j})=0$$

of the Hamilton-Jacobi equation

$$\partial_t u+H(x,Du)=0.$$

3.1 Stability properties of the scheme

We set $Q_n^{h,\Delta t}=\mathbb{T}_h^2\times\{0,\dots,n\Delta t\}$ and we denote by $\mathcal{G}$ the space of the grid functions on $\mathbb{T}^2_h$ and by $\mathcal{G}^n$, $n=0,\dots,N$, the set of the grid function on $Q_n^{h,\Delta t}$, i.e.

$$\mathcal{G}^n=\left\{U=\{U^m\}_{m=0}^n \,|\, U^m:\mathbb{T}^2_h\to\mathbb{R}\right\}.$$

Moreover, we set $\|U\|_\infty=\sup_{i,j}|U_{i,j}|$ for $U=\{U_{i,j}\}_{i,j=0}^{1/h}\in\mathcal{G}$, and $\|U\|_\infty=\sup_{m=0,\dots,n}\|U^m\|_\infty$ for $U=\{U^m\}_{m=0}^n\in\mathcal{G}^n$.
For $n\in\{0,\dots,N-1\}$,, we define a map $G^n:\mathcal{G}^n\to\mathcal{G}$ by

$$G^n(U)_{i,j}=\sum_{m=0}^n c_m^{n+1}U^m_{i,j}-\rho_\alpha S(x_{i,j},h,U^n_{i,j},[U^n]_{i,j}). \quad (3.9)$$

Hence the scheme (3.6) can be rewritten in the equivalent iterative form

$$U^{n+1}_{i,j}=G^n(U)_{i,j}, \qquad i,j=1,\dots,\frac{1}{h},\ n=0,\dots,N-1 \quad (3.10)$$

Definition 3.3. We say that the scheme (3.10) is *monotone* if, for any $n=0,\dots,N-1$, $U,V\in\mathcal{G}^n$, we have that

$$U^m\le V^m,\ m=0,\dots,n, \quad \Longrightarrow \quad G^n(U)\le G^n(V),$$

where the previous inequalities are intended in the sense of the comparison of components.

Since the scheme (3.10) is explicit, for the monotonicity, we need some restriction on the approximation steps h and Δt, as we will discuss later on.

Proposition 3.4. Assume that the scheme (3.6) is monotone. Then, for $n=0,\dots,N-1$, we have

(i) $G^n(U+\lambda) = G^n(U) + \lambda$ for any $\lambda \in \mathbb{R}$, $U \in \mathcal{G}^n$ (where we identify λ with the constant function on $\mathbb{T}_h^2$);

(ii) $\|G^n(U) - G^n(V)\|_\infty \le \|U - V\|_\infty$ for any $U, V \in \mathcal{G}^n$;

(iii) $\|D_h G^n(U)\|_\infty \le \|D_h U\|_\infty$ for any $U \in \mathcal{G}^n$;

(iv) for any $U \in \mathcal{G}^{n+1}$

$$\|G^{n+1}(U) - G^n(U)\|_\infty \le (1 - c_0^{n+2}) \sup_{m=0,\dots,n} \|U^{m+1} - U^m\|_\infty + 2\Gamma(2-\alpha)\Delta t^\alpha K,$$

where $K = \sup_{m=0,\dots,n} \|g(x, D_h U^m)\|_\infty$;

(v) for any $U \in \mathcal{G}^n$

$$\|G^n(U)\|_\infty \le \|U\|_\infty + \Gamma(2-\alpha)\Delta t^\alpha \|H(x,0)\|_\infty.$$

Proof. (i) By Lemma 3.2, we have

$$\begin{aligned} G^n(U+\lambda)_{i,j} &= \sum_{m=0}^{n} c_m^{n+1}(U^m+\lambda)_{i,j} - \rho_\alpha S(x_{i,j}, h, U_{i,j}^n + \lambda, [U^n+\lambda]_{i,j}) \\ &= \sum_{m=0}^{n} c_m^{n+1}(U^m) + \lambda - \rho_\alpha S(x_{i,j}, h, U_{i,j}^n, [U^n]_{i,j}) = G^n(U) + \lambda. \end{aligned}$$

(ii) Let $U, V \in C$ and $\lambda = \|(U-V)^+\|_\infty$. We have, in the sense of the comparison of components,

$$U = V + (U - V) \le V + \|(U-V)^+\|_\infty = V + \lambda.$$

By monotonicity and commutativity,

$$G^n(U) \le G^n(V+\lambda) = G^n(V) + \lambda.$$

Hence, $G^n(U) - G^n(V) \le \|(U-V)^+\|_\infty$, and we get the reverse inequality analogously.

(iii) Let τ be a translation operator in space, that is, $\tau_l U_{i,j} = U_{i+l_1, j+l_2}$ for $l = (l_1, l_2) \in \mathbb{Z}^2$. Then $\tau_l G^n(U) = G^n(\tau_l U)$ for all $U \in \mathcal{G}^n$. Hence, by property (ii)

$$\begin{aligned} \|D_+^1 G^n(U)\|_\infty &= \left\| \frac{\tau_{(1,0)} G^n(U) - G^n(U)}{h} \right\|_\infty = \left\| \frac{G^n(\tau_{(1,0)} U) - G^n(U)}{h} \right\|_\infty \\ &\le \left\| \frac{\tau_{(1,0)} U - U}{h} \right\|_\infty = \|D_+^1 U\|_\infty \end{aligned}$$

and similarly for the other components of $D_h G(U)$. Note that the previous property implies that, if $\|D_h U\|_\infty \le R$, then $\|D_+ G^n(U)\|_\infty \le R$.

