Finite-Time Estimation Algorithms for LPV Discrete-Time Systems with Application to Output Feedback Stabilization

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Abstract

This paper deals with new finite-time estimation algorithms for Linear Parameter Varying (LPV) discrete-time systems and their application to output feedback stabilization. Two exact finite-time estimation schemes are proposed. The first scheme provides a direct and explicit estimation algorithm based on the use of delayed outputs, while the second scheme uses two combined asymptotic observers, connected by a condition of invertibility of a certain time-varying matrix, to recover in a finite-time the solution of the LPV system. Furthermore, two stabilization strategies are proposed. The first strategy, called Delayed Inputs/Outputs Feedback (DIOF) stabilization method, is based on the use of the explicit estimation algorithm. The second technique, called Two Connected Observers Feedback (2-COF) stabilization method, is based on the use of two combined observers providing exact finite-time estimation. A numerical example and simulations are given to show the validity and effectiveness of the proposed algorithms.

Key words: Estimation; observer design; LMI approach; LPV systems; output feedback stabilization.

1 Introduction

State estimation has many applications in control system design. The estimates of state are needed for implementation of control laws and also for fault diagnosis (Gao and Ho, 2006), (Alcorta-Garcia and Frank, 1997), (Marino and Tomei, 1995), (Arcak and Kokotovic, 2001). Nonlinear estimation is more complex and lacks general and systematic methodology. Several methods have been proposed recently in the literature to improve observer design aiming at covering a wider class of nonlinear systems, but this issue still remains a challenge from nonlinear observer design point of view (Zemouche *et al.*, 2017), (Alessandri and Rossi, 2015), (Açikmese and Corless, 2011), (Wang *et al.*, 2017), (Kao *et al.*, 2015).

Among the widely used and investigated class of nonlinear systems is the Linear Parameters Varying (LPV) class (Heemels *et al.*, 2010), (Wu *et al.*, n.d.), (Wu, 2001), (Song and Yang, 2011). This particular structure of nonlinearity attracts the automatic control community for two reasons:

- LPV systems represent a wide class of nonlinear real-

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world models, such as wind turbines models, vehicle models, biogas processes, and wastewater treatment models (Gilbert *et al.*, 1999), (Rajamani, 2012).

- Stabilizing LPV systems from an observer-based feedback control point of view is not an easy task due to the difficulty in obtaining non conservative sufficient conditions ensuring the exponential convergence (Jetto and Orsini, 2010), (Heemels *et al.*, 2010). For instance, from Linear Matrix Inequality (LMI) point of view, all the available design methods for this class of systems provide conservative LMI conditions. Although a recent and new technique has been proposed in (Bibi *et al.*, 2017) to reduce the conservatism of these LMIs, the issue is far from being definitely solved.

In this paper, first, we focus on exact and finite-time state estimation for a class of LPV systems in discrete-time, then an extension to output feedback stabilization is provided. Based on the ideas given in (Engel and Kreisselmeier, 2002; Sauvage *et al.*, 2007; Mazenc *et al.*, 2015) for linear continuous-time systems, we propose a generalization to LPV systems, which is not obvious in the continuous-time case. Indeed, in continuous-time case, it is difficult to integrate the differential equations and get explicit solutions of the system where the complexity of the computation will lead to differentiating the time-varying parameters of the LPV system. The parameters' derivatives require additional and conservative assumptions and constraints on the system (such as boundedness of the derivatives of the parameters, for instance). However, in discrete-time systems, only delayed values of the LPV parameters appear in the derivation of the explicit solution. This paper proposes two finite-time estimation algorithms:

- *A Direct and explicit estimation:* The proposed algorithm for the three above mentioned classes of systems, provides an explicit state estimation in finite-time and is based on the use of delayed outputs to recover the solution of the system.
- *Two observers-based estimation:* This technique consists in combining two asymptotic state observers to reconstruct the solution of the system in finite-time.

Moreover, two numerical algorithms are proposed to design the parameters of the above two estimation methods. The first algorithm is based on poly-quadratic stability, which consists in determining the parameters by solving a set of LMIs, while the second method is used the standard pole assignment.

To demonstrate the role of the proposed exact finite-time estimation algorithms, an extension to output feedback stabilization is provided where two direct stabilization methods are proposed. The first method is based on the use of delayed inputs/outputs of the system (DIOF), while the second approach uses two connected observers feedback (2-COF). It is shown that both stabilization methods avoid solving Bilinear Matrix Inequalities (BMIs), which are not suitable for numerical solvers.

Compared to the short preliminary version of the work accepted in (Chaib-Draa *et al.*, 2019), this extended journal version contains several further contributions, namely additional remarks and comments on the estimation algorithms; a detailed section on the design of the estimation parameters where two design algorithms are proposed; an additional output feedback stabilization technique; and extended numerical illustrations of both estimation and stabilization algorithms.

The rest of this paper is organized as follows: Section 2 is devoted to the development of two exact finite-time estimation algorithms. Section 3 provides two numerical procedures to show how to design the parameters of the two estimation methods presented in Section 2. Extension and application of the proposed estimation algorithms are presented in Section 4. To show the validity and effectiveness of the proposed design algorithms, an illustrative example is given in Section 5. Finally, Section 6 concludes this work.

2 Exact Finite-Time Estimation of LPV systems

This section is dedicated to the development of two exact finite-time estimation algorithms for a class of LPV systems. We consider the class of LPV systems defined by the following set of equations:

$$x_{k+1} = A(\rho_k)x_k + Bu_k \quad (1a)$$
$$y_k = Cx_k, \quad (1b)$$

where $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^p$ is the output measurement and $u_k \in \mathbb{R}^m$ is the control input vector, $\rho_k \in \Theta \subset \mathbb{R}^r$ is a bounded time-varying parameter. B and C are constant matrices of appropriate dimensions.

