Computation of Electromagnetic Fields Scattered From Dielectric Objects of Uncertain Shapes Using MLMC

A. Litvinenko\textsuperscript{1}, A. C. Yucel\textsuperscript{3}, H. Bagci\textsuperscript{2}, J. Oppelstrup\textsuperscript{4}, E. Michielssen\textsuperscript{5}, R. Tempone\textsuperscript{1,2}

\textsuperscript{1}RWTH Aachen, \textsuperscript{2}KAUST, \textsuperscript{3}Nanyang Technological University in Singapore, \textsuperscript{4}KTH Royal Institute of Technology, \textsuperscript{5}University of Michigan

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Motivation

Efficient computation tools for characterizing scattering from objects of uncertain shapes are needed in the fields of electromagnetics, optics, and photonics.

**How:** Use CMLMC method (advanced version of Multi-Level Monte Carlo).

CMLMC optimally balances statistical and discretization errors. It requires very few samples on fine meshes and more on coarse.

taken from wiki, reddit.com, EMCoS
Plan:

1. Scattering problem setup
2. Deterministic solver
3. Generation of random shapes
4. Shape transformation
5. QoI on perturbed shape
6. Continuation Multi Level Monte Carlo (CMLMC)
7. Results (time, work vs. TOL, weak and strong convergences)
8. Conclusion
Scattering problem

Input: randomly perturbed shape
Output: radar and scattering cross sections, electric and magnetic surface current densities
Previous works

- Monte Carlo (N. Garcia, Jandhyala, Michielssen,)
- surrogate methods (A. Yucel, H. Bagci, L. Gomez, L.H. Garcia)
- stochastic collocation ([C. Chauviere, J. Hesthaven, K. Wilcox’07], [D. Xiu, J. Hesthaven’07], [Zh. Zeng, J. M. Jin’07])
Deterministic solver

Electromagnetic scattering from dielectric objects is analyzed by using the Poggio-Miller-Chan-Harrington-Wu-Tsai surface integral equation (PMCHWT-SIE) solver. The PMCHWT-SIE is discretized using the method of moments (MoM) and the iterative solution of the resulting matrix system is accelerated using a (parallelized) fast multipole method (FMM) - fast Fourier transform (FFT) scheme (FMM-FFT).

Input uncertainties: position, orientation, roughness, and shape of scatterers, as well as internal and/or external excitation characteristics such as the frequency, amplitude, and angle of arrival.
Generation of random shapes

Perturbed shape $v(\vartheta_m, \varphi_m)$ is defined as

$$v(\vartheta_m, \varphi_m) \approx \tilde{v}(\vartheta_m, \varphi_m) + \sum_{k=1}^{K} a_k \kappa_k(\vartheta_m, \varphi_m).$$

(1)

where $\vartheta_m$ and $\varphi_m$ are angular coordinates of node $m$, $\tilde{v}(\vartheta_m, \varphi_m) = 1$ is unperturbed radial coordinate on the unit sphere. $\kappa_k(\vartheta, \varphi)$ obtained from spherical harmonics by re-scaling their arguments, $\kappa_1(\vartheta, \varphi) = \cos(\alpha_1 \vartheta)$, $\kappa_2(\vartheta, \varphi) = \sin(\alpha_2 \vartheta) \sin(\alpha_3 \varphi)$, where $\alpha_1, \alpha_2, \alpha_3 > 0$. 
Some fun (un)realistic uncertain shapes
Mesh transformation

The perturbed mesh $P_0$ is also rotated and scaled using the following transformation

$$
\begin{bmatrix}
  x_m \\
  y_m \\
  z_m
\end{bmatrix} := L(l_x, l_y, l_z)R_x(\varphi_x)R_y(\varphi_y)R_z(\varphi_z)
\begin{bmatrix}
  x_m \\
  y_m \\
  z_m
\end{bmatrix},
$$

(2)

matrices $R_x(\varphi_x)$, $R_y(\varphi_y)$, and $R_z(\varphi_z)$ perform rotations around $x$, $y$, and $z$ axes by angles $\varphi_x$, $\varphi_y$, and $\varphi_z$,

matrix $L(l_x, l_y, l_z)$ implements scaling along $x$, $y$, and $z$ axes by $l_x$, $l_y$, and $l_z$, respectively.
Random rotation, stretching and expanding

rotations around axes \(x\), \(y\), and \(z\) by angles \(\varphi_x\), \(\varphi_y\), and \(\varphi_z\):

\[
R_x(\varphi_x) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \varphi_x & -\sin \varphi_x \\
0 & \sin \varphi_x & \cos \varphi_x
\end{bmatrix}
\]