(iv) Using Lemma 3.2, we have

$$
\begin{aligned}
|G^{n+1}(U)_{i,j} - G^n(U)_{i,j}| &= \Bigg|\sum_{m=0}^{n}(c_m^{n+2} - c_m^{n+1})U_{i,j}^m + c_{n+1}^{n+2}U_{i,j}^{n+1} \\
&\quad -\rho_\alpha\Big(S(x_{i,j},h,U_{i,j}^{n+1},[U^{n+1}]_{i,j}) - S(x_{i,j},h,U_{i,j}^n,[U^n]_{i,j})\Big)\Bigg| \\
&= \Bigg|c_0^{n+2}U_{i,j}^0 - c_0^{n+1}U_{i,j}^0 + \sum_{m=0}^{n} c_{m+1}^{n+2}U_{i,j}^{m+1} - \sum_{m=1}^{n} c_m^{n+1}U_{i,j}^m \\
&\quad - \rho_\alpha\Big(S(x_{i,j},h,U_{i,j}^{n+1},[U^{n+1}]_{i,j}) - S(x_{i,j},h,U_{i,j}^n,[U^n]_{i,j})\Big)\Bigg| \\
&\le \left|\sum_{m=0}^{n} c_{m+1}^{n+2}(U_{i,j}^{m+1} - U_{i,j}^m)\right| \\
&\quad + \rho_\alpha\left|S(x_{i,j},h,U_{i,j}^{n+1},[U^{n+1}]_{i,j}) - S(x_{i,j},h,U_{i,j}^n,[U^n]_{i,j})\right| \\
&\le (1 - c_0^{n+2})\|U^{n+1} - U^n\|_\infty + 2\Gamma(2-\alpha)\Delta t^\alpha K.
\end{aligned}
$$

(v) By the consistency of scheme, it follows that $G^n(0) = -\rho_\alpha H(x_{i,j}, 0)$. Hence, by property (ii), we have

$$
\|G^n(U)\|_\infty \le \|G^n(U) - G^n(0)\|_\infty + \|G^n(0)\|_\infty \le \|U\|_\infty + \Gamma(2-\alpha)\Delta t^\alpha\|H(x,0)\|_\infty.
$$

□

Proposition 3.5. Let $\{U^n\}$ be the sequence generated by scheme (3.6) with the initial condition (3.8). Then

$$
\|U^n - U^0\|_\infty \le \frac{K\Gamma(2-\alpha)}{\alpha(1-\alpha)}(n\Delta t)^\alpha, \tag{3.11}
$$

where $K = \sup\{|g(x,q)| : x \in \mathbb{T}^2,\ |q| \le R\}$ and $R = \sup_{i,j}|[D_h U_0]_{i,j}|$.

Proof. For $n = 1$, (3.11) is true since $U^1 = U^0 - \rho_\alpha S(x_{i,j}, h, U_{i,j}^0, [U^0]_{i,j})$ with S defined as in (3.7). Arguing by induction, assume now that (3.11) is true for $j \le n$. Then by Lemma 3.2, (iv) and Proposition 3.4, (iii), we have

$$
\begin{aligned}
|U_{i,j}^{n+1} - U_{i,j}^0| &= \left|\sum_{m=0}^{n} c_m^{n+1}(U_{i,j}^m - U_{i,j}^0) - \rho_\alpha S(x_{i,j},h,U_{i,j}^n,[U^n]_{i,j})\right| \\
&\le \left|\sum_{m=0}^{n} c_m^{n+1}(U_{i,j}^m - U_{i,j}^0)\right| + K\Gamma(2-\alpha)\Delta t^\alpha \qquad (3.12)\\
&\le \left(\frac{1}{\alpha(1-\alpha)}\sum_{m=0}^{n} c_m^{n+1}m^\alpha + 1\right) K\Gamma(2-\alpha)\Delta t^\alpha.
\end{aligned}
$$

We observe that

$$
\sum_{m=0}^{n} c_m^{n+1}m^\alpha = (n+1)^\alpha - \sum_{m=0}^{n}((n+1-m)^{(1-\alpha)} - (n-m)^{(1-\alpha)})((m+1)^\alpha - m^\alpha).
$$

Moreover, by the inequality $(r+1)^\beta - r^\beta \geq \beta r^{-(1-\beta)}$ for $r > 0$ and $\beta \in (0,1)$, we get

$$
\begin{aligned}
&(n+1-m)^{(1-\alpha)} - (n-m)^{(1-\alpha)} \geq \frac{1-\alpha}{(n+1)^\alpha}, \\
&(m+1)^\alpha - m^\alpha \geq \frac{\alpha}{(n+1)^{1-\alpha}}.
\end{aligned}
$$

Hence

$$
\sum_{m=0}^{n} c_m^{n+1} m^\alpha \leq (n+1)^\alpha - \alpha(1-\alpha),
$$

and replacing the previous inequality in (3.12), we get estimate (3.11). □

We discuss some classical examples of approximation scheme for Hamilton-Jacobi equations adapted to the fractional case. We consider the equation

$$
\partial_t^\alpha u(t,x) + H(Du(t,x)) = 0 \quad \text{for } (t,x) \in (0,T] \times \mathbb{R} \tag{3.13}
$$

with periodic boundary condition.