We introduce the following assumptions:

- the parameter ρ_k is known and bounded;
- the matrix $A(\rho_k) \in \mathrm{Co}(A_1,...,A_N)$, can be written in the form
$$A(\rho_k) = A_0 + \sum_{i=1}^{i=n_\rho} \xi^i(\rho_k)A_i \quad (2)$$
where $\xi^i(\rho_k) \geq 0$ and $\sum_{i=1}^{i=n_\rho} \xi^i(\rho_k) = 1$;
- the pairs (A_i, C) are observable for all $i = 1,\dots,n_\rho$.

2.1 Explicit Solutions Using Delayed Outputs

Before proposing the first algorithm that provides an exact finite-time estimation of the state x_k, the following useful lemma is stated.

Lemma 1 *Assume that the pairs (A_j, C) are observable for all $j = 0,\dots,n_\rho$. Then there exist L_j, K_j, $j = 0,\dots,n_\rho$, and*

$m \geq 1$ *such that the matrix*

$$\mathbb{E}_m(k) \triangleq \left[\prod_{i=1}^{m}\left(\sum_{j=0}^{n_\rho} \xi_{k-i}^{j}\big(A_j - L_jC\big)\right)\right]^{-1} - \left[\prod_{i=1}^{m}\left(\sum_{j=0}^{n_\rho} \xi_{k-i}^{j}\big(A_j - K_jC\big)\right)\right]^{-1} \quad (3)$$

exists and invertible for all $k \geq 1$*, where* $\xi^0(\rho_k) \triangleq 1$ *and* $\xi^i(\rho_k) \triangleq \xi_k^i$ *for* $i \geq 1$*.*

PROOF. Since the parameters are bounded and (A_j, C) are observable, then it is always possible to find L_j, K_j, and $m \geq 1$ such that the matrix $\mathbb{E}_m(k)$ remains invertible for all $k \geq m$. Indeed, we can choose the eigenvalues of $\big(A_j - L_jC\big)$ inside the ball $\mathscr{B}(0,\delta)$ and those of $\big(A_j - K_jC\big)$ inside $\mathscr{B}(0,1)\backslash\mathscr{B}(0,\delta)$, where $0 < \delta < 1$. The detailed proof is omitted from this preliminary version of the paper. Algorithms 1 and 2 in Section 3 provide numerical procedures to design L_j and K_j for $j = 0, \ldots, n_\rho$.

Consequently, we can provide a direct and exact estimation of the state x_k, presented in the following theorem.

Theorem 2 *Assume that there exist*

$$\mathbb{L}(\rho_k) = L_0 + \sum_{i=1}^{i=n_\rho} \xi^i(\rho_k) L_i,\ \mathbb{K}(\rho_k) = K_0 + \sum_{i=1}^{i=n_\rho} \xi^i(\rho_k) K_i$$

and $m \geq 1$ *so that matrix* $\mathbb{E}_m(k)$*, defined in* (4)*, exists and invertible for all* $k \geq m$*:*

$$\mathbb{E}_m(k) \triangleq \left[\prod_{i=1}^{m}\left(\sum_{j=0}^{n_\rho} \xi_{k-i}^{j}\big(A_j - L_jC\big)\right)\right]^{-1} - \left[\prod_{i=1}^{m}\left(\sum_{j=0}^{n_\rho} \xi_{k-i}^{j}\big(A_j - K_jC\big)\right)\right]^{-1}. \quad (4)$$

Then a direct and exact estimation of the state x_k *can be computed as in* (5)*.*

PROOF. It is easy to show iteratively that x_k can be written under the forms (6) and (7). Then, by subtracting (7), after multiplication by $\left(\prod_{i=1}^{m}\Big(\mathbb{A}(\rho_{k-i}) - \mathbb{K}(\rho_{k-i})C\Big)\right)^{-1}$, from (6) multiplied by $\left(\prod_{i=1}^{m}\Big(\mathbb{A}(\rho_{k-i}) - \mathbb{L}(\rho_{k-i})C\Big)\right)^{-1}$, we get easily (5) by using the inverse of $\mathbb{E}_m(k)$.

2.2 *Estimation By Using Two Combined Observers*

Unlike the previous section where a sum of delayed outputs weighted by powers of $A - LC$ and $A - KC$ have been used, this section is devoted to state estimate using two different asymptotic state observers. By using tools borrowed from the continuous-time results in (Engel and Kreisselmeier, 2002) and (Mazenc *et al.*, 2015), we get an exact estimation of the solution without using explicitly the delayed outputs. Indeed, the delayed output measurements are hidden and appear implicitly in the states of the intermediate observers. This way to provide an exact estimation of the state x_k is more suitable from a practical point of view.

Considering the class of systems (1), then an exact estimation of x_k may be obtained by using two combined asymptotic observers, instead of using directly an explicit solution. The result is summarized in the following theorem.

Theorem 3 *Assume that the gain matrices* L_i *and* K_i *are selected such that:*

i) all the eigenvalues of $\big(A_i - L_iC\big)$ *and* $\big(A_i - K_iC\big)$ *are nonzero and less than one;*
ii) there exists $m \geq 1$ *so that the matrix* $\mathbb{E}_m(k)$ *exists and invertible.*

Then the extended state dynamic system

$$\zeta_{k+1} = A(\rho_k)\zeta_k + Bu_k + \mathbb{L}(\rho_k)\Big(y_k - C\zeta_k\Big) \quad (8a)$$

$$\eta_{k+1} = A(\rho_k)\eta_k + Bu_k + \mathbb{K}(\rho_k)\Big(y_k - C\eta_k\Big) \quad (8b)$$

$$\hat{x}_k = \mathbb{E}_m^{-1}(k)\left[\left(\prod_{i=1}^{m}\left(\sum_{l=0}^{n_\rho} \xi_{k-i}^{l}\big(A_l - L_lC\big)\right)\right)^{-1}\zeta_k - \left(\prod_{i=1}^{m}\left(\sum_{l=0}^{n_\rho} \xi_{k-i}^{l}\big(A_l - K_lC\big)\right)\right)^{-1}\eta_k + \eta_{k-m} - \zeta_{k-m}\right] \quad (8c)$$

is an observer for system (1)*, which converges in finite time* $m \geq 1$*.*

PROOF. The proof exploits the explicit solution technique. Indeed, by analogy to (6) and (7), we get (9) and (10). Hence, by substituting (9) and (10) in (5) and using the definition of $\mathbb{E}_m(k)$, we get from (8c) that $\hat{x}_k = x_k, \forall k \geq m$.

