THE RELAXATION LIMIT OF BIPOLAR FLUID MODELS

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Abstract. This work establishes the relaxation limit from a bipolar Euler-Poisson system with friction towards a bipolar drift-diffusion system. A weak-strong formalism is developed and, within this framework, a dissipative weak solution of the bipolar Euler-Poisson system converges in the high-friction regime to a conservative, bounded away from vacuum, strong solution of the bipolar drift-diffusion system. This limiting process is based on a relative entropy identity for the bipolar fluid system.

1. Introduction

This article studies the emergence of the bipolar drift-diffusion system

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \mathbf{u}) &= 0 \\
(\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) &= -\frac{1}{\varepsilon} \rho \nabla \delta E_{\delta \rho} - \frac{1}{\varepsilon} \rho \mathbf{u} \\
n_t + \nabla \cdot (n \mathbf{v}) &= 0 \\
(n \mathbf{v})_t + \nabla \cdot (n \mathbf{v} \otimes \mathbf{v}) &= -\frac{1}{\varepsilon} n \nabla \delta E_{\delta n} - \frac{1}{\varepsilon} n \mathbf{v} \\
-\Delta \phi &= \rho - n
\end{align*}
\]

as a relaxation limit of the bipolar Euler-Poisson system:

\[
\begin{align*}
\rho_t &= \nabla \cdot (\rho \nabla \delta E_{\delta \rho}) \\
(\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) &= -\frac{1}{\varepsilon} \rho \nabla \delta E_{\delta \rho} - \frac{1}{\varepsilon} \rho \mathbf{u} \\
n_t &= \nabla \cdot (n \nabla \delta E_{\delta n}) \\
-\Delta \phi &= \rho - n
\end{align*}
\]

in the space-time domain \([0, T] \times \Omega\), where \(T > 0\) is a fixed time horizon and \(\Omega\) is a smooth bounded domain of \(\mathbb{R}^d\) with smooth boundary \(\partial \Omega\), where \(d \in \mathbb{N} \setminus \{1, 2\}\). The systems are expressed using the formalism of [7]. This work is restricted to the classical hydrodynamical case by selecting the functional

\[
\mathcal{E}(\rho, n) = \int_{\Omega} h_1(\rho) + h_2(n) + \frac{1}{2} |\nabla \phi|^2 \, dx, \tag{1.3}
\]

\[
-\Delta \phi = \rho - n, \tag{1.4}
\]

while \(\delta \mathcal{E}_{\delta \rho}, \delta \mathcal{E}_{\delta n}\) stand for the functional derivatives of (1.3) subject to the Poisson equation (1.4) computed by

\[
\begin{align*}
\frac{\delta \mathcal{E}}{\delta \rho} &= h_1'(\rho) + \phi, \\
\frac{\delta \mathcal{E}}{\delta n} &= h_2'(n) - \phi. \tag{1.5}
\end{align*}
\]
The models (1.1) and (1.2) (subject to (1.3)-(1.4)) describe two charged fluid systems interacting through an electrostatic potential, and are basic models for applications in semi-conductor devices or plasma physics. [10, 17, Chapter 3, Subsection 3.3.7]. Our objective is to describe the relation between the two models, thus extending the framework of convergence developed in [14] for a single fluid-system. The technical tool for the comparison is a relative entropy identity for bipolar fluid models, which is developed here and is a counterpart of the general theory for abstract dispersive fluid systems [8]. The functions \( h_1, h_2 \in C^3([0, +\infty) \cap C([0, +\infty]) \) represent the internal energies of the fluids, and the electrostatic potential \( \phi \) is obtained from the fluid densities \( \rho, n \) via the elliptic equation \(-\Delta \phi = \rho - n\). The solution of the Poisson equation is expressed as 
\[
\phi(t, x) = (N * (\rho - n))(t, x) := \int_{\Omega} N(x, y) (\rho(t, y) - n(t, y)) dy,
\]
and its spatial gradient is understood as 
\[
\nabla \phi(t, x) = (\nabla_x N * (\rho - n))(t, x) := \int_{\Omega} \nabla_x N(x, y) (\rho(t, y) - n(t, y)) dy,
\]
where \( N \) is the Neumann function [11]. Using the symmetry of \( N \), one derives the formulas (1.5) for the functional derivatives \( \frac{\delta E}{\delta \rho}, \frac{\delta E}{\delta n} \). Introducing (1.5) to (1.2) leads to (2.4), where \( p_i \) are the pressures connected to the internal energies via the usual thermodynamic formulas (2.1). The formal relaxation limit of (1.2) is the system (1.1); establishing this limit is the objective of the present work.

Relaxation problems arise in physics and chemistry, and from a mathematical viewpoint they have been analysed in several contexts. Compensated compactness methods have been used to perform the relaxation limit of single-species hydrodynamic models towards a drift-diffusion equation in one and three spatial dimensions [16, 13]. Refer to [18] for an interesting analysis leading to existence of weak solutions for the bipolar Euler-Poisson in one-space dimension.

The relative entropy method is used here to perform this limiting process for strong solutions of (1.1) in several space dimensions. This approach was successful for the relaxation limit in single-species fluid models [14, 3], as well as for certain (weakly coupled through friction) multicomponent systems [9]. The relative entropy method provides an efficient mathematical mechanism for stability analysis and establishing limiting processes; see [5] for early developments, [2, 14] and references therein for applications to diffusive relaxation. Here, we consider bipolar fluid models in a bounded domain, which is closer to the actual physical situation and requires handling the boundary conditions. No-flux boundary conditions are applied to the electric field and the velocities for the system (1.2), and to the gradient of the densities for the system (1.1).

In order to compare a solution of the limiting system (1.2) with a solution of (1.1), one calculates the evolution of a relative energy functional, see section 3. This calculation is effected between a dissipative weak solution of (1.2) and a conservative, bounded and away from vacuum, strong solution of (1.1); here “strong” refers to the boundedness of its derivatives. The precise notions of solutions are stated in section 4. The main convergence analysis is the content of section 5. To this end, the solution of (1.1) is regarded as an approximate solution of (1.2) and the relative entropy identity is calculated for solutions of the appropriate regularity classes in Proposition 5.7. The technical part amounts to bound the error terms in the relative energy identity. The only term requiring attention is the one associated with the electric field, and the desired bound is reached using a result for Riesz potentials [20] together with properties of the Neumann function [11]. A Gronwall inequality then yields the relaxation convergence as a stability result. The latter is the main result of this work, Theorem 5.3 and shows that if a strong solution of (1.1) is bounded away from vacuum and the initial data converge at the initial time then this convergence is preserved for all \( t \in [0, T] \).
2. Bipolar fluid models

The systems of equations considered here describe the dynamics of fluids formed by charged particles. Such models are common in semiconductor devices (electrons and holes), or in modeling of plasmas (negative and positive charged ions), and play a significant role in various technological contexts related to semiconductors or plasma physics. Both systems can be derived from the semi-classical bipolar Boltzmann model [10].

It is expedient to introduce the monotone increasing pressure functions

\[ p_1, p_2 \in C^2([0, +\infty)) \cap C([0, +\infty]) \]

which are connected to the internal energy functions \( h_1, h_2 \) through the thermodynamic consistency relations

\[ rh_i''(r) = p_i'(r), \quad rh_i'(r) = p_i(r) + h_i(r), \quad \text{with } p_i'(r) > 0; \forall r > 0, \quad (2.1) \]

for \( i = 1, 2 \). Observe that in these conditions \( h_i''(r) > 0 \) for all \( r > 0 \), which corresponds to the monotonicity of the pressures.

2.1. Bipolar drift-diffusion. For energies given by (1.3)-(1.4) the system (1.1) is composed by two drift-diffusion equations for the densities, coupled with a Poisson equation for the electrostatic potential,

\[
\begin{align*}
\rho_t &= \nabla \cdot (\nabla p_1(\rho) + \rho \nabla \phi) \\
n_t &= \nabla \cdot (\nabla p_2(n) - n \nabla \phi) \\
-\Delta \phi &= \rho - n.
\end{align*}
\]

(2.2)

Drift-diffusion equations are commonly used to model semiconductor devices [17]. From a mathematical standpoint, it is well known that drift-diffusion equations incorporate a gradient flow structure induced by the Wasserstein distance. Refer to [19] for a work on these geometric properties of diffusive equations, and to [12] for the role of that geometric structure in an existence theory of global weak solutions for the bipolar drift-diffusion system. In the one-dimensional case, refer to [18] for a theory of weak solutions to a bipolar drift-diffusion system as the limit of a scaled sequence of entropy weak solutions of a bipolar Euler-Poisson system, and to [6] for long-time asymptotics. In addition, [1] provides uniform \( L^\infty \) bounds for approximate solutions to a bipolar drift-diffusion system.

2.2. Bipolar Euler-Poisson. The bipolar Euler-Poisson system (1.2) describes the motion of two-species charged isentropic fluids subjected to an electric field. The system is formed by a pair of continuity equations for the densities, two momentum equations with friction, and a coupling Poisson equation for the electric field. The friction term is responsible for a damping force that gives rise to energy dissipation.

For a theory of existence of weak solutions to a bipolar Euler-Poisson system in the one-dimensional case, refer to [18]. There, compensated compactness is used to establish the existence of weak entropy solutions as the limit of a numerical approximation based on a modified fractional step Lax-Friedrich scheme. There it is also proved that these solutions satisfy \( L^\infty \) and \( L^2 \) bounds.

Observe that the frictional coefficient \( 1/\varepsilon \) also multiplies the internal energy and electric field terms of the momentum equations of system (1.2). From the bipolar Boltzmann-Poisson model with a Lenard-Bernstein collision operator [10, 15, 17], one formally derives the following system
\[
\begin{aligned}
\rho_t + \nabla \cdot (\rho u) &= 0 \\
(pu)_t + \nabla \cdot (pu \otimes u) + \nabla p_1(\rho) &= -\rho \nabla \phi - \frac{1}{\tau} \rho u \\
(nu)_t + \nabla \cdot (nu) &= 0 \\
(nv)_t + \nabla \cdot (nv \otimes v) + \nabla p_2(n) &= n \nabla \phi - \frac{1}{\tau} nv \\
-\Delta \phi &= \rho - n,
\end{aligned}
\]
(2.3)

where \( \rho \) and \( n \) represent the densities of the fluids, \( \rho u \) and \( nv \) represent the momentums, \( \phi \) stands for the electrostatic potential, and \( \tau \) is the collision time.

The formal limit as \( \tau \to 0 \) of this system is trivial in the hyperbolic scale. In the assumed Eulerian frame of reference, the collision time is usually much smaller than the observational time. For this reason one considers the time scaling \( \partial / \partial t \to \tau \partial / \partial t \) (the so-called diffusion scaling), so that the observational time is measured in multiple units of the collision time. A change of scale is also applied for the velocities \( u' = \frac{u}{\tau}, \quad v' = \frac{v}{\tau} \). After dropping the primes \( (u' \to u, \quad v' \to v) \) and setting \( \varepsilon = \frac{\tau^2}{2} \) one obtains the system

\[
\begin{aligned}
\rho_t + \nabla \cdot (\rho u) &= 0 \\
(pu)_t + \nabla \cdot (pu \otimes u) + \frac{1}{\varepsilon} \nabla p_1(\rho) &= -\frac{1}{\varepsilon} \rho \nabla \phi - \frac{1}{\varepsilon} \rho u \\
(nu)_t + \nabla \cdot (nu) &= 0 \\
(nv)_t + \nabla \cdot (nv \otimes v) + \frac{1}{\varepsilon} \nabla p_2(n) &= \frac{1}{\varepsilon} n \nabla \phi - \frac{1}{\varepsilon} nv \\
-\Delta \phi &= \rho - n,
\end{aligned}
\]
(2.4)

which is system (1.2) thanks to (1.5) and (2.1).

The collision time squared \( \tau^2 = \varepsilon \) is called the momentum relaxation time of system (2.4). The limit of system (2.4) as \( \varepsilon \to 0 \) is called the relaxation, overdamped or high-friction limit.

3. Relative entropy

This section presents some considerations related to the relative energy functionals and relative entropy identities of systems (1.1) and (2.4). The relative energy identity is first produced by an exact calculation between smooth solutions of (1.1) and (2.4). Then, in Proposition 5.7, we outline an argument that produces the relative entropy calculation for weaker differentiability classes.

Given a continuously differentiable function \( h = h(r) \), the relative quantity \( h(r|\bar{r}) \) is defined by

\[
h(r|\bar{r}) := h(r) - h(\bar{r}) - h'(\bar{r})(r - \bar{r}).
\]

3.1. Relative entropy for the bipolar drift-diffusion system. Let \( (\rho, n) \) together with \( \phi = N*(\rho - n) \) be a smooth solution of system (1.1), and set \( u := -\nabla (h_1'(\rho) + \phi) \) and \( v := -\nabla (h_2'(n) - \phi) \). Multiplying the previous expressions by \( \rho u \) and \( nv \), respectively, and adding the resulting expressions it yields

\[
\frac{\partial}{\partial t} E + \nabla \cdot F + G = 0,
\]
(3.1)

where

\[
\begin{aligned}
E &= h_1(\rho) + h_2(n) + \frac{1}{2} |\nabla \phi|^2, \\
F &= h_1'(\rho) \rho u + h_2'(n) nv + (\rho u - nv) \phi + \phi \nabla \phi, \\
G &= \rho |u|^2 + n |v|^2.
\end{aligned}
\]
No kinetic energy contributions are present, hence the total energy of the system is only formed by potential energy. Thus, the total energy functional of system (1.1) is simply the potential energy functional $E$.