Upwind scheme
Simple upwind schemes for the equation (3.13) are

$$
U_j^{n+1} = \sum_{m=0}^{n} c_m^{n+1} U_j^m - \rho_\alpha H\left(\frac{U_{j+1}^n - U_j^n}{h}\right) \tag{3.14}
$$

if H is non-increasing, or

$$
U_j^{n+1} = \sum_{m=0}^{n} c_m^{n+1} U_j^m - \rho_\alpha H\left(\frac{U_j^n - U_{j-1}^n}{h}\right) \tag{3.15}
$$

if H is non-decreasing. The numerical Hamiltonian is given by $g(q_1,q_2) = H(q_1)$, in the first case, and by $g(q_1,q_2) = H(q_2)$ in the second case. In both cases, g is monotone, consistent and regular if H is locally Lipschitz. For the monotonicity of the previous schemes, since by Lemma 3.2 all the coefficients c_m^{n+1} are positive, the map G^n is increasing with respect to the variable U^m, $m = 0,\dots,n-1$. Moreover, (3.14) is monotone with respect to U_j^n if $c_n^{n+1} - \frac{\rho_\alpha}{h} H'(p) \geq 0$. Recalling that $c_n^{n+1} = 2 - 2^{1-\alpha}$, we get the CFL condition

$$
\frac{\Delta t^\alpha}{h} |H'(p)| \leq \frac{2-2^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.16}
$$

The same condition is necessary also for (3.15).

Lax-Friedrichs scheme
The Lax-Friedrichs scheme is given by

$$
U_j^{n+1} = \sum_{m=0}^{n} c_m^{n+1} U_j^m - \rho_\alpha \left[H\Big(\frac{U_{j+1}^n - U_{j-1}^n}{2h}\Big) - \frac{(U_{j+1}^n + U_{j-1}^n - 2U_j^n)\theta}{\rho_\alpha} \right], \tag{3.17}
$$

where θ has to be chosen in order to satisfy the CFL condition. Therefore, the numerical Hamiltonian g is

$$g(a,b) = H\Big(\frac{a+b}{2}\Big) - \frac{(a-b)\theta}{\lambda^x}$$

for $\lambda^x = \rho_\alpha/h$ and $a, b \in \mathbb{R}$. For the monotonicity of the scheme with respect to U_j^n, we need the condition

$$c_n^{n+1} - 2\theta \geq 0.$$

and, for the monotonicity with respect to $U_{j\pm1}^n$,

$$\theta - \rho_\alpha \frac{|H'(p)|}{2h} \geq 0.$$

Then the monotonicity of the scheme is implied by the CFL condition

$$\frac{\rho_\alpha |H'(p)|}{2h} \leq \theta \leq 1 - 2^{-\alpha} \tag{3.18}$$

Remark 3.6. The CFL conditions (3.16) and (3.18) reduce to the classical ones for $\alpha = 1$. In general, they become more and more restrictive for α decreasing to 0^+. This phenomenum has been also observed in [10] in the study of approximation schemes for time-fractional conservation laws.

4 A convergence result for the finite difference scheme

In this section, we prove the convergence of the scheme (3.6) following the classical stability argument in [4], where it is proved that a *monotone*, *stable* and *consistent* approximation scheme converges to the unique solution of the continuous Hamilton-Jacobi equations.

We recall the definition of the relaxed limit for a locally bounded sequence $\{u_\rho\}_{\rho>0}$. The upper relaxed limit is given by

$$(\limsup_{\rho\to0^+}{}^* u_\rho)(t,x) = \lim_{\delta\to0} \sup \Big\{ u_\rho(s,y) \,:\, (s,y) \in Q_T \cap \overline{B_\delta(t,x)},\, 0 < \rho < \delta \Big\},$$

while the lower relaxed limit by $\liminf^*_{\rho\to0^+} u_\rho = -\limsup^*_{\rho\to0^+}(-u_\rho)$.
We set $\rho = (\Delta t, h)$ and we denote with u_ρ the piecewise constant extension to Q_T of the solution of the approximation scheme (3.6) corresponding to the parameter ρ.

Theorem 4.1. We assume that the scheme (3.6) is monotone, the numerical Hamiltonian g satisfies (G1)-(G3) and u_0 is Lipschitz continuous. As $\rho \to 0^+$, the sequence $\{u_\rho\}_{\rho>0}$ given by the scheme (3.6) converges uniformly to the unique viscosity solution u of (1.1).

