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State Observer Design Method for a Class of Nonlinear Systems

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Abstract: In this paper, we develop a new high gain observer design method for nonlinear systems. This new structure provides a lower gain compared to both the high gain and the enhanced high gain observer. The idea is to combine the improved high gain methodology with the LMI-based observer design technique to build a more general observer that allows us to exploit the benefits of both approaches.

Keywords: Observer design, enhanced high-gain technique, Lipschitz systems, LMIs.

1. INTRODUCTION

Observer design for nonlinear systems have been investigated for many decades. This is due to its important role in control design systems, diagnosis, health monitoring, and other modern applications like synchronization of multi-agent systems and cyber-attacks detection. There are several methods developed in the literature which we may classify in three categories: Extended Kalman Filter; Luenberger observer; high-gain observer methodology; and LMI-based techniques. However, this paper focuses on high-gain observers only.

The design of high gain observers was essentially motivated by its simplicity to implement due to the use of only single tuning parameter. However, there are three limitations that make high gain observer weak and difficult to be used in sensitive industrial applications. The first limitation is related to numerical issues concerning large systems as high values of the observer gain are required. The second limitation is the sensitivity to measurement noise because high values of the observer gains amplify the noise. The third and last limitation is the peaking phenomenon characterized by large amplitudes of the estimated states in the transient.

To overcome this restrictions, several solutions have been proposed in the literature. The main solutions are generally based on a time-varying gain that is appropriately updated by taking into account the stability and convergence requirements (Ahrens and Khalil, 2009), (Boizot et al., 2010), (Oueder et al., 2012), (Sanfelice and Praly, 2011) and (Alessandri and Rossi, 2015). Recently, a new high gain observer, called a low power high gain observer, has been proposed in Astolfi and Marconi (2015). Their contribution consists in limiting the power of the tuning parameter to 2. However, the dimension of the observer is equal to $2(n-1)$ where n is the dimension of the original system, and the power n is only distributed between different additional state variables injected in the observer. Then this power n reappears in the bound of the estimation error when the system is subject to measurement noise. This particular design has been reconsidered in Astolfi et al. (June 2016) and in Teel (2016) by including saturations to avoid the peaking phenomenon. Another recent high gain observer with the same dimension as the original system and where the observer's gain power is limited to 1 was proposed in Khalil (2017) for the same class of systems considered in Astolfi et al. (June 2016). As in Astolfi et al. (June 2016) and in Teel (2016), nested saturation functions have been used to limit the peaking phenomenon. In Zemouche et al. (2019) a new structure of observers, called HG/LMI observer, has been developed by combining the standard high-gain methodology with the LPV/LMI technique (Zemouche and Boutayeb, 2013). This new observer has the advantage to provide lower tuning parameter compared to the previous high-gain observers, without using saturation functions or filtering.

In this paper, we develop a new state observer design for systems with multi-nonlinearities in triangular form or any system that can be transformed into a triangular structure. The proposed observer has the advantage of allowing more possibilities of choosing the design parameters. The idea consists in combining the enhanced high gain methodology (Alessandri and Zemouche, 2016) with the LPV/LMI methodology in order to reduce more

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the value of the observer gains. This structure has the advantage to use multiple tuning parameters. Indeed, the observer in Zemouche et al. (2019) becomes a particular case of the proposed observer in this paper by a special choice of the design parameters. It is shown through a simple example that the proposed technique reduces the peaking phenomenon and decreases the sensitivity to high-frequency measurement noise as compared with the high gain observer.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Preliminaries

Before formulating the problem, we introduce some useful preliminaries for the developed approach. We will recall two lemmas, which are necessary for the mathematical developments given in the next section.

Lemma 1. (Alessandri and Zemouche (2016)). Let X and Y be two matrices of adequate dimensions. Then the following inequality holds for any symmetric and nonsingular matrix S of appropriate dimension:

$$X^\top Y + Y^\top X \leq \frac{1}{2}(X+SY)^\top S^{-1}(X+SY)$$

This lemma will be with great helpful to linearize bilinear matrix inequalities encountered during the observer synthesis.

As for the next lemma, it will be used to decompose any Lipschitz nonlinear function in a convenient way. Such decompositions play an important role in the observer synthesis and allow enhancing the standard high-gain observer. However, before stating the lemma, the following definition is needed.