\[
R_y(\varphi_y) = \begin{bmatrix}
\cos \varphi_y & 0 & \sin \varphi_y \\
0 & 1 & 0 \\
-\sin \varphi_y & 0 & \cos \varphi_y
\end{bmatrix}
\]

\[
R_z(\varphi_z) = \begin{bmatrix}
\cos \varphi_z & -\sin \varphi_z & 0 \\
\sin \varphi_z & \cos \varphi_z & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\(L(l_x, l_y, l_z)\) implements scaling along axes \(x, y, z\) by factors \(l_x\), \(l_y\), and \(l_z\):

\[
\bar{L}(l_x, l_y, l_z) = \begin{bmatrix}
1/l_x & 0 & 0 \\
0 & 1/l_y & 0 \\
0 & 0 & 1/l_z
\end{bmatrix}
\]
RVs used in generating the coarsest perturbed mesh $P_0$ are:

1. perturbation weights $a_k, k = 1, \ldots, K$,
2. rotation angles $\varphi_x, \varphi_y, \text{and} \varphi_z$,
3. scaling factors $l_x, l_y, \text{and} l_z$.

Thus, random input parameter vector:

$$\xi = (a_1, \ldots, a_K, \varphi_x, \varphi_y, \varphi_z, l_x, l_y, l_z) \in \mathbb{R}^{K+6}$$

defines the perturbed shape.
Mesh refinement

Mesh $P_0$: the coarsest discretisation of the sphere (e.g., icosahedron)
Mesh $P_{\ell=1}$ is generated by refining each triangle of the perturbed $P_0$
into four (by halving all three edges and connecting mid-points).
Mesh $P_2$ is generated in the same way from $P_1$.
All meshes $P_\ell$ at all levels $\ell = 1, \ldots, L$ are nested discretizations of $P_0$.

(!!!) No uncertainties are added on meshes $P_\ell$, $\ell > 0$; the uncertainty is introduced only at level $\ell = 0$. 
Refinement of the perturbed shape

4 nested meshes with \(\{320, 1280, 5120, 20480\}\) triangular elements.
Electric (left) and magnetic (right) surface current densities

Amplitudes: a) $\mathbf{J}(\mathbf{r})$; b) $\mathbf{M}(\mathbf{r})$ (sphere); c) $\mathbf{J}(\mathbf{r})$; d) $\mathbf{M}(\mathbf{r})$ (perturbed shape).
Electric (left) and magnetic (right) surface current densities

Amplitudes of (a) $\mathbf{J}(\mathbf{r})$ and (b) $\mathbf{M}(\mathbf{r})$ induced on the unit sphere under excitation by an $\hat{x}$-polarized plane wave propagating in $-\hat{z}$ direction at 300 MHz.

Amplitudes of (c) $\mathbf{J}(\mathbf{r})$ and (d) $\mathbf{M}(\mathbf{r})$ induced on the perturbed shape under excitation by the same plane wave. For all figures, amplitudes are normalized to 1 and plotted in $dB$ scale.
QoI: RCS and SCS

To compute RCS and SCS, the scatterer is excited by a plane wave $E^{\text{inc}}(r)$.

$$\sigma_{\text{rcs}}(\vartheta, \varphi) = \frac{|F(\vartheta, \varphi)|^2}{4\pi |E_0|^2},$$  \hspace{1cm} (4)

$F(\vartheta, \varphi)$ is the scattered electric field pattern in the far field. The SCS $C^{\text{sca}}(\Omega)$ is obtained by integrating $\sigma_{\text{rcs}}(\vartheta, \varphi)$ over the angle $\Omega$:

$$C^{\text{sca}}(\Omega) = \frac{1}{4\pi} \int_{\Omega} \sigma_{\text{rcs}}(\vartheta, \varphi) \sin \vartheta d\vartheta d\varphi.$$  \hspace{1cm} (5)
RCS of unit sphere and perturbed shape

RCS is computed on

(top) $xz$
(bottom) $yz$ planes

under excitation by an $\hat{x}$-polarized plane wave propagating in $-\hat{z}$ direction at 300 MHz.

(top) $\varphi = 0$ and $\varphi = \pi$ rad in the first and second halves of the horizontal axis, respectively.

(bottom) $\varphi = \pi/2$ rad and $\varphi = 3\pi/2$ rad in the first and second halves of the horizontal axis.
Multilevel Monte Carlo Algorithm

**Aim:** to approximate the mean $\mathbb{E}(g(u))$ of QoI $g(u)$ to a given accuracy $\varepsilon := \text{TOL}$, where $u = u(\omega)$ random shape.

**Input:** a hierarchy of $L + 1$ meshes $\{h_\ell\}_{\ell=0}^L$, $h_\ell := h_0 \beta^{-\ell}$ for each realization of random domain.

