Contact Linearizability of Scalar Ordinary Differential Equations of Arbitrary Order

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Abstract. We consider the problem of the exact linearization of scalar nonlinear ordinary differential equations by contact transformations. This contribution is extending the previous work by Lyakhov, Gerdt, and Michels addressing linearizability by means of point transformations. We have restricted ourselves to quasi-linear equations solved for the highest derivative with a rational dependence on the occurring variables. As in the case of point transformations, our algorithm is based on simple operations on Lie algebras such as computing the derived algebra and the dimension of the symmetry algebra. The linearization test is an efficient algorithmic procedure while finding the linearization transformation requires the computation of at least one solution of the corresponding system of the Bluman-Kumei equation.

Key words: contact symmetry, differential Thomas decomposition, exact linearization, nonlinear ordinary differential equations, symbolic computation.

1 Introduction

Symmetry analysis as a systematic method was discovered by Sophus Lie more than 150 years ago and then rediscovered by Ovsyannikov and his colleagues in the 20 century. Sophus Lie himself considered groups of point and contact transformations to integrate systems of partial differential equations. His key idea was to obtain first infinitesimal generators of one-parameter symmetry subgroups and then to construct the full symmetry group. The study of symmetries of differential equations allows one to gain insights into the structure of the problem they describe. Existence of symmetry group allows to decrease the order of differential equation, reduce from partial to ordinary differential equations, construct particular exact solutions or sometimes even general solutions.

In contrast to Lie, recently Lyakhov, Gerdt, and Michels discovered [7,8] that such kind of properties like exact linearizations could be detected completely algorithmically without solving the determining system. It relies strongly on differential algebra and symbolic manipulations with differential equations. Their work in this field was inspired by Ibragimov and Meleshko [2,18]. We want to exclude obtaining of explicit expressions (as they are really large and not really meaningful) and instead of it obtain an algorithm to test the exact linearization property.

This paper is organized as follows. In Sect. 2, we briefly describe the mathematical objects we deal with and the former result on linearization by point transformation [7]. In Sect. 3, we introduce contact symmetry and prove the main theorem of our paper. The implementation of algorithms and its application is illustrated in Sect. 4 by several examples. Finally, we provide a conclusion in Sect. 5.

2 Point Symmetry

We consider an arbitrary order ordinary differential equation (ODE) of the form

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \quad y^{(k)} := \frac{d^k y}{dx^k} \tag{1}$$

with a rational right-hand side which is solved with respect to the highest derivative.

If an ODE of the form (1) admits transformation into a linear nth order homogeneous equation

$$u^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\, u^{(k)}(t) = 0\,, \quad u^{(k)} := \frac{d^k u}{dt^k}$$

by means of functions[1]

$$u = \phi(x, y)\,, \quad t = \psi(x, y)\,, \tag{2}$$

then we say that (1) admits exact linearization or is linearizable by point transformation.

The invertibility of (2) is provided by the local differential condition

$$J := \phi_x \psi_y - \phi_y \psi_x \neq 0\,.$$

Our way to check the linearizability of Eq. (1) is based on Lie's approach [4]. We study the symmetry properties of (1) under the *infinitesimal* transformation

$$\tilde{x} = x + \varepsilon\, \xi(x, y) + \mathcal{O}(\varepsilon^2)\,, \quad \tilde{y} = y + \varepsilon\, \eta(x, y) + \mathcal{O}(\varepsilon^2)\,. \tag{3}$$

The *invariance condition* for (1) under the transformation (3) is given by the equality

$$\mathcal{X}(y^{(n)} - f(x, y, ..., y^{(n-1)}))|_{y^{(n)}=f\left(x,y,\ldots,y^{(n-1)}\right)} = 0\,,$$

[1] Please note, that we assumed analytical homeomorphisms.

where the *symmetry operator* reads

$$\mathcal{X} := \xi\,\partial_x + \sum_{k=0}^{n} \eta^{(k)}\partial_{y^{(k)}}\,,\ \eta^{(k)} := D_x\eta^{(k-1)} - y^{(k)}D_x\,\xi\,,$$

$\eta^{(0)} := \eta$, and $D_x := \partial_x + \sum_{k\geq 0} y^{(k+1)}\partial_{y^{(k)}}$ is the total derivative operator with respect to x.