Integrating (3.1) over space provides the dissipation of total energy of system (1.1)

$$\frac{d}{dt} \mathcal{E}(\rho, u) = - \int_{\Omega} \rho|u|^2 + n|v|^2 \, dx.$$  \hspace{1cm} (3.2)

**Definition 3.1.** The relative total energy functional of system (1.1) is the functional given by

$$\mathcal{E}(\rho, n|\bar{\rho}, \bar{n}) = \mathcal{E}(\rho, n) - \mathcal{E}(\bar{\rho}, \bar{n}) - \langle \frac{\delta \mathcal{E}}{\delta \rho}(\bar{\rho}, \bar{n}), \rho - \bar{\rho} \rangle - \langle \frac{\delta \mathcal{E}}{\delta n}(\bar{\rho}, \bar{n}), n - \bar{n} \rangle.$$  

A straightforward computation gives

$$\mathcal{E}(\rho, n|\bar{\rho}, \bar{n}) = \int_{\Omega} h_1(\rho|\bar{\rho}) + h_2(n|\bar{n}) + \frac{1}{2} |\nabla(\phi - \bar{\phi})|^2 \, dx.$$  

Let $P$ be the density of the relative energy functional of system (1.1), i.e.,

$$P = h_1(\rho|\bar{\rho}) + h_2(n|\bar{n}) + \frac{1}{2} |\nabla(\phi - \bar{\phi})|^2.$$  

The relative entropy identity for the bipolar drift-diffusion system can be obtained by taking the time derivative of $P$. Regarding $(\rho, n)$ and $(\bar{\rho}, \bar{n})$ together with $\phi = N*(\rho - n)$ and $\bar{\phi} = N*(\bar{\rho} - \bar{n})$ as a pair of smooth solutions of system (1.1), and setting as before

$$(u, v, \bar{u}, \bar{v}) = \left( -\nabla(h_1'(\rho) + \phi), -\nabla(h_2'(n) - \phi), -\nabla(h_1'(\rho) + \bar{\phi}), -\nabla(h_2'(n) - \bar{\phi}) \right)$$

one has

$$\frac{\partial}{\partial t} h_1(\rho|\bar{\rho}) = -\nabla \cdot \left( h_1(\rho|\bar{\rho})u + \rho(h_1'(\rho) - h_1'(\bar{\rho}))(u - \bar{u}) \right) - p_1(\rho|\bar{\rho})\nabla \cdot \bar{u},$$  

$$- \rho|u - \bar{u}|^2 - \rho(u - \bar{u}) \cdot \nabla(\phi - \bar{\phi}),$$

$$\frac{\partial}{\partial t} h_2(n|\bar{n}) = -\nabla \cdot \left( h_2(n|\bar{n})v + n(h_2'(n) - h_2'(\bar{n}))(v - \bar{v}) \right) - p_2(n|\bar{n})\nabla \cdot \bar{v},$$  

$$- n|v - \bar{v}|^2 + n(v - \bar{v}) \cdot \nabla(\phi - \bar{\phi}),$$

$$\frac{\partial}{\partial t} \left( \frac{1}{2} |\nabla(\phi - \bar{\phi})|^2 \right) = \nabla \cdot \left( (\phi - \bar{\phi})\nabla(\phi - \bar{\phi}) - (\phi - \bar{\phi})(\rho u - n \bar{v} - \rho \bar{u} + n \bar{v}) \right)$$  

$$+ (\rho u - n \bar{v} - \rho \bar{u} + n \bar{v}) \cdot \nabla(\phi - \bar{\phi}).$$

Putting this together yields

$$\frac{\partial}{\partial t} P + \nabla \cdot Q + R + D = 0,$$  \hspace{1cm} (3.3)

where

$$Q = h_1(\rho|\bar{\rho})u + \rho(h_1'(\rho) - h_1'(\bar{\rho}))(u - \bar{u}) + h_2(n|\bar{n})v + n(h_2'(n) - h_2'(\bar{n}))(v - \bar{v})$$

$$- (\phi - \bar{\phi})\nabla(\phi - \bar{\phi}) + (\phi - \bar{\phi})(\rho u - n \bar{v} - \rho \bar{u} + n \bar{v})$$

is the relative energy flux for (1.1),

$$R = p_1(\rho|\bar{\rho})\nabla \cdot \bar{u} + p_2(n|\bar{n})\nabla \cdot \bar{v} - \left( (\rho - \bar{\rho})\bar{u} + (n - \bar{n})\bar{v} \right) \cdot \nabla(\phi - \bar{\phi})$$

is the term that represents the error between these two smooth solutions, and

$$D = \rho|u - \bar{u}|^2 + n|v - \bar{v}|^2$$

relates to dissipation of relative energy. Expression (3.3) is called the relative entropy identity for system (1.1).
Integrating the relative entropy identity \((\rho, n) = (\rho, n) = (\rho, n)\) over space provides the evolution of the relative total energy of system \((1.1)\)

\[
\frac{d}{dt} E(\rho, n) = -\int \rho u^2 + p_2(n) - n v^2 dx \\
+ \int \rho u - \bar{u}^2 + n v - \bar{v}^2 dx.
\]

\[\tag{3.4}\]

3.2. Relative entropy for the bipolar Euler-Poisson system. Let \((\rho, \rho, n, n)\) together with \(\phi = N \ast (\rho - n)\) be a smooth solution of \((2.4)\). Multiplying the second and fourth equations of system \((2.4)\) by \(u\) and \(v\), respectively, by letting

\[
\varepsilon = \varepsilon \\text{respectively, provides}
\]

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} |u|^2 + \frac{1}{\varepsilon} |\rho - n|^2 \right) + \nabla \cdot \left( \frac{1}{2} |u|^2 \rho u + \frac{1}{\varepsilon} h_1(\rho) \rho u + \frac{1}{\varepsilon} \rho u^2 - \frac{1}{\varepsilon} \rho \phi \right) = 0
\]

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} |v|^2 + \frac{1}{\varepsilon} h_2(n) - \frac{1}{\varepsilon} n \phi \right) + \nabla \cdot \left( \frac{1}{2} |v|^2 n v + \frac{1}{\varepsilon} h_2(n) n v - \frac{1}{\varepsilon} n \phi v \right) + \frac{1}{\varepsilon} |n|^2 + \frac{1}{\varepsilon} n \phi = 0.
\]

Adding together the two previous equations leads to

\[
\frac{\partial}{\partial t} \left( K + \frac{1}{\varepsilon} \hat{E} \right) + \nabla \cdot \left( L + \frac{1}{\varepsilon} \hat{F} \right) + \frac{1}{\varepsilon} \hat{G} = 0, \tag{3.5}\]

where

\[
K = \frac{1}{2} \rho |u|^2 + \frac{1}{2} n |v|^2,
\]

\[
\hat{E} = h_1(\rho) + h_2(n) + \frac{1}{2} |\nabla \phi|^2,
\]

\[
L = \frac{1}{2} |u|^2 \rho u + \frac{1}{2} |v|^2 n v,
\]

\[
\hat{F} = h_1'(\rho) \rho u + h_2'(n) n v + (\rho u - n v) \phi + \phi \nabla \phi,
\]

\[
\hat{G} = \rho |u|^2 + n |v|^2.
\]

The total energy functional of system \((2.4)\) is the functional \(\mathcal{K} + \frac{1}{\varepsilon} \mathcal{E}\), where

\[
\mathcal{K}(\rho, \rho, n, n) = \int \frac{1}{2} \rho |u|^2 + \frac{1}{2} n |v|^2 dx
\]

represents the kinetic energy, and

\[
\frac{1}{\varepsilon} \mathcal{E}(\rho, n) = \frac{1}{\varepsilon} \int h_1(\rho) + h_2(n) + \frac{1}{2} |\nabla \phi|^2 dx
\]

represents the potential energy.

Integrating \((3.5)\) over space yields the dissipation of total energy of system \((2.4)\)

\[
\frac{d}{dt} \left( \mathcal{K}(\rho, \rho, n, n) + \frac{1}{\varepsilon} \mathcal{E}(\rho, n) \right) = -\frac{1}{\varepsilon} \int \rho |u|^2 + n |v|^2 dx. \tag{3.6}\]

Observe that the energy identities \((3.1)\) and \((3.2)\) are formally recovered from \((3.5)\) and \((3.6)\), respectively, by letting \(\varepsilon \to 0\).

The relative kinetic energy functional of system \((2.4)\) is the functional given by

\[
\mathcal{K}(\rho, \rho, n, n, \rho, \rho, n, n) = \mathcal{K}(\rho, \rho, n, n) - \mathcal{K}(\bar{\rho}, \bar{\rho}, \bar{n}, \bar{n})
\]

\[
- \left\langle \frac{\delta \mathcal{K}}{\delta \rho} (\bar{\rho}, \bar{\rho}, \bar{n}, \bar{n}), \rho - \bar{\rho} \right\rangle - \left\langle \frac{\delta \mathcal{K}}{\delta (\rho u)} (\bar{\rho}, \bar{\rho}, \bar{n}, \bar{n}), \rho u - \bar{\rho} u \right\rangle
\]

\[
- \left\langle \frac{\delta \mathcal{K}}{\delta \bar{n}} (\bar{\rho}, \bar{\rho}, \bar{n}, \bar{n}), \bar{n} - \bar{n} \right\rangle - \left\langle \frac{\delta \mathcal{K}}{\delta (n v)} (\bar{\rho}, \bar{\rho}, \bar{n}, \bar{n}), n v - \bar{n} v \right\rangle.
\]
A simple calculation gives

\[
\frac{\delta K}{\delta \rho}(\bar{\rho}, \bar{\rho}u, \bar{n}, \bar{\bar{n}}v) = -\frac{1}{2}|\bar{u}|^2,
\]

\[
\frac{\delta K}{\delta (\rho u)}(\bar{\rho}, \bar{\rho}u, \bar{n}, \bar{\bar{n}}v) = \bar{u},
\]

\[
\frac{\delta K}{\delta n}(\bar{\rho}, \bar{\rho}u, \bar{n}, \bar{\bar{n}}v) = -\frac{1}{2}|v|^2,
\]

\[
\frac{\delta K}{\delta (n v)}(\bar{\rho}, \bar{\rho}u, \bar{n}, \bar{\bar{n}}v) = \bar{v},
\]

wherefrom

\[
K(\rho, \rho u, n, n v|\bar{\rho}, \bar{\rho}u, \bar{n}, \bar{\bar{n}}v) = \int_{\Omega} \frac{1}{2}|\bar{u}|^2 + \frac{1}{2}|v|^2 \, dx.
\]

The relative potential energy functional of system \((2.4)\) is the functional given by

\[
\frac{1}{\varepsilon} E(\rho, n|\bar{\rho}, \bar{n}) = \frac{1}{\varepsilon} E(\rho, n) - \frac{1}{\varepsilon} E(\bar{\rho}, \bar{n}) - \frac{1}{\varepsilon} \langle \delta E(\bar{\rho}, \bar{n}), \rho - \bar{\rho} \rangle - \frac{1}{\varepsilon} \langle \delta E(\bar{\rho}, \bar{n}), n - \bar{n} \rangle.
\]

Repeating the computations that lead to the relative total energy functional of system \((1.1)\) one has

\[
\frac{1}{\varepsilon} E(\rho, n|\bar{\rho}, \bar{n}) = \int_{\Omega} \frac{1}{\varepsilon} h_{1}(\rho|\bar{\rho}) + \frac{1}{\varepsilon} h_{2}(n|\bar{n}) + \frac{1}{2\varepsilon} |\nabla(\phi - \bar{\phi})|^{2} \, dx.
\]

Summing the relative kinetic energy and the relative potential energy functionals yields the relative total energy functional.

**Definition 3.2.** The relative total energy functional of system \((2.4)\) is the functional given by

\[
K(\rho, \rho u, n, n v|\bar{\rho}, \bar{\rho}u, \bar{n}, \bar{\bar{n}}v) = \int_{\Omega} \frac{1}{2}|\bar{u}|^2 + \frac{1}{2}|v|^2 \, dx
\]

\[
+ \frac{1}{\varepsilon} \int_{\Omega} h_{1}(\rho|\bar{\rho}) + h_{2}(n|\bar{n}) + \frac{1}{2\varepsilon} |\nabla(\phi - \bar{\phi})|^{2} \, dx.
\]

Let \(H\) and \(\bar{P}\) be the densities of the relative kinetic energy and the relative potential energy functionals, respectively, i.e.,

\[
H = \frac{1}{2}|\bar{u}|^2 + \frac{1}{2}|v|^2,
\]

\[
\bar{P} = h_{1}(\rho|\bar{\rho}) + h_{2}(n|\bar{n}) + \frac{1}{2} |\nabla(\phi - \bar{\phi})|^{2},
\]

and regard \((\rho, \rho u, n, n v)\), \((\bar{\rho}, \bar{\rho}u, \bar{n}, \bar{\bar{n}}v)\) together with \(\phi = N * (\rho - n)\), \(\bar{\phi} = N * (\bar{\rho} - \bar{n})\) as two smooth solutions of \((2.4)\). Similarly as before, the relative entropy identity that is pursued can be obtained by computing the time derivative of \(H\) and \(\bar{P}\).

