An Algebraic Approach to Exact Linearization

Yang Liu[1], Dmitry Lyakhov[1] and Dominik L. Michels[1]

Computational Sciences Group, KAUST, Bldg 1, Thuwal 23955-6900, KSA

Abstract. An algebraic framework is proposed for determining if a scalar ordinary differential equation could be mapped into a linear one. We call this property linearizability or exact linearization. The problem is split into two parts: (I) obtain a certificate which ensures the existence of a linearizing mapping and (II) construct the determining system for it. The complexity bottleneck of this technique is the completion to a standard basis (Riquier basis of symmetry infinitesimals for problem (I)) and the Thomas Decomposition of the nonlinear determining system of the linearizing mapping for problem (II). In both cases, the numbers of dependent and independent variables remain fixed which defines an upper complexity bound of the corresponding algorithms. The main difference is the transformation of the infinitesimal symmetry generators by means of Bluman-Kumei equations. We prove a theorem on the construction of finite-dimensional linearizing systems, which is important in theory and application. Moreover, we illustrate our approach with several examples which admit exact linearization by point or contact transformations.

Key words: Point Symmetry, Contact Symmetry, Lie Algebra of Vector Fields, Exact Linearization, Nonlinear Ordinary Differential Equations, Symbolic Computation.

1 Introduction

Our paper devotes to the old problem of determination of exact linearizable differential systems in natural sciences ([10]). Fundamental mathematical method is to study object which remains unchanged after operations of a certain type. Starting from initially given differential equation we define equivalent class of all possible equations which could be obtained by action of some invertible point transformation. Which mathematical object does remain these mappings? Naturally, it is structure of its Lie algebra of symmetry. Linearizability (or exact linearization) is one of the properties of this Lie algebra. The theory described here is an extension of obtained in older ones ([15], [18]) based on algebraic manipulations with Lie Algebras and Bluman-Kumei equations of equivalent mappings.

From the point of view of differential algebra ([3]) the linearizability could be rewritten as transformation between two differential polynomials, one of it is linear. This approach was implemented as **LinearizationTestII**([15], [18]) and it suffers from the complexity of decomposition algorithms in differential algebra

([4]). Indeed, upper bounds grow double exponentially by number of dependent and independent variables.

Here we introduce theory based on Bluman-Kumei equations which prepares linearizing mapping system without including unknown coefficients $a_k(t)$ to the set of dependent variables and using block-elimination ranking. It allows to fix number of dependent and independent variables in decomposition algorithms to $(2,2)$ for point and $(2,3)$ for contact transformation respectively. Moreover, in the case of constant coefficients $a_k(t) = const$, we can construct characteristic polynomial algorithmically.

2 Mathematical Background

In this paper we consider ODEs of the form

$$y^{(n)} + f(x, y, y', \ldots, y^{(n-1)}) = 0\,, \quad y^{(k)} := \frac{d^k y}{dx^k}\,, \tag{1}$$

where f is a rational function of its arguments.

Remark 1. We can also assume that f is a rational function of its arguments $y', \ldots, y^{(n-1)}$ with coefficients $\alpha(x, y)$ which satisfy some given polynomially nonlinear differential equations.

Starting from ODE of the form (1), we want to check the existence of an invertible (point) transformation[1]

$$t = \phi(x, y)\,, \quad u = \psi(x, y) \tag{2}$$

which maps (1) into a linear n-th order homogeneous equation

$$u^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\, u^{(k)}(t) = 0\,, \quad u^{(k)} := \frac{d^k u}{dt^k}\,. \tag{3}$$

The local invertibility of (2) is provided by the inequation

$$J := \phi_x \psi_y - \phi_y \psi_x \neq 0\,, \tag{4}$$

where we use the standard notion for derivatives

$$\phi_x = \frac{\partial \phi}{\partial x}, \quad \psi_y = \frac{\partial \psi}{\partial y}.$$

Definition 1. *Assume that $(t, u) = T_\alpha(x, y) = (f(x, y, \alpha), g(x, y, \alpha))$ is a point transformation with a parameter α such that*

- ⋄ *$T_\alpha(x, y)$ maps Eq. (1) to itself for any chosen of α;*
- ⋄ *$T_{\alpha+\beta} = T_\alpha \circ T_\beta \, \forall \alpha, \beta$;*

[1] More precisely, we consider local linearizability, hereafter we assume that all functions are analytical homeomorphism in the vicinity of the linearization point.

