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# High-Gain Observer Design for Nonlinear Systems with Delayed Outputs ⋆

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**Abstract:** This paper deals with high-gain nonlinear observer design for a class of triangular systems with delayed output measurements. Based on a recent high-gain like observer design method, called HG/LMI observer, a larger bound of the time-delay is allowed compared to that obtained by using the standard high-gain methodology. Such a HG/LMI observer leads to a significantly lower tuning parameter, which reduces the values of the observer gains and increases the maximum bound of the delay allowed to ensure exponential convergence. Indeed, an explicit relation between the maximum bound of the delay and the observer tuning parameter is inferred by using a Lyapunov-Krasovskii functional jointly with the Halanay inequality. Such a relation shows clearly the superiority of the use of the HG/LMI observer design methodology.



*Keywords:* Nonlinear systems, high-gain observer, delayed output measurements, linear matrix inequalities (LMIs), Lyapunov-Krasovskii functionals.

## 1. INTRODUCTION

For many decades, the interest of automatic control community to nonlinear observers continues to grow because of their crucial role in the design of control schemes, namely trajectory tracking, fault diagnosis, and health monitoring (Parisini, 1997), (Alcorta-Garcia and Frank, 1997), (Gao and Ho, 2006). Recently, due to the introduction of new technologies and the complexity of novel industrial infrastructures, the use of nonlinear observers has been emerged in modern applications such as synchronization of multi-agent systems, cyber-attacks detection and control of cyber-physical systems (Zhu and Basar, 2011), (Teixeira et al., 2010), (An and Liu, 2014). More interesting also, nonlinear observers continue to evolve towards the artificial intelligence by introducing data-driven or learning-based neuro-adaptive observers (Chakrabarty et al., 2019), (Koga et al., 2019), (Liu et al., 2018).

Due to such success of nonlinear observers, several methodologies have been developed in the literature, namely the extended Kalman observer (Kalman et al., 1960), Luenberger observer (Luenberger, 1971), (Huong et al., 2019), high-gain observer (Gauthier and Kupka, 1994), sliding mode observer (Alessandri, 1999), and LMI-based observers (Zemouche and Boutayeb, 2013). Although all these techniques provide solutions to observer design for large classes of nonlinear systems, there is no general solution and this challenge still remains open until now. Hence there are many possibilities for improvements, however, in this paper we will focus only on high-gain observers. The high-gain observer is particularly interesting due to its easy implementation because it depends on only one single tuning parameter, which requires a specific condition, to ensure exponential convergence. Despite this simplicity of implementation, the high-gain observer is far from being a perfect solution to nonlinear estimation, and it has three limitations that should be highlighted: 1) numerical problems because for high dimensional systems due to the high values of the observer gain; 2) the presence of peaking phenomenon; and 3) the high sensitivity to output disturbances (measurement noise, delayed outputs, sampled data,...). Many research activities have been paid to this research area aiming to propose solutions overcoming such drawbacks of the high-gain observer (Zemouche et al., 2019), (Alessandri and Rossi, 2015), (Astolfi and Marconi, 2015), (Khalil, 2017), (Boizot et al., 2010). In this paper, we focus only on the use of high-gain methodologies for systems with delayed output measurements. Indeed, such a problem is complex when the goal is to provide an observer with a maximum allowable value of the time-delay, which is the main motivation of this paper. Many efficient solutions, based on high-gain methodology, have been proposed in the literature to cope with this issue (Ahmed-Ali et al., 2009), (Assche et al., 2011). Since systems

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with sampled output measurements can be rewritten in an equivalent form as systems with delayed outputs, similar results have been proposed in the literature (Ahmed-Ali et al., 2012), (Ahmed-Ali et al., 2013), (Bouraoui et al., 2015), (Zhang and Shen, 2017).