In this case one has

\[
\frac{\partial}{\partial t} \left( \frac{1}{2}|\bar{u}|^2 \right) = - \nabla \cdot \left( \frac{1}{2}|\bar{u}|^2 \rho u \right) - \nabla \bar{u} : \rho (u - \bar{u}) \otimes (u - \bar{u})
\]

\[
- \frac{1}{\varepsilon}|\bar{u}|^2 - \frac{1}{\varepsilon} \rho (u - \bar{u}) \cdot \nabla \left( h_{1}'(\rho) - h_{1}'(\bar{\rho}) \right)
\]

\[
- \frac{1}{\varepsilon} \rho (u - \bar{u}) \cdot \nabla (\phi - \bar{\phi}),
\]

\[
\frac{\partial}{\partial t} \left( \frac{1}{2}|v|^2 \right) = - \nabla \cdot \left( \frac{1}{2}|v|^2 n v \right) - \nabla \bar{v} : n (v - \bar{v}) \otimes (v - \bar{v})
\]

\[
- \frac{1}{\varepsilon}|v|^2 - \frac{1}{\varepsilon} n (v - \bar{v}) \cdot \nabla \left( h_{2}'(n) - h_{2}'(\bar{n}) \right)
\]

\[
+ \frac{1}{\varepsilon} n (v - \bar{v}) \cdot \nabla (\phi - \bar{\phi}),
\]
\[
\frac{\partial}{\partial t}\left(\frac{1}{\varepsilon} h_1(\rho | \tilde{\rho})\right) = -\frac{1}{\varepsilon} \nabla \cdot \left( h_1(\rho | \tilde{\rho}) \tilde{u} + \rho (h'_1(\rho) - h'_1(\tilde{\rho}))(u - \bar{u}) \right) - \frac{1}{\varepsilon} p_1(\rho | \tilde{\rho}) \nabla \cdot \tilde{u} \\
+ \frac{1}{\varepsilon} \rho (u - \bar{u}) \cdot \nabla (h'_1(\rho) - h'_1(\tilde{\rho})),
\]

\[
\frac{\partial}{\partial t}\left(\frac{1}{\varepsilon} h_2(n | \tilde{n})\right) = -\frac{1}{\varepsilon} \nabla \cdot \left( h_2(n | \tilde{n}) \bar{v} + n (h'_2(n) - h'_2(\tilde{n}))(v - \bar{v}) \right) - \frac{1}{\varepsilon} p_2(n | \tilde{n}) \nabla \cdot \bar{v} \\
+ \frac{1}{\varepsilon} \rho (v - \bar{v}) \cdot \nabla (h'_2(n) - h'_2(\tilde{n})),
\]

\[
\frac{\partial}{\partial t}\left(\frac{1}{2\varepsilon} |\phi - \tilde{\phi}|^2\right) = \frac{1}{\varepsilon} \nabla \cdot \left( (\phi - \tilde{\phi}) \nabla (\phi_1 - \tilde{\phi}_1) - (\phi - \tilde{\phi})(\rho u - n v - \tilde{\rho} \bar{u} + \tilde{n} \bar{v}) \right) \\
+ \frac{1}{\varepsilon} (\rho u - n v - \tilde{\rho} \bar{u} + \tilde{n} \bar{v}) \cdot \nabla (\phi - \tilde{\phi}).
\]

Putting this together provides

\[
\frac{\partial}{\partial t} \left( H + \frac{1}{\varepsilon} \tilde{\rho} \right) + \nabla \cdot \left( I + \frac{1}{\varepsilon} \tilde{Q} \right) + \left( J + \frac{1}{\varepsilon} \tilde{R} \right) + \frac{1}{\varepsilon} \tilde{D} = 0, \tag{3.7}
\]

where

\[
I = \frac{1}{2} |u - \bar{u}|^2 \rho u + \frac{1}{2} |v - \bar{v}|^2 n v,
\]

\[
\tilde{Q} = h_1(\rho | \tilde{\rho}) \tilde{u} + \rho (h'_1(\rho) - h'_1(\tilde{\rho}))(u - \bar{u}) + h_2(n | \tilde{n}) \bar{v} + n (h'_2(n) - h'_2(\tilde{n}))(v - \bar{v}) \\
- (\phi - \tilde{\phi}) \nabla (\phi_1 - \tilde{\phi}_1) + (\phi - \tilde{\phi})(\rho u - n v - \tilde{\rho} \bar{u} + \tilde{n} \bar{v}),
\]

\[
J = \nabla \tilde{u} : \rho (u - \bar{u}) \otimes (u - \bar{u}) + \nabla \bar{v} : n (v - \bar{v}) \otimes (v - \bar{v}),
\]

\[
\tilde{R} = p_1(\rho | \tilde{\rho}) \nabla \cdot \tilde{u} + p_2(n | \tilde{n}) \nabla \cdot \bar{v} - ((\rho - \tilde{\rho}) \tilde{u} - (n - \tilde{n}) \bar{v}) \cdot \nabla (\phi - \tilde{\phi}),
\]

\[
\tilde{D} = \rho |u - \bar{u}|^2 + n |v - \bar{v}|^2.
\]

The term \( I + \frac{1}{\varepsilon} Q \) is the relative energy flux for (2.4); \( J + \frac{1}{\varepsilon} R \) stands for the error between these two smooth solutions, and \( \frac{1}{\varepsilon} D \) refers to the relative energy dissipation due to friction.

Expression (3.7) is called the relative entropy identity for system (2.4), and integrating it over space renders the evolution of the relative total energy of system (2.4)

\[
\frac{d}{dt} \left( \mathcal{K}(\rho, \rho u, n, n v | \tilde{\rho}, \tilde{\rho} u, \tilde{\rho} n v) + \frac{1}{\varepsilon} \mathcal{E}(\rho, n | \tilde{\rho}, \tilde{n}) \right) = \\
= - \int_{\Omega} \nabla \tilde{u} : \rho (u - \bar{u}) \otimes (u - \bar{u}) dx - \int_{\Omega} \nabla \tilde{v} : n (v - \bar{v}) \otimes (v - \bar{v}) dx \\
- \frac{1}{\varepsilon} \int_{\Omega} p_1(\rho | \tilde{\rho}) \nabla \cdot \tilde{u} + \rho |u - \bar{u}|^2 dx - \frac{1}{\varepsilon} \int_{\Omega} p_2(n | \tilde{n}) \nabla \cdot \bar{v} + n |v - \bar{v}|^2 dx \\
+ \frac{1}{\varepsilon} \int_{\Omega} ((\rho - \tilde{\rho}) \tilde{u} - (n - \tilde{n}) \bar{v}) \cdot \nabla (\phi - \tilde{\phi}) dx. \tag{3.8}
\]

Note that as \( \varepsilon \to 0 \), the relative entropy identity (3.7) and expression (3.8) formally become the relative entropy identity (3.3) and the evolution of relative total energy of system (1.1), respectively.
4. Notions of solutions

In this section, various notions of solutions of (2.4) and (1.1) are defined and will be used in the study of the relaxation convergence.

Regarding the internal energy and pressure functions, it is assumed that there are positive constants \( k_i, \hat{k}_i, i = 1, 2 \), such that
\[
\lim_{r \to +\infty} \frac{h_i(r)}{r^{\gamma_i}} = \frac{k_i}{\gamma_i - 1},
\]
where \( \gamma_1, \gamma_2 \geq 2 - \frac{1}{d} \), and
\[
|p''_i(r)| \leq \hat{k}_i \frac{p'_i(r)}{r} \forall r > 0.
\]

Throughout the rest of this work, the internal energy functions \( h_1, h_2 \in C^3([0, +\infty]) \cap C([0, +\infty]) \) and the pressure functions \( p_1, p_2 \in C^2([0, +\infty]) \cap C([0, +\infty]) \) are assumed to be fixed, satisfying (2.1), (4.1), and (4.2). The prototypical pressure and internal energy functions that satisfy these conditions are
\[
p(r) = kr^{\gamma},
\]
\[
h(r) = \frac{k}{\gamma - 1} r^{\gamma},
\]
where \( \gamma \geq 2 - \frac{1}{d} \) and \( k > 0 \).

Remark 4.1. The condition \( \gamma_1, \gamma_2 \geq 2 - \frac{1}{d} \) is purely technical and it is a necessary ingredient in Lemma 5.12.

Throughout this document \( C \) will always denote an unspecified positive constant unless stated otherwise.

4.1. Weak solutions to the bipolar Euler-Poisson system. For the system (2.4), no-flux boundary conditions are prescribed for the velocities and the electric field, i.e.,
\[
\begin{align*}
  u \cdot \nu &= 0, \\
  v \cdot \nu &= \frac{\partial \phi}{\partial \nu} = 0 \text{ on } [0, T] \times \partial \Omega,
\end{align*}
\]
where \( \nu \) is any outer normal vector to \( \partial \Omega \). Physically, these boundary conditions mean that the fluids do not exit the confined space where they move, and that the system is electrically isolated.

For the initial datum \( (\rho_0, u_0, n_0, v_0) \) we assume that \( \rho_0, n_0 \) are non-negative and satisfy
\[
\int_{\Omega} \rho_0 \, dx = \int_{\Omega} n_0 \, dx = M < +\infty;
\]
while \( u_0, v_0 \) satisfy the no-flux condition at the boundary. From the elliptic equation one observes that condition \([4.4]\) is preserved for all times \( t \in [0, T] \).

**Definition 4.2.** A tuple of functions \((\rho, \rho u, n, nv)\) such that

\[
\rho \in C([0, T]; L^1(\Omega)) \cap L^\infty(\Omega)),
\]
\[
n \in C([0, T]; L^1(\Omega)) \cap L^\infty(\Omega)),
\]
\[
\rho u, nv \in C\left([0, T]; (L^1(\Omega))^d\right),
\]
\[
\rho|u|^2, n|v|^2 \in L^\infty([0, T]; L^1(\Omega)),
\]
\[
\rho, n \geq 0,
\]
together with \( \phi = N * (\rho - n) \) is called a weak solution of \([2.4]\) provided that:

(i) \((\rho, \rho u, n, nv)\) satisfies \([2.4]\) in the weak sense of

\[
- \int_0^T \int_\Omega \varphi_t \rho \, dx \, dt - \int_0^T \int_\Omega \nabla \varphi \cdot (\rho u) \, dx \, dt - \int_\Omega \varphi \rho |_{t=0} \, dx = 0,
\]

\[
- \int_0^T \int_\Omega \tilde{\varphi}_t \cdot (\rho u) \, dx \, dt - \int_0^T \int_\Omega \nabla \tilde{\varphi} : \rho u \otimes u \, dx \, dt - \frac{1}{\varepsilon} \int_0^T \int_\Omega (\nabla \cdot \tilde{\varphi})p_1(\rho) \, dx \, dt
\]

\[
- \int \Omega \tilde{\varphi} \cdot (\rho u) |_{t=0} \, dx = - \frac{1}{\varepsilon} \int_0^T \int_\Omega \tilde{\varphi} \cdot (\rho \nabla \phi) \, dx \, dt - \frac{1}{\varepsilon} \int_0^T \int_\Omega \tilde{\varphi} \cdot (\rho u) \, dx \, dt,
\]

\[
- \int_0^T \int_\Omega \psi_t n \, dx \, dt - \int_0^T \int_\Omega \nabla \psi \cdot (nv) \, dx \, dt - \int \Omega \psi n |_{t=0} \, dx = 0,
\]

\[
- \int_0^T \int_\Omega \tilde{\psi}_t \cdot (nv) \, dx \, dt - \int_0^T \int_\Omega \nabla \tilde{\psi} : nv \otimes v \, dx \, dt - \frac{1}{\varepsilon} \int_0^T \int_\Omega (\nabla \cdot \tilde{\psi})p_2(n) \, dx \, dt
\]

\[
- \int \Omega \tilde{\psi} \cdot (nv) |_{t=0} \, dx = - \frac{1}{\varepsilon} \int_0^T \int_\Omega \tilde{\psi} \cdot (n \nabla \phi) \, dx \, dt - \frac{1}{\varepsilon} \int_0^T \int_\Omega \tilde{\psi} \cdot (nv) \, dx \, dt,
\]

for all Lipschitz test functions \( \varphi, \psi : [0, T] \times \Omega \rightarrow \mathbb{R} \), \( \tilde{\varphi}, \tilde{\psi} : [0, T] \times \Omega \rightarrow \mathbb{R}^d \) compactly supported in time with \( \tilde{\varphi}, \tilde{\psi} \) satisfying the no-flux property on the boundary;

(ii) \((\rho, \rho u, n, nv)\) is equipped with the following bounds:

\[
\int_\Omega \rho \, dx = \int_\Omega n \, dx = M < +\infty, \forall t \in [0, T],
\]

\[
\sup_{\Omega} \frac{1}{T} \int_0^T \frac{1}{2} |\rho|u|^2 + \frac{1}{2} |n|v|^2 + \frac{1}{\varepsilon} h_1(\rho) + \frac{1}{\varepsilon} h_2(n) + \frac{1}{2\varepsilon} |\nabla \phi|^2 \, dx < +\infty;
\]

(iii) \((u, v)\) satisfy the no-flux boundary condition.

**Remark 4.3.** Property \([4.9]\) represents the conservation of mass of \( \rho \) and \( n \), whereas \([4.10]\) asserts that the total energy is finite.