◇ $T_0(x,y) = (x,y)$.

The above conditions make $\{T_\alpha\}$ *a group via composition, we call it a one parameter group of Eq. (1).*

Moreover, set

$$(\xi(x,y), \eta(x,y)) = (f_\alpha(x,y,\alpha), g_\alpha(x,y,\alpha))|_{\alpha=0}.$$

We call (ξ,η) *a point symmetry generator of Eq. (1).*

Since (ϕ,ψ) are always assumed to be analytical, (ξ,η) are their first derivatives in Taylor expansions respect to α (see Example 1).

A first-order ODE $y' = f(x,y)$ is always linearizable outside of singular points because of rectification theorem, but its linearization procedure is equivalent to integration of the equation ([1], Ch. 2, Thm. 1). For $n = 2$ any homogeneous linear equation

$$y''(x) + a(x)\,y'(x) + b(x)\,y(x) = 0$$

can be transformed by a substitution

$$t = \mu(x), \quad \mu'(x) \neq 0, \quad u = \sigma(x)\,y, \quad \sigma(x) \neq 0$$

to the simplest second order equation ([8], Thm. 3.3.1)

$$u''(t) = 0\,. \tag{5}$$

Our method for determining exact linearization property of Eq. (1) is to perform symmetry analysis without explicit integration of determining system. Following Sophus Lie, we define symmetry as transformation which maps Eq. (1) into itself locally. The set of all these transformations defines Lie group for operation of mapping composition.

Example 1. As a simple example let consider the simplest second-order ODE

$$y''(x) = 0\,. \tag{6}$$

By a substitution of the arbitrary point transformation

$$[t = f(x,y), \quad u = g(x,y), \quad f_x g_y - f_y g_x \neq 0]\,,$$

it is transformed into

$$u''(t) + A_3 \cdot (u')^3 + A_2 \cdot (u')^2 + A_1 \cdot u' + A_0 = 0\,.$$

The symmetry property implies

$$A_3 = -\frac{\partial^2 f}{\partial y^2}\frac{\partial g}{\partial y} + \frac{\partial^2 g}{\partial y^2}\frac{\partial f}{\partial y} = 0\,,$$

$$A_2 = -\frac{\partial^2 f}{\partial y^2}\frac{\partial g}{\partial x} + \frac{\partial^2 g}{\partial x \partial y}\frac{\partial f}{\partial y} + 2\frac{\partial^2 g}{\partial y \partial y}\frac{\partial f}{\partial x} - 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial g}{\partial y} = 0\,,$$

$$A_1 = -\frac{\partial^2 f}{\partial x^2}\frac{\partial g}{\partial y} + \frac{\partial^2 g}{\partial x^2}\frac{\partial f}{\partial y} + 2\frac{\partial^2 g}{\partial x \partial y}\frac{\partial f}{\partial x} - 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial g}{\partial x} = 0\,,$$

$$A_0 = -\frac{\partial^2 f}{\partial x^2}\frac{\partial g}{\partial x} + \frac{\partial^2 g}{\partial x^2}\frac{\partial f}{\partial x} = 0\,.$$

First and last equations are simple to solve in terms of f and g, but substitution of solution into A_2, A_1 leads to bulky expressions which is hard to analyze by elementary methods.

Instead of finding the Lie group of symmetry itself we will follow the approach of Sophus Lie. It consists of seeking all one-parameter subgroups of symmetry transformations (which connected with unit element - identity). Expansion by group parameter a in Taylor series

$$f(x,y) := x + a \cdot \xi(x,y) + o(a)\,,$$
$$g(x,y) := y + a \cdot \eta(x,y) + o(a)\,,$$

and substitution to symmetry condition leads to the linear PDE system when $a \to 0$

$$\frac{\partial^2 \eta}{\partial x^2} = 0\,,\; -\frac{\partial^2 \xi}{\partial x^2} + 2\frac{\partial^2 \eta}{\partial x \partial y} = 0\,,\; \frac{\partial^2 \xi}{\partial y^2} = 0\,,\; -\frac{\partial^2 \eta}{\partial y^2} + 2\frac{\partial^2 \xi}{\partial x \partial y} = 0\,. \tag{7}$$

It can be easily solved by hand:

$$\xi(x,y) = (C_7 \cdot x + C_8) \cdot y + C_5 \cdot x^2 + C_3 \cdot x + C_4\,,$$
$$\eta(x,y) = (C_5 \cdot y + C_6) \cdot x + C_7 \cdot y^2 + C_1 \cdot y + C_2\,.$$

Obviously it corresponds to the 8-parameter Lie group of symmetries, which contains all linear mappings of plane, projections, and shifts. Geometrically, the symmetry group represents all possible transformations of the set of straight lines on a plane into itself that is given by solutions of Eq. (6).