Proof. In order to apply the Barles-Souganidis' convergence result, we define for $(t,x) \in Q_T$

$$\overline{u}(t,x) = (\limsup_{\rho\to0^+}{}^* u_\rho)(t,x),$$
$$\underline{u}(t,x) = (\liminf_{\rho\to0^+}{}^* u_\rho)(t,x).$$

Note that, by definition, $\underline{u}(t,x) \le \overline{u}(t,x)$. We claim that $\overline{u}$, $\underline{u}$ are, respectively, a viscosity subsolution and a viscosity supersolution of (1.1) such that $\overline{u}(0,x) \le \underline{u}(0,x)$ for $x \in \mathbb{T}^2$. If the claim holds, then from Theorem 2.2 it follows that $\overline{u}(t,x) \le \underline{u}(t,x)$ and therefore $u = \overline{u} \equiv \underline{u}$ is the unique viscosity solution of (1.1) in Q_T. Moreover, the definition of $\overline{u}$, $\underline{u}$ implies the uniform convergence of $\{u_\rho\}_{\rho>0}$ to u.
To prove the claim, we first observe that (3.11) and the continuity of u_0 implies that $\underline{u} = \overline{u} = u_0(x)$ for $x \in \mathbb{T}^2$. Clearly, by (3.8) we have $\underline{u} \le u_0 \le \overline{u}$. Moreover, if $(s_\rho, y_\rho) \to (0,x)$ for $\rho = (\Delta t_\rho, h_\rho) \to 0$, then define $n_\rho = [s_\rho/\Delta t_\rho]$ and let i_ρ, j_ρ be such that $y_\rho \in [(i_\rho - 1/2)h_\rho, (i_\rho + 1/2)h_\rho] \times [(j_\rho - 1/2)h_\rho, (j_\rho - 1/2)h_\rho]$. We have

$$u_\rho(s_\rho, y_\rho) = U^{n_\rho}_{i_\rho, j_\rho} \le U^0_{i_\rho, j_\rho} + 2K\Gamma(2-\alpha)(n_\rho \Delta t)^\alpha \le u_0(x) + L_0|x - (ih, jh)|,$$

where L_0 is the Lipschitz constant of u_0 and $K = \sup\{|g(x,q)| : x \in \mathbb{T}^2,\ |q| \le L_0\}$. Passing to the limit in the previous inequality for $\rho \to 0^+$, we get $\limsup_\rho u_\rho(s_\rho, y_\rho) \le u_0(x)$ which implies, for the arbitrariness of the sequence (s_ρ, y_ρ), $\overline{u}(0,x) \le u_0(x)$. We prove similarly that $\underline{u}(0,x) \ge u_0(x)$.
The stability of the scheme (3.10), i.e. the sequence $\{u_\rho\}_{\rho>0}$ bounded uniformly in ρ, is clearly implied by property (v) in Prop. 3.4.
To prove the consistency of the scheme, we claim that, given a test function φ and a sequence $(t_\rho, x_\rho) = (n_\rho \Delta t_\rho, (i_\rho h_\rho, j_\rho h_\rho))$ converging to $(t,x) \in Q_T$ for $\rho \to 0$, then we have

$$\lim_{\rho \to 0} D^\alpha_{\Delta t}\varphi(t_\rho, x_\rho) + S\left(x_\rho, h_\rho, \varphi(t_\rho, x_\rho), [\varphi(t_\rho, \cdot)]_{(i_\rho, j_\rho)}\right) = \partial^\alpha_t \varphi(t,x) + H(x, D\varphi(t,x)), \quad (4.1)$$

where

$$D^\alpha_{\Delta t}\varphi(t_\rho, x_\rho) = \frac{1}{\rho_\alpha}\left(\varphi(t_\rho + \Delta t_\rho, x_\rho) - \sum_{m=0}^{n_\rho} c^{n_\rho+1}_m \varphi(t_\rho - (n_\rho - m)\Delta t_\rho, x_\rho)\right)$$

Since, by the assumptions (G1)-(G3) for the numerical Hamiltonian g, it is straightforward to prove that

$$\lim_{\rho \to 0} S\left(x_\rho, h_\rho, \varphi(t_\rho, x_\rho), [\varphi(t_\rho, \cdot)]_{i_\rho, j_\rho}\right) = H(x, D\varphi(t,x)),$$

we focus on proving the convergence of the discrete time-derivative to the continuous one. To simplify the notation, since in this argument only the time variable is involved, we omit the dependence of φ on x. Because of the continuity of the Caputo derivative of φ with respect to t (see [13, Prop. 2.1]), it is sufficient to prove that

$$\lim_{\rho \to 0} \left(D^\alpha_{\Delta t}\varphi(t_\rho) - \partial^\alpha_t \varphi(t_\rho)\right) = 0.$$

Moreover, for a test function φ, the Caputo derivative can be defined in the standard way, see (2.1). In the rest of the proof, we omit the index ρ and we write t, n and in place of t_ρ, n_ρ. Fix $\eta > 0$ such that $t > 2\eta$ and let $\bar{n} < n$ be the greatest integer such that $\bar{n}\Delta t \le \eta$.