In Ahmed-Ali et al. (2009), the authors proposed a cascade high-gain observer for a triangular system, where they provided conditions ensuring the exponential convergence of the observer. Then the results have been extended in Assche et al. (2011) to systems with time-varying delayed measurements. Conditions on the delay and the tuning parameter of the proposed observer have been explicitly introduced. Although the maximum value of the allowable delay is improved, however, it still remains small due to the high value of the tuning parameter required by the standard high-gain observer. The proposed work in this paper has been motivated by this issue, namely establishing a high-gain like design method with a low tuning parameter, which leads to higher maximum value of the delay. To this end, a recent high-gain like observer design method, called HG/LMI observer, is exploited. Indeed, compared to the standard high-gain observer, the HG/LMI observer leads to a lower tuning parameter and then provides a higher bound of the allowable time-delay, while ensuring exponential convergence of the observer. The convergence analysis is performed by using a Lyapunov-Krasovskii functional jointly with a Halanay inequality. The obtained results show explicitly, thanks to a mathematical relation between the tuning parameter of the observer and the maximum bound of the delay, the superiority of the proposed HG/LMI observer-based technique.

The rest of the paper is organized as follows. In section 2, the problem is formulated and the class of systems under consideration is presented. The high gain observer is then recalled and its convergence is analyzed in section 3. Section 4 presents and discusses the proposed observer design strategy using the HG/LMI observer with a lower tuning parameter. Finally, we end the paper by a conclusion summarizing the main contributions.

## 2. PROBLEM FORMULATION

We consider the class of nonlinear systems described by

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \\ f(x(t)) \end{bmatrix} \\ y(t) \qquad = x_1(t-\tau), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the system and $y(t) \in \mathbb{R}$ is the measured output . We assume that the delay $\tau$ is known and there is $\tau_M$ such that $\tau \in [0, \tau_M]$. The function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the Lipschitz property formulated under the following form:

$$|f(x_1+\Delta_1,\ldots,x_n+\Delta_n) - f(x_1,\ldots,x_n)| \le \gamma_f \sum_{j=1}^{n} |\Delta_j| \tag{2}$$

where $\gamma_f$ is the Lipschitz constant and $\Delta_j \in \mathbb{R}, \forall j = 1,\ldots,n$.

For simplicity of the presentation, system (1) can be rewritten under the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bf(x(t)) \\ y(t) = Cx(t-\tau), \end{cases} \tag{3}$$

where

$$B = [\,0\ \ldots\ 0\ 1\,]^T, C = [\,1\ 0\ \ldots\ 0\,], \tag{4}$$

and the state matrix $A$ is defined by

$$(A)_{i,j} = \begin{cases} 1 & \text{if} \quad j = i+1 \\ 0 & \text{if} \quad j \neq i+1. \end{cases} \tag{5}$$

Let us introduce the following candidate Luenberger observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bf(\hat{x}(t)) + L[y(t) - C\hat{x}(t-\tau)], \tag{6}$$

where $\hat{x}$ represents the state estimation and $L$ is the observer gain.

The dynamics of the estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$ is given by

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B[f(x(t)) - f(\hat{x}(t))] - LC\tilde{x}(t-\tau). \tag{7}$$

The objective consists in designing a high-gain observer for system (1) that provides stability of the estimation error. We also provide an expression of the maximum bound of the allowable delay and the design parameter under which the proposed observer converges exponentially.

To establish the exponential convergence of the estimation error, the following lemma is useful to demonstrate the main results.

*Lemma 1.* (Halanay (1966)). If there exists a positive Lyapunov-Krasovskii functional $V(t)$ such that

$$\frac{d}{dt}V(t) \le -\alpha V(t) + \beta \sup_{s\in[t-\tau,t]} V(s), \tag{8}$$

where $\alpha > \beta > 0$, then there exits $\eta > 0$ and $\delta > 0$ such that

$$V(t) \le \eta e^{-\delta(t-t_0)} \quad \text{for} \quad t \ge t_0. \tag{9}$$

We also recall some useful inequalities needed in the proof of some results.