Bluman-Kumei (B-K) equations is one of the most useful results if one apply point transformations on symmetry algebras. Let assume a point transformation given in Eq. (2) maps $y^{(n)} + f(x, y, y', \ldots, y^{(n-1)}) = 0$ to $u^{(n)} +$

$g(t, u, u', \ldots, u^{(n-1)}) = 0$ and these two ODEs admit symmetry algebras $\mathcal{L}$ and $\mathcal{L}'$ respectively. Then (ϕ, ψ) induces an isomorphism between Lie algebras

$$\mathcal{L} \to \mathcal{L}'\,; \quad (\xi, \eta) \mapsto (\phi_x\xi + \phi_y\eta, \psi_x\xi + \psi_y\eta).^2$$

It can be easily verified (we recommend to use software) that the above linear mapping preserves Lie brackets. Then we only need to demonstrate that $(\phi_x\xi + \phi_y\eta, \psi_x\xi + \psi_y\eta)$ is actually a generator for the new equation. By Definition 1, (ξ, η) must be associated with some point transformation $T_\alpha(x, y)$. Also we denote (ϕ, ψ) by $H(x, y)$. Thus $H(T_\alpha(H^{-1}(t, u)))$ satisfies the conditions in Definition 1, i.e.

$$(\phi_x\xi + \phi_y\eta, \psi_x\xi + \psi_y\eta) = \frac{\partial}{\partial\alpha} H(T_\alpha(H^{-1}(t, u)))|_{\alpha=0}$$

is a point symmetry generator of $u^{(n)} + g(t, u, u', \ldots, u^{(n-1)}) = 0$.

3 Determining System for Point Transformation

Vector field (ξ, η) naturally defines first-order differential operator

$$\mathcal{X} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$$

on plane.

Proposition 1. *Differential operator $\mathcal{X}$ is a symmetry generator of Eq. (1) if and only if its nth prolongation, say $\mathcal{X}$, satisfies:*

$$\mathcal{X}(y^{(n)} + f(x, y, ..., y^{(n-1)}))\,|_{y^{(n)}+f\left(x,y,\ldots,y^{(n-1)}\right)=0} = 0, \tag{8}$$

where the symmetry operator *reads*

$$\mathcal{X} := \xi\,\partial_x + \sum_{k=0}^{n} \eta^{[k]}\partial_{y^{(k)}}\,,\ \ \eta^{[k]} := D_x\eta^{[k-1]} - y^{(k)} D_x\,\xi, \tag{9}$$

and $\eta^{[0]} := \eta$ and $D_x := \partial_x + \sum_{k\geq 0} y^{(k+1)}\partial_{y^{(k)}}$ is the total derivative operator with respect to x. Here we use the commonly accepted standard notion in symmetry analysis ([8], Sect. 1.4), where all differential functions depend only on a finite number of differential variables $y^{(k)}$, and

$$\partial_x = \frac{\partial}{\partial x}\,, \quad \partial_{y^{(k)}} = \frac{\partial}{\partial y^{(k)}}\,.$$

Elementary proof could be found in [8].

[2] The new generator in $\mathcal{L}'$ is expressed in old coordinates.

The invariance condition (8) means that its left-hand side vanishes when Eq. (1) holds. Then the application of (9) to the left-hand side of Eq. ((1)). and the substitution of

$$y^{(n)} = -f(x, y, ..., y^{(n-1)})$$

in the resulting expression leads to the equality $g = 0$ with the polynomial dependence of g on the derivatives $y', \ldots, y^{(n-1)}$. Since, by the transformation (2), the functions ξ and η do not depend on these derivatives, the equality $g = 0$ holds if and only if all coefficients in $y', \ldots, y^{(n-1)}$ are equal to zero. This leads to an overdetermined system of linear PDEs in ξ and η called *determining system.* Its solution yields a set of symmetry operators $\mathbf{X}$.