This set of symmetry operators forms a basis of the *Lie symmetry algebra*

$$[\mathcal{X}_i, \mathcal{X}_j] = \sum_{k=1}^{m} C^k_{i,j}\mathcal{X}_k\,, \quad 1 \leq i < j \leq m\,. \tag{4}$$

Let L denote the Lie symmetry algebra and $m = \dim(L)$. An important role for the analysis plays the *derived algebra* $L' \subset L$ which by definition is a subalgebra that consists of all commutators of pairs of elements in L.

Lie showed ([5], Ch. 12, p. 298, "Satz" 3) that the Lie point symmetry algebra of an n-order ODE has a dimension m satisfying

$$n = 1,\ m = \infty;\quad n = 2,\ m \leq 8;\quad n \geq 3,\ m \leq n + 4\,.$$

Interrelations between n and m ensure the linearizability of the differential equation (1) by point transformation. Here we present the two theorems that describe such interrelations and form the basis of our exact linearization test.

Theorem 1. ([9], Thm. 1) *A necessary and sufficient condition for the linearization of* (1) *with $n \geq 3$ via a point transformation is the existence of an n-dimensional abelian subalgebra of* (4)*.*

The proof is based on the following lemma which is important for further discussions of contact symmetries.

Lemma 1. *Let us suppose three linear independent operators $X_i = f_i(x, y)\frac{\partial}{\partial x} + g_i(x, y)\frac{\partial}{\partial y}, i = 1, 2, 3$ commuting each other. Then, there exists an appropriate point transformation which maps X_i onto $\bar{X}_i = \bar{f}_i(t)\frac{\partial}{\partial u}$.*

Proof. By rectification the theorem for the non-singular point, we can also find a point transformation to map one operator (e.g. X_1) to shift $\frac{\partial}{\partial u}$. Then,

$$\bar{X}_i = \bar{g}_i(t)\frac{\partial}{\partial t} + \bar{f}_i(t)\frac{\partial}{\partial u}, i = 2, 3\,.$$

Since $[X_2, X_3] = 0$, direct calculations show that

$$\bar{g}_2(t)\bar{f}_3'(t) - \bar{g}_3(t)\bar{f}_2'(t) = 0,\ \bar{g}_2(t)\bar{g}_3'(t) - \bar{g}_3(t)\bar{g}_2'(t) = 0\,.$$

One of two possibilities may hold: either $f_2'g_3' - f_3'g_2' = 0$ or both $g_2 = g_3 = 0$. The first case is not possible without $g_2 = g_3 = 0$. Otherwise it contradicts the linear independence of the operators.

Corollary. Lemma 1 could be easily generalized to an arbitrary number of operators more than 3.

The main result for point symmetry is based on the following theorem, which forms a basis for the algebraic test linearization.

Theorem 2. ([7]) *Eq.* (1) *with* $n \geq 2$ *is linearizable by a point transformation if and only if one of the following conditions is fulfilled:*

1. $n = 2$, $m = 8$;
2. $n \geq 3$, $m = n + 4$;
3. $n \geq 3$, $m \in \{n+1, n+2\}$ *and the derived algebra of* (4) *is abelian of dimension* n.

This theorem shows that the verification of linearizability requires only checking of dimensions and also finding the derived algebra, which is simple from a computational point of view and abstract theory of finite-dimensional Lie algebra.

3 Contact Symmetry

The most general smooth invertible transformation of variables for an ODE is a contact transformation. It is a local diffeomorphism of the jet bundle $J^1\pi$ into itself defined in standard coordinates by the formulas

$$X = X(x, y, p)\,,\ Y = Y(x, y, p)\,,\ P = Y_p/X_p\,.$$

Here, we use the standard notation $y' = p$ and $Y' = P$. Also, as a contact transformation, $X_p(Y_x + pY_y) = Y_p(X_x + pX_y)$ is required. One should be aware that the third formula is only valid for nontrivial contact transformations (i.e. $X_p \neq 0$). We use $P = (Y_x + pY_y)/(X_x + pX_y)$ instead for point transformations. The notion of contact transformation was introduced in Lie's doctoral dissertation first. One-parameter group of a contact symmetry is a flow of contact transformation[2]

$$\bar{x} = \bar{x}(x, y, p, a)\,,\ \bar{y} = \bar{y}(x, y, p, a)\,,\ \bar{p} = \bar{p}(x, y, p, a)\,.$$