*Lemma 2.* (Jensen's Inequality). [Gu (2000)] For any constant symmetric positive matrix $M \in \mathcal{R}^{n\times n}$, scalars $t_1, t_2$ and vector function $v : [t_1, t_2] \to \mathcal{R}^n$, then the following inequality holds:

$$\left(\int_{t_1}^{t_2} v(\beta)d\beta\right)^T M \left(\int_{t_1}^{t_2} v(\beta)d\beta\right) \le (t_2 - t_1)\left(\int_{t_1}^{t_2} v^T(\beta)Mv(\beta)d\beta\right). \tag{10}$$

*Lemma 3.* (Young's Inequality). [Kheloufi et al. (2015)] Let $X$ and $Y$ be two matrices of appropriate dimensions. Then, for every invertible matrix $S$ and scalar $\mu > 0$, we have

$$X^TY + Y^TX \le \mu X^TSX + \frac{1}{\mu}Y^TS^{-1}Y. \tag{11}$$

## 3. STANDARD HIGH-GAIN OBSERVER BASED DESIGN

This section is devoted to the design of high-gain observer for a class of nonlinear systems with the presence of a delay in the output measurements. We recall the standard high gain design, where usually the observer gain $L$ is written as follows [Zemouche et al. (2019)]:

$$L := T(\theta)K; \quad \theta \geq 1 \tag{12}$$

where

$$T(\theta) := \text{diag}(\theta, \dots, \theta^n) \quad \text{and} \quad K \in \mathbb{R}^n. \tag{13}$$

Moreover, the estimation error is transformed into

$$\bar{x} := T^{-1}(\theta)\tilde{x}, \tag{14}$$

where $T^{-1}(\theta)$ is the inverse of $T(\theta)$ given by

$$T^{-1}(\theta) := \text{diag}(\frac{1}{\theta}, \dots, \frac{1}{\theta^n}). \tag{15}$$

The dynamics of the transformed error is given by

$$\begin{aligned}\dot{\bar{x}}(t) = \theta(A - KC)\bar{x}(t) + T^{-1}(\theta)B\Delta f \\ -\theta KC(\bar{x}(t-\tau) - \bar{x}(t)),\end{aligned} \tag{16}$$

with

$$\Delta f := f(x) - f(x - T(\theta)\bar{x}). \tag{17}$$

From the fact that $\theta \geq 1$ and by using the Lipschitz condition (2), it was shown in Alessandri and Rossi (2013) that there exists a positive constant $k_f$, independently of $\theta$, such that

$$\|T^{-1}(\theta)B\Delta f)\| \leq k_f\|\bar{x}\|. \tag{18}$$

Now we can state the preliminary result summarized in the following theorem providing synthesis conditions to ensure exponential convergence of the observer.

*Theorem 4.* If there exists a positive definite matrix $P$ and a matrix $\mathcal{Y}$ of appropriate dimension and a real constants $\mu_1, \lambda, \tau_M > 0$ so that the following conditions hold:

$$\begin{bmatrix} \mathcal{H}e\Big\{PA - \mathcal{Y}^\top C\Big\} + \tau_M\mathcal{R}^T\mathcal{R} + \lambda I & \mathcal{Y}^T \\ \mathcal{Y} & -\mu_1 \end{bmatrix} \leq 0, \tag{19}$$

$$\theta < \sqrt{\frac{\lambda_{\min}(P)}{2\tau_M\mu_1(\lambda_{\min}(P) + K_1^2\tau_M^2)}}, \tag{20}$$

$$\theta > \max\left(1, \frac{2k_f\lambda_{max}^2(P)}{\lambda}\right), \tag{21}$$

$$\tau_M \in \left[0, \sqrt{\frac{\lambda\lambda_{\min}(P)}{2\mu_1K_1^2\lambda_{max}(P)\theta^2} - \frac{k_f\lambda_{\min}(P)\lambda_{\max}(P)}{\mu_1K_1^2\theta^3}}\right] \tag{22}$$

where $\mathcal{H}e\Big\{\mathcal{S}\Big\} := \mathcal{S} + \mathcal{S}^\top$ and

$$\begin{cases} \mathcal{R} = [0 \quad 1 \quad 0_{1\times n-2}], \\ K = P^{-1}\mathcal{Y}^T = [K_1 \dots K_n]^T. \end{cases} \tag{23}$$

Then the observer (6) is exponentially convergent.