Definition 1. (Zemouche and Boutayeb (2013)). Consider two vectors

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad Z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n.$$

For all $i = 0, ..., n$, we define an auxiliary vector $X^{Z_i} \in \mathbb{R}^n$ corresponding to X and Z as follows:

$$\begin{cases} X^{Z_i} = \begin{pmatrix} z_1 \\ \vdots \\ z_i \\ x_{i+1} \\ \vdots \\ x_n \end{pmatrix} \quad \text{for } i = 1, ..., n \\ X^{Z_0} = X. \end{cases} \tag{1}$$

Lemma 2. (Zemouche and Boutayeb (2013)). Consider a function $\Psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Then, the two following claims are equivalent:

- Ψ is γ_Ψ-Lipschitz with respect to its argument, i.e.:
$$\Big\|\Psi(X) - \Psi(Z)\Big\| \leq \gamma_\Psi \Big\|X - Z\Big\|, \quad \forall\, X, Z \in \mathbb{R}^n; \tag{2}$$
- for all $i, j = 1, ..., n$, there exist functions
$$\psi_{ij} \;:\; \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$
and constants $\underline{\gamma}_{\psi_{ij}} \leq 0$, $\bar{\gamma}_{\psi_{ij}} \geq 0$, so that $\forall\, X, Z \in \mathbb{R}^n$,
$$\Psi(X) - \Psi(Z) = \sum_{i=1}^{i=n}\sum_{j=1}^{j=n} \psi_{ij} H_{ij}\Big(X - Z\Big), \tag{3}$$
and
$$-\gamma_\Psi \leq \underline{\gamma}_{\psi_{ij}} \leq \psi_{ij} \leq \bar{\gamma}_{\psi_{ij}} \leq \gamma_\Psi, \tag{4}$$
where
$$\psi_{ij} \triangleq \psi_{ij}\Big(X^{Z_{j-1}}, X^{Z_j}\Big) \quad \text{and} \quad H_{ij} = \mathrm{e}_n(i)\mathrm{e}_n^\top(j).$$

2.2 Problem formulation

Let us consider the class of nonlinear systems described by the set of equations:

$$\begin{cases} \dot{x} = Ax + f(x), \\ y = Cx. \end{cases} \tag{5}$$

Where $x \in \mathbb{R}^n$ is the state vector and $y \in \mathbb{R}$ is the measured output. The matrices $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{1\times n}$, and the nonlinear function $f : \mathbb{R}^n \to \mathbb{R}^n$ are defined as follows:

$$A \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad C \triangleq [\,1\;0\;\cdots\;0\,],$$

$$f(x) \triangleq \begin{bmatrix} f_1(x_1) \\ f_2(x_1,\, x_2) \\ \vdots \\ f_{n-1}(x_1, x_2,\, \cdots,\, x_{n-1}) \\ f_n(x_1, x_2,\, \cdots,\, x_n) \end{bmatrix}.$$

As usually done for this class of systems, we introduce the following Lipschitz assumption on f.

Assumption 1. The function f satisfies the global Lipschitz condition, i.e., there exists $L \in \mathbb{R}^n_{\geq 0}$ such that

$$\begin{aligned} &|f_i(x_1 + w_1,\, x_2 + w_2,\, \cdots + x_i + w_i) \\ &- f_i(x_1,\, x_2,\, \cdots,\, x_i)| \leq \sum_{j=1}^{i} L_i |w_j| \end{aligned}$$

for all x, $w \in \mathbb{R}^n$, where L_i is the i^{th} component of L.

Instead of standard high-gain observer structure, we use in this paper the enhanced structure previously proposed in Alessandri and Zemouche (2016). This structure has the advantage to use multiple tuning parameters.