3 Design of the Estimation Parameters

This section is devoted to the numerical implementation of the proposed exact finite-time estimation methods. We

$$x_k = \mathbb{E}_m^{-1}(k) \sum_{j=1}^{m} \Bigg[\left(\prod_{i=1}^{m} \left(\sum_{l=0}^{n_\rho} \xi_{k-i}^l (A_l - L_l C) \right) \right)^{-1} \left(\sum_{l=0}^{n_\rho} \xi_{k-j}^l (A_l - L_l C) \right)^{j-1} \left(\sum_{l=0}^{n_\rho} \xi_{k-j}^l L_l \right)$$
$$- \left(\prod_{i=1}^{m} \left(\sum_{l=0}^{n_\rho} \xi_{k-i}^l (A_l - K_l C) \right) \right)^{-1} \left(\sum_{l=0}^{n_\rho} \xi_{k-j}^l (A_l - K_l C) \right)^{j-1} \left(\sum_{l=0}^{n_\rho} \xi_{k-j}^l K_l \right) \Bigg] y_{k-j}$$
$$+ \mathbb{E}_m^{-1}(k) \sum_{j=1}^{m} \Bigg[\left(\prod_{i=1}^{m} \left(\sum_{l=0}^{n_\rho} \xi_{k-i}^l (A_l - L_l C) \right) \right)^{-1} \left(\sum_{l=0}^{n_\rho} \xi_{k-j}^l (A_l - L_l C) \right)^{j-1}$$
$$- \left(\prod_{i=1}^{m} \left(\sum_{l=0}^{n_\rho} \xi_{k-i}^l (A_l - K_l C) \right) \right)^{-1} \left(\sum_{l=0}^{n_\rho} \xi_{k-j}^l (A_l - K_l C) \right)^{j-1} \Bigg] B u_{k-j} \tag{5}$$

$$x_k = \left(\prod_{i=1}^{m} \left[\sum_{l=0}^{n_\rho} \xi_{k-j}^l (A_l - L_l C) \right] \right) x_{k-m}$$
$$+ \sum_{j=1}^{m} \left(\left[\sum_{l=0}^{n_\rho} \xi_{k-j}^l (A_l - L_l C) \right]^{j-1} \right) \left[\left(\sum_{l=0}^{n_\rho} \xi_{k-j}^l L_l \right) y_{k-j} + B u_{k-j} \right] \tag{6}$$

and

$$x_k = \left(\prod_{i=1}^{m} \left[\sum_{l=0}^{n_\rho} \xi_{k-j}^l (A_l - K_l C) \right] \right) x_{k-m}$$
$$+ \sum_{j=1}^{m} \left(\left[\sum_{l=0}^{n_\rho} \xi_{k-j}^l (A_l - K_l C) \right]^{j-1} \right) \left[\left(\sum_{l=0}^{n_\rho} \xi_{k-j}^l K_l \right) y_{k-j} + B u_{k-j} \right] \tag{7}$$

$$\zeta_k = \left(\prod_{i=1}^{m} \left[\sum_{l=0}^{n_\rho} \xi_{k-j}^l (A_l - L_l C) \right] \right) \zeta_{k-m}$$
$$+ \sum_{j=1}^{m} \left(\left[\sum_{l=0}^{n_\rho} \xi_{k-j}^l (A_l - L_l C) \right]^{j-1} \right) \left[\left(\sum_{l=0}^{n_\rho} \xi_{k-j}^l L_l \right) y_{k-j} + B u_{k-j} \right], \tag{9}$$

$$\eta_k = \left(\prod_{i=1}^{m} \left[\sum_{l=0}^{n_\rho} \xi_{k-j}^l (A_l - K_l C) \right] \right) \eta_{k-m}$$
$$+ \sum_{j=1}^{m} \left(\left[\sum_{l=0}^{n_\rho} \xi_{k-j}^l (A_l - K_l C) \right]^{j-1} \right) \left[\left(\sum_{l=0}^{n_\rho} \xi_{k-j}^l K_l \right) y_{k-j} + B u_{k-j} \right]. \tag{10}$$

provide a structured algorithm for computing the values of L_i and K_i for $i = 0, \ldots, n_\rho$.

Since the LPV parameter ρ_k (or equivalently ξ_k) is known, then designing a single observer under the form (8a) or (8b) is not a difficult task. We can use some observer design techniques for LPV systems available in the literature. In this section, we use the well known poly-quadratic stability (Pandey and de Oliveira, 2018), (Bara *et al.*, 2001), (Daafouz and Bernussou, 2001). However, we need to slightly modify the standard LMIs to get different gains L_i and K_i for each i. The goal is to add a scalar parameter to place the eigenvalues of $A_i - L_iC$ and $A_i - K_iC$, respectively, in different regions in order to get $m \geq 1$ satisfying the invertibility of the matrix $\mathbb{E}_m(k)$ defined in (4), for all $k \geq m$.