**Proof.** Define the following Lyapunov-Krasovskii candidate functional

$$V(t) = V(\bar{x}(t)) = V_1(t) + \theta V_2(t), \tag{24}$$

where

$$V_1(t) = \bar{x}^T(t)P\bar{x}(t), \tag{25}$$

and

$$V_2(t) = \int_{t-\tau}^{t}\int_{s}^{t}(\bar{x}_2(s))^2 ds d\xi. \tag{26}$$

First, let us compute the derivative of $V_1$ along the trajectories of (16). We obtain

$$\begin{aligned}\frac{d}{dt}V_1(t) = \theta\bar{x}^T(t)\left[(A-KC)^TP + P(A-KC)\right]\bar{x}(t) \\ + \left(\frac{1}{\theta^n}B\Delta f\right)^T P\bar{x}(t) + \bar{x}^T(t)P\left(\frac{1}{\theta^n}B\Delta f\right) + \Gamma_1,\end{aligned}$$

where

$$\begin{aligned}\Gamma_1 := \Bigg(\underbrace{\bar{x}^T(t)\mathcal{Y}\sqrt{\theta}}_{Y^T}\underbrace{\sqrt{\theta}C(\bar{x}(t) - \bar{x}(t-\tau))}_{X}\Bigg)^T \\ + \Bigg(\underbrace{\bar{x}^T(t)\mathcal{Y}\sqrt{\theta}}_{Y^T}\underbrace{\sqrt{\theta}C(\bar{x}(t) - \bar{x}(t-\tau))}_{X}\Bigg),\end{aligned}$$

and

$$\mathcal{Y} = K^TP. \tag{27}$$

By applying Young's inequality on $\Gamma_1$, we obtain

$$\begin{aligned}\Gamma_1 \leq \mu_1\theta\left(\bar{x}(t) - \bar{x}(t-\tau)\right)^T C^TC\left(\bar{x}(t) - \bar{x}(t-\tau)\right) \\ + \frac{1}{\mu_1}\theta\bar{x}^T(t)\mathcal{Y}^T\mathcal{Y}\bar{x}(t),\end{aligned} \tag{28}$$

where $\mu_1$ is a given positive scalar.
Using the Leibniz integration formula

$$\bar{x}(t) - \bar{x}(t-\tau) = \int_{t-\tau}^{t}\dot{\bar{x}}(s)ds \tag{29}$$

then we can rewrite (28) as follows

$$\begin{aligned}\Gamma_1 \leq \underbrace{\mu_1\theta\left(\int_{t-\tau}^{t}\dot{\bar{x}}(s)ds\right)^T C^TC\left(\int_{t-\tau}^{t}\dot{\bar{x}}(s)ds\right)}_{\Gamma_2} \\ + \frac{1}{\mu_1}\theta\bar{x}^T(t)\mathcal{Y}^T\mathcal{Y}\bar{x}(t).\end{aligned} \tag{30}$$

If we apply Jensen's inequality on the term $\Gamma_2$, we get

$$\begin{aligned}\Gamma_2 &\leq \tau_M\int_{t-\tau}^{t}\dot{\bar{x}}(s)^TC^TC\dot{\bar{x}}(s)ds \leq \tau_M\int_{t-\tau}^{t}\|\dot{\bar{x}}_1(s)\|^2ds \\ &\leq \tau_M\int_{t-\tau}^{t}\|\theta\bar{x}_2(s) - \theta K_1\bar{x}_1(s-\tau)\|^2ds \\ &\leq 2\tau_M\left(\int_{t-\tau}^{t}(\theta\bar{x}_2(s))^2ds + \int_{t-\tau}^{t}\left(\theta K_1\bar{x}_1(s-\tau)\right)^2ds\right).\end{aligned}$$

From (30), we will have

$$\begin{aligned}\Gamma_1 \leq \frac{1}{\mu_1}\theta\bar{x}^T(t)\mathcal{Y}^T\mathcal{Y}\bar{x}(t) + 2\mu_1\theta^3\tau_M\int_{t-\tau}^{t}(\bar{x}_2(s))^2ds \\ + 2\mu_1\theta^3\tau_M\int_{t-\tau}^{t}\left(K_1\bar{x}_1(s-\tau)\right)^2ds.\end{aligned}$$