It defines in a similar way as for point transformation the infinitesimal operator

$$\mathcal{X} := \xi(x, y, p)\,\partial_x + \eta(x, y, p)\,\partial_y + \eta^{[1]}(x, y, p)\partial_p\,,$$

which is an appropriate derivation of the one-parameter group at $a = 0$. But contact transformation comes with additional constraints on the components of the generator $\mathcal{X}$:

$$\eta_p - p\xi_p = 0,\ \eta^{[1]} = \eta_x + p(\eta_y - \xi_x) - p^2\eta_y\,.$$

[2] It defines an identity transformation if $a = 0$.

Definition 1. *A Lie algebra of contact vector fields is reducible if there exists a local contact transformation around a non-singular point which maps these vector fields onto the first prolongations of point vector fields. Otherwise, it is irreducible.*

The beautiful property of contact symmetries is that except the three specific Lie algebras on plane, all other ones are reducible. Moreover, the following theorem clarifies it.

Theorem 3. ([13], page 134; [17]) *Finite-dimensional irreducible Lie algebras of contact transformations in the complex plane (x, y), where x and y are in general complex numbers, belong to one of the following three classes modulo local contact transformations: L_6, L_7, and L_{10}, which dimensions are* 6, 7 *and* 10.

The direct computation of the derived algebra shows that

$$[L_6, L_6] = L_6,\ [L_7, L_7] = L_6,\ [L_{10},\, L_{10}] = L_{10}\,.$$

This leads to an interesting observation. Any abelian contact Lie algebra possesses the zero derived algebra by definition, thus it is reducible. Then, transforming to basis when it is merely a prolongation of the point Lie algebra, it is possible to apply Lemma 1 and the corollary, which immediately leads to the following theorem.

Theorem 4. *A necessary and sufficient condition for the linearizability of* (1) *with $n \geq 3$ via a contact transformation is the existence of an n-dimensional abelian subalgebra in the contact symmetry algebra.*

Proof. By reducibility, this subalgebra can be taken as a point Lie algebra. In the light of Lemma 1, its n generators under new variables (t, u) imply that the n-dimensional symmetry group acts on some solution $u_0(t)$ by rule

$$u(t) = u_0 + \sum_{i=1}^{n} C_i f_i(t)\,,$$

where C_i are group parameters. Without loss of generality, we can assume that $u_0 = 0$, otherwise we apply one more transformation of variables by the rule $U = u - u_0, T = t$. Every solution $u(t)$ of an nth order scalar ODE is defined completely by its initial conditions $u(t_0), u'(t_0), \ldots, u^{(n-1)}(t_0)$ at some point t_0. Varying C_i, it is possible to get any set of initial conditions from the list, because it has a non-zero Wronskian determinant. Finally, if $v(t)$ and $w(t)$ are solutions, then $u(t) = v(t) + w(t)$ is also a solution. This concludes the proof.

Important results for contact symmetries of linear ODE were obtained by Svirshchevskii and Yumaguzhin described by the two following theorems.

Theorem 5. ([19]) *A linear ODE of kth order with $k \geq 4$ does not possess nontrivial (non-point) contact symmetries.*

Theorem 6. ([20]) *The dimension of the contact symmetry algebra of any third order linear ODE is equal to one of the following numbers: 4, 5, and 10. Moreover,*

1. *any third order linear ODE with a 10-dimensional contact symmetry algebra is equivalent to the trivial equation* $y''' = 0$ *,*
2. *any third order linear ODE with 5-dimensional contact symmetry algebra is equivalent to one of the equations of the form* $y''' = Ky' + y, K = \mathsf{const}$*.*

Theorem 6, together with Theorem 2 and Theorem 5, shows that except for $y''' = 0$, all other linear cases do not posses nontrivial contact symmetries. Thus, the dimensions are

1. $n = 3\,,\; m = 10\,,$
2. $n \geq 4\,,\; m = n + 4\,,$
3. $n \geq 3\,,\; m \in \{n+1, n+2\}\,.$

The first two items characterize the case of a maximal symmetry dimension. A remarkable point is that it implies linearizability like it was shown by Lie.