Example 2. Let illustrate the proposition on differential equation from Example 1. Assume that it admits some operator $\mathcal{X}$ as a symmetry generator, so the 2nd prolongation of $\mathcal{X}$ would be

$$\mathcal{X} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta^{[1]} \frac{\partial}{\partial y'} + \eta^{[2]} \frac{\partial}{\partial y''}.$$

By Proposition 1 and ignoring the terms of y'' in $\eta^{[2]}$ we have

$$\mathcal{X}(y'')\mid_{y''=0} = \eta^{[2]}|_{y''=0} = -\xi_{yy} {y'}^3 + (\eta_{yy} - 2\xi_{xy}) {y'}^2 + (2\eta_{xy} - \xi_{xx}) y' + \eta_{xx} = 0.$$

Since y' as a initial data could be arbitrarily valued, all the coefficients must be zero. It leads to four partial differential equations, which coincides with 7.

Corollary 1. *A linear ODE in the form of Eq. (1) admits the operator $\mathcal{X} := \partial_x$ if and only if its coefficients are constants, and, it admits the operator $\mathcal{Y} := y\partial_y$ if and only if it is homogeneous.*

Proof. This is just a simple application of Proposition 1, only need to notice that any prolongation of $\mathcal{X}$ is still itself and the nth prolongation of $\mathcal{Y}$ is $\sum_{k=0}^{n} y^{(k)} \partial_{y^{(k)}}$. ∎

Theorem 1. *Point symmetry algebra of scalar ordinary differential equation* (1) *is always finite-dimensional if* $n > 1$.

Proof. Assuming right-hand side of second-order ODE is analytical function, let expand it in Taylor series

$$f(x, y, y') = \sum_{i=0}^{\infty} f_i(x, y)(y')^i.$$

Formal procedure based on Proposition 1 leads to infinite number of equations, where the most important are first four equations:

$$\frac{\partial^2 \eta}{\partial x^2} = F_1(\xi_x, \xi_y, \eta_x, \eta_y, \xi, \eta),$$

$$-\frac{\partial^2\xi}{\partial x^2} + 2\frac{\partial^2\eta}{\partial x\partial y} = F_2(\xi_x, \xi_y, \eta_x, \eta_y, \xi, \eta),$$

$$\frac{\partial^2\xi}{\partial y^2} = F_3(\xi_x, \xi_y, \eta_x, \eta_y, \xi, \eta),$$

$$-\frac{\partial^2\eta}{\partial y^2} + 2\frac{\partial^2\xi}{\partial x\partial y} = F_4(\xi_x, \xi_y, \eta_x, \eta_y, \xi, \eta).$$

Differentiation by x and y, linear combination leads to all possible third-order derivatives of ξ and η:

$$(\xi,\eta)_{xxx} = (\bar{F}_1, \bar{G}_1),\ (\xi,\eta)_{xxy} = (\bar{F}_2, \bar{G}_2),\ (\xi,\eta)_{xyy} = (\bar{F}_3, \bar{G}_3),\ (\xi,\eta)_{yyy} = (\bar{F}_4, \bar{G}_4).$$

It is finite-dimensional system with maximal dimension of 12. Adding four constraints implies that dimension of solution space is not higher than 8 (it is possible to achieve as it is clearly seen from Example 1). For higher orders the proof is similar, we refer to [25].

Remarkable property of solutions of determining system is that they form Lie algebra. If it is finite-dimensional, then expansion yields

$$[\mathcal{X}_i, \mathcal{X}_j] = \sum_{k=1}^{m} C_{i,j}^k \mathcal{X}_k\,, \quad 1 \le i < j \le m\,, \tag{10}$$

where $C_{i,j}^k$ is structure constant tensor.

Definition 2. *If an ODE of the form (1) admits mapping (2) into a linear nth order homogeneous equation (3), then we say that (1) admits exact linearization or is linearizable by point transformation (P-linearizable for shortness).*

3.1 Lie Algebras of Vector Fields

Lie Algebras of Vector Fields (LAVF[3]), an algorithmic technique designed by Ian G. Lisle [14], is the core of our exact linearization theory. It provides various algebraic information by direct operations on L without integrating the system purely by means of linear algebra. We will give an introduction on two functions which are frequently used in our algorithms.