The derivative of $V_1$ becomes

$$\begin{aligned}\frac{d}{dt}V_1 \leq \theta\bar{x}^T(t)[A^TP + PA - C^T\mathcal{Y} - \mathcal{Y}^TC \\ + \frac{1}{\mu_1}\mathcal{Y}^T\mathcal{Y}]\bar{x}(t) + \left(\frac{1}{\theta^n}B\Delta f\right)^T P\bar{x}(t) \\ + \bar{x}^T(t)P\left(\frac{1}{\theta^n}B\Delta f\right) + 2\mu_1\theta^3\tau_M\int_{t-\tau}^{t}(\bar{x}_2(s))^2ds \\ + 2\mu_1\theta^3\tau_M\int_{t-\tau}^{t}(K_1\bar{x}_1(s-\tau))^2ds.\end{aligned}$$

Now, let us compute the derivative of $V_2$ along the trajectories of (16):

$$\frac{d}{dt}V_2(t) \le \tau_M(\bar{x}_2(t))^2 - \int_{t-\tau}^{t}(\bar{x}_2(s))^2 ds.$$

Thus we have

$$\begin{aligned}\frac{d}{dt}V(t) \le{}& \theta\bar{x}^T(t)[A^TP + PA - C^T\mathcal{Y} - \mathcal{Y}^TC \\ &+ \frac{1}{\mu_1}\mathcal{Y}^T\mathcal{Y} + \tau_M\mathcal{R}^T\mathcal{R}]\bar{x}(t) + \left(\frac{1}{\theta^n}B\Delta f\right)^T P\bar{x}(t) \\ &+ \bar{x}^T(t)P\left(\frac{1}{\theta^n}B\Delta f\right) \\ &- \theta\tau_M\left(\frac{1}{\tau_M} - 2\mu_1\theta^2\right)\int_{t-\tau}^{t}(\bar{x}_2(s))^2 ds \\ &+ 2\mu_1\theta^3\tau_M\int_{t-\tau}^{t}(K_1\bar{x}_1(s-\tau))^2 ds,\end{aligned}$$

where the matrix $\mathcal{R}$ is defined as $\mathcal{R} = [0 \quad 1 \quad 0_{1\times n-2}]$.
By choosing $\theta < \sqrt{\frac{1}{2\mu_1\tau_M}}$ and from the fact that

$$V_2(t) \le \tau_M\int_{t-\tau}^{t}(\bar{x}_2(s))^2 ds. \tag{31}$$

We obtain

$$\begin{aligned}-\theta\tau_M\left(\frac{1}{\tau_M} - 2\mu_1\theta^2\right)&\int_{t-\tau}^{t}(\bar{x}_2(s))^2 ds \\ &\le -\theta\left(\frac{1}{\tau_M} - 2\mu_1\theta^2\right)V_2(t).\end{aligned}$$

Then the derivative of $V$ becomes

$$\begin{aligned}\frac{d}{dt}V(t) \le{}& \theta\bar{x}^T(t)[A^TP + PA - C^T\mathcal{Y} - \mathcal{Y}^TC \\ &+ \frac{1}{\mu_1}\mathcal{Y}^T\mathcal{Y} + \tau_M\mathcal{R}^T\mathcal{R}]\bar{x}(t) + \left(\frac{1}{\theta^n}B\Delta f\right)^T P\bar{x}(t) \\ &+ \bar{x}^T(t)P\left(\frac{1}{\theta^n}B\Delta f\right) - \theta\left(\frac{1}{\tau_M} - 2\mu_1\theta^2\right)V_2(t) \\ &+ 2\mu_1\theta^3\tau_M\int_{t-2\tau}^{t-\tau}(K_1\bar{x}_1(s))^2 ds.\end{aligned}$$

Now, let

$$A^TP + PA - C^T\mathcal{Y} - \mathcal{Y}^TC + \frac{1}{\mu_1}\mathcal{Y}^T\mathcal{Y} + \tau_M\mathcal{R}^T\mathcal{R} < -\lambda I \tag{32}$$