Theorem 7. ([4,5]) *Let Equation* (1) *be an* n*th order scalar ODE.*

1. *If* $n = 3$*, then Eq.* (1) *admits at most a ten-dimensional symmetry group of contact transformations. Moreover, the symmetry group is ten-dimensional if and only if Eq.* (1) *is equivalent (up to a local contact transformation) to* $u^{(3)}(t) = 0$*.*
2. *If* $n \geq 4$*, then Eq.* (1) *admits at most an* $(n + 4)$*-parameter symmetry group of contact transformations. In addition, the symmetry group is* $(n+4)$*-dimensional if and only if Eq.* (1) *is equivalent (up to a local contact transformation) to* $u^{(n)}(t) = 0$*.*

According to Theorem 5 and Theorem 6, a linear equation which is not trivializable should correspond to the third case (i.e., $n \geq 3$ and $m \in \{n+1, n+2\}$). Thus, its derived algebra is abelian and has the dimension n. Vice versa, following Theorem 4, this is also a sufficient condition.

Theorem 8. *Eq.* (1) *with* $n \geq 3$ *is linearizable by a contact transformation if and only if one of the following conditions is fulfilled:*

1. $n = 3\,,\; m = 10$ *or* $n \geq 4\,,\; m = n + 4$ *(maximal dimension),*
2. $n \geq 3\,,\; m = n + 1$ *or* $n + 2$ *and the derived algebra of contact symmetry is abelian of dimension* n*.*

4 Algorithm and Examples

The main result of this paper is Theorem 8, which serves as the foundation for the algebraic test for exact linearizability by contact transformations. Once a system of determining equations for symmetry generators is given, we can complete them to involution [15,16], and then by computing the differential Hilbert polynomial find the dimension of the symmetry algebra [3]. In this regard, we prefer to use the differential Thomas decomposition [1] which already showed its convenience and superiority for such kind of tasks. There are a lot of packages for finding particular solutions of determining systems based on some heuristics. It is of great relevance in geometry and physics. Unfortunately, there is no algorithm to solve completely any determining system of symmetries for scalar ODEs, because the existence of this algorithm would immediately imply the ability to solve any linear ODE.

A beautiful property of the finite-dimensional Lie symmetry algebra is that the structure constants could be found exactly without any heuristics. We follow here an approach proposed by Reid [14]. Any N linear independent solutions of the determining system span an N-dimensional Lie symmetry algebra. They could be expressed via a power series solution for the determining system in involution. Substitution of these expressions into (4) leads to an infinite system of linear equations for a finite number of structure constants $C^k_{i,j}$. This system is always equivalent to some truncated version, which leads to an efficient procedure for obtaining structure constants given by Algorithm 1.[3].

Algorithm 1 ***Contact Linearization Test***

Input: q, a nonlinear differential equation of form (1) .
Output: `True`, if q is linearizable, and `False`, otherwise .

n := ***DifferentialOrder*** (q);
DS := ***DeterminingSystem*** (q);
IDS := ***InvolutiveDeterminingSystem*** (DS);
m := dim(***LieSymmetryAlgebra*** (IDS));
if $(n = 3 \wedge m = 10) \vee (n > 3 \wedge m = n + 4)$ **then**
    return `True`;
else if $n \geq 3 \wedge (m = n + 1 \vee m = n + 2)$ **then**
    SC := ***StructureConstants*** (IDS);
    DA := ***DerivedAlgebra*** (SC);
    if DA is abelian and $\dim(DA) = n$ **then**
        return `True`;
    end if
end if
return `False`;

We illustrate our theory by presenting the following two examples.