Then we write

$$
\begin{aligned}
D^{\alpha}_{\Delta t}\varphi(t_\rho) - \partial^{\alpha}_t \varphi(t_\rho) &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \Big(\frac{\varphi(t_{j+1}) - \varphi(t_j)}{\Delta t (t-s)^{\alpha}} - \frac{\varphi'(s)}{(t-s)^{\alpha}} \Big) ds \\
= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{\bar{n}-1} & \left(\int_{t_j}^{t_{j+1}} \frac{\varphi(t_{j+1}) - \varphi(t_j)}{\Delta t (t-s)^{\alpha}} ds - \int_{t_j}^{t_{j+1}} \frac{\varphi'(s)}{(t-s)^{\alpha}} ds \right) \\
+ \frac{1}{\Gamma(1-\alpha)} \sum_{j=\bar{n}}^{n} & \left(\int_{t_j}^{t_{j+1}} \frac{\varphi(t_{j+1}) - \varphi(t_j)}{\Delta t (t-s)^{\alpha}} ds - \int_{t_j}^{t_{j+1}} \frac{\varphi'(s)}{(t-s)^{\alpha}} ds \right).
\end{aligned}
\tag{4.2}
$$

We estimate the two sums (multiplied by $\Gamma(1-\alpha)$) in (4.2) separately. For $0 \le j \le \bar{n}-1$, use the integration by parts to get

$$
\begin{aligned}
\int_{t_j}^{t_{j+1}} \frac{\varphi'(s)}{(t-s)^{\alpha}} ds &= \Big[\frac{\varphi(t_{j+1})}{(t-t_{j+1})^{\alpha}} - \frac{\varphi(t_j)}{(t-t_j)^{\alpha}} \Big] - \alpha \int_{t_j}^{t_{j+1}} \frac{\varphi(s)}{(t-s)^{\alpha+1}} ds \\
&= \frac{\varphi(t_{j+1}) - \varphi(t_j)}{(t-t_j)^{\alpha}} + \alpha \int_{t_j}^{t_{j+1}} \frac{\varphi(t_{j+1}) - \varphi(s)}{(t-s)^{\alpha+1}} ds.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_{t_j}^{t_{j+1}} \frac{\varphi(t_{j+1}) - \varphi(t_j)}{\Delta t (t-s)^{\alpha}} ds - \int_{t_j}^{t_{j+1}} \frac{\varphi'(s)}{(t-s)^{\alpha}} ds \\
&= \frac{\varphi(t_{j+1}) - \varphi(t_j)}{\Delta t} \int_{t_j}^{t_{j+1}} \Big[\frac{1}{(t-s)^{\alpha}} - \frac{1}{(t-t_j)^{\alpha}} \Big] ds - \alpha \int_{t_j}^{t_{j+1}} \frac{\varphi(t_{j+1}) - \varphi(s)}{(t-s)^{\alpha+1}} ds.
\end{aligned}
\tag{4.3}
$$

Observe that $\frac{1}{t-s} \le \frac{1}{t-t_{j+1}} \le \frac{1}{t-\eta}$ for $t_j \le s \le t_{j+1}$. For the first term of (4.3), using the inequality $t^{\alpha} - s^{\alpha} \le (t-s)^{\alpha}$ for $0 \le s \le t$, we have the following estimate:

$$
\begin{aligned}
&\frac{\varphi(t_{j+1}) - \varphi(t_j)}{\Delta t} \int_{t_j}^{t_{j+1}} \Big[\frac{1}{(t-s)^{\alpha}} - \frac{1}{(t-t_j)^{\alpha}} \Big] ds \\
&\le \frac{\int_{t_j}^{t_{j+1}} |\varphi'(s)| ds}{\Delta t} \int_{t_j}^{t_{j+1}} \frac{(t-t_j)^{\alpha} - (t-s)^{\alpha}}{(t-s)^{\alpha}(t-t_j)^{\alpha}} ds \\
&\le \frac{\int_{t_j}^{t_{j+1}} |\varphi'(s)| ds}{\Delta t} \int_{t_j}^{t_{j+1}} \frac{(s-t_j)^{\alpha}}{(t-s)^{\alpha}(t-t_j)^{\alpha}} ds \\
&\le \frac{\int_{t_j}^{t_{j+1}} |\varphi'(s)| ds}{\cancel{\Delta t}} \frac{(\Delta t)^{\alpha}}{(t-t_j)^{\alpha}(t-t_{j+1})^{\alpha}} \cancel{\Delta t} \\
&\le \frac{1}{(t-\eta)^{2\alpha}} (\Delta t)^{\alpha} \int_{t_j}^{t_{j+1}} |\varphi'(s)| ds.
\end{aligned}
$$

For the second term of (4.3), we have the following estimate:

$$
\alpha \int_{t_j}^{t_{j+1}} \frac{\varphi(t_{j+1}) - \varphi(s)}{(t-s)^{\alpha+1}} ds \le \alpha \frac{\omega_{\varphi}(\Delta t)}{(t-t_{j+1})^{\alpha+1}} \Delta t \le \alpha \frac{\omega_{\varphi}(\Delta t)}{(t-\eta)^{\alpha+1}} \Delta t,
$$

where ω_φ is a modulus of continuity of φ. Thus,

$$
\begin{aligned}
&\Big|\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{\bar{n}-1}\int_{t_j}^{t_{j+1}}\frac{\varphi(t_{j+1})-\varphi(t_j)}{\Delta t(t-s)^\alpha}ds-\int_{t_j}^{t_{j+1}}\frac{\varphi'(s)}{(t-s)^\alpha}ds\Big|\\
&\le \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{\bar{n}-1}\frac{1}{(t-\eta)^{2\alpha}}(\Delta t)^\alpha\int_{t_j}^{t_{j+1}}|\varphi'(s)|ds+\alpha\frac{\omega_\varphi(\Delta t)}{(t-\eta)^{\alpha+1}}\Delta t\\
&\le \frac{1}{\Gamma(1-\alpha)}\frac{1}{(t-\eta)^{2\alpha}}(\Delta t)^\alpha\int_0^\eta|\varphi'(s)|ds+\frac{\alpha}{\Gamma(1-\alpha)}\frac{\omega_\varphi(\Delta t)}{(t-\eta)^{\alpha+1}}\eta.
\end{aligned}
$$