Consider the following state observer Alessandri and Zemouche (2016):

$$\dot{\widehat{x}} = A\hat{x} + f(\widehat{x}) + G(\gamma,\, K)\big(y - C\widehat{x}\big) \tag{6}$$

where $\widehat{x} \in \mathbb{R}^n$ is the estimate of x and

$$G(\gamma,\, K) \triangleq \begin{bmatrix} \gamma_1 k_1 \\ \gamma_2 k_2 \\ \vdots \\ \gamma_n k_n \end{bmatrix} \triangleq \mathrm{T}(\gamma)K$$

with

$$\gamma \triangleq \begin{bmatrix}\gamma_1\\ \gamma_2\\ \vdots\\ \gamma_n\end{bmatrix} \in \mathbb{R}^n_{>0},\ K \triangleq \begin{bmatrix}k_1\\ k_2\\ \vdots\\ k_n\end{bmatrix} \in \mathbb{R}^n$$

and

$$\mathrm{T}(\gamma) = \mathrm{diag}(\gamma_1,\, \gamma_2, \ldots,\, \gamma_n).$$

As usually in the high-gain methodology, we consider the transformed error

$$\tilde{x} \triangleq \mathrm{T}^{-1}(\gamma)e, \tag{7}$$

where $e(t) \triangleq x(t) - \hat{x}(t)$ is the estimation error vector. After developing the computations, it follows that

$$\dot{\tilde{x}} = \gamma_1\Big(A - KC + \Omega(\gamma)\Big)\tilde{x} + \mathrm{T}^{-1}(\gamma)\left[f(x) - f\left(x - \mathrm{T}(\gamma)\tilde{x}\right)\right] \tag{8}$$

where

$$\Omega(\gamma) \triangleq \begin{bmatrix} 0 & z_1 & 0 & \cdots & 0\\ 0 & 0 & z_2 & & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & z_{n-1}\\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

with

$$z_i \triangleq \frac{\gamma_{i+1}}{\gamma_1\gamma_i} - 1,\ i = 1,\, 2, \ldots,\, n-1. \tag{9}$$

The aim consists in synthesizing the observer parameters γ_i and k_i such that the error $\tilde{x}$ converges exponentially to zero. Usually, this problem is solved by using the standard high-gain observer methodology Gauthier et al. (1992). However, in some situations (for instance larger Lipschitz constants, high dimension of the systems), it leads to extremely high values of the gains, which render the observer very sensitive to high frequency measurement noise and causes the picking phenomenon in the transient. To overcome this obstacle, various improvements have been established in the literature, proposing high-gain observers with constant or time-varying gains. Limiting our study, in this paper, to observers with constant gains, some recent methods proposed considerable solutions (Alessandri and Zemouche, 2016), (Astolfi and Marconi, 2015), (Zemouche et al., 2019), nevertheless, the problem remains still open for further improvements. In this paper, we will combine between (Alessandri and Zemouche, 2016) and (Zemouche et al., 2019) to propose a new approach. To tackle this problem, a convenient decomposition of the nonlinearity and introduction of additional parameters are required. This is the goal of the next section.

Before stating the observer design methodology proposed in this paper, let us define the set

$$\Gamma^+ \triangleq \Big\{\gamma = (\gamma_i)_{1\le i\le n} \in \mathbb{R}^n,\ 0 < \gamma_1 \le \gamma_2 \le \ldots \le \gamma_n\Big\} \tag{10}$$

for any fixed $\gamma = [\gamma_1,\, \gamma_2, \ldots,\, \gamma_n]^\top$.

3. OBSERVER SYNTHESIS METHODOLOGY

3.1 Preliminary transformations

As stated above, the idea is to exploit the results of Alessandri and Zemouche (2016) and Zemouche et al. (2019) to improve the solution of the observer gains. Borrowed from (Zemouche et al., 2019, Eq. (54)), the first step consists in decomposing the nonlinearity of the system into two parts. From Lemma 2 and after some rearrangements, there exist functions ψ_{ij}, scalars $\underline{\nu}_{\psi_{ik_i(j)}} \le 0$ and $\bar{\nu}_{\psi_{ik_i(j)}} \ge 0$ such that

$$f(x) - f\left(x - \mathrm{T}(\gamma)\tilde{x}\right) = \Delta f_1 + \Delta f_2,$$

with

$$\Delta f_1 \triangleq \sum_{i=1}^{n}\sum_{j=1}^{i-j_i}\gamma_j\psi_{ij}e_n(i)\tilde{x}_j,$$

$$\Delta f_2 \triangleq \sum_{i=1}^{n}\sum_{j=1}^{j_i}\gamma_{k_i(j)}\psi_{ik_i(j)}e_n(i)\tilde{x}_{k_i(j)},$$

$$k_i(j) \triangleq i - (j_i - j),\ 0 \le j_i \le i$$

and

$$\underline{\nu}_{\psi_{ik_i(j)}} \le \psi_{ij} \le \bar{\nu}_{\psi_{ik_i(j)}}.$$

By analogy to Zemouche et al. (2019), the first term Δf_1 will be handled by the EHGO approach in Alessandri and Zemouche (2016), while the second one, Δf_2, will be associated to the linear part and will be processed by the LPV/LMI method Zemouche and Boutayeb (2013) as in Zemouche et al. (2019).