Thus

$$\begin{aligned}\frac{d}{dt}V(t) \le{}& -\lambda\theta\bar{x}^T(t)\bar{x}(t) + \left(\frac{1}{\theta^n}B\Delta f\right)^T P\bar{x}(t) \\ &+ \bar{x}^T(t)P\left(\frac{1}{\theta^n}B\Delta f\right) - \theta\left(\frac{1}{\tau_M} - 2\mu_1\theta^2\right)V_2(t) \\ &+ \frac{2\mu_1\theta^3K_1^2\tau_M^2}{\lambda_{\min}(P)}\sup_{[t-2\tau,t]}V(s)\end{aligned} \tag{33}$$

which can be rewritten as

$$\frac{d}{dt}V(t) \le -\alpha_1 V_1(t) - \alpha_2\theta V_2(t) + \beta\sup_{[t-2\tau,t]}V(s), \tag{34}$$

where

$$\alpha_1 = \frac{\theta\lambda}{\lambda_{\max}(P)} - 2k_f\lambda_{\max}(P) > 0, \tag{35}$$

$$\alpha_2 = \frac{1}{\tau_M} - 2\mu_1\theta^2 > 0 \text{ and } \beta = \frac{2\mu_1\theta^3K_1^2\tau_M^2}{\lambda_{\min}(P)} > 0, \tag{36}$$

where $\lambda_{\max}(P)$ (resp. $\lambda_{\min}(P)$ is the largest (resp. smallest) eigenvalue of $P$.
According to *Lemma*1, we have

$$V(t) \le \eta e^{-\delta(t-t_0)} \quad \text{for} \quad t \ge t_0, \tag{37}$$

where $\beta < \alpha$ and $\alpha = \min\{\alpha_1, \alpha_2\}$ which is fulfilled under the following conditions:
$\beta < \alpha_1$ implies that

$$\tau_M < \sqrt{\frac{\lambda\lambda_{\min}(P)}{2\mu_1K_1^2\lambda_{max}(P)\theta^2} - \frac{k_f\lambda_{\min}(P)\lambda_{\max}(P)}{\mu_1K_1^2\theta^3}},$$

and

$$\beta < \alpha_2 \Longrightarrow \theta < \sqrt{\frac{\lambda_{\min}(P)}{2\tau_M\mu_1(\lambda_{\min}(P) + K_1^2\tau_M^2)}}.$$

This ends the proof.

As shown previously, we can see that the upper bound of the delay can be very small for high values of $\theta$. This means that for relatively important delays, the considered observer cannot guarantee the exponential convergence. In the next section, to overcome this problem, we use a novel high gain observer with lower tuning parameter introduced in Zemouche et al. (2019).

## 4. HG/LMI OBSERVER BASED DESIGN

In this section, we recall the HG/LMI observer methodology developed in Zemouche et al. (2019) for nonlinear systems. We apply this method for the system (1) which allows to improve the high gain results supposing the asymptotic stability given.

### *4.1 Transformations based on HG/LMI technique*

By the use of the LPV/LMI method in Zemouche and Boutayeb (2013), $\Delta f$ in (16) can be reformulated under the following form:

$$\Delta f = \underbrace{\overbrace{\sum_{j=1}^{n-j_0}\theta^j\psi_j\bar{x}_j}^{\text{for HG}}}_{\Delta f_1} + \overbrace{\sum_{j=1}^{j_0}\theta^{k(j)}\psi_{k(j)}\bar{x}_{k(j)}}^{\text{for LPV/LMI}}, \tag{38}$$

where

$$k(j) = n - (j_0 - j), \quad 0 \le j_0 \le n. \tag{39}$$

Consequently, the error dynamics (16) is rewritten as follows:

$$\begin{aligned}\dot{\bar{x}}(t) = \theta(\mathcal{A}(\Psi^\theta) - KC)\bar{x}(t) + \frac{1}{\theta^n}B\Delta f_1 \\ -\theta KC(\bar{x}(t-\tau) - \bar{x}(t)),\end{aligned} \tag{40}$$