3.1 Poly-quadratic stability based algorithm

To have a consistent and well-structured algorithm, we propose to design the gains L_i (or K_i) stabilizing asymptotically the error $\zeta_k - \hat{x}_k$ (or $\eta_k - \hat{x}_k$). The overall LMI synthesis conditions providing the observer gains are given in the following proposition.

Proposition 1 *There exist symmetric positive definite matrices $\mathbb{P}_i$ and square matrices $\mathbb{X}_i$ of appropriate dimensions, such that the following LMI conditions hold:*

$$\begin{bmatrix} \mathbb{P}_j - \mathbb{X}_i - \mathbb{X}_i^\top & \mathbb{X}_iA_i - \mathbb{Z}_iC \\ \left(\mathbb{X}_iA_i - \mathbb{Z}_iC\right)^\top & -\mathbb{P}_i \end{bmatrix} < 0, \forall i, j = 1, \ldots, n_\rho. \tag{11}$$

Then the estimation error $\zeta_k - \hat{x}_k$ converges asymptotically to zero for $L_i = \mathbb{X}_i^{-1}\mathbb{Z}_i$.

PROOF. The proof is straightforward and can be easily obtained from (Pandey and de Oliveira, 2018) by using the poly-quadratic Lyapunov function

$$\vartheta_k(\tilde{x}_k) = \sum_{i=0}^{n_\rho} \xi_k^i \tilde{x}_k^\top \mathbb{P}_i \tilde{x}_k, \quad \tilde{x}_k = \zeta_k - \hat{x}_k.$$

The same LMIs (11) ensure asymptotic stability of $\eta_k - \hat{x}_k$ with $K_i = \mathbb{X}_i^{-1}\mathbb{Z}_i$, since ζ_k and η_k have the same dynamics. However, to achieve exact finite-time estimation, we need invertibility of the matrix $\mathbb{E}_m(k)$, which may not be satisfied if the eigenvalues of $A_i - L_iC$ and $A_i - K_iC$ are close to each other. To overcome this issue and augment the possibility to get invertibility of $\mathbb{E}_m(k)$, we propose to slightly modify (11) by including a positive parameter $\alpha_i < 1, i = 1, \ldots, n_\rho$. That is, LMI (11) is replaced by the following one:

$$\begin{bmatrix} \mathbb{P}_j - \mathbb{X}_i - \mathbb{X}_i^\top & \alpha_i\left(\mathbb{X}_iA_i - \mathbb{Z}_iC\right) \\ \alpha_i\left(\mathbb{X}_iA_i - \mathbb{Z}_iC\right)^\top & -\mathbb{P}_i \end{bmatrix} < 0. \tag{12}$$

Hence, if need the eigenvalues of $A_i - L_iC$ to be less than α_i, we should solve LMIs (12). Moreover, if we need the eigenvalues of $A_i - K_iC$ to be greater than α_i while ensuring poly-quadratic convergence, we should solve (11) together with (13), given as follows:

$$\begin{bmatrix} \mathbb{P}_j - \mathbb{X}_i - \mathbb{X}_i^\top & \alpha_i\left(\mathbb{X}_iA_i - \mathbb{Z}_iC\right) \\ \alpha_i\left(\mathbb{X}_iA_i - \mathbb{Z}_iC\right)^\top & -\mathbb{P}_i \end{bmatrix} > 0. \tag{13}$$

Once the gains L_i and K_i are computed, we check existence and invertibility of the matrix $\mathbb{E}_m(k)$, for $1 \leq m \leq m^\star$, where $m^\star$ is a prescribed integer representing the maximum of the desired finite-time to achieve exact estimation. If $\mathbb{E}_m(k)$ is not invertible for all $m \leq m^\star$, then we will change the values of $\alpha_i, i = 1, \ldots, n_\rho$. We summarize the numerical design procedure in the following Algorithm.

Algorithm 1: Poly-quadratic stability based algorithm

Step 1. Choose $m^\star \geq 1$ and small values $\alpha_i, i = 1, \ldots, n_\rho$.
Step 2. Solve LMIs (12) and compute the gains:
- $L_i = \mathbb{X}_i^{-1}\mathbb{Z}_i$.

Step 3. Solve jointly LMIs (11) and (13) and compute the gains:
- $K_i = \mathbb{X}_i^{-1}\mathbb{Z}_i$.

Step 4. Check invertibility of the matrix $\mathbb{E}_\ell(k)$:
for $\ell \leftarrow 1$ *to* $m^\star$ *and* $k \geq 1$ **do**
if $\mathbb{E}_\ell(k)$ *is invertible* **then**
  return $m \leftarrow \ell$; **break**;
else
  Increase the values of α_i and go to **Step 2** to generate new observer gains L_i and K_i.

Remark 4 *Algorithm 1 gives a global view of the numerical procedure to design the parameters of the proposed finite-time exact estimator. It does not consider the detailed procedure, where for instance, we encounter infeasibility of one of the LMIs* (11)-(13). *The well-structured and complete procedure is to be done by the user depending on the software at hand.*

3.2 Pole placement based algorithm

The previous poly-quadratic stability based algorithm is based on the feasibility of the sufficient LMIs (11)-(13), which are not always easy to tune more suitable eigenvalues of $A_i - L_iC$ and $A_i - K_iC$. To overcome this obstacle, we proposed a second algorithm based directly on fixing the eigenvalues of $A_i - L_iC$ and $A_i - K_iC$ in such a way that they will not be close. To this end, we propose to introduce scalar variables $\sigma_i, i = 1, \ldots, n_\rho$. Then we compute the eigenvalues of $A_i - L_iC$ and $A_i - K_iC$, respectively, so that

$$\max_{j=1,\ldots,n} |\lambda_j (A_i - L_iC)| < \sigma_i < \min_{j=1,\ldots,n} |\lambda_j (A_i - K_iC)|, \quad \forall i = 1, \ldots, n_\rho \tag{14}$$

where $\lambda_j(A)$ is the j^{th} eigenvalue of the matrix A. Note that since we study discrete-time systems, we have also $\max_{j=1,\ldots,n} |\lambda_j (A_i - K_iC)| < 1$. The idea consists in assigning eigenvalues satisfying (14) and such that for a prescribed $m^\star \geq 1$, there exists $m \leq m^\star$ for wich $\mathbb{E}_m(k)$ is invertible. Therefore, if such a property is not satisfied, we propose to decrease $\lambda_j (A_i - L_iC)$ and increase $\lambda_j (A_i - K_iC)$ until $\mathbb{E}_m(k)$ is invertible.