Notice that the term $\mathrm{T}^{-1}(\gamma)\Delta f_2$ can be rewritten as:

$$\mathrm{T}^{-1}(\gamma)\Delta f_2 = \sum_{i=1}^{n}\sum_{j=1}^{j_i}\frac{\gamma_{k_i(j)}}{\gamma_i}\psi_{ik_i(j)}e_n(i)\tilde{x}_{k_i(j)},$$

Now, we will introduce some notations needed to rewrite system (8) under a suitable structure to apply the ideas of Zemouche and Boutayeb (2013) and Zemouche et al. (2019). Let us introduce the following matrix function:

$$\mathcal{A}(\Psi^\gamma) \triangleq A + \sum_{i=1}^{n}\sum_{j=1}^{j_i}\psi^\gamma_{ij}e_n(i)e_n^\top(k_i(j)), \tag{11}$$

where

$$\Psi^\gamma \triangleq \begin{pmatrix}\psi^\gamma_{11}\\ \vdots\\ \psi^\gamma_{ij_1}\\ \vdots\\ \psi^\gamma_{ij_i}\\ \vdots\\ \psi^\gamma_{nj_i}\\ \vdots\\ \psi^\gamma_{nj_n}\end{pmatrix} \in \mathbb{R}^d \tag{12}$$

and

$$\psi^\gamma_{ij} \triangleq \frac{\gamma_{k_i(j)}}{\gamma_1\gamma_i}\psi_{ik_i(j)},\ d \triangleq \sum_{i=1}^{n} j_i.$$

Consequently, system (8) can be expressed as follows:

$$\dot{\tilde{x}} = \gamma_1\Big(\mathcal{A}(\Psi^\gamma) - KC + \Omega(\gamma)\Big)\tilde{x} + \mathrm{T}^{-1}(\gamma)\Delta f_1. \tag{13}$$

3.2 Preliminary results

Before stating the observer design conditions ensuring the exponential convergence of the proposed state observer, we start by introducing some preliminary results, which are necessary for the proposed design procedure. We first define the convex set for any fixed $\gamma \in \Gamma^+$:

$$\tilde{\mathcal{H}}^\gamma \triangleq \left\{ \Phi \in \mathbb{R}^d : \frac{\gamma_{k_i(j)} \underline{\nu}_{\psi_{ik_i(j)}}}{\gamma_1 \gamma_i} \le \Phi_{ij} \le \frac{\gamma_{k_i(j)} \bar{\nu}_{\psi_{ik_i(j)}}}{\gamma_1 \gamma_i} \right\}. \quad (14)$$

It is obvious that $\tilde{\mathcal{H}}^\gamma$ is a bounded convex. Indeed, from the fact that the function f is Lipschitz, the scalars $\underline{\nu}_{\psi_{ik_i(j)}}$ and $\bar{\nu}_{\psi_{ik_i(j)}}$ are bounded. On the other hand, since $\gamma \in \Gamma^+$, we have $\frac{\gamma_{k_i(j)}}{\gamma_1 \gamma_i} \le \frac{1}{\gamma_1}$. It follows that $\frac{\underline{\nu}_{\psi_{ik_i(j)}}}{\gamma_1} \le \Phi_{ij} \le \frac{\bar{\nu}_{\psi_{ik_i(j)}}}{\gamma_1}$., which means that Φ_{ij} is bounded since $\gamma_1 > 0$.

At this stage, the bounded convex set $\tilde{\mathcal{H}}^\gamma$ is not exploitable in an LMI framework because the set of vertices depends on all the parameters $\gamma_i, i = 1, \dots, n$. In other word, it depends on the decision variables $z_i, i = 1, \dots, n-1$. To overcome this obstacle, we need to define a new bounded and convex hyper-rectangle independent from all these observer parameters. Before introducing such a set, we first state the following lemma.