**Compute:**

\[
\mathbb{E}(g_L) = \sum_{\ell=0}^L \mathbb{E}(g_\ell(\omega) - g_{\ell-1}(\omega)) =: \sum_{\ell=0}^L \mathbb{E}(G_\ell) \approx \sum_{\ell=0}^L \mathbb{E}(\tilde{G}_\ell),
\]

where $\tilde{G}_\ell = M^{-1}_\ell \sum_{m=0}^{M_\ell} G_\ell(\omega_{\ell,m})$.

**Output:** $A \approx \mathbb{E}(g(u)) \approx \sum_{\ell=0}^L \tilde{G}_\ell$.

**Cost of one sample of $\tilde{G}_\ell$:** $W_\ell \propto h^{\gamma_\ell} = (h_0 \beta^{-\ell})^{-\gamma}$.

**Total work of estimation $A$:** $W = \sum_{\ell=0}^L M_\ell W_\ell$.

**Estimator $A$ satisfies a tolerance with a prescribed failure probability $0 < \nu \leq 1$, i.e.,**

\[
P[|\mathbb{E}(g) - A| \leq \text{TOL}] \geq 1 - \nu \tag{6}
\]

while minimizing $W$.
CMLMC numerical tests

The QoI is the SCS over a user-defined solid angle
\[ \Omega = \left[ \frac{1}{6}, \frac{11}{36} \right] \pi \text{ rad} \times \left[ \frac{5}{12}, \frac{19}{36} \right] \pi \text{ rad} \] (i.e., a measure of far-field scattered power in a cone).

Uniform RVs are:
- \( a_1, a_2 \sim U[-0.14, 0.14] \) m,
- \( \varphi_x, \varphi_y, \varphi_z \sim U[0.2, 3] \) rad,
- \( l_x, l_y, l_z \sim U[0.9, 1.1] \);

CMLMC runs for TOL ranging from 0.2 to 0.008.
At TOL \( \approx 0.008 \), CMLMC requires \( L = 5 \) meshes with
\{320, 1280, 5120, 20480, 81920\} triangles.
Probability density functions of \((g_{\ell} - g_{\ell-1})\)

(a) \(\ell = 1\) and (b) \(\ell = \{2, 3\}\).
The experiment is repeated 15 times independently and the obtained values are shown as error bars on the curves.
Work estimate vs. TOL

- - - - TOL$^{-2}$
CMLMC
MC Estimate
Time required to compute $G_\ell$ vs. $\ell$. 

![Graph showing the time required to compute $G_\ell$ vs. $\ell$ with a linear increase on a log-log scale.](image-url)
$E_{\ell} = \mathbb{E}(G_{\ell})$ vs. $\ell$ (weak convergence)

assumed weak convergence curve $2^{-3\ell}$ ($q_1 = 3$).
\[ V_\ell = \text{Var}[G_\ell] \text{ vs. } \ell \text{ (strong convergence)} \]

assumed strong convergence curve \(2^{-5\ell} (q_2 = 5)\).
Value of $\theta$
Best practices for applying CMLMC method to CEM problems

- Download CMLMC: https://github.com/StochasticNumerics/mimclib.git (or use MLMC from M. Giles)
- Implement interface to couple CMLMC and your deterministic solver
- Generate a hierarchy of meshes (minimum 3), nested are better
- Generate 5-7 random shapes on first 3 meshes
- Estimate the strong and weak convergence rates, $q_1$, $q_2$, (later they will be corrected by CMLMC algorithm)
- Run CMLMC solver and check visually the automatically generated plots
Conclusion (what is done)

- Used CMLMC method to characterize EM wave scattering from dielectric objects with uncertain shapes.
- Researched how uncertainties in the shape propagate to the solution.
- Demonstrated that the CMLMC algorithm can be 10 times faster than MC.
- To increase the efficiency further, each of the simulations is carried out using the FMM-FFT accelerated PMCHWT-SIE solver.
- Confirmed that the known advantages of the CMLMC algorithm can be observed when it is applied to EM wave scattering: non-intrusiveness, dimension independence, better convergence rates compared to the classical MC method, and higher immunity to irregularity w.r.t. uncertain parameters, than, for example, sparse grid methods.
Conclusion

Some random perturbations may affect the convergence rates in CMLMC.

With difficult-to-predict convergence rates, it is hard for CMLMC to estimate:

- computational cost $W$,
- number of levels $L$,
- number of samples on each level $M_\ell$,
- computation time,
- parameter $\theta$,
- variance in QoI.