**Definition 4.4.** A weak solution \((\rho, \rho u, n, nv)\) of \([2.4]\) together with \( \phi = N * (\rho - n) \) is called dissipative if

\[
- \int_0^T \int_\Omega \left( \frac{1}{2} |\rho|u|^2 + \frac{1}{2} |n|v|^2 + \frac{1}{\varepsilon} h_1(\rho) + \frac{1}{\varepsilon} h_2(n) + \frac{1}{2\varepsilon} |\nabla \phi|^2 \right) \theta(t) \, dx \, dt
\]

\[
+ \int_0^T \int_\Omega \left( \frac{1}{\varepsilon} |\rho|u|^2 + \frac{1}{\varepsilon} |n|v|^2 \right) \theta(t) \, dx \, dt
\]

\[
\leq \int_\Omega \left( \frac{1}{2} |\rho|u|^2 + \frac{1}{2} |n|v|^2 + \frac{1}{\varepsilon} h_1(\rho) + \frac{1}{\varepsilon} h_2(n) + \frac{1}{2\varepsilon} |\nabla \phi|^2 \right) |_{t=0} \theta(0) \, dx
\]
for any non-negative $\theta \in W^{1,\infty}(0, T]$ with compact support.

**Definition 4.5.** A weak solution $(\rho, \rho u, n, n v)$ of (2.4) together with $\phi = N * (\rho - n)$ is called conservative if it satisfies (4.11) as an equality for any non-negative $\theta \in W^{1,\infty}(0, T]$ with compact support.

For the study of the relaxation convergence one will be focused on dissipative weak solutions.

**Remark 4.6.** It is clear that a weak solution of (2.4) depends on $\varepsilon$, $(\rho, \rho u, n, n v) = (\rho^\varepsilon, \rho^\varepsilon u, n^\varepsilon, n^\varepsilon v)$, however, for simplicity, $\varepsilon$ is dropped from the notation.

### 4.2. Strong solutions to the bipolar drift-diffusion system.

The physically relevant boundary conditions in this case are the no-flux boundary condition for the gradient of the densities and the no-flux boundary condition for the gradient of the electrostatic potential. Precisely, it is imposed that

$$\frac{\partial \rho}{\partial \nu} = \frac{\partial n}{\partial \nu} = \frac{\partial \phi}{\partial \nu} = 0 \text{ on } [0, T] \times \partial \Omega, \quad \int_{\partial \Omega} \phi \, dx = 0. \tag{4.12}$$

System (1.1) is also provided with non-negative initial datum $(\bar{\rho}_0, \bar{n}_0)$ that satisfy

$$\int_{\Omega} \bar{\rho}_0 \, dx = \int_{\Omega} \bar{n}_0 \, dx = \bar{M} < +\infty. \tag{4.13}$$

As before, one observes that condition (4.13) is formally preserved for all $t \in [0, T]$.

**Definition 4.7.** A pair of Lipschitz functions $(\bar{\rho}, \bar{n})$ defined in $[0, T] \times \Omega$ such that

$$\bar{\rho}, \bar{n} \geq 0,$$

together with $\bar{\phi} = N * (\bar{\rho} - \bar{n})$, is called a strong solution of (1.1) provided that:

(i) for all $i, j = 1, \ldots, d$, the derivatives

$$\frac{\partial^2 \bar{\rho}}{\partial x_i \partial t}, \frac{\partial^2 \bar{\phi}}{\partial x_i \partial t}, \frac{\partial^2 \bar{n}}{\partial x_i \partial t}, \frac{\partial^2 \bar{\rho}}{\partial x_i \partial x_j}, \frac{\partial^2 \bar{n}}{\partial x_i \partial x_j}, \frac{\partial^2 \bar{\phi}}{\partial x_i \partial x_j}, \frac{\partial^3 \bar{\phi}}{\partial x_i \partial x_j \partial t}$$

exist in the weak sense and belong to $L^\infty([0, T] \times \Omega)$;

(ii) $(\bar{\rho}, \bar{n})$ satisfies (1.1) in the weak sense of

$$\bar{\rho}_t = \nabla \cdot (\bar{\rho} \nabla (h'_1(\bar{\rho}) + \bar{\phi})) \quad \text{a.e. in } [0, T] \times \Omega, \tag{4.14}$$

$$\bar{n}_t = \nabla \cdot (\bar{\rho} \nabla (h'_2(\bar{n}) - \bar{\phi})) \quad \text{a.e. in } [0, T] \times \Omega; \tag{4.15}$$

(iii) $(\bar{\rho}, \bar{n})$ is equipped with the following bounds:

$$\int_{\Omega} \bar{\rho} \, dx = \int_{\Omega} \bar{n} \, dx = \bar{M} < +\infty, \quad \forall t \in [0, T], \tag{4.16}$$

$$\sup_{[0, T]} \int_{\Omega} \bar{h}_1(\bar{\rho}) + \bar{h}_2(\bar{n}) + \frac{1}{2} |\nabla \bar{\phi}|^2 \, dx < +\infty; \tag{4.17}$$

(iv) $(\bar{\rho}, \bar{n})$ satisfies boundary conditions (4.12).

**Definition 4.8.** A strong solution $(\bar{\rho}, \bar{n})$ of (1.1) is said to be bounded and away from vacuum if $\bar{\rho}, \bar{n} \in L^\infty([0, T] \times \Omega)$ and there exist $\delta_1, \delta_2 > 0$ and $M_1, M_2 < +\infty$ such that $\bar{\rho}(t, x) \in [\delta_1, M_1]$ and $\bar{n}(t, x) \in [\delta_2, M_2]$ for a.e. $(t, x) \in [0, T] \times \Omega.$
Definition 4.9. A tuple of functions \((\bar{\rho}, \bar{\rho}u, \bar{n}, \bar{n}v)\), where
\[
\bar{u} = -\nabla (h_1'(\bar{\rho}) + \bar{\phi}),
\]
\[
\bar{v} = -\nabla (h_2'(\bar{n}) - \bar{\phi}),
\]
with \(\bar{\phi} = N * (\bar{\rho} - \bar{n})\), is called a strong solution of (1.1) if \((\bar{\rho}, \bar{n})\) is a strong solution of (1.1).

If \((\bar{\rho}, \bar{\rho}u, \bar{n}, \bar{n}v)\) is a strong solution of (1.1) then
\[
\bar{\rho}_t + \nabla \cdot (\bar{\rho}u) = 0 \text{ a.e. in } [0, T] \times \Omega,
\]
\[
\bar{n}_t + \nabla \cdot (\bar{n}v) = 0 \text{ a.e. in } [0, T] \times \Omega.
\]
Moreover, if \((\bar{\rho}, \bar{\rho}u, \bar{n}, \bar{n}v)\) is a strong solution of (1.1) bounded and away from vacuum, then
\[
\bar{u}, \bar{v} \in L^\infty([0, T], (W^{1, \infty}(\Omega))^d),
\]
and
\[
\bar{u} \cdot \nu = \bar{v} \cdot \nu = 0
\]
for every \(\nu\), an outer normal vector to \(\partial \Omega\).

Definition 4.10. A strong solution \((\bar{\rho}, \bar{\rho}u, \bar{n}, \bar{n}v)\) of (1.1) together with \(\bar{\phi} = N * (\bar{\rho} - \bar{n})\) is called conservative if
\[
- \int_0^T \int_{\Omega} \left( h_1(\bar{\rho}) + h_2(\bar{n}) + \frac{1}{2} |\nabla \bar{\phi}|^2 \right) \bar{\theta}(t) \, dx \, dt + \int_0^T \int_{\Omega} (\bar{\rho}|\bar{u}|^2 + \bar{n}|\bar{v}|^2) \theta(t) \, dx \, dt
= \int_{\Omega} \left( h_1(\bar{\rho}) + h_2(\bar{n}) + \frac{1}{2} |\nabla \bar{\phi}|^2 \right) \bigg|_{t=0} \theta(0) \, dx
\]
for any non-negative \(\theta \in W^{1, \infty}([0, T])\) with compact support.

5. Convergence in the relaxation limit

This work compares the overdamped limit of a dissipative weak solution \((\rho, \rho u, n, n v)\) of (2.4) together with \(\phi = N * (\rho - n)\) with a bounded and away from vacuum, conservative strong solution \((\bar{\rho}, \bar{\rho}u, \bar{n}, \bar{n}v)\) of (1.1) with \(\bar{\phi} = N * (\bar{\rho} - \bar{n})\). Such strong solution can be regarded as an approximate solution of (2.4). Setting
\[
\bar{e}_1 := (\bar{\rho}u)_t + \nabla \cdot (\bar{\rho}u \otimes \bar{u}),
\]
\[
\bar{e}_2 := (\bar{n}v)_t + \nabla \cdot (\bar{n}v \otimes \bar{v}),
\]
the equilibrium system (1.1) can be rewritten as
\[
\begin{cases}
\bar{\rho}_t + \nabla \cdot (\bar{\rho}u) = 0 \\
(\bar{\rho}u)_t + \nabla \cdot (\bar{\rho}u \otimes \bar{u}) = -\frac{1}{\varepsilon} \bar{\rho} \nabla (h_1'(\bar{\rho}) + \bar{\phi}) - \frac{1}{\varepsilon} \bar{\rho} \bar{u} + \bar{e}_1 \\
\bar{n}_t + \nabla \cdot (\bar{n}v) = 0 \\
(\bar{n}v)_t + \nabla \cdot (\bar{n}v \otimes \bar{v}) = -\frac{1}{\varepsilon} \bar{n} \nabla (h_2'(\bar{n}) - \bar{\phi}) - \frac{1}{\varepsilon} \bar{n} \bar{v} + \bar{e}_2 \\
- \Delta \bar{\phi} = \bar{\rho} - \bar{n}.
\end{cases}
\]

The boundedness of a strong solution of (1.1) implies
\[
\bar{e}_1, \bar{e}_2 \in (L^\infty([0, T] \times \Omega))^d,
\]
and after a straightforward calculation one obtains
\[
\int_0^t \int_{\Omega} \bar{u} \cdot \bar{e}_1 \, dx \, \tau = \int_0^t \left( \frac{1}{2} \bar{\rho}|\bar{u}|^2 \right) \bigg|_{\tau=0} \, dx,
\]
\[
\int_0^t \int_{\Omega} \bar{v} \cdot \bar{e}_2 \, dx \, \tau = \int_0^t \left( \frac{1}{2} \bar{n}|\bar{v}|^2 \right) \bigg|_{\tau=0} \, dx.
\]
Definition 5.1. Let \((\rho, \rho u, n, nv)\) together with \(\phi = N \ast (\rho - n)\) be a weak solution of \((2.4)\), and let \((\bar{\rho}, \bar{\rho}u, \bar{n}, \bar{nv})\) with \(\bar{\phi} = N \ast (\bar{\rho} - \bar{n})\) be a strong solution of \((1.1)\). The function \(\Theta : [0, T] \rightarrow \mathbb{R}\) given by
\[
\Theta(t) = \int_{\Omega} \frac{1}{2} \rho |u - \bar{u}|^2 + \frac{1}{2} n |v - \bar{v}|^2 + \frac{1}{\varepsilon} h_1(\rho |\bar{\rho}|) + \frac{1}{\varepsilon} h_2(n |\bar{n}|) + \frac{1}{2\varepsilon} |\nabla(\phi - \bar{\phi})|^2 \, dx
\]
is called the relative energy function between these two solutions.

Definition 5.2. The function \(\Psi : [0, T] \rightarrow \mathbb{R}\) given by
\[
\Psi(t) = \varepsilon \Theta(t)
\]
is called the reduced relative energy function between a weak solution \((\rho, \rho u, n, nv)\) of \((2.4)\) together with \(\phi = N \ast (\rho - n)\) and a strong solution \((\bar{\rho}, \bar{\rho}u, \bar{n}, \bar{nv})\) of \((1.1)\) with \(\bar{\phi} = N \ast (\bar{\rho} - \bar{n})\).

The reduced relative energy function \(\Psi\) will serve as a yardstick for the comparison between a dissipative weak solution of \((2.4)\) and a conservative, bounded and away from vacuum, strong solution of \((1.1)\). Precisely, the main result is:

Theorem 5.3. Let \((\rho, \rho u, n, nv)\) together with \(\phi = N \ast (\rho - n)\) be a dissipative weak solution of \((2.4)\) with \(\gamma_1, \gamma_2 \geq 2 - \frac{1}{d}\) and let \((\bar{\rho}, \bar{\rho}u, \bar{n}, \bar{nv})\) with \(\bar{\phi} = N \ast (\bar{\rho} - \bar{n})\) be a conservative, bounded and away from vacuum, strong solution of \((1.1)\); where \(d \in \mathbb{N} \setminus \{1, 2\}\). Then, there exists \(C > 0\) such that for each \(t \in [0, T]\), the reduced relative energy function \(\Psi\) between these two solutions satisfies the stability estimate
\[
\Psi(t) \leq e^{CT} (\Psi(0) + \varepsilon^2).
\]
Therefore if \(\Psi(0) \rightarrow 0\) as \(\varepsilon \rightarrow 0\), then \(\Psi(t) \rightarrow 0\) as \(\varepsilon \rightarrow 0\) for every \(t \in [0, T]\).

5.1. Derivation of the relative entropy inequality. The relative entropy inequality satisfied by the relative energy function is now derived within the regularity class of the compared solutions. This result is fundamental for the achievement of convergence in the relaxation limit.