If $L = \{e_1, ..., e_s\}$ is a linear homogeneous PDE system, with a fixed ranking of derivatives which satisfies two following conditions:

$$\partial^j u \le \partial^i(\partial^j u)\ \forall i, j$$

and

$$\partial^{j_1} u \le \partial^{j_2} u \Rightarrow \partial^i(\partial^{j_1})u \le \partial^i(\partial^{j_2} u)\ \forall i, j_1, j_2,$$

[3] Since its 2020 version, the tool is available as a built-in package of the standard Maple™ distribution.

the highest-ranking term (leading derivative) in PDE with respect to $\leq$ is called $Leader$. We may assume that each e_i is monic with its leading derivative on the left-hand side and others on the right-hand side. Reduction (denoted as "$\rightarrow$") of PDE f modulo system L means repeatedly finding a derivative in f of the form $\partial_j(Leader(e_i))$ and replacing it by the tail of $\partial_j e_i$. The resulting reduced form of f consists entirely of terms that are not a derivative of any leader $Leader(e_i)$ of $e_i \in L$.

Definition 3. *A system L is called Riquier basis, if for all e_i, e_j, l, k*

$$Leader(\partial_l e_j) = Leader(\partial_k e_i)$$

implies reduction to zero of integrability conditions $\partial_l e_j - \partial_k e_i \rightarrow 0$.

Roughly speaking, completion of arbitrary system of linear PDE with polynomial coefficients consists of reduction steps and addition of integrability conditions could be achieved by finite number of steps ([28]).

Moreover L could be transformed into reduced Riquier basis (see Definition 2.2 in [14]), which means the derivatives on both-hand sides are disjoint. We call the derivatives from the right-hand side the parametric derivatives. For any given solution (ξ, η) of L, the linear functional Par^k $(k = 1, ..., r)$ maps (ξ, η) to its kth parametric derivative and $\mathrm{Par}^k_{x_0}$ stands for the evaluation operator at a given point x_0.

Example 3. Consider the system

$$\{\xi_{xx} = \frac{-2}{y-x}\xi_x + \frac{2}{(x+y)^2}(\eta - \xi)\,, \quad \xi_y = 0\,, \quad \eta_x = 0\,, \quad \eta_y = -\xi_x + \frac{2}{x+y}(\eta - \xi)\}.$$

The parametric derivatives are $\{\xi\,, \eta\,, \xi_x\}$. Therefore $\mathrm{Par}^3(\xi, \eta) = \xi_x$ and $\mathrm{Par}^1_{(1,0)}(\xi, \eta) = \xi(1, 0)$.

The correctness of LAVF package is based on the following theorem.

Theorem 2. *(Riquier existence-uniqueness). Let L be a Riquier basis for a linear homogeneous PDE system with r parametric derivatives. Let $x_0 \in \mathbb{C}$ be a regular point of L (we call x_0 a base point), and let $(a^1, ..., a^r) \in \mathbb{C}^r$ be initial data values. Then there is a unique formal power series solution $u(x)$ of L at x_0 such that $Par^k_{x_0}(u) = a^k$ for each $k = 1, ..., r$. Also, every solution to L at x_0 may be obtained in this way for some initial data values.*

The proof can be found in ([28]). In fact, it can be shown that the formal power series are convergent, so that the space of initial data is isomorphic to the solution space $\mathcal{L}_{x_0}$ for any given base point x_0. It immediately follows that $\mathcal{L}_{x_0}$ is r-dimensional and $\{\mathrm{Par}^k_{x_0}\}^r_{k=1}$ forms a basis of the dual space $\mathcal{L}^*_{x_0}$.

Now it is enough for us to introduce the method how to extract structure coefficients of $\mathcal{L}$ from its determining system L. Here we use the generalized version for structure coefficients:

Definition 4. *Let* $\mathcal{L}, \mathcal{M}, \mathcal{N}$ *be Lie algebras which satisfy* $[\mathcal{L}, \mathcal{M}] \subset \mathcal{N}$*.* $\{X_{\mathcal{L}i}\}, \{X_{\mathcal{M}j}\}, \{X_{\mathcal{N}k}\}$ *are the bases of* $\mathcal{L}, \mathcal{M}, \mathcal{N}$ *respectively. The structure coefficients (also known as structure constants) are a set of constants* $\{f_{ij}^k\}$ *such that* $[X_{\mathcal{L}i}, X_{\mathcal{M}j}] = \sum_k f_{ij}^k X_{\mathcal{N}k}$*. Specially, one can get the structure coefficients of* $\mathcal{L}$ *by setting* $\mathcal{M} = \mathcal{N} = \mathcal{L}$*.*