Hence, we are ready to propose a second algorithm, which is more easier and simpler than Algorithm 1.

Algorithm 2: Eigenvalues assignment based algorithm

Step 1. Choose $m^\star \geq 1$ and $\sigma_i, i = 1, \ldots, n_\rho$.
Step 2. Assign eigenvalues for $A_i - L_iC$ and $A_i - K_iC$ according to (14);
Step 3. Compute the corresponding gains L_i and K_i, respectively;
Step 4. Check invertibility of the matrix $\mathbb{E}_\ell(k)$:
for $\ell \leftarrow 1$ *to* $m^\star$ *and* $k \geq 1$ **do**
if $\mathbb{E}_\ell(k)$ *is invertible* **then**
  return $m \leftarrow \ell$; **break**;
else
  Decrease the eigenvalues of $A_i - L_iC$ and increase those of $A_i - K_iC$, and go to **Step 3** to generate new observer gains L_i and K_i.

Remark 5 *As in the previous subsection, Algorithm 2 gives a global view of the numerical procedure to design the parameters of the proposed finite-time exact estimator by using eigenvalues assignment. For instance, to increase and decrease the eigenvalues of $A_i - L_iC$ and $A_i - K_iC$, we can introduce a small scalar parameter ε that we will increase at each iteration, and put*

$$\lambda_j (A_i - L_iC) \leftarrow \lambda_j (A_i - L_iC) - \varepsilon,$$

$$\lambda_j (A_i - K_iC) \leftarrow \lambda_j (A_i - K_iC) + \varepsilon.$$

4 Output Feedback Stabilization of LPV Systems

In this section, we propose two different output feedback stabilization methods. Both methods are based on the exact finite-time estimation methodologies proposed in the previous sections.

4.1 2-COF stabilization method

In this paper, due to the exact finite-time estimation of the system state, we will propose necessary and sufficient LMI conditions ensuring poly-quadratic stabilization of the system state.

Theorem 6 *Assume that the gain matrices L_i, K_i and F_i are selected such that:*

i) L_i are solutions of LMIs (12) *and K_i are solutions of* (11) *and* (13)*, respectively, for prescribed $\alpha_i, i = 1, \ldots, n_\rho$;*
ii) there exists $m \geq 1$ so that the matrix $\mathbb{E}_m(k)$ exists and invertible;
iii) there exist matrices $\mathbb{P}_i = \mathbb{P}_i^{-1} > 0, i = 1, \ldots, n_\rho$ and matrices $\mathbb{X}_i, i = 1, \ldots, n_\rho$ of appropriate dimensions such that the following LMI conditions hold:

$$\begin{bmatrix} -\mathbb{P}_j & A_i\mathbb{P}_i - B\mathbb{X}_i \\ \mathbb{P}_iA_i^\top - \mathbb{X}_i^\top B^\top & -\mathbb{P}_i \end{bmatrix} < 0, \forall i, j = 1, \ldots, n_\rho. \tag{15}$$

Then the following observer-based controller

$$\zeta_{k+1} = A(\rho_k)\zeta_k + Bu_k + \mathbb{L}(\rho_k)\Big(y_k - C\zeta_k\Big) \tag{16a}$$

$$\eta_{k+1} = A(\rho_k)\eta_k + Bu_k + \mathbb{K}(\rho_k)\Big(y_k - C\eta_k\Big) \tag{16b}$$

$$\hat{x}_k = \mathbb{E}_m^{-1}(k)\left[\left(\prod_{i=1}^{m}\left(\sum_{l=0}^{n_\rho}\xi_{k-i}^l(A_l - L_lC)\right)\right)^{-1}\zeta_k - \left(\prod_{i=1}^{m}\left(\sum_{l=0}^{n_\rho}\xi_{k-i}^l(A_l - K_lC)\right)\right)^{-1}\eta_k + \eta_{k-m} - \zeta_{k-m}\right] \tag{16c}$$

$$u_k = -\mathbb{F}(\rho_k)\hat{x}_k \tag{16d}$$

with

$$\mathbb{F}(\rho_k) \triangleq F_0 + \sum_{i=1}^{i=n_\rho} \xi^i(\rho_k)F_i, \; F_i = \mathbb{X}_i\mathbb{P}_i^{-1} \tag{17}$$

stabilizes globally asymptotically the system (1).