In order to deal with the electrostatic potential \(\phi\) one needs to recall the notion of Riesz potential. Given a function \(f : \mathbb{R}^d \rightarrow \mathbb{R}\), the Riesz potential of \(f\) is the function \(I_\alpha(f)\) given by
\[
I_\alpha(f)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d - \alpha}} \, dy,
\]
with \(0 < \alpha < d\). Regarding this potentials one has the following result [20, Chapter V, Section 1]:

Proposition 5.4. Let \(0 < \alpha < d\) and \(1 < p < d/\alpha\). If \(f \in L^p(\mathbb{R}^d)\), then \(I_\alpha(f)(x)\) converges absolutely for a.e. \(x \in \mathbb{R}^d\) and
\[
\|I_\alpha(f)\|_{L^\frac{dp}{d-\alpha}(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)},
\]
for some positive constant \(C = C(\alpha, d, p)\).

An integration by parts formula for the electrostatic potential \(\phi = N \ast (\rho - n)\) can be derived using this previous proposition.

Proposition 5.5. Let \(d \in \mathbb{N} \setminus \{1, 2\}\), \(f \in L^1(\Omega) \cap L^\gamma(\Omega)\), where \(\Omega \subseteq \mathbb{R}^d\) is a smooth bounded domain with smooth boundary and \(\gamma \geq 2d/(d + 2)\), \(\phi = N \ast f\), and \(\nabla \phi = \nabla N \ast f\), where \(N\) is the Neumann function. Then:
(i) \(\phi \in L^{\frac{2d}{d+2}}(\Omega)\),
(ii) \(\nabla \phi \in L^2(\Omega)\),
(iii) \(\int_{\Omega} f \phi dx = \int_{\Omega} \int_{\Omega} f(x)N(x, y)f(y)dx dy\),
(iv) \( \int_{\Omega} f \phi dx = \int_{\Omega} |\nabla \phi|^2 dx \).

Proof. Set \( p = \frac{2d}{d+2} \) and observe that since \( d > 2 \) one has \( 1 < p < d/2 \). Let \( \tilde{f} \) be given by

\[
\tilde{f}(x) = \begin{cases} 
  f(x), & \text{if } x \in \Omega, \\
  0, & \text{otherwise}.
\end{cases}
\]

Clearly \( \tilde{f} \in L^1(\mathbb{R}^d) \cap L^\gamma(\mathbb{R}^d) \), and since \( \gamma \geq p = \frac{2d}{d+2} \) interpolation gives that \( \tilde{f} \in L^p(\mathbb{R}^d) \).

(i) From the properties of the Neumann function one deduces that

\[
|\phi(x)| \leq \int_{\Omega} |N(x,y)||f(y)|dy 
\leq C \int_{\Omega} \frac{|f(y)|}{|x-y|^{d-2}}dy 
\leq C \int_{\mathbb{R}^d} \frac{\tilde{f}(y)}{|x-y|^{d-2}}dy 
= CI_2(|\tilde{f}|)(x), \quad x \in \Omega.
\]

Using Proposition 5.4 with \( \alpha = 2, \quad p = \frac{2d}{d+2} \), one obtains

\[
||\phi||_{L^\frac{2d}{d-2}(\Omega)} \leq C||I_2(\tilde{f})||_{L^\frac{2d}{d-2}(\Omega)} 
\leq C||I_2(\tilde{f})||_{L^\frac{2d}{d-2}(\mathbb{R}^d)} 
\leq C||\tilde{f}||_{L^p(\mathbb{R}^d)} 
= C||f||_{L^p(\Omega)}.
\]

(ii) Similarly,

\[
|\nabla \phi(x)| \leq CI_1(|\tilde{f}|)(x), \quad x \in \Omega,
\]

hence

\[
||\nabla \phi||_{L^2(\Omega)} \leq C||I_1(\tilde{f})||_{L^2(\Omega)} \leq C||f||_{L^p(\Omega)},
\]

where we used Proposition 5.4 with \( \alpha = 1, \quad p = \frac{2d}{d+2} \).

(iii) Simply note that \( p' = 2d/(d-2) \), so

\[
\int_{\Omega} \phi f dx \leq \left( \int_{\Omega} |\phi|^{p'} dx \right)^{\frac{1}{p'}} \left( \int_{\Omega} |f|^p dx \right)^{\frac{1}{p}} < +\infty,
\]

and Fubini’s theorem yields the desired identity.

(iv) Since \( f \in L^p(\Omega) \), there exists a sequence \( (f_n)_{n \in \mathbb{N}} \subseteq C_c^\infty(\Omega) \) such that

\[ f_n \to f \text{ in } L^p(\Omega). \]

Let \( \phi_n = N * f_n \). Then, from (i) and (ii) one has

\[
||\phi - \phi_n||_{L^{p'}(\Omega)} \leq C||f - f_n||_{L^p(\Omega)} \to 0 \text{ as } n \to +\infty,
\]

and

\[
||\nabla \phi - \nabla \phi_n||_{L^2(\Omega)} \leq C||f - f_n||_{L^p(\Omega)} \to 0 \text{ as } n \to +\infty.
\]
In other words, $\phi_n \to \phi$ in $L^p(\Omega)$ and $\nabla \phi_n \to \nabla \phi$ in $L^2(\Omega)$.
Thus,
\[
\left| \int_{\Omega} \phi_n f_n dx - \int_{\Omega} \phi f dx \right| \leq \int_{\Omega} |\phi_n f_n - \phi f| dx \\
\leq \int_{\Omega} |f_n| |\phi_n - \phi| dx + \int_{\Omega} |\phi||f_n - f| dx \\
\leq ||f_n||_{L^p(\Omega)} ||\phi_n - \phi||_{L^p(\Omega)} + ||\phi||_{L^p(\Omega)} ||f_n - f||_{L^p(\Omega)} \\
\to 0 \text{ as } n \to +\infty,
\]
and
\[
||\nabla \phi||_{L^2(\Omega)} - ||\nabla \phi_n||_{L^2(\Omega)} \leq ||\nabla \phi - \nabla \phi_n||_{L^2(\Omega)} \to 0 \text{ as } n \to +\infty.
\]
Observing that $f_n, \phi_n$ satisfy
\[
\int_{\Omega} f_n \phi_n dx = \int_{\Omega} |\nabla \phi_n|^2 dx,
\]
after letting $n \to +\infty$ one obtains the desired integration by parts formula. \hfill \Box

Similarly as in (iii) above, it holds that if $f, g \in L^4(\Omega) \cap L^7(\Omega)$, with $\gamma \geq 2d/(d + 2)$, $\phi = N * f$, and $\varphi = N * g$, then
\[
\int_{\Omega} f \varphi dx = \int_{\Omega} g \varphi dx = \int_{\Omega} \int_{\Omega} f(x)N(x,y)g(y)dx dy. \quad (5.3)
\]

**Remark 5.6.** Observe that since $d > 1$, one has $\frac{2d}{d+2} < 2 - \frac{1}{d}$. Thus, if $\gamma_1, \gamma_2 \geq 2 - \frac{1}{d}$, then the function $f$ in Proposition 5.5 can be taken as $\rho - n$, where $\rho$ and $n$ are part of a weak solution of (2.4).

Now the important relative entropy inequality is presented.

**Proposition 5.7.** Let $(\rho, \rho u, n, nu)$ together with $\phi = N * (\rho - n)$ be a dissipative weak solution of (2.4), and let $(\bar{\rho}, \bar{\rho} u, n, \bar{n} v)$ with $\bar{\phi} = N * (\bar{\rho} - \bar{n})$ be a conservative strong solution of (1.1). Then, for each $t \in [0, T]$, the relative energy function $\Theta$ between these two solutions satisfies the following relative entropy inequality:
\[
\Theta(t) - \Theta(0) + \frac{1}{\varepsilon} \int_{\Omega} \int_{0}^{t} \rho |\bar{u} - u|^2 + n |v - \bar{v}|^2 dx \leq J_1(t) + J_2(t) + J_3(t) + J_4(t), \quad (5.4)
\]
where
\[
J_1(t) = -\int_{0}^{t} \int_{\Omega} \nabla \bar{u} : \rho (u - \bar{u}) \otimes (u - \bar{u}) + \nabla \bar{v} : n (v - \bar{v}) \otimes (v - \bar{v}) dx dr,
\]
\[
J_2(t) = -\frac{1}{\varepsilon} \int_{0}^{t} \int_{\Omega} (\nabla \cdot \bar{u}) p_1 (\rho \bar{\rho}) + (\nabla \cdot \bar{v}) p_2 (n \bar{n}) dx dr,
\]
\[
J_3(t) = \frac{1}{\varepsilon} \int_{0}^{t} \int_{\Omega} \left( (\rho - \bar{\rho}) \bar{u} - (n - \bar{n}) \bar{v} \right) \cdot \nabla (\phi - \bar{\phi}) dx dr,
\]
\[
J_4(t) = -\int_{0}^{t} \int_{\Omega} \frac{\hat{\rho}}{\hat{n}} (\bar{u} - u) + \frac{\bar{n}}{n} (\bar{v} - v) dx dr.
\]

**Proof.** Fix $t \in [0, T]$, let $\kappa$ be such that $t + \kappa < T$, and define $\theta : [0, T] \to \mathbb{R}$ by
\[
\theta(\tau) = \begin{cases} 
1, & \text{if } 0 \leq \tau < t, \\
\frac{t - \tau}{\kappa} + 1, & \text{if } t \leq \tau < t + \kappa, \\
0, & \text{if } t + \kappa \leq \tau < T.
\end{cases}
\]
Using this choice of $\theta$ in (4.11) yields
\[
\int_{t}^{t+\kappa} \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{1}{2} \eta |v|^2 + \frac{1}{\varepsilon} h_1(\rho) + \frac{1}{\varepsilon} h_2(n) + \frac{1}{2\varepsilon} |\nabla \phi|^2 \right) dx \, dr \\
+ \int_{t}^{t+\kappa} \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{1}{\varepsilon} |v|^2 \right) dx \, dr + \int_{\Omega} \int_{t}^{t+\kappa} \left( \frac{1}{\varepsilon} |u|^2 + \frac{1}{\varepsilon} n |v|^2 \right) dx \, dr \\
\leq \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{1}{2} n |v|^2 + \frac{1}{\varepsilon} h_1(\rho) + \frac{1}{\varepsilon} h_2(n) + \frac{1}{\varepsilon} (\rho - n) \phi - \frac{1}{2\varepsilon} |\nabla \phi|^2 \right) \bigg|_{t=0} \, dx.
\]

Letting $\kappa \to 0^+$ above one deduces
\[
\int_{t}^{t+\kappa} \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{1}{2} n |v|^2 + \frac{1}{\varepsilon} h_1(\rho) + \frac{1}{\varepsilon} h_2(n) + \frac{1}{2\varepsilon} |\nabla \phi|^2 \right) \bigg|_{t=0} \, dx \\
\leq - \int_{t}^{t+\kappa} \int_{\Omega} \frac{1}{\varepsilon} |u|^2 + \frac{1}{\varepsilon} n |v|^2 \, dx \, dr.
\]

Similarly, using this $\theta$ in (4.18) together with (5.2) gives
\[
\int_{t}^{t+\kappa} \int_{\Omega} \left( \frac{1}{2} \rho |\bar{v}|^2 + \frac{1}{2} \eta |\bar{v}|^2 + \frac{1}{\varepsilon} h_1(\bar{\rho}) + \frac{1}{\varepsilon} h_2(\bar{n}) + \frac{1}{\varepsilon} (\bar{\rho} - \bar{n}) \bar{\phi} - \frac{1}{2\varepsilon} |\nabla \bar{\phi}|^2 \right) \bigg|_{t=0} \, dx \\
= - \int_{t}^{t+\kappa} \int_{\Omega} \frac{1}{\varepsilon} |\bar{v}|^2 + \frac{1}{\varepsilon} n |\bar{v}|^2 \, dx \, dr + \int_{t}^{t+\kappa} \int_{\Omega} \bar{v} \cdot \bar{e}_1 + \bar{\bar{v}} \cdot \bar{e}_2 \, dx \, dr.
\]

Regarding the difference $(\rho - \bar{\rho}, p - \bar{\rho}, n - \bar{n}, u - \bar{u}, n v - \bar{n} \bar{v})$ between a weak solution of (2.4) and a strong solution of (1.1), one has the following:
\[
- \int_{0}^{T} \int_{\Omega} \varphi_1 (\rho - \bar{\rho}) \, dx \, dt - \int_{0}^{T} \int_{\Omega} \nabla \varphi \cdot (p - \bar{\rho}) \, dx \, dt - \int_{0}^{T} \int_{\Omega} \varphi (\rho - \bar{\rho}) \bigg|_{t=0} \, dx = 0,
\]
\[
- \int_{0}^{T} \int_{\Omega} \bar{\varphi}_1 (p - \bar{\rho}) \, dx \, dt - \int_{0}^{T} \int_{\Omega} \nabla \bar{\varphi} : (p - \bar{\rho}) \, dx \, dt - \int_{0}^{T} \int_{\Omega} \bar{\varphi} (p - \bar{\rho}) \bigg|_{t=0} \, dx = 0,
\]
\[
- \int_{0}^{T} \int_{\Omega} \bar{\bar{\varphi}}_1 (n v - \bar{n} \bar{v}) \, dx \, dt - \int_{0}^{T} \int_{\Omega} \nabla \bar{\bar{\varphi}} : (n v - \bar{n} \bar{v}) \, dx \, dt - \int_{0}^{T} \int_{\Omega} \bar{\bar{\varphi}} (n v - \bar{n} \bar{v}) \bigg|_{t=0} \, dx = 0,
\]
\[
- \int_{0}^{T} \int_{\Omega} \bar{\bar{\bar{\varphi}}}_1 (n v - \bar{n} \bar{v}) \, dx \, dt - \int_{0}^{T} \int_{\Omega} \nabla \bar{\bar{\bar{\varphi}}} : (n v - \bar{n} \bar{v}) \, dx \, dt - \int_{0}^{T} \int_{\Omega} \bar{\bar{\bar{\varphi}}} (n v - \bar{n} \bar{v}) \bigg|_{t=0} \, dx = 0,
\]
for all Lipschitz test functions $\varphi, \psi, \bar{\varphi}, \bar{\psi}, \bar{\bar{\varphi}}, \bar{\bar{\psi}} : [0, T] \times \Omega \to \mathbb{R}$, $\bar{\varphi}, \bar{\bar{\varphi}}, \bar{\bar{\bar{\varphi}}} : [0, T] \times \Omega \to \mathbb{R}^d$ compactly supported in time and with $\bar{\varphi}, \bar{\psi}$ satisfying the no-flux boundary condition.