Example 1. ([18]) We start with a classical example. Equation

$$y''' = \frac{3y''^2}{2y'}$$

[3] A modern package for calculations of determining systems is discussed in the literature [6].

describes the family of hyperbolas. As it was shown by Lie, it could be transformed into the simplest equation $y''' = 0$ using a Legendre transformation. Computation of the Hilbert dimension polynomial for the determining system for contact symmetries shows that the dimension is 10, which by Theorem 8 immediately implies trivialization. Let us compute the same for the point transformation. The dimension of symmetry group then is 6, which corresponds to the case in which the linearization is not possible. Thus, it is essentially a contact transformation.

Example 2. ([18]) Let us consider

$$-16y'^2y''y^{(4)} + 48y'^2y'''^2 + y'y''^5x - 48y'y''^2y''' - y''^5y + 12y''^4 = 0 \tag{5}$$

This example also passes our linearization test with dimension $m = 6$. It requires also the computation of the derived algebra which is 4-dimensional and abelian.

In order to recover the linearizing mapping, we will use an analog of the method described in ([10], [11], [12]). We will briefly discuss it here. Our analysis is based on the Bluman-Kumei equations:

$$\begin{aligned}
&\xi(x,y,p)\,\tfrac{\partial X}{\partial x} + \eta(x,y,p)\,\tfrac{\partial X}{\partial y} + \eta^{[1]}(x,y,p)\tfrac{\partial X}{\partial p} = \bar{\xi}(x,y,p)\,,\\
&\xi(x,y,p)\,\tfrac{\partial Y}{\partial x} + \eta(x,y,p)\,\tfrac{\partial Y}{\partial y} + \eta^{[1]}(x,y,p)\tfrac{\partial Y}{\partial p} = \bar{\eta}(x,y,p)\,,\\
&\xi(x,y,p)\,\tfrac{\partial P}{\partial x} + \eta(x,y,p)\,\tfrac{\partial P}{\partial y} + \eta^{[1]}(x,y,p)\tfrac{\partial P}{\partial p} = \bar{\eta}^{[1]}(x,y,p)\,,
\end{aligned}$$

where $\bar{\xi}(x,y,p), \bar{\eta}(x,y,p), \bar{\eta}^{[1]}(x,y,p)$ are generators mapped by contact transformation (X, Y, P) being expressed in old coordinates. So, in Example 1, we

1. solve the system of contact symmetry of $y^{(3)} = 0$ (i.e. $\bar{\xi}, \bar{\eta}, \bar{\eta}^{[1]}$ are expressed in polynomials of X, Y, P);
2. solve ξ, η from Bluman-Kumei equations (as a linear system);
3. substitute the solution from the previous step into the system of ξ, η (i.e., the determining system of contact symmetry algebra of $y''' = \frac{3y''^2}{2y'}$).

Completion to involution forms giant nonlinear determining system[4]. It has one particular solution $X = \sqrt{p}, Y = xp - y$ which coincides with [18].

Example 2 corresponds to the constant coefficient case, and requires different consideration. Since every linear constant coefficient ODE admits only trivial contact symmetries (point symmetries), by Lemma 1, all elements in derived algebra (DA) have the form $f(X)\frac{\partial}{\partial Y} + f'(X)\frac{\partial}{\partial P}$ (being expressed in new coordinates), thus we

1. set $\bar{\xi} = 0$, $\frac{\bar{\eta}_x}{X_x} = \frac{\bar{\eta}_y}{X_y} = \frac{\bar{\eta}_p}{X_p}$;
2. reduce the equations in step 1 with the system of DA;
3. vanish all the coefficients of parametric derivatives in step 2.

[4] We will not write it down for brevity.

Completion to involution gives the system of differential equations and inequations

$$\{X_x = 0, X_y = 0, Y_{xx} = 0, Y_{xy} = 0, Y_x + pY_y = 0\}, \{X_p \neq 0, Y_p \neq 0, Y_x \neq 0\}$$

which forms basis of linearizing mappings of (5).

5 Conclusison

We constructed a new algebraic linearization test for scalar ordinary differential equations by contact transformation. It indicates that this approach could be applied to large classes of differential equations including systems of ordinary and partial differential equations. Of course, in the case of partial differential equations the main problem is the infinite-dimensionality of their symmetry algebras. Thus, devising a general scheme for the detection of infinite-dimensional abelian symmetry subalgebras is still a challenge.

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