Clearly, both terms converge to 0 as $\Delta t\to 0$.
Now, we estimate the second sum (4.2). , Since $\varphi\in C^1([\eta,t])$, we have

$$\varphi(t_{j+1})=\varphi(s)+\varphi'(s)(t_{j+1}-s)+h(t_{j+1}-s)(t_{j+1}-s)$$

and

$$\varphi(t_j)=\varphi(s)+\varphi'(s)(t_j-s)+g(t_j-s)(t_j-s)$$

with $h(s),g(s)$ continuous functions such that $h(s),g(s)\to 0$ as $s\to 0$. Thus, using that $t_{j+1}-t_j=\Delta t$, we get

$$\varphi(t_{j+1})-\varphi(t_j)-\Delta t\varphi'(s)=h(t_{j+1}-s)(t_{j+1}-s)-g(t_j-s)(t_j-s).$$

Hence,

$$
\begin{aligned}
&\Big|\int_{t_j}^{t_{j+1}}\frac{\varphi(t_{j+1})-\varphi(t_j)-\Delta t\varphi'(s)}{\Delta t(t-s)^\alpha}ds\Big|\\
&=\Big|\int_{t_j}^{t_{j+1}}\frac{h(t_{j+1}-s)(t_{j+1}-s)-g(t_j-s)(t_j-s)}{\Delta t(t-s)^\alpha}ds\Big|\\
&\le\Big|\int_{t_j}^{t_{j+1}}\frac{h(t_{j+1}-s)(t_{j+1}-s)}{\Delta t(t-s)^\alpha}ds\Big|+\Big|\int_{t_j}^{t_{j+1}}\frac{g(t_j-s)(t_j-s)}{\Delta t(t-s)^\alpha}ds\Big|\\
&=\Big|\int_{t_j}^{t_{j+1}}\frac{o(t_{j+1}-s)}{\Delta t(t-s)^\alpha}ds\Big|+\Big|\int_{t_j}^{t_{j+1}}\frac{o(s-t_j)}{\Delta t(t-s)^\alpha}ds\Big|\\
&=\Big|\int_{t_j}^{t_{j+1}}\frac{o(\Delta t)}{\Delta t(t-s)^\alpha}ds\Big|+\Big|\int_{t_j}^{t_{j+1}}\frac{o(\Delta t)}{\Delta t(t-s)^\alpha}ds\Big|\\
&=2\frac{o(\Delta t)}{\Delta t}\int_{t_j}^{t_{j+1}}\frac{1}{(t-s)^\alpha}ds\\
&=2\frac{o(\Delta t)}{\Delta t(1-\alpha)}[(t-t_j)^{1-\alpha}-(t-t_{j+1})^{1-\alpha}].
\end{aligned}
$$

Hence

$$\left|\sum_{j=\bar{n}}^{n}\int_{t_j}^{t_{j+1}}\frac{\varphi(t_{j+1})-\varphi(t_j)}{\Delta t(t-s)^\alpha}ds-\int_{t_j}^{t_{j+1}}\frac{\varphi'(s)}{(t-s)^\alpha}ds\right|=2\frac{o(\Delta t)}{\Delta t}(t-t_{\bar{n}})^{1-\alpha}\to 0$$

as $\Delta t \to 0$. Hence the claim (4.1) holds.
We conclude that the scheme (3.10) is monotone, stable and consistent with the continuous equation (1.1). Moreover the continuous problem (1.1) satisfies a Comparison Principle, see Theorem 2.2. Hence, arguing as in Theorem 2.1 in [4], it follows that $\overline{u}$, $\underline{u}$ are, respectively, a viscosity subsolution and a viscosity supersolution of (1.1) and the uniform convergence of the sequence $\{u_\rho\}_{\rho>0}$ to the unique viscosity solution of (1.1). □

5 Explicit solutions and numerical tests

In this section, we implement upwind and Lax-Friedrichs schemes to test the convergence.