Set
\[
(\varphi, \bar{\varphi}, \bar{\psi}, \bar{\bar{\psi}}) = \left( \theta \left( \frac{1}{\varepsilon} h_1'(\bar{\rho}) + \frac{1}{2} \bar{\phi} - \frac{1}{2} |\bar{u}|^2 \right), \theta \bar{\rho}, \theta \left( \frac{1}{\varepsilon} h_2'(\bar{n}) - \frac{1}{\varepsilon} \bar{\phi} - \frac{1}{2} |\bar{v}|^2 \right), \theta \bar{v} \right),
\]
where $\theta$ is as before. Using this on the previous expressions, and letting $\kappa \to 0^+$, it gives:

\[
\int_{\Omega} \left( \frac{1}{\varepsilon} h_1' (\bar{\rho}) + \frac{1}{\varepsilon} \phi - \frac{1}{2} |\bar{u}|^2 \right) (\rho - \bar{\rho}) |^{t \to 0}_{t=0} dx \\
- \int_{0}^{t} \int_{\Omega} \partial_{\tau} \left( \frac{1}{\varepsilon} h_1' (\bar{\rho}) + \frac{1}{\varepsilon} \phi - \frac{1}{2} |\bar{u}|^2 \right) (\rho - \bar{\rho}) dx d\tau \\
- \int_{0}^{t} \int_{\Omega} \nabla \left( \frac{1}{\varepsilon} h_1' (\bar{\rho}) + \frac{1}{\varepsilon} \phi - \frac{1}{2} |\bar{u}|^2 \right) \cdot (\rho u - \bar{\rho} \bar{u}) dx d\tau \\
= 0,
\] (5.7)

\[
\int_{\Omega} \bar{u} \cdot (\rho u - \bar{\rho} \bar{u}) |^{t \to 0}_{t=0} dx - \int_{0}^{t} \int_{\Omega} (\partial_{\tau} \bar{u}) \cdot (\rho u - \bar{\rho} \bar{u}) dx d\tau \\
- \int_{0}^{t} \int_{\Omega} \nabla \bar{u} : (\rho u \otimes u - \bar{\rho} \bar{u} \otimes \bar{u}) dx d\tau - \frac{1}{\varepsilon} \int_{0}^{t} \int_{\Omega} (\nabla \cdot \bar{u}) (p_1 (\rho) - p_1 (\bar{\rho})) dx d\tau \\
= - \frac{1}{\varepsilon} \int_{0}^{t} \int_{\Omega} \bar{u} \cdot (\rho \nabla \phi - \bar{\rho} \nabla \bar{\phi}) dx d\tau - \frac{1}{\varepsilon} \int_{0}^{t} \int_{\Omega} \bar{u} \cdot (\rho u - \bar{\rho} \bar{u}) dx d\tau \\
- \int_{0}^{t} \int_{\Omega} \bar{u} \cdot \bar{e}_1 dx d\tau,
\] (5.8)

\[
\int_{\Omega} \left( \frac{1}{\varepsilon} h_2' (\bar{n}) - \frac{1}{\varepsilon} \phi - \frac{1}{2} |\bar{v}|^2 \right) (n - \bar{n}) |^{t \to 0}_{t=0} dx \\
- \int_{0}^{t} \int_{\Omega} \partial_{\tau} \left( \frac{1}{\varepsilon} h_2' (\bar{n}) - \frac{1}{\varepsilon} \phi - \frac{1}{2} |\bar{v}|^2 \right) (n - \bar{n}) dx d\tau \\
- \int_{0}^{t} \int_{\Omega} \nabla \left( \frac{1}{\varepsilon} h_2' (\bar{n}) - \frac{1}{\varepsilon} \phi - \frac{1}{2} |\bar{v}|^2 \right) \cdot (n v - \bar{n} \bar{v}) dx d\tau \\
= 0,
\] (5.9)

\[
\int_{\Omega} \bar{v} \cdot (n v - \bar{n} \bar{v}) |^{t \to 0}_{t=0} dx - \int_{0}^{t} \int_{\Omega} (\partial_{\tau} \bar{v}) \cdot (n v - \bar{n} \bar{v}) dx d\tau \\
- \int_{0}^{t} \int_{\Omega} \nabla \bar{v} : (n v \otimes v - \bar{n} \bar{v} \otimes \bar{v}) dx d\tau - \frac{1}{\varepsilon} \int_{0}^{t} \int_{\Omega} (\nabla \cdot \bar{v}) (p_2 (n) - p_2 (\bar{n})) dx d\tau \\
= \frac{1}{\varepsilon} \int_{0}^{t} \int_{\Omega} \bar{v} \cdot (\rho \nabla \phi - \bar{\rho} \nabla \bar{\phi}) dx d\tau - \frac{1}{\varepsilon} \int_{0}^{t} \int_{\Omega} \bar{v} \cdot (n v - \bar{n} \bar{v}) dx d\tau \\
- \int_{0}^{t} \int_{\Omega} \bar{v} \cdot \bar{e}_2 dx d\tau.
\] (5.10)
From the computation \((5.5) - \(5.6) - (5.7) + (5.8) + (5.9) + (5.10)\) it follows that

\[
\begin{aligned}
\int_{\Omega} \left( \frac{1}{2} \rho (u - \bar{u})^2 + \frac{1}{2} n (v - \bar{v})^2 + \frac{1}{\varepsilon} \epsilon_h (\rho \bar{\phi}) + \frac{1}{2} n h_2 (n \bar{n}) + \frac{1}{2\varepsilon} |\nabla (\phi - \bar{\phi})|^2 \right) \bigg|_{t=0}^{t} \, dx \\
\leq - \frac{1}{\varepsilon} \int_{0}^{t} \int_{\Omega} \rho |u|^2 - \bar{\rho} |\bar{u}|^2 - \bar{u} \cdot (\rho u - \bar{\rho} \bar{u}) \, dx \, d\tau \\
- \frac{1}{\varepsilon} \int_{0}^{t} \int_{\Omega} \rho |v|^2 - \bar{\rho} |\bar{v}|^2 - \bar{v} \cdot (\rho v - \bar{\rho} \bar{v}) \, dx \, d\tau \\
- \int_{0}^{t} \int_{\Omega} \partial_t \left( \frac{1}{\varepsilon} h_1 (\rho) + \frac{1}{\varepsilon} \epsilon \phi - \frac{1}{2} |\bar{u}|^2 \right) (\rho - \bar{\rho}) \, dx \, d\tau \\
- \int_{0}^{t} \int_{\Omega} \partial_t \left( \frac{1}{\varepsilon} h_2 (n) - \frac{1}{\varepsilon} \epsilon \phi - \frac{1}{2} |\bar{v}|^2 \right) (n - \bar{n}) \, dx \, d\tau \\
- \int_{0}^{t} \int_{\Omega} (\partial_t \bar{u}) \cdot (\rho u - \bar{\rho} \bar{u}) \, dx \, d\tau - \int_{0}^{t} \int_{\Omega} (\partial_t \bar{v}) \cdot (nv - \bar{n} \bar{v}) \, dx \, d\tau \\
- \int_{0}^{t} \int_{\Omega} \nabla (\frac{1}{\varepsilon} h_1 (\rho) + \frac{1}{\varepsilon} \epsilon \phi - \frac{1}{2} |\bar{u}|^2) \cdot (\rho u - \bar{\rho} \bar{u}) \, dx \, d\tau \\
- \int_{0}^{t} \int_{\Omega} \nabla (\frac{1}{\varepsilon} h_2 (n) - \frac{1}{\varepsilon} \epsilon \phi - \frac{1}{2} |\bar{v}|^2) \cdot (n - \bar{n}) \, dx \, d\tau \\
- \frac{1}{\varepsilon} \int_{0}^{t} \int_{\Omega} (\rho \nabla \phi - \bar{\rho} \nabla \bar{\phi}) \, dx \, d\tau - \frac{1}{\varepsilon} \int_{0}^{t} \int_{\Omega} \nabla \cdot (n \nabla \phi - \bar{n} \nabla \bar{\phi}) \, dx \, d\tau.
\end{aligned}
\]

(5.11)

If \((\rho, \rho \bar{u}, n, \bar{n} \bar{v})\) together with \(\bar{\phi} = N * (\rho - \bar{n})\) is a strong solution of (1.1) bounded and away from vacuum, then \(\bar{u}\) and \(\bar{v}\) satisfy the following system

\[
\begin{aligned}
\bar{u}_t + \bar{u} \cdot \nabla \bar{u} &= - \frac{1}{\varepsilon} \nabla (h_1 (\rho) + \epsilon \phi) - \frac{1}{\varepsilon} \bar{u} + \frac{\epsilon_1}{\rho}, \\
\bar{v}_t + \bar{v} \cdot \nabla \bar{v} &= - \frac{1}{\varepsilon} \nabla (h_2 (n) - \epsilon \phi) - \frac{1}{\varepsilon} \bar{v} + \frac{\epsilon_2}{n}.
\end{aligned}
\]

(5.12)

Multiplying the first and second equations above by \(\rho (u - \bar{u})\) and \(n (v - \bar{v})\), respectively, yields:

\[
\begin{aligned}
&\frac{\partial}{\partial t} \left( - \frac{1}{2} |\bar{u}|^2 \right) (\rho - \bar{\rho}) + \bar{u}_t \cdot (\rho u - \bar{\rho} \bar{u}) + \nabla \left( - \frac{1}{2} |\bar{u}|^2 \right) \cdot (\rho u - \bar{\rho} \bar{u}) \\
&+ \nabla \bar{u} : (\rho u \otimes u - \bar{\rho} \bar{u} \otimes \bar{u}) \\
&= - \frac{1}{\varepsilon} \rho \nabla h_1 (\rho) \cdot (u - \bar{u}) - \frac{1}{\varepsilon} \rho \nabla \phi \cdot (u - \bar{u}) - \frac{1}{\varepsilon} \rho \bar{u} \cdot (u - \bar{u}) \\
&+ \rho \nabla \bar{u} : (u - \bar{u}) \otimes (u - \bar{u}) + \frac{\rho}{\epsilon} \epsilon_1 \cdot (u - \bar{u}), \\
\end{aligned}
\]

(5.13)

\[
\begin{aligned}
&\frac{\partial}{\partial t} \left( - \frac{1}{2} |\bar{v}|^2 \right) (n - \bar{n}) + \bar{v}_t \cdot (nv - \bar{n} \bar{v}) + \nabla \left( - \frac{1}{2} |\bar{v}|^2 \right) \cdot (nv - \bar{n} \bar{v}) \\
&+ \nabla \bar{v} : (nv \otimes v - \bar{n} \bar{v} \otimes \bar{v}) \\
&= - \frac{1}{\varepsilon} n \nabla h_2 (n) \cdot (v - \bar{v}) + \frac{1}{\varepsilon} n \nabla \phi \cdot (v - \bar{v}) - \frac{1}{\varepsilon} n \bar{v} \cdot (v - \bar{v}) \\
&+ n \nabla \bar{v} : (v - \bar{v}) \otimes (v - \bar{v}) + \frac{n}{\epsilon} \epsilon_2 \cdot (v - \bar{v}).
\end{aligned}
\]
Additionally, given that $\bar{\rho} + \nabla \cdot (\bar{\rho}u) = 0$ and $\bar{n} + \nabla \cdot (\bar{n}\bar{v}) = 0$ a.e. in $[0, T] \times \Omega$, one derives

$$-\frac{1}{\varepsilon} \int_0^t \int_{\Omega} \left( \rho |u|^2 - \bar{\rho} |\bar{u}|^2 - \bar{u} \cdot (\rho u - \bar{\rho} \bar{u}) - \rho \bar{u} \cdot (u - \bar{u}) \right) dx dt = -\frac{1}{\varepsilon} \int_0^t \int_{\Omega} \rho |u - \bar{u}|^2 dx dt, \quad (5.15)$$

$$-\frac{1}{\varepsilon} \int_0^t \int_{\Omega} \left( n |v|^2 - \bar{n} |\bar{v}|^2 - \bar{v} \cdot (n v - \bar{n} \bar{v}) - n \bar{v} \cdot (v - \bar{v}) \right) dx dt = -\frac{1}{\varepsilon} \int_0^t \int_{\Omega} n |v - \bar{v}|^2 dx dt. \quad (5.16)$$