where

$$\mathcal{A}(\Psi^\theta) = A + B\sum_{j=1}^{j_0}\psi_j^\theta e_n^T(k(j)), \tag{41}$$

$$\Psi^\theta = \begin{pmatrix} \psi_1^\theta \\ \vdots \\ \psi_{j_0}^\theta \end{pmatrix} \in \mathbb{R}^{j_0}, \quad (42)$$

$$\psi_j^\theta = \frac{\psi_{k(j)}}{\theta^{1+(j_0-j)}}. \quad (43)$$

Define the convex bounded set

$$\mathcal{H}_{j_0}^\sigma = \left\{ \Phi \in \mathbb{R}^{j_0} : \frac{\underline{\gamma}_{\gamma_{k(j)}}}{\sigma^{1+(j_0-j)}} \le \Phi_j \le \frac{\bar{\gamma}_{\gamma_{k(j)}}}{\sigma^{1+(j_0-j)}} \right\} \quad (44)$$

for which the set of vertices is defined by

$$\mathcal{V}_{\mathcal{H}_{j_0}^\sigma} = \left\{ \Phi \in \mathbb{R}^{j_0} : \Phi_j \in \left\{ \frac{\underline{\gamma}_{\gamma_{k(j)}}}{\sigma^{1+(j_0-j)}}, \frac{\bar{\gamma}_{\gamma_{k(j)}}}{\sigma^{1+(j_0-j)}} \right\} \right\} \quad (45)$$

Since $\bar{\gamma}_{\gamma_{k(j)}}$ and $\underline{\gamma}_{\gamma_{k(j)}} \le 0$, then it is evident that for two positive scalars $\sigma_1$,$\sigma_2$, we have the implication:

$$\sigma_1 \le \sigma_2 \Longrightarrow \mathcal{H}_{j_0}^{\sigma_1} \supset \mathcal{H}_{j_0}^{\sigma_2}. \quad (46)$$

It follows that

$$\lim_{\sigma \to +\infty} (\mathcal{H}_{j_0}^\sigma) = \{0_{\mathbb{R}^{j_0}}\} \quad (47)$$

Furthermore , we can prove the existence of a positive real number $k_{j_0} \le k_f$ so that $\Delta f_1$ satisfies

$$\|T^{-1}(\theta)B\Delta f_1\| \le \frac{k_{j_0}}{\theta^{j_0}}\|\bar{x}\| \quad (48)$$

### *4.2 Synthesis conditions based on HG/LMI technique*

This section is devoted to the main theorem, which provides sufficient synthesis conditions guaranteeing exponential convergence of the estimation error. The design is based on the use of the HG/LMI technique.

*Theorem 5.* If there exist a positive definite matrix $P$ and a matrix $\mathcal{Y}$ of appropriate dimensions and real constants $\mu_1, \lambda, \tau_M > 0$ so that the following conditions hold for all $\Psi \in \mathcal{V}_{\mathcal{H}_{j_0}^\sigma}$ :

$$\begin{bmatrix} \mathcal{H}e\Big\{P\mathcal{A}(\Psi) - \mathcal{Y}^\top C\Big\} + \tau_M \mathcal{R}^T\mathcal{R} + \lambda I & \mathcal{Y}^T \\ \mathcal{Y} & -\mu_1 \end{bmatrix} \le 0, \quad (49)$$

$$\theta < \sqrt{\frac{\lambda_{\min}(P)}{2\tau_M\mu_1(\lambda_{\min}(P) + K_1^2\tau_M^2)}}, \quad (50)$$

$$\theta > \max\left(\sigma, \sqrt[1+j_0]{\frac{2k_{j_0}\lambda_{max}^2(P)}{\lambda}}\right), \quad (51)$$

$$\tau_M \in \left[0, \sqrt{\frac{\lambda\lambda_{\min}(P)}{2\mu_1 K_1^2\lambda_{max}(P)\theta^2} - \frac{k_{j_0}\lambda_{\min}(P)\lambda_{\max}(P)}{\mu_1 K_1^2\theta^{3+j_0}}}\right] \quad (52)$$

where

$$\begin{cases} \mathcal{R} = [0 \quad 1 \quad 0_{1\times n-2}] \\ K = P^{-1}\mathcal{Y}^T = [K_1 \dots K_n]^T. \end{cases} \quad (53)$$

Then the observer (6) is exponentially convergent.