PROOF. From Theorem 3, we know that if *i)* and *ii)* of Theorem 6 are satisfied, then (16c) provides an exact and finite-time estimation of x_k. that is $\hat{x}_k = x_k, \forall k \geq m$. It follows that for $k \geq m$, equation (16d) becomes

$$u_k = -\mathbb{F}(\rho_k)x_k.$$

Consequently, for $k \geq m$, system (1) is rewritten after feedback as:

$$x_{k+1} = \Big(A(\rho_k) - B\mathbb{F}(\rho_k)\Big)x_k = \sum_{j=0}^{n_\rho} \xi_k^j \big(A_j - BF_j\big)x_k \qquad (18)$$

which is globally asymptotically stable if there exists a Lyapunov function

$$\vartheta_k = \sum_{i=0}^{n_\rho} \xi_k^i x_k^\top \mathbb{P}_i^{-1} x_k$$

such that $\Delta\vartheta \triangleq \vartheta_{k+1} - \vartheta_k < 0, \forall x_k \neq 0$. By developing the $\Delta\vartheta$, we get

$$\Delta\vartheta = \sum_{i=0}^{n_\rho}\sum_{j=0}^{n_\rho} \xi_{k+1}^i \xi_k^j x_k^\top \left[\Big(A_j - BF_j\Big)^\top \mathbb{P}_i^{-1} \times \Big(A_j - BF_j\Big) - \mathbb{P}_j^{-1}\right] x_k. \quad (19)$$

It follows that $\Delta\vartheta < 0, \forall x_k \neq 0$ if the following inequalities hold, $\forall i, j = 1, \ldots, n_\rho$:

$$\Big(A_j - BF_j\Big)^\top \mathbb{P}_i^{-1} \Big(A_j - BF_j\Big) - \mathbb{P}_j^{-1} < 0. \qquad (20)$$

On the other hand, (20) is equivalent to (15) by using the congruence principle and Schur lemma. This ends the proof.

Remark 7 *It is worth to notice that LMIs* (15) *are necessary and sufficient conditions for the global poly-quadratic stabilization of system* (1)*, however, they are only sufficient for its global asymptotic stabilization. Indeed, according to (Daafouz and Bernussou, 2001, Definition 2), the notion of poly-quadratic stability is stronger than asymptotic stability. Poly-quadratic stability is basically, by definition, a sufficient criterion to ensure asymptotic stability.*

Remark 8 *In the presence of uncertainties, the proof of convergence is different and the LMIs* (15) *are not sufficient to ensure poly-quadratic stability of the system. This issue is one of the future work we aim to tackle. Especially, we aim to investigate the class of LPV systems with inexact parameters. That is the case where $\rho_k = \rho_k^0 + \Delta\rho_k$, with ρ_k^0 is known and $\Delta\rho_k$ unknown but bounded. Solving this problem allows to generalize the methodology to a class of nonlinear systems.*

4.2 DIOF stabilization method

This section is dedicated to a new stabilization technique, called DIOF stabilization method, which allows overcoming the issue of static output feedback (SOF) stabilization problem. Although SOF controller is simple, from LMI point of view, it is not so obvious because of resulting Bilinear Matrix Inequalities (BMIs), which are not easy to solve (from complexity point of view) by using available convex optimization algorithms. SOF controller consists in stabilizing (1) by using $u_k = -\mathbb{F}(\rho_k)y_k$, which leads to

$$x_{k+1} = \sum_{j=0}^{n_\rho} \xi_k^j \big(A_j - BF_jC\big)x_k. \qquad (21)$$

However, by following the steps in Section 4.1, equation (21) leads to the BMIs

$$\begin{bmatrix} -\mathbb{P}_j & \Big(A_i - BF_iC\Big)\mathbb{P}_i \\ \mathbb{P}_i\Big(A_i - BF_iC\Big)^\top & -\mathbb{P}_i \end{bmatrix} < 0, \quad \forall i, j = 1, \ldots, n_\rho, \quad (22)$$

which are not easy to linearize. Several techniques have been proposed in the literature, but the challenge of obtaining less conservative LMIs is still open. To have a precise idea on the difficulty of this problem in linear case, we refer to (Huynh *et al.*, 2019) and the references therein. On the other hand, by exploiting the first exact estimation methodology proposed in Section 2.1, we are able to stabilize (1) by using delayed inputs/outputs. Indeed, to overcome the BMIs (22), we propose the following output feedback controller, which uses only measured quantities:

$$u_k = -\mathbb{F}(\rho_k)\Theta_k(\rho_k); \qquad (23)$$

$$\Theta_k \triangleq x_k \text{ given by } (5), \forall k \geq m; \qquad (24)$$

$$u_k = u_k^m, \forall k = \overline{1, \ldots, m-1}. \qquad (25)$$

It follows that with (23)-(24), instead of system (21), we get (18). Consequently, instead of facing the complicated BMIs (22), we only need to solve LMIs (15).

Remark 9 *This section allows both consolidating Section 4 and showing that also the explicit estimation technique proposed in this paper can be applied to output feedback stabilization issue. Nevertheless, we need the value of u_k^m for $k = \overline{1, \ldots, m-1}$, which is necessary from* (23)-(25)*. We propose to fix these values by putting $u_k^m = -F\zeta_k$, where ζ_k is defined in* (8a)*. Since generally, we have small value of m (we can fix small values of m for appropriate L and K), hence we can select $u_k^m = 0$ for $k = \overline{1, \ldots, m-1}$.*

Remark 10 *Notice that both stabilization methods DIOF and 2-COF use the same synthesis conditions to determine the parameters L_j, K_j, and F_j because both techniques give $u_k = -\mathbb{F}(\rho_k)x_k$ for $k \geq m$.*

5 Illustrative Example

This section is devoted to illustrate the theoretical contributions presented in the previous sections. Due to lack of space, only the methodology based on the use of two combined observers will be illustrated.

5.0.1 System description

As an example, consider the LPV system described by the following equations (Heemels *et al.*, 2010):

$$x_{k+1} = \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.6+\rho_k \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u_k \tag{26a}$$

$$y_k = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} x_k \tag{26b}$$

with $\rho_k \in \begin{bmatrix} 0 & 0.5 \end{bmatrix}, k \in \mathbb{N}$. In this case, we can take the functions $\xi^1(\rho_k) = (0.5 - \rho_k)/0.5$ and $\xi^2(\rho_k) = \rho_k/0.5$ and $A_1 = A(0)$, $A_2 = A(0.5)$.