A set of simple calculations provide

$$-\frac{1}{\varepsilon} \int_0^t \int_{\Omega} \rho |u|^2 - \bar{\rho} |\bar{u}|^2 - \bar{u} \cdot (\rho u - \bar{\rho} \bar{u}) - \rho \bar{u} \cdot (u - \bar{u}) dx dt = -\frac{1}{\varepsilon} \int_0^t \int_{\Omega} \rho |u - \bar{u}|^2 dx dt, \quad (5.15)$$

$$-\frac{1}{\varepsilon} \int_0^t \int_{\Omega} n |v|^2 - \bar{n} |\bar{v}|^2 - \bar{v} \cdot (n v - \bar{n} \bar{v}) - n \bar{v} \cdot (v - \bar{v}) dx dt = -\frac{1}{\varepsilon} \int_0^t \int_{\Omega} n |v - \bar{v}|^2 dx dt. \quad (5.16)$$
Moreover, identity (5.19) and the no-flux boundary condition imply that
\[
\begin{aligned}
&\frac{1}{\varepsilon} \int_0^t \int_{\Omega} \left( -\partial_t \phi \right) (\rho - \tilde{\rho}) - \nabla \phi \cdot (\rho u - \tilde{\rho} \tilde{u}) \, dx \, dt \\
&+ \frac{1}{\varepsilon} \int_0^t \int_{\Omega} \left( -\varepsilon \right) \left( \phi \nabla \phi \right) \cdot \nabla \phi \cdot \left( u - \tilde{u} \right) \, dx \, dt \\
&- \frac{1}{\varepsilon} \int_0^t \int_{\Omega} \left( -\partial_t \phi \right) \left( n - \tilde{n} \right) - \nabla \phi \cdot \left( n \bar{v} - \tilde{n} \tilde{v} \right) \, dx \, dt \\
&\leq \frac{1}{\varepsilon} \int_0^t \int_{\Omega} \left( \rho \left| u - \tilde{u} \right|^2 + \frac{1}{2} n \left| v - \tilde{v} \right|^2 \right) \, dx \, dt \\
&\leq - \frac{1}{\varepsilon} \int_0^t \int_{\Omega} \rho \left| u - \tilde{u} \right|^2 + n \left| v - \tilde{v} \right|^2 \, dx \\
&- \int_0^t \int_{\Omega} \nabla \bar{u} \cdot \rho (u - \tilde{u}) \otimes (u - \tilde{u}) + \nabla \bar{v} : n (v - \tilde{v}) \otimes (v - \tilde{v}) \, dx \\
&- \frac{1}{\varepsilon} \int_0^t \int_{\Omega} \left( \phi \nabla \phi \right) \cdot \left( \rho \tilde{u} + (\rho \tilde{v}) \rho \right) \, dx \\
&+ \frac{1}{\varepsilon} \int_0^t \int_{\Omega} \left( \rho - \tilde{\rho} \right) \left( n - \tilde{n} \right) \bar{v} \cdot \nabla (\phi - \tilde{\phi}) \, dx \\
&\leq \int_0^t \int_{\Omega} \rho \left| u - \tilde{u} \right|^2 + \frac{1}{n} \varepsilon \left| v - \tilde{v} \right|^2 \, dx \\
&\leq \int_0^t \int_{\Omega} \left( \rho \left| u - \tilde{u} \right|^2 + \frac{1}{n} \varepsilon \left| v - \tilde{v} \right|^2 \right) \, dx \\
&\leq C \int_0^t \Theta(\tau) d\tau.
\end{aligned}
\]

which completes the proof. \qed

Remark 5.8. Inequality (5.4) can be made an equality if instead of choosing a dissipative weak solution of (2.4) one opts to use a conservative one.

5.2. Bounds in terms of the relative energy function.

Lemma 5.9. Under the same conditions as Proposition 5.4 one has
\[
J_1(t) \leq C \int_0^t \Theta(\tau) d\tau \quad \forall t \in [0, T]
\]
for some positive constant C.

Proof. Note that for \( t \in [0, T] \),
\[
J_1(t) = - \int_0^t \int_{\Omega} \nabla \bar{u} : \rho (u - \tilde{u}) \otimes (u - \tilde{u}) + \nabla \bar{v} : n (v - \tilde{v}) \otimes (v - \tilde{v}) \, dx \\
\leq (||\nabla \bar{u}||_\infty + ||\nabla \bar{v}||_\infty) \int_0^t \int_{\Omega} \rho \left| u - \tilde{u} \right|^2 + n \left| v - \tilde{v} \right|^2 \, dx \\
\leq C \int_0^t \Theta(\tau) d\tau.
\]
Lemma 5.10. Under the same conditions as Proposition 5.7, one has
\[ \mathcal{J}_2(t) \leq C \int_0^t \Theta(\tau)d\tau \quad \forall t \in [0,T] \]
for some positive constant \( C \).

Proof. From condition (4.2) it follows that
\[
p_2(r|\tilde{r}) = (r - \tilde{r})^2 \int_0^1 \int_\Omega (\nabla \cdot \tilde{u})p_1(\rho|\tilde{\rho}) + (\nabla \cdot \tilde{v})p_2(\bar{n}|\tilde{\bar{n}})dx d\tau
\]
\[
\leq (r - \tilde{r})^2 \int_0^1 \int_\Omega \left( \tilde{k}_1 h'_{\bar{\bar{\bar{n}}} \bar{\bar{n}}} \right) d\tau \]
\[
\leq \tilde{k}_1 h'_\bar{n}(r|\tilde{r}).
\]
Thus, for \( t \in [0,T] \),
\[
\mathcal{J}_2(t) = -\frac{1}{\varepsilon} \int_0^t \int_\Omega (\nabla \cdot \tilde{u})p_1(\rho|\tilde{\rho}) + (\nabla \cdot \tilde{v})p_2(\bar{n}|\tilde{\bar{n}})dx d\tau
\]
\[
\leq (|\nabla \cdot \tilde{u}|_{\infty} + |\nabla \cdot \tilde{v}|_{\infty})(\tilde{k}_1 + \tilde{k}_2) \int_0^1 \int_\Omega \frac{1}{\varepsilon} h_1(\rho|\tilde{\rho}) + \frac{1}{\varepsilon} h_2(\bar{n}|\tilde{\bar{n}})dx d\tau
\]
\[
\leq C \int_0^t \Theta(\tau)d\tau.
\]
\( \square \)

Lemma 5.11. Let \( h \in C^2([0, +\infty[) \cap C([0, +\infty[) \) be such that \( h''(r) > 0 \forall r > 0 \), and \( \lim_{r \to +\infty} \frac{h(r)}{r^\gamma} = \frac{k}{\gamma - 1} \) for some \( k > 0 \) and \( \gamma > 1 \). Assume that \( \tilde{r} \in [\delta, M] \), where \( \delta > 0 \) and \( M < +\infty \). Then, there exists \( R \geq M + 1 \) and positive constants \( C_1, C_2 \) such that
\[
h(r|\tilde{r}) \geq \begin{cases} C_1 |r - \tilde{r}|^2, & \text{if } (r, \tilde{r}) \in [0, R] \times [\delta, M] \\ C_2 |r - \tilde{r}|^\gamma, & \text{if } (r, \tilde{r}) \in ]R, +\infty[ \times [\delta, M]. \end{cases}
\]
Furthermore, if \( \gamma \geq 2 \), then \( h(r|\tilde{r}) \geq C|r - \tilde{r}|^2 \) for every \( (r, \tilde{r}) \in [0, +\infty[ \times [\delta, M], \) where \( C = \min\{C_1, C_2\} \).

Proof. Let \( \tilde{A} = \max_{[\delta, M]} h \), \( B = \max\{\max_{[\delta, M]} h', 1\} \), \( \tilde{C} = \min_{r \in [\delta, M]} \tilde{r} h'(\tilde{r}) \), and \( A = \max\{\tilde{A} - \tilde{C}, 1\} \). Then, \( A, B > 0 \) and
\[
h(r|\tilde{r}) \geq h(r) - A - Br.
\]
Consider \( f : [0, +\infty[ \to \mathbb{R} \) given by
\[
f(r) = \frac{h(r) - A - Br}{r^\gamma},
\]
and note that
\[
\forall \xi > 0 \quad \exists R_\xi > 0 \quad \forall r > R_\xi \quad \left| f(r) - \frac{k}{\gamma - 1} \right| \leq \xi.
\]
Choosing \( \xi = \frac{k}{2 \gamma - 1} \) and taking \( R = \max\{R_\xi, M + 1\} \) gives
\[
f(r) \geq \frac{1}{2 \gamma - 1} \quad \forall r > R.
\]
Noticing that for \( r > R \) one has \( |r - \tilde{r}| \leq r^\gamma \), setting \( C_2 := \frac{k}{2 \gamma - 1} \) yields
\[
h(r|\tilde{r}) \geq C_2 |r - \tilde{r}|^\gamma \quad \forall r > R.
\]
Now, assume that \( r \in [0, R], \) and consider \( g : [0, R] \times [\delta, M] \to \mathbb{R} \) given by
\[
g(r, \tilde{r}) = \frac{h(r|\tilde{r})}{|r - \tilde{r}|^2}.
\]
Note that
\[ g(r, \tilde{r}) = \int_0^1 \int_0^T h''(sr + (1 - s)\tilde{r}) dsd\tau \]
hence \( g > 0 \), and since
\[ g(r, \tilde{r}) \to \frac{h''(\tilde{r})}{2} > 0 \quad \text{(as } r \to \tilde{r}) \]
g is continuous at \( r = \tilde{r} \); therefore continuous in all its domain. Consequently, from the continuity of \( g \) and the compactness of \([0, R] \times [\delta, M] \), \( g \geq \min_{[0, R] \times [\delta, M]} g =: C_1 > 0 \), i.e.,
\[ h(r|\tilde{r}| \geq C_1 |r - \tilde{r}|^2 \quad \forall (r, \tilde{r}) \in [0, R] \times [\delta, M]. \]

Finally, if \( \gamma \geq 2 \) and \( r \in ]R, +\infty[ \), then \( |r - \tilde{r}| > 1 \), so \(|r - \tilde{r}|^\gamma \geq |r - \tilde{r}|^2\), whence
\[ h(r|\tilde{r}| \geq C|r - \tilde{r}|^2 \quad \forall (r, \tilde{r}) \in [0, +\infty[ \times [\delta, M], \]
where \( C = \min\{C_1, C_2\} \).

**Lemma 5.12.** Under the same conditions as Theorem 5.3 one has
\[ J_3(t) \leq C \int_0^t \Theta(\tau) d\tau \quad \forall t \in [0, T] \]
for some positive constant \( C \).

**Proof.** Let \( \gamma = \min\{\gamma_1, \gamma_2\} \). The proof is divided into two cases: \( \gamma \geq 2 \) and \( \gamma \in [2 - \frac{1}{q}, 2] \).

**Case \( \gamma \geq 2 \):** Using inequality \( ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2 \) and Lemma 5.11 one derives
\[
J_3(t) = \frac{1}{\varepsilon} \int_0^t \int_\Omega \left( (\rho - \bar{\rho}) \bar{u} - (n - \bar{n}) \bar{v} \right) \cdot \nabla (\phi - \bar{\phi}) dx \, d\tau \\
\leq \left( ||\bar{u}||_{\infty} + ||\bar{v}||_{\infty} \right) \int_0^t \int_\Omega \frac{1}{\varepsilon} (|\rho - \bar{\rho}| + |n - \bar{n}|) ||\nabla (\phi - \bar{\phi})|| dx \, d\tau \\
\leq C \int_0^t \int_\Omega \frac{1}{\varepsilon} \left( |\rho - \bar{\rho}|^2 + |n - \bar{n}|^2 + ||\nabla (\phi - \bar{\phi})||^2 \right) dx \, d\tau \\
\leq C \int_0^t \int_\Omega \frac{1}{\varepsilon} h_1(\rho |\bar{\rho}| + \frac{1}{\varepsilon} h_2(n |\bar{n}| + \frac{1}{\varepsilon} ||\nabla (\phi - \bar{\phi})||^2 dx \, d\tau \\
\leq C \int_0^t \int_\Omega \Theta(\tau) dx \, d\tau,
\]
for every \( t \in [0, T] \).

**Case \( \gamma \in [2 - \frac{1}{q}, 2] \):**
Fix \( t \in [0, T] \). Let \( q = \frac{2}{3 - \gamma} \), \( q' = \frac{q}{q - 1} \), and \( p = \frac{2d}{d(\gamma - 1) + 2} \), so that \( q' = \frac{dp}{d - p} \). Since \( \gamma \in [2 - \frac{1}{q}, 2] \), then \( 1 < p = q < \gamma < 2 \).

Set \( J(t) = \int_\Omega ((\rho - \bar{\rho}) \bar{u} - (n - \bar{n}) \bar{v} \right) \cdot \nabla (\phi - \bar{\phi}) dx \), and note that
\[
J(t) \leq (||\bar{u}||_{\infty} + ||\bar{v}||_{\infty}) \left( (\rho - \bar{\rho}) + (n - \bar{n}) \right) ||\nabla (\phi - \bar{\phi})|| dx
\leq C \left( \int_\Omega ((\rho - \bar{\rho}) + (n - \bar{n}))^q dx \right)^{\frac{1}{q}} \left( \int_\Omega ||\nabla (\phi - \bar{\phi})||^p dx \right)^{\frac{1}{p}}
\leq C \left( \int_\Omega ((\rho - \bar{\rho}) + (n - \bar{n}))^q dx \right)^{\frac{1}{q}} \left( \int_\Omega ||\rho - \rho + \bar{n}||^p dx \right)^{\frac{1}{p}},
\]
where we used Proposition 5.5.