**Proof.** The proof is similar to the previous result. By analogy to the proof of Theorem (4), the derivative of $V$ along the trajectories of (40) satisfies

$$\begin{aligned} \frac{d}{dt}V(t) \le\ & \theta\bar{x}^T(t)[\mathcal{A}(\Psi^\theta)^T P + P\mathcal{A}(\Psi^\theta) - C^T\mathcal{Y} - \mathcal{Y}^T C \\ & + \frac{1}{\mu_1}\mathcal{Y}^T\mathcal{Y} + \tau_M\mathcal{R}^T\mathcal{R}]\bar{x}(t) + \left(\frac{1}{\theta^n}B\Delta f_1\right)^T P\bar{x}(t) \\ & + \bar{x}^T(t)P\left(\frac{1}{\theta^n}B\Delta f_1\right) - \theta\left(\frac{1}{\tau_M} - 2\mu_1\theta^2\right)V_2(t) \\ & + \frac{2\mu_1\theta^3K_1^2\tau_M^2}{\lambda_{\min}(P)}\sup_{[t-2\tau,t]} V(s) \end{aligned} \quad (54)$$

Let

$$\begin{aligned} \mathcal{A}(\Psi^\theta)^T P + P\mathcal{A}(\Psi^\theta) - C^T\mathcal{Y} - \mathcal{Y}^T C + \frac{1}{\mu_1}\mathcal{Y}^T\mathcal{Y} \\ + \tau_M\mathcal{R}^T\mathcal{R} \le -\lambda I, \forall \Psi^\theta \in \mathcal{H}_{j_0}^\theta \end{aligned} \quad (55)$$

The inequality (55) is not exploitable because its dependence on $\theta$. Nevertheless, using the implication (46), we obtain $\Psi^\theta \in \mathcal{H}_{j_0}^\sigma$ for all $\theta \ge \sigma$. Then from the convexity principle, inequality (55) holds only if the following inequality is fulfilled

$$\begin{aligned} \mathcal{A}(\Psi^\sigma)^T P + P\mathcal{A}(\Psi^\sigma) - C^T\mathcal{Y} - \mathcal{Y}^T C + \frac{1}{\mu_1}\mathcal{Y}^T\mathcal{Y} \\ + \tau_M\mathcal{R}^T\mathcal{R} \le -\lambda I, \forall \Psi^\sigma \in \mathcal{V}_{\mathcal{H}_{j_0}^\sigma} \end{aligned} \quad (56)$$

This ends the proof.

*Remark 6.* The previous proof shows the role of the "compromise index" $j_0$. It allows reducing the tuning parameter of the observer. Consequently, the value of the maximum delay, $\tau_M$, allowed becomes higher.

## 5. CONCLUSION

In this paper we addressed the problem of observer design for a class of nonlinear systems with delayed measurements. The delay is assumed to be known and bounded. The objective was to develop a state observer allowing a maximum bound of the delay as high as possible while ensuring exponential convergence. To this end, we used the HG/LMI observer (Zemouche et al., 2019), instead of the standard high-gain methodology, which led to a considerably higher allowable maximum bound on the delay. The convergence analysis was performed by using a Lyapunov-Krasovskii functional, depending on the tuning parameter of the observer, jointly with the Halanay inequality. Due to the lack of space, there are no numerical examples to illustrate the superiority of the proposed observer design procedure. However, on the other hand, the explicit relation between the tuning parameter of the observer and the maximum bound of the delay provided in this note, shows analytically the superiority of the proposed method.

As future works, we aim to improve more the result by exploring new ideas on high-gain observers, namely the introduction of specific nonlinear transformations to decrease the value of the tuning parameter. The goal consists in applying the theoretical results to real-world applications like wastewater treatment models and anaerobic digestion processes.

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