Lemma 3. Let $\gamma \in \Gamma^+$ and z_i given by (9). If z_i satisfies $z_i \le 0$, then there exists $\alpha \in]0, 1]$ such that inequality (15) below holds:

$$\frac{\gamma_{k_i(j)}}{\gamma_1 \gamma_i} \le \frac{1}{(\alpha \gamma_1)^{1+(j_i - j)}}. \quad (15)$$

Proof.

From the definition of the variables z_i in Section 2.2 and the assumption $z_i \le 0$, we get

$$\gamma_i = \gamma_1^i \prod_{k=1}^{i-1} (z_k + 1), \quad i = 2, ..., n$$

and

$$0 < 1 + z_k \le 1, \quad i = 1, ..., n-1.$$

It follows that

$$\frac{\gamma_{k_i(j)}}{\gamma_1 \gamma_i} \le \frac{1}{\gamma_1^{1+(j_i - j)} \prod\limits_{k=i-(j_i-j)}^{i} (z_k + 1)} \quad (16)$$

and from the Archimedean property, we deduce that there exists $\alpha \in]0, 1]$ so that

$$0 < \alpha \le z_k + 1 \le 1. \quad (17)$$

Hence, by substituting (17) in (16) as

$$\frac{1}{1 + z_k} \le \frac{1}{\alpha}$$

the inequality (15) is straightforwardly inferred.

Now we are ready to introduce a new bounded convex set parameterized by two scalar variables, namely α given as in Lemma 3 and a new tuning parameter $\sigma > 0$ to be included later in the observer design procedure.

Let α and σ be two positive scalars. Then, we define the bounded convex set

$$\mathcal{H}_\alpha^\sigma \triangleq \left\{ \Phi \in \mathbb{R}^d : \frac{\underline{\nu}_{\psi_{ik_i(j)}}}{(\sigma\alpha)^{1+(j_i - j)}} \le \Phi_{ij} \le \frac{\bar{\nu}_{\psi_{ik_i(j)}}}{(\sigma\alpha)^{1+(j_i - j)}} \right\}. \quad (18)$$

for which the set of vertices, $\mathcal{H}_\alpha^\sigma$, is given by:

$$\mathcal{V}_{\mathcal{H}_\alpha^\sigma} \triangleq \left\{ \Phi \in \mathbb{R}^d : \Phi_{ij} \in \left\{ \frac{\underline{\nu}_{\psi_{ik_i(j)}}}{(\sigma\alpha)^{1+(j_i - j)}}, \frac{\bar{\nu}_{\psi_{ik_i(j)}}}{(\sigma\alpha)^{1+(j_i - j)}} \right\} \right\}. \quad (19)$$

The next lemma is useful and plays an important role in the design procedure we proposed in this paper.

Lemma 4. Let $\gamma \in \Gamma^+$ and $\sigma > 0$ such that $\gamma_1 \ge \sigma$. Let $\alpha \in]0, 1]$ be a positive scalar given by (17). Then the following inclusion holds:

$$\tilde{\mathcal{H}}^\gamma \subseteq \mathcal{H}_\alpha^\sigma. \quad (20)$$

Proof. The proof is straightforward by using Lemma 3 and the fact that the quantities $\underline{\nu}_{\psi_{ik_i(j)}}$ and $\bar{\nu}_{\psi_{ik_i(j)}}$ are negative and positive, respectively. The inequality $\frac{1}{\gamma_1} \le \frac{1}{\sigma}$ is also used and substituted in (15).

The next section is devoted to the stability analysis of the estimation error dynamics. By using Lyapunov arguments and the preliminary results provided above, new high-gain like synthesis conditions will be established.