Let \( r > 0 \) be such that \( \frac{1}{r} = \frac{1}{p} - \frac{1}{q} \). Then
\[
||\rho - \rho + \bar{n}||_{L^r(\Omega)} \leq ||\rho - \rho + \bar{n}||_{L^q(\Omega)}.
\]
whence

\[ J(t) \leq C \left( \int_{\Omega} (|\rho - \bar{\rho}| + |n - \bar{n}|)^q dx \right)^{\frac{1}{q}} \left( \int_{\Omega} (|\rho - \bar{\rho} - n + \bar{n}|)^q dx \right)^{\frac{1}{q}}. \]

\[ \leq C \left( \int_{\Omega} (|\rho - \bar{\rho}| + |n - \bar{n}|)^q dx \right)^{\frac{1}{q}} + C \left( \int_{\Omega} |n - \bar{n}|^q dx \right)^{\frac{1}{q}}. \]

Let \( B(t) = \{ x \in \Omega \mid 0 \leq \rho \leq R \} \) and \( U(t) = \{ x \in \Omega \mid \rho > R \} \), where \( R > M_1 + 1 \) is as in Lemma 5.11. Note that \( \frac{1}{\gamma} = \frac{q}{2} + (1 - \theta) \) with \( 2\theta = \gamma \). Therefore, the inclusion \( L^2(\Omega) \subseteq L^q(\Omega) \), interpolation, conservation of mass, and Lemma 5.11 imply that

\[ \left( \int_{\Omega} |\rho - \bar{\rho}|^q dx \right)^{\frac{1}{q}} \leq C \left( \int_{B(t)} |\rho - \bar{\rho}|^q dx \right)^{\frac{1}{q}} + C \left( \int_{U(t)} |\rho - \bar{\rho}|^q dx \right)^{\frac{1}{q}} \]

\[ \leq C \int_{B(t)} |\rho - \bar{\rho}|^2 dx + C \int_{U(t)} |\rho - \bar{\rho}|^\gamma dx \]

\[ \leq C \int_{\Omega} h_1(\rho |\bar{\rho}|) dx + C \int_{U(t)} |\rho - \bar{\rho}|^\gamma dx. \]

If \( \gamma = \gamma_1 \), then

\[ \int_{U(t)} |\rho - \bar{\rho}|^\gamma dx \leq C \int_{\Omega} h_1(\rho |\bar{\rho}|) dx \]

immediately follows from Lemma 5.11.

If \( \gamma = \gamma_2 \), then \( \gamma \leq \gamma_1 \), and since \( |\rho - \bar{\rho}| > 1 \) in \( U(t) \), again by Lemma 5.11 one obtains

\[ \int_{U(t)} |\rho - \bar{\rho}|^\gamma dx \leq \int_{U(t)} |\rho - \bar{\rho}|^{\gamma_1} dx \leq C \int_{\Omega} h_1(\rho |\bar{\rho}|) dx. \]

Consequently,

\[ \left( \int_{\Omega} |\rho - \bar{\rho}|^q dx \right)^{\frac{1}{q}} \leq C \int_{\Omega} h_1(\rho |\bar{\rho}|) dx. \]

Analogously,

\[ \left( \int_{\Omega} |n - \bar{n}|^q dx \right)^{\frac{1}{q}} \leq C \int_{\Omega} h_2(n |\bar{n}|) dx. \]

Thus

\[ J(t) \leq C \int_{\Omega} h_1(\rho |\bar{\rho}|) + h_2(n |\bar{n}|) dx \leq C \varepsilon \Theta(t), \]

wherefrom

\[ J_3(t) \leq C \int_0^t \Theta(\tau) d\tau. \]

\[ \square \]

**Lemma 5.13.** Under the same conditions as Proposition 5.7 one has

\[ J_4(t) \leq \frac{1}{2C} \int_0^t \int_{\Omega} |\rho u - \bar{u}|^2 + |n v - \bar{v}|^2 dx d\tau + C \varepsilon t \quad \forall t \in [0, T], \]

for some positive constant \( C \).
Proof. The inequality \( ab \leq \frac{a^2}{2} + \frac{b^2}{2} \), the boundedness of \( \dot{c}_1 \) and \( \dot{c}_2 \), and the conservation of mass of the densities imply, for \( t \in [0, T] \), that
\[
J_4(t) = -\int_0^t \int_\Omega \rho \dot{c}_1 \cdot (u - \bar{u}) + \frac{n}{\bar{n}} \dot{c}_2 \cdot (v - \bar{v}) \, dx \, dt
\]
\[
\leq -\int_0^t \int_\Omega \rho \frac{\dot{c}_1}{\rho} |u - \bar{u}| + \frac{n}{\bar{n}} \dot{c}_2 |v - \bar{v}| \, dx \, dt
\]
\[
\leq \frac{1}{2\varepsilon} \int_0^t \int_\Omega \rho |u - \bar{u}|^2 + n|v - \bar{v}|^2 \, dx \, dt + \frac{\varepsilon}{2} \int_0^t \int_\Omega \rho \left( \frac{\dot{c}_1}{\rho} \right)^2 + \frac{n}{\bar{n}} \dot{c}_2^2 \, dx \, dt
\]
\[
\leq \frac{1}{2\varepsilon} \int_0^t \int_\Omega \rho |u - \bar{u}|^2 + n|v - \bar{v}|^2 \, dx \, dt + C\varepsilon t
\]
for some positive constant \( C \).

5.3. Proof of main theorem. The following result is a consequence of Proposition 5.7 and the previous Lemmas.

Proposition 5.14. Let \((\rho, \rho u, n, n v)\) together with \( \phi = N (\rho - n) \) be a dissipative weak solution of (2.4), and let \((\bar{\rho}, \bar{\rho} u, \bar{n}, \bar{n} v)\) with \( \bar{\phi} = N (\bar{\rho} - \bar{n}) \) be a conservative, bounded and away from vacuum, strong solution of (1.1). Then, the relative energy function \( \Theta \) between these two solutions satisfies
\[
\Theta(t) + \frac{1}{2\varepsilon} \int_0^t \int_\Omega \rho |u - \bar{u}|^2 + n|v - \bar{v}|^2 \, dx \, dt \leq \Theta(0) + C \int_0^t \Theta(\tau) \, d\tau + C\varepsilon t \quad \forall t \in [0, T] \quad (5.20)
\]
for some positive constant \( C \).

With this, one can demonstrate the main result. The proof of the theorem is fairly classical and follows easily from the previous proposition using a Gronwall inequality type argument; nevertheless, this is presented for illustrative purposes.

Proof of Theorem (5.3). Let \( f : [0, T] \rightarrow [0, +\infty] \) be given by
\[
f(t) = \int_0^t \Theta(\tau) \, d\tau,
\]
and let \( g : [0, T] \rightarrow [0, +\infty] \) be given by
\[
g(t) = \frac{1}{2\varepsilon} \int_0^t \int_\Omega \rho |u - \bar{u}|^2 + n|v - \bar{v}|^2 \, dx \, dt.
\]
Then, expression (5.20) becomes
\[
f'(t) + g(t) \leq \Theta(0) + C f(t) + C\varepsilon t,
\]
from which it can be deduced that
\[
f(t) \leq \int_0^t e^{-C(\tau-t)}(\Theta(0) - g(\tau) + C\varepsilon t) \, d\tau.
\]

Thus
\[
f'(t) + g(t) \leq \Theta(0) + C \int_0^t e^{-C(\tau-t)}(\Theta(0) - g(\tau) + C\varepsilon t) \, d\tau + C\varepsilon t,
\]
and since \(-g(t) - C \int_0^t e^{-C(\tau-t)}g(\tau) \, d\tau \leq 0\) it follows that
\[
f'(t) \leq \Theta(0) + C \int_0^t e^{-C(\tau-t)}(\Theta(0)) \, d\tau + C^2 \varepsilon \int_0^t e^{-C(\tau-t)} \, d\tau + C\varepsilon t
\]
\[
= \Theta(0) + \Theta(0)(e^{Ct} - 1) + \varepsilon(-Ct + e^{Ct} - 1) + C\varepsilon t
\]
whence
\[
\Theta(t) \leq e^{CT} (\Theta(0) + \varepsilon).
\]
Multiplying both sides of the above expression by \( \varepsilon \) gives the desired conclusion. \( \square \)
Remark 5.15. The positive constant $C$ in the main result depends on $d$, $\Omega$, $\gamma_1$, $\gamma_2$, $k_1$, $k_2$, $\delta_1$, $\delta_2$, $M_1$, $M_2$, $||\bar{u}||_\infty$, $||\bar{v}||_\infty$, $||\nabla \bar{u}||_\infty$, $||\nabla \bar{v}||_\infty$, $||\nabla \cdot \bar{u}||_\infty$, $||\bar{e}_1||_\infty$ and $||\bar{e}_2||_\infty$.

Appendix A.

This section provides a formal derivation of system (2.3) from the bipolar Boltzmann-Poisson model with friction terms adapted from [15].

Consider the system

\[
\begin{align*}
\frac{\partial f}{\partial t} + w \cdot \nabla f - \nabla \phi \cdot \nabla w f &= \nabla w \cdot \left( \frac{1}{\tau} wf \right) \\
\frac{\partial g}{\partial t} + w \cdot \nabla g + \nabla \phi \cdot \nabla w g &= \nabla w \cdot \left( \frac{1}{\tau} wg \right) \\
-\Delta \phi &= \int fdw - \int gdw
\end{align*}
\]  

(A.1)

in the phase space-time $[0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d$, where $w \in \mathbb{R}^d$ is called the pseudo-wave vector, and $\nabla_w$ is the gradient operator with respect to $w$. One can visualize the functions $f = f(t, x, w)$ and $g = g(t, x, w)$ as distribution functions of sets of electrons and holes, respectively. It is assumed that $f$ and $g$ satisfy

\[
\lim_{|w| \to +\infty} f(t, x, w) = \lim_{|w| \to +\infty} g(t, x, w) = 0. 
\]  

(A.2)

Moreover, $\phi = \phi(t, x)$ is the electrostatic potential of the system and $1/\tau > 0$ is the effective collision frequency.

Set

\[
\rho = \int f dw, \quad u = \frac{1}{\rho} \int w f dw,
\]

and

\[
n = \int g dw, \quad v = \frac{1}{n} \int w g dw.
\]

Integrating the first equation of (A.1) with respect to the variable $w$ gives

\[
\frac{\partial}{\partial t} \int f dw + \nabla \cdot \int w f dw = \nabla \phi \cdot \int \nabla_w f dw + \int \nabla_w \cdot \left( \frac{1}{\tau} w f \right) dw,
\]

whence, using (A.2),

\[
\rho_t + \nabla \cdot (\rho u) = 0. 
\]  

(A.3)

Similarly one obtains

\[
n_t + \nabla \cdot (nv) = 0. 
\]  

(A.4)

Next, the momentum equations are deduced. Multiplying the first equation of (A.1) by $w_j$ and integrating with respect to $w$ yields

\[
\frac{\partial}{\partial t} \int w_j f dw + \sum_{i=1}^d \frac{\partial}{\partial x_i} \left( \int w_j w_i f dw \right) = \nabla \phi \cdot \int w_j \nabla_w f dw + \int w_j \nabla_w \cdot \left( \frac{1}{\tau} w f \right) dw. 
\]  

(A.5)

Setting

\[
z_{ij} = \frac{1}{\rho} \int w_j w_i f dw,
\]

one can rewrite (A.5) as

\[
(\rho u_j)_t + \sum_{i=1}^d \frac{\partial}{\partial x_i} (\rho z_{ij}) = -\rho \frac{\partial \phi}{\partial x_j} - \frac{1}{\tau} \rho u_j. 
\]  

(A.6)

Defining $\sigma_{ij} = z_{ij} - u_i u_j$, after subtracting equation

\[
u_j \left( \rho_t + \sum_{i=1}^d \frac{\partial}{\partial x_i} (\rho u_i) \right) = 0
\]

...
from equation (A.6), one obtains
\[
\rho \frac{\partial u_j}{\partial t} + \rho u_j \cdot \nabla u_j + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (\rho \sigma_{ij}) = -\rho \frac{\partial \phi}{\partial x_j} - \frac{1}{\tau} \rho u_j, \tag{A.7}
\]
Assume that \(\rho \sigma_{ij}\) is a stress tensor that represents a pressure. Precisely, \(\rho \sigma_{ij} = \delta_{ij} p_1(\rho)\) where \(p_1\) is a function that symbolizes the pressure. Thus,
\[
\sum_{i=1}^{d} \frac{\partial}{\partial x_i} (\rho \sigma_{ij}) = \frac{\partial}{\partial x_j} (p_1(\rho))
\]
and the vector form of (A.7) can be written as
\[
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla p_1(\rho) = -\rho \nabla \phi - \frac{1}{\tau} \rho u. \tag{A.8}
\]
Analogously,
\[
(nv)_t + \nabla \cdot (nv \otimes v) + \nabla p_2(n) = n \nabla \phi - \frac{1}{\tau} nv, \tag{A.9}
\]
where again \(p_2\) represents a pressure.

The third equation of (A.1) together with equations (A.3), (A.4), (A.8) and (A.9) give system (2.3).

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