All these may result in a sub-optimal performance.
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Results are published:

A. Litvinenko, A. C. Yucel, H. Bagci, J. Oppelstrup, E. Michielssen, R. Tempone,
Computation of Electromagnetic Fields Scattered From Objects With Uncertain Shapes Using Multilevel Monte Carlo Method,
https://arxiv.org/abs/1809.00362
Main idea of (C)MLMC method

Let \( \{ P_\ell \}_{\ell=0}^L \) be sequences of meshes with \( h_\ell = h_0 \beta^{-\ell} \), \( \beta > 1 \). Let \( g_\ell(\xi) \) represent the approximation to \( g(\xi) \) computed using mesh \( P_\ell \).

\[
E[g_L] = \sum_{\ell=0}^L E[G_\ell] \quad (7)
\]

where \( G_\ell \) is defined as

\[
G_\ell = \begin{cases} 
  g_0 & \text{if } \ell = 0 \\
  g_\ell - g_{\ell-1} & \text{if } \ell > 0
\end{cases} \quad (8)
\]

Note that \( g_\ell \) and \( g_{\ell-1} \) are computed using the same input random parameter \( \xi \).
Main idea of (C)MLMC method

\[ E[G_\ell] \approx \tilde{G}_\ell = M_\ell^{-1} \sum_{m=1}^{M_\ell} G_{\ell,m}, \]

\[ E[g - g_\ell] \approx Q_W h_{\ell}^{q_1} \]
\[ \text{Var}[g_\ell - g_{\ell-1}] \approx Q_S h_{\ell-1}^{q_2} \]

(9a) (9b)

for \( Q_W \neq 0, Q_S > 0, q_1 > 0, \) and \( 0 < q_2 \leq 2q_1. \)

QoI \( A = \sum_{\ell=0}^{L} \tilde{G}_\ell. \)

Let the average cost of generating one sample of \( G_\ell \) (cost of one deterministic simulation for one random realization) be

\[ W_\ell \propto h_{\ell}^{-d\gamma} = h_0^{-d\gamma} \beta^{\ell d\gamma} \]

(10)
Main idea of (C)MLMC method

The total CMLMC computational cost is

$$W = \sum_{\ell=0}^{L} M_{\ell} W_{\ell}. \quad (11)$$

The estimator $\mathcal{A}$ satisfies a tolerance with a prescribed failure probability $0 < \nu \leq 1$, i.e.,

$$P[|E[g] - \mathcal{A}| \leq \text{TOL}] \geq 1 - \nu \quad (12)$$

while minimizing $W$. The total error is split into bias and statistical error,

$$|E[g] - \mathcal{A}| \leq |E[g - \mathcal{A}]| + |E[\mathcal{A}] - \mathcal{A}|$$

Bias Statistical error
Main idea of (C)MLMC method

Let \( \theta \in (0, 1) \) be a splitting parameter, so that

\[
TOL = \underbrace{(1 - \theta)TOL} \quad \text{Bias tolerance} + \underbrace{\theta TOL} \quad \text{Statistical error tolerance}.
\]

The CMLMC algorithm bounds the bias, \( B = |E[g - A]| \), and the statistical error as

\[
B = |E[g - A]| \leq (1 - \theta)TOL \quad (14)
\]

\[
|E[A] - A| \leq \theta TOL \quad (15)
\]

where the latter bound holds with probability \( 1 - \nu \).

To satisfy condition in (15) we require:

\[
\text{Var}[A] \leq \left(\frac{\theta TOL}{C_{\nu}}\right)^2 \quad (16)
\]

for some given confidence parameter, \( C_{\nu} \), such that \( \Phi(C_{\nu}) = 1 - \frac{\nu}{2} \), \( \Phi \) is the cdf of a standard normal random variable.
Main idea of (C)MLMC method

By construction of the MLMC estimator, \( E[A] = E[g_L] \), and by independence \( \text{Var}[A] = \sum_{\ell=0}^{L} V_\ell M_\ell^{-1} \), where \( V_\ell = \text{Var}[G_\ell] \).

Given \( L, \text{TOL}, \) and \( 0 < \theta < 1 \), and by minimizing \( W \) obtain the following optimal number of samples per level \( \ell \):

\[
M_\ell = \left( \frac{C_\nu \theta}{TOL} \right)^2 \sqrt{\frac{V_\ell}{W_\ell} \left( \sum_{\ell=0}^{L} \sqrt{V_\ell W_\ell} \right)}.
\]  

(17)

Summing the optimal numbers of samples over all levels yields the following expression for the total optimal computational cost in terms of \( \text{TOL} \):

\[
W(\text{TOL}, L) = \left( \frac{C_\nu}{\theta TOL} \right)^2 \left( \sum_{\ell=0}^{L} \sqrt{V_\ell W_\ell} \right)^2.
\]  

(18)


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9. A Litvinenko, Y Sun, MG Genton, DE Keyes, Likelihood approximation with hierarchical matrices for large spatial datasets, Computational Statistics & Data Analysis 137, 115-132, 2019


Thank you!