3.3 Stability analysis

To investigate the stability analysis of the estimation error dynamics, we consider the Lyapunov function

$$V(\tilde{x}) = \tilde{x}^\top P \tilde{x}$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric and positive definite matrix. First, let us consider the change of variable $\tilde{K} = PK$. Therefore, after developing the derivative of the function $V(\tilde{x})$ along the trajectories of (13), we obtain

$$\dot{V}(\tilde{x}) = \gamma_1 \tilde{x}^\top \Big[\mathcal{A}(\Psi^\gamma)^\top P + P\mathcal{A}(\Psi^\gamma) - C^\top \tilde{K}^\top - \tilde{K} C + \Omega^\top(\gamma) P + P\Omega(\gamma) \Big] \tilde{x} + 2\tilde{x}^\top P \mathrm{T}^{-1}(\gamma) \Delta f_1. \quad (21)$$

Before presenting the stability conditions ensuring the exponential convergence of the estimation error $\tilde{x}$ to zero, we provide some informations on the term Δf_1. This term will be handled by using the high-gain methodology.

Lemma 5. Assume that $\gamma \in \Gamma^+$, and $z_i \le 0, i = 1, \dots, n-1$. Then there exist bounded scalars $k_{f_1} > 0$ and $\alpha \in]0, 1]$, independent from γ, such that

$$\left\| \mathrm{T}^{-1}(\gamma) \Delta f_1 \right\| \le \frac{1}{(\alpha \gamma_1)^{j_{\min}}} k_{f_1} \|\tilde{x}\| \quad (22)$$

where

$$j_{\min} = \min_{j_i \neq i} j_i.$$

Proof. We have

$$\Delta f_1 = \sum_{i=1}^{n}\left(\sum_{j=1}^{i-j_i}\gamma_j\psi_{ij}\tilde{x}_j\right)e_n(i)$$

$$T^{-1}(\gamma)\Delta f_1 = \sum_{i=1}^{n}\left(\sum_{j=1}^{i-j_i}\frac{\gamma_j}{\gamma_i}\psi_{ij}\tilde{x}_j\right)e_n(i).$$

Therefore

$$\|T^{-1}(\gamma)\Delta f_1\|^2 = \sum_{i=1}^{n}\left(\sum_{j=1}^{i-j_i}\frac{\gamma_j}{\gamma_i}\psi_{ij}\tilde{x}_j\right)^2.$$

Using Assumption 1 and Holder's inequality, it follows that:

$$\begin{aligned}\|T^{-1}(\gamma)\Delta f_1\|^2 &\leq \sum_{i=1}^{n}\left(\sum_{j=1}^{i-j_i}L_j|\tilde{x}_j|\frac{\gamma_j}{\gamma_i}\right)^2\\ &\leq \sum_{i=1}^{n}\left[\left(\max_{1\leq j\leq i-j_i}L_j\right)^2\left(\frac{\gamma_{i-j_i}}{\gamma_i}\right)^2\left(\sum_{j=1}^{i-j_i}|\tilde{x}_j|\right)^2\right]\\ &\leq (\max_{1\leq j\leq n}L_j)^2\sum_{i=1}^{n}(i-j_i)\left(\frac{\gamma_{i-j_i}}{\gamma_i}\right)^2\|\tilde{x}\|^2\\ &\leq \left[k_{f_1}\max_{1\leq i\leq n}\frac{\gamma_{i-j_i}}{\gamma_i}\right]^2\|\tilde{x}\|^2\end{aligned} \tag{23}$$

where

$$k_{f_1} = \bar{L}\sqrt{\left(\frac{n(n+1)}{2} - \sum_{i=1}^{n}j_i\right)},\ \bar{L} = \max_{1\leq j\leq n}L_j.$$

Since $z_k \leq 0, \forall k = 1,\dots,n-1$, then from Lemma 3, there exists $\alpha \in]0,1]$ such that:

$$\frac{\gamma_{i-j_i}}{\gamma_1\gamma_i} \leq \frac{1}{(\alpha\gamma_1)^{1+j_i}}.$$

By putting $j_{\min} = \min_{1\leq i\leq n} j_i$, we get

$$\max_{1\leq i\leq n}\frac{\gamma_{i-j_i}}{\gamma_i} \leq \frac{1}{(\alpha\gamma_1)^{j_{\min}}}.$$

Then inequality (22) is inferred. This ends the proof.

Now we are ready to state the first theorem which provides sufficient design conditions ensuring exponential convergence of the estimation error to zero.