5.0.2 Estimation without feedback stabilization

By using Algorithm 1, we obtain the following solutions:

$$L_0 = -\begin{bmatrix} -0.0474 \\ 0.0007 \\ 0.3637 \end{bmatrix}, L_1 = -\begin{bmatrix} -0.0134 \\ -0.0009 \\ 0.6372 \end{bmatrix},$$

and

$$K_0 = -\begin{bmatrix} -0.0314 \\ -0.0010 \\ 0.3260 \end{bmatrix}, K_1 = -\begin{bmatrix} -0.0638 \\ 0.0033 \\ 0.7513 \end{bmatrix}.$$

with $m = 3$. The matrix $\mathbb{E}_m(k)$ in (4) exists and found invertible for any $k \geq 0$.

For simulations, we use

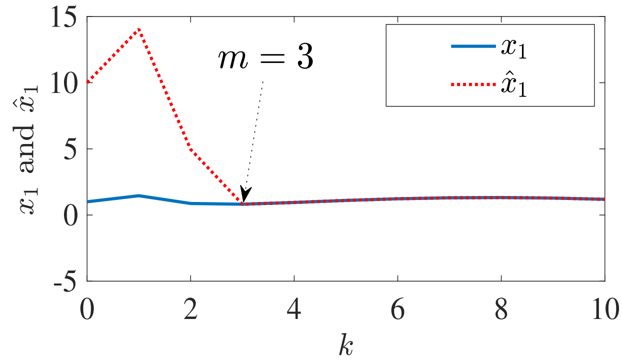
$$\rho_k = \frac{1}{2} \mid \sin(\frac{\pi}{10}k) \mid \text{ and } u_k = \sin(\frac{\pi}{15}k)).$$

The initial state of the system is $x_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$. As for ζ_0 and η_0 in (8a)-(8b) are given by $\begin{bmatrix} 10 & 10 & 10 \end{bmatrix}^\top$ and $\begin{bmatrix} 5 & 5 & 5 \end{bmatrix}^\top$, respectively. We also use $\hat{x}_k = \zeta_k$ for $k = 0, \ldots, m-1$. It is quite clear from Figure 1 that the estimation $\hat{x}_k$ given by (8c) reaches exactly the solution x_k of (26a) in finite-time.

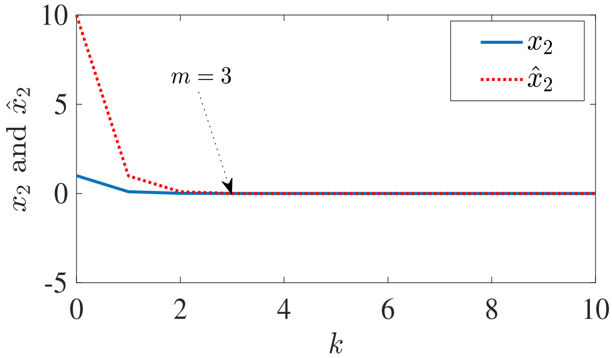
To show, in simulation, the performance of the proposed estimation algorithm, we add a measurement noise. The output y_k is assumed to be disturbed by a Gaussian noise with mean zero and standard deviation $\sigma = 0.3$. The simulation results are depicted in Figure 2.

5.0.3 Observer-based feedback stabilization

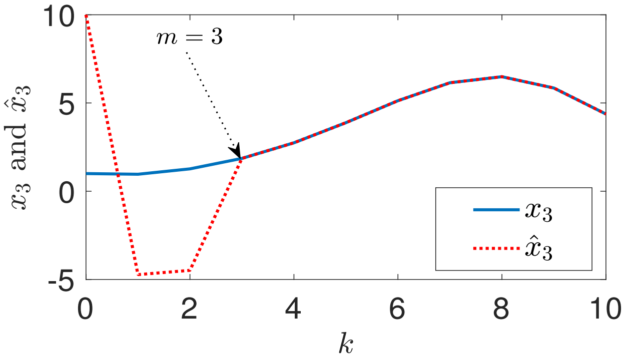
This subsection is devoted to show the effectiveness of the exact estimation based output feedback control method



(a) x_1 and its exact estimation $\hat{x}_1$



(b) x_2 and its exact estimation $\hat{x}_2$



(c) x_3 and its exact estimation $\hat{x}_3$

Fig. 1. Behavior of the states and their estimates.

proposed in Section 4. The parameters related to the exact estimation are those obtained in the previous subsection devoted to estimation only. Furthermore, to compute the controller parameters $F_i, i = 1, \ldots, n_\rho$, we need to solve LMIs (15). Hence, by using Matlab LMI toolbox with Yalmip interface, we get the following solutions:

$$F_0 = \begin{bmatrix} -0.0203 \\ -0.2058 \\ 0.6427 \end{bmatrix}^\top, F_1 = \begin{bmatrix} -0.0143 \\ -0.0442 \\ 1.1651 \end{bmatrix}^\top.$$

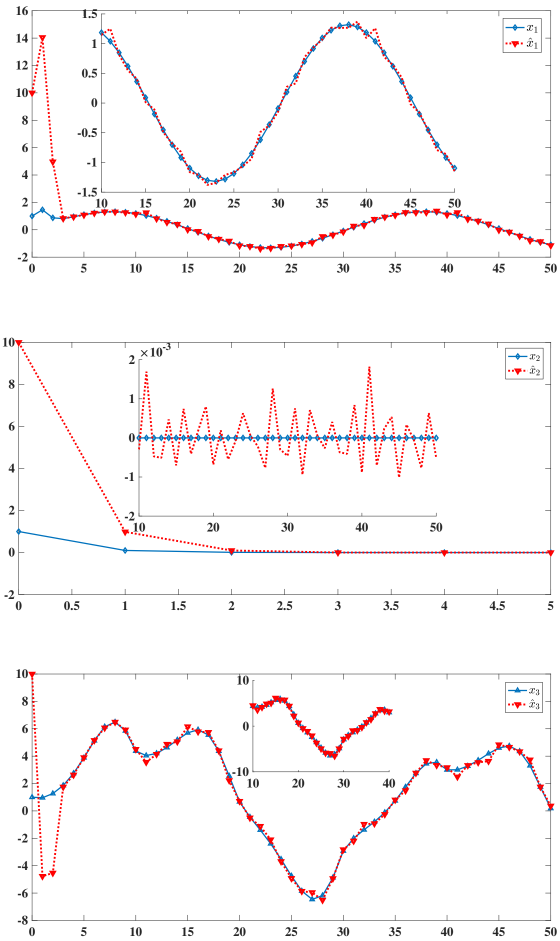


Fig. 2. Estimation results with measurement noise.