5.1 Test 1

First, we consider the following Hamilton-Jacobi equation:

$$\begin{cases} \partial_t^\alpha u + \frac{|Du|^2}{2} = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u_0(x) = \min\{0, |x|^2 - 1\}, & x \in \mathbb{R}^d. \end{cases} \tag{5.1}$$

It is easy to verify that, if $\alpha = 1$, then the unique viscosity solution of (5.1) is given by

$$u(t,x) = \min\left\{0, \frac{|x|^2}{1+2t} - 1\right\}.$$

We claim that a solution of (5.1) for $\alpha \in (0,1)$ is given by

$$u(t,x) = \min\left\{0, |x|^2 f(t) - 1\right\} \tag{5.2}$$

with $f(t)$ non-negative function to be determined. Replacing into the equation (5.1) for $|x| \le \sqrt{1/f(t)}$ and taking into account the initial datum, we find that the function $f(t)$ has to satisfy the fractional differential equation

$$\begin{cases} \partial_t^\alpha f + 2f(t)^2 = 0, \\ f(0) = 1. \end{cases} \tag{5.3}$$

We look for a solution of (5.3) in the form of a power series $f(t) = \sum_{n=0}^\infty f_n t^{\alpha n}$. Replacing in the equation (5.3) and observing that $\partial_t^\alpha t^0 = 0$, we have

$$\sum_{n=1}^\infty f_n \partial_t^\alpha t^{\alpha n} + 2\sum_{n=0}^\infty \sum_{m=0}^\infty f_n f_m t^{\alpha(n+m)} = 0. \tag{5.4}$$

A straightforward computation gives

$$\partial_t^\alpha t^{\alpha n} = \beta_n t^{\alpha(n-1)},$$

where $\beta_n = \Gamma(\alpha n + 1)/\Gamma(\alpha(n-1)+1)$. Replacing the previous identity in the equation (5.4), we get

$$\sum_{n=1}^\infty f_n \beta_n t^{\alpha(n-1)} + 2\sum_{n=0}^\infty \sum_{m=0}^\infty f_n f_m t^{\alpha(n+m)} = 0.$$

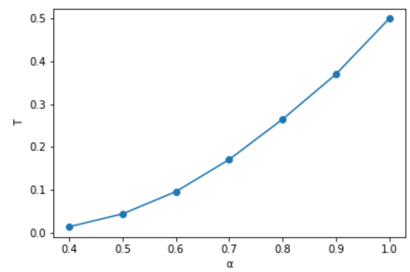


Figure 1: Critical time for which the power series $f(t)$ converges

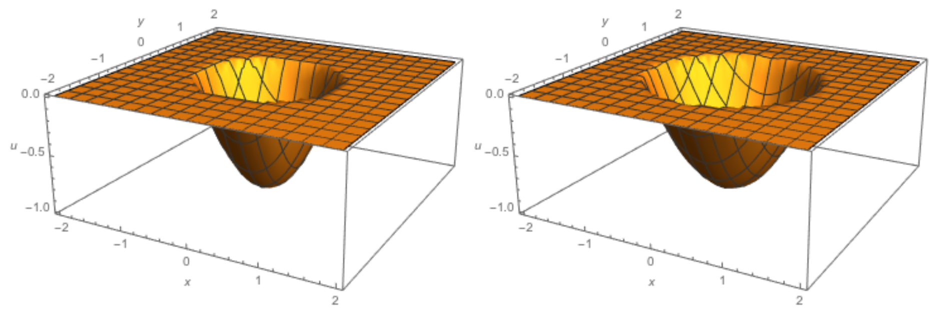


Figure 2: (A) Initial condition (B) Numercial solution at $t = 0.2$

Collecting the terms of the same order and recalling that, by the initial condition in (5.3), $f_0 = 1$, we find

$$\begin{cases} \beta_1 f_1 + 2f_0^2 = 0 & \iff \quad f_1 = -2\frac{\Gamma(1)}{\Gamma(\alpha+1)}, \\ \beta_2 f_2 + 2(f_0 f_1 + f_1 f_0) = 0 & \iff \quad f_2 = 8\frac{\Gamma(1)}{\Gamma(2\alpha+1)}, \\ \beta_3 f_3 + 2(f_0 f_2 + f_1^2 + f_2 f_0) = 0 & \iff \quad f_3 = -2(f_1^2 + 2f_2)\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}, \\ \vdots & \qquad\quad \vdots \\ \beta_n f_n + 2\sum_{i+j=n-1} f_i f_j = 0 & \iff \quad f_n = -\frac{2}{\beta_n}\sum_{i+j=n-1} f_i f_j. \end{cases}$$

From the previous relations, we can iteratively compute the coefficients of the power series $f(t)$ and we replace in (5.2). Note that for $\alpha = 1$, we get the power series of $\frac{1}{1+2t}$. However, for each $\alpha \in (0, 1]$ there is a critical time T for which the power series $f(t)$ converges if $t \leq T$ and diverges if $t > T$. The dependence of T on α is presented in Fig. 1.

The numerical solution at $t = 0.2$ of (5.1) where $\alpha = 0.8$ and $d = 2$ computed by the upwind scheme with $h = 10^{-1}$ and $\Delta t = 10^{-3}$ is provided in Fig. 2. We plot numerical solutions at $t = 0.2$ for different values of α in Fig. 3 (A) for $d = 1$. We observe the convergent behavior of the solutions as $\alpha \to 1$. These solutions eventually converge to the solution of the classical case.

For the convergence test, we use l^∞ error defined by the maximum difference between the exact and numerical solutions over all nodes. From Fig. 3 (B), we determine that the convergence for the upwind scheme under the CFL condition is linear. We note that the Lax-Friedrichs scheme also implies similar results.