Theorem 6. Assume there exist $P = P^\top > 0$, $\lambda > 0$, $\tilde{K} \in \mathbb{R}^n$, $\gamma \in \Gamma^+$ and $\sigma > 0$ such that:

$$\Big\{\mathcal{A}(\Psi)^\top P + P\mathcal{A}(\Psi) - C^\top\tilde{K}^\top - \tilde{K}C + \Omega^\top(\gamma)P + P\Omega(\gamma)\Big\} + \lambda I < 0, \forall\Psi \in \mathcal{V}_{\mathcal{H}_\alpha^\sigma} \tag{24}$$

and

$$\gamma_1 > \max\left(\sigma, \left[\frac{2k_{f_1}\lambda_{\max}(P)}{\lambda\alpha^{j_{\min}}}\right]^{\frac{1}{1+j_{\min}}}, \frac{1}{\alpha}\right). \tag{25}$$

Then the estimation error $\tilde{x}(t)$ is exponentially stable.

Proof. From the convexity principle and the inclusion (20) for $\gamma_1 \geq \sigma$, if (24) hold, we deduce that

$$\dot{V}(\tilde{x}) \leq -\gamma_1\lambda\|\tilde{x}\|^2 + 2\tilde{x}^\top P\mathrm{T}^{-1}(\gamma)\Delta f_1. \tag{26}$$

Let $\alpha \in]0,1]$ satisfying (17). Then, from Lemma 5, we have

$$\begin{aligned}2\tilde{x}^\top P\mathrm{T}^{-1}(\gamma)\Delta f_1 &\leq 2\lambda_{\max}(P)\|\tilde{x}\|\left\|\mathrm{T}^{-1}(\gamma)\Delta f_1\right\|\\ &\leq \frac{2\lambda_{\max}(P)}{(\alpha\gamma_1)^{j_{\min}}}k_{f_1}\|\tilde{x}\|^2.\end{aligned} \tag{27}$$

It follows that

$$\dot{V}(\tilde{x}) \leq -\left(\gamma_1\lambda - \frac{2\lambda_{\max}(P)}{(\alpha\gamma_1)^{j_{\min}}}k_{f_1}\right)\|\tilde{x}\|^2. \tag{28}$$

From the definition of $\dot{V}(\tilde{x})$ and after integrating from 0 to t, we obtain

$$\|\tilde{x}(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\|\tilde{x}_0\|e^{-\left(\gamma_1\lambda - \frac{2\lambda_{\max}(P)}{(\alpha\gamma_1)^{j_{\min}}}k_{f_1}\right)t} \tag{29}$$

which means that $\tilde{x}(t)$ converges exponentially to zero if $\gamma_1\lambda - \frac{2k_{f_1}\lambda_{\max}(P)}{(\alpha\gamma_1)^{j_{\min}}} > 0$. On the other hand, to guarantee $\gamma \in \Gamma^+$, we need to have $\gamma_1 \geq \frac{1}{\alpha}$, since γ_1 satisfies (17). These conditions on γ_1 lead to inequality (25). To sum up, the conditions on γ_1 are requird for the following reasons:

(1) $\gamma_1 \geq \sigma$ is needed to ensure feasibility of the inequalities (24). It is justified by the inclusion (20);
(2) $\gamma_1 > \frac{1}{\alpha}$ is needed to guarantee $\gamma \in \Gamma^+$;
(3) $\gamma_1 > \left[\frac{2k_{f_1}\lambda_{\max}(P)}{\lambda\alpha^{j_{\min}}}\right]^{\frac{1}{1+j_{\min}}}$ is required to ensure $\tilde{x}(t)$ converges exponentially to zero, according to (29).

This ends the proof.

3.4 LMI formalization

Although Theorem 6 provides sufficient conditions to guarantee the design of the observer parameters K and γ, it still not fully exploitable at this stage because the matrix inequalities (24) are not numerically tractable. Indeed, (24) are not LMIs and depend on the parameter γ multiplied by the Lyapunov matrix P. To linearize (24) and to render it independent of γ, we should separate the coupling $P\Omega(\gamma)$ and use some mathematical tools to make γ vanish from inequality (24). To start the linearization procedure, we consider the following decomposition of $\Omega(\gamma)$:

$$\Omega(\gamma) \triangleq \Omega(Z) = A_1 Z A_2,$$

where

$$Z = \mathrm{diag}(z_1, \dots, z_{n-1}) \in \mathbb{R}^{(n-1)\times(n-1)}$$

and

$$A_1 \triangleq \begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & 1\\ 0 & \cdots & 0 & 0\end{bmatrix} \in \mathbb{R}^{n\times(n-1)},$$

$$A_2 \triangleq \begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 1\end{bmatrix} \in \mathbb{R}^{(n-1)\times n}.$$