If we set $\{\theta_{\mathcal{L}}^i\}$ be the dual basis in $\mathcal{L}_{x_0}^*$ such that $\theta_{\mathcal{L}}^i(X_{\mathcal{L}j}) = \delta_j^i$ and similar settings on $\{\theta_{\mathcal{M}}^j\}, \{\theta_{\mathcal{N}}^k\}$, a simple calculation shows

$$\theta_{\mathcal{N}}^k([V_{\mathcal{L}}, V_{\mathcal{M}}]) = \sum_{i,j} f_{ij}^k(x_0)\theta_{\mathcal{L}}^i(V_{\mathcal{L}})\theta_{\mathcal{M}}^j(V_{\mathcal{M}}), \quad \text{for all } V_{\mathcal{L}} \in \mathcal{L}_{x_0}, V_{\mathcal{M}} \in \mathcal{M}_{x_0}.$$[4]

The above equation is valid for any chosen bases so we may simply take $\{\mathrm{Par}_{\mathcal{L},x_0}^i\}, \{\mathrm{Par}_{\mathcal{M},x_0}^j\}, \{\mathrm{Par}_{\mathcal{N},x_0}^k\}$ as the dual bases and the above equation turns to

$$\mathrm{Par}_{\mathcal{N},x_0}^k|_{[\mathcal{L}_{x_0},\mathcal{M}_{x_0}]} = \sum_{i,j} f_{ij}^k(x_0)\mathrm{Par}_{\mathcal{L},x_0}^i \cdot \mathrm{Par}_{\mathcal{M},x_0}^j \tag{11}$$

By applying $\{\mathrm{Par}_{\mathcal{N}}^k\}$ on elements in $[L, M]$,[5] one may collect the structure coefficients (i.e. coefficients of $\mathrm{Par}_{\mathcal{L}}^i \cdot \mathrm{Par}_{\mathcal{M}}^j$) after reducing the results by L and M. For a formal algorithm please see [28].

The other function we are going to discuss is the construction algorithm for determining system of derived algebra. More general, for Lie algebras $\mathcal{L}, \mathcal{M}, \mathcal{N}$ such that $[\mathcal{L}, \mathcal{M}] \subset \mathcal{N}$, LAVF provides a simple method to get the determining system for the subalgebra $[\mathcal{L}, \mathcal{M}] = \mathcal{N}'$. Instead of $\mathcal{N}'$, the tricky thing here is to find the annihilating space of $\mathcal{N}'$ in $\mathcal{N}^*$. This is an equivalent approach from the view of linear algebra. Now if $\theta = \sum_k \alpha_k \mathrm{Par}_{\mathcal{N},x_0}^k$ is an annihilator of $\mathcal{N}_{x_0}'$ in $\mathcal{N}_{x_0}^*$, combined with Eq. (11) we get a linear system

$$\sum_{k=1}^{r_N} f_{ij}^k(x_0)\alpha_k = 0, \quad i = 1, ..., r_L, j = 1, ..., r_M$$

where r_L, r_M, r_N are the numbers of parametric derivatives in L, M, N respectively. Let $\{v^1, ..., v^l\}$ be a basis of solution space of the above system, so the annihilating space of $\mathcal{N}_{x_0}'$ is spanned by $\{\sum_k v_k^i \mathrm{Par}_{\mathcal{N},x_0}^k\}_{i=1}^l$. Thus for any $(\xi, \eta) \in \mathcal{N}$, $(\xi, \eta) \in \mathcal{N}'$ if and only if $\sum_k v_k^i \mathrm{Par}_{\mathcal{N}}^k(\xi, \eta) = 0$ for all $i = 1, ..., l$. i.e. $N' = N \cup \{\sum_k v_k^i \mathrm{Par}_{\mathcal{N}}^k(\xi, \eta) = 0 : i = 1, ..., l\}$.

3.2 Exact Linearization Certificate

In this section, we will use the following notations. The Lie symmetry algebra of Eq. (1) is denoted by $\mathcal{L}$ and $m = \dim(\mathcal{L})$. The important role in our theory plays

[4] The value of f_{ij}^k depends on the base point x_0 because the bases may change when x_0 varies.

[5] We are not using $[\mathcal{L}, \mathcal{M}]$ because the elements are expressed in terms of unknown functions from the determining systems.

derived algebra $\mathcal{D} \subset \mathcal{L}$ which is a subalgebra that consists of all commutators of pairs of elements in $\mathcal{L}$ (i.e. $\mathcal{D}$=[$\mathcal{L}$,