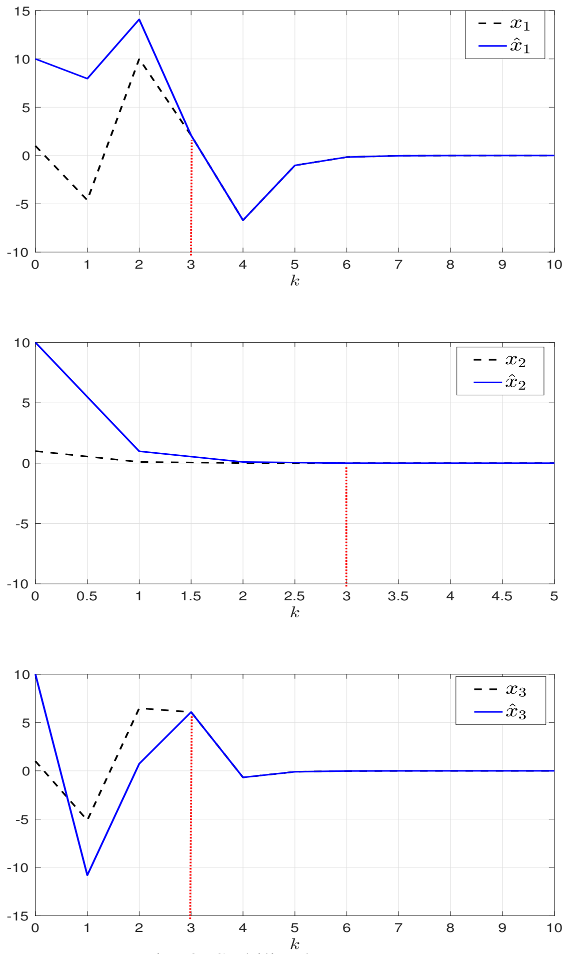


Fig. 3. Stabilized states.

It is quite clear from Figure 3 that with these parameters, the exact finite-time estimation based controller (16) ensures asymptotic stabilization of system (1). Real states and their estimation are depicted in Figure 3 to show, at the same time, that the exact estimation is done in finite-time with the new control input. Additional simulations are presented in Figure 4 to show the performance of the proposed stabilization scheme. Both the output $y_k$and each component of the system are assumed to be disturbed by a Gaussian noise with mean zero and standard deviation $\sigma = 0.1$.

6 Conclusion

This paper provided powerful state estimation algorithms for LPV discrete-time systems. Two new estimation procedures are proposed. The first one allows computing explicitly the solution of the system through delayed outputs/inputs, while the second one use the strategy of two connected asymptotic observers. Due to the exact estimation in finite-time, the problem of output feedback stabilization of LPV systems is solved by mean of simple and non conservative LMI conditions. Therefore, two novel control design strategies are proposed and two well-structured algorithms are given to design the parameters of the estimation and stabilization schemes. A numerical example is provided to show the effectiveness of the proposed exact finite-time estimation algorithms and their application to output feedback stabilization. As a future work, we aim to generalize the results given in this paper to systems with unknown parameters ($\rho_k = \rho_k^0 + \Delta\rho_k$) in order to provide robust stabilization schemes.

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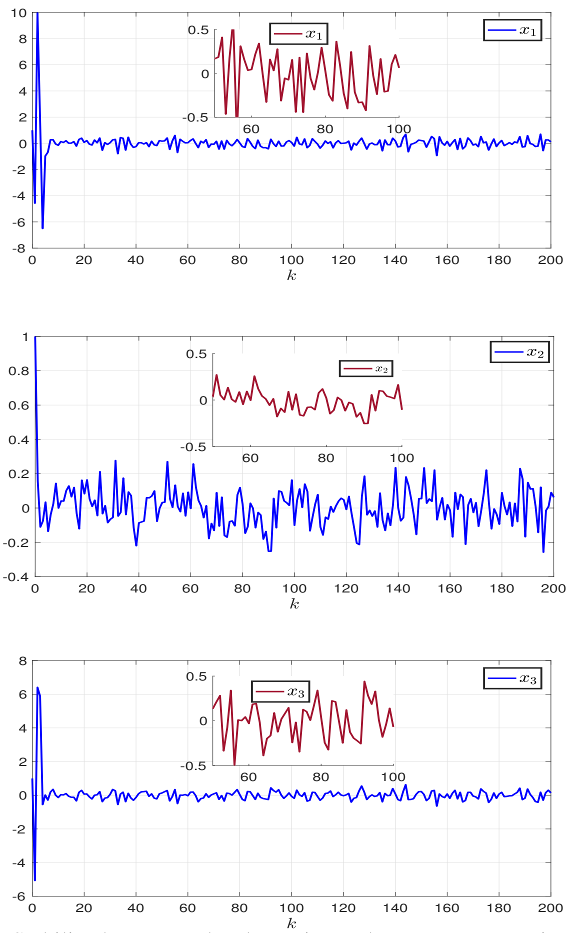

Fig. 4. Stabilized states under dynamics and measurement noise.

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