Then a simple use of Lemma 1 leads to separate P from $\Omega(Z)$. To satisfy (17), we should include the constraints:

$$Z \leq 0, \tag{30a}$$

$$(\alpha-1)I_{n-1}-Z\leq 0. \tag{30b}$$

Hence we are ready to state the following main theorem, which provides LMI-based synthesis conditions ensuring the exponential convergence of the observer.

Theorem 7. Assume that there exist positive scalars λ, α, σ, and γ_1, a symmetric positive definite matrix P, diagonal matrices $S>0$ and $W\leq 0$, such that for all $\psi\in\mathcal{V}_{\mathcal{H}_\alpha^\sigma}$ the following conditions are fulfilled:

$$\begin{bmatrix}\mathcal{A}(\psi)^\top P+P\mathcal{A}(\psi)-C^\top\tilde{K}^\top-\tilde{K}C+\lambda I & (*)\\ A_1^\top P+WA_2 & -2S\end{bmatrix}<0 \tag{31}$$

$$(\alpha-1)S-W\leq 0 \tag{32}$$

$$\gamma_1>\max\left(\sigma,\left[\frac{2k_{f_1}\lambda_{\max}(P)}{\lambda\alpha^{j_{\min}}}\right]^{\frac{1}{1+j_{\min}}},\frac{1}{\alpha}\right). \tag{33}$$

Then, the estimation error $e(t)$ converges exponentially to zero if the observer parameters are selected as follows:

$$K=P^{-1}\tilde{K},\ Z=S^{-1}W, \tag{34a}$$

$$\gamma_i=\gamma_1^i\prod_{k=1}^{i-1}(z_k+1),\quad i=2,\ldots,n. \tag{34b}$$

Proof. First, to get (31), we apply Lemma 1 on the inequality (24) of Theorem 6. Indeed, from Lemma 1, we have

$$\begin{aligned}\Omega^\top P+P\Omega &= (A_1P)^\top(ZA_2)+(ZA_2)^\top(A_1P)\\ &\leq \frac{1}{2}(A_1^\top P+SZA_2)^\top S^{-1}(A_1^\top P+SZA_2).\end{aligned}$$

Then, after using Schur lemma and the change of variables $W=SZ$, $\tilde{K}=PK$, we get (31). Also, condition $W\leq 0$ comes from (30a). As for the inequality (32), it stems from (30b) after multiplying by S. This ends the proof.

Remark 8. The proposed observer design method is more general than those proposed in the literature and related to high-gain methodology with constant observer gain. Indeed, for particular cases, the design is reduced to some recent methods. We summarize the particular cases in the following items:

(1) If we take $j_i=0$ and $z_k<0$, we will get exactly the enhanced high gain proposed in Alessandri and Zemouche (2016). Indeed, in such a case, we have $j_{\min}=0$, $k_{f_1}=k_f$, and $\mathcal{A}(\psi)\equiv A$.
(2) Likewise, if we have $j_i=0$ (then $j_{\min}=0$) and $z_k=0$, we get the standard high gain and Theorem 7 will be reduced to the main theorem of the standard high-gain observer Gauthier et al. (1992).
(3) Notice also that the HG/LMI observer proposed in Zemouche et al. (2019) is a particular case of the result in Theorem 7 corresponding to $j_{\min}\geq 1$ and $z_k\equiv 0$.

4. CONCLUSION

What we proposed in this paper is a general structure that comprises many of methods discussed in the literature that we can recover by making particular choices on the observer design parameters. Especially, we generalized the work presented in Zemouche et al. (2019) with more many possibilities of choosing the design parameters of the gain. The stability of the estimation error is shown using a Lyapunov function after having successfully established new high-gain like synthesis conditions.

As future work, we will investigate more the methodology by adding illustrative examples and establish comparisons with other existing methods in the literature. We mainly aim to apply this methodology on estimation of key variables in an anaerobic digestion process with experimental data and simulations. The experimentation will be integrated in an Arduino-based experimental platform we will create.

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