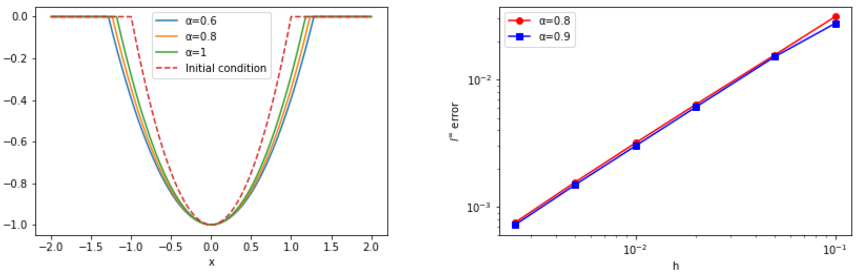


Figure 3: (A) Numerical solutions at $t = 0.2$ for different values of α (B) Convergence test for $t = 0.2$

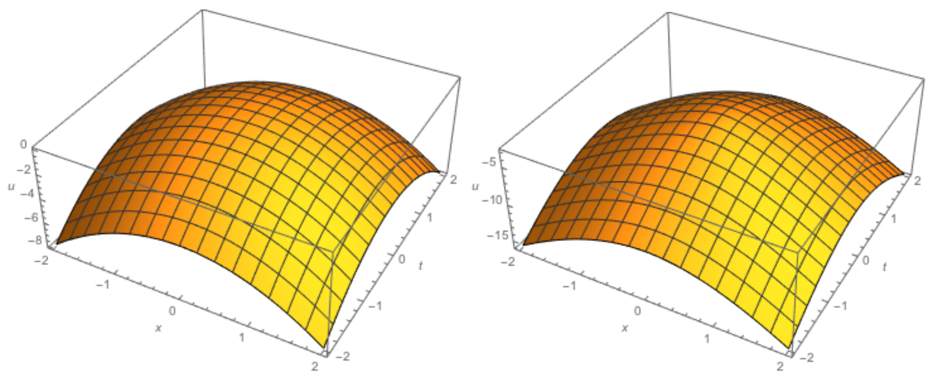


Figure 4: (A) Initial condition of (5.5) for $\alpha = 0.8$ (B) Numerical solutions of (5.5) when $\alpha = 0.8$ at $t = 0.2$

5.2 Test 2

In this part, we present numerical results for the Lax-Friedrichs scheme. Here, we consider the Hamilton-Jacobi equation of the form

$$\begin{cases} \partial_t^\alpha u + |Du| = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u_0(x) = -|x|^2, & x \in \mathbb{R}^d. \end{cases} \tag{5.5}$$

For $\alpha = 1$, the unique classical solution of (5.5) is

$$u(t,x) = -(|x| + t)^2.$$

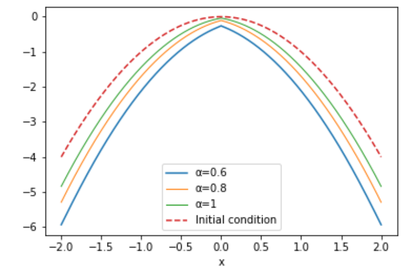
For $\alpha = 1/2$, we look for a solution in the form

$$u_{\frac{1}{2}}(t,x) = -|x|^2 + \gamma t - 2\beta |x| t^{1/2}. \tag{5.6}$$

Computing the derivatives

$$Du_{\frac{1}{2}}(t,x) = -2(|x| + \beta t^{1/2}) \frac{x}{|x|}$$

$$\partial_t^{\frac{1}{2}} u_{\frac{1}{2}}(t,x) = -\frac{2\gamma}{\sqrt{\pi}} \sqrt{t} - \beta |x| \pi$$

(A) 

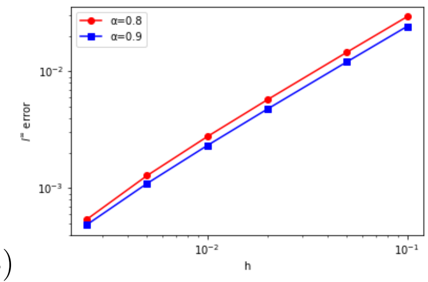
(B) 

Figure 5: Lax-Friedrichs scheme: (A) Numerical solutions of (5.5) at $T = 0.2$ for different values of α (B) Convergence test for the fixed $\Delta t = 10^{-3}$

and replacing in the equation (5.5), we get

$$\beta = \frac{2}{\sqrt{\pi}}, \quad \gamma = 2,$$

and therefore

$$u_{\frac{1}{2}}(t,x) = -|x|^2 - 2t - \frac{4}{\sqrt{\pi}}|x|t^{1/2}.$$

For $\alpha \in (0,1)$, a similar computation gives that the solution of (5.5) is given by

$$u_\alpha(t,x) = -|x|^2 - \frac{1}{\alpha\Gamma(2\alpha)}t^{2\alpha} - \frac{2}{\alpha\Gamma(\alpha)}t^\alpha|x|.$$

Fig. 4 depicts the initial condition and the numerical solution at $t = 0.2$ for $\alpha = 0.8$ obtained using the Lax-Friedrichs scheme in 2 dimensions. The numerical solutions corresponding to different values of α are plotted in Fig. 5 (A) for $d = 1$. We can see the same convergent behavior of the solutions as $\alpha \to 1$ as in the previous part. Moreover, from the convergence test in Fig. 5 (B), we observe that the convergence to the exact solution is linear. The upwind scheme gives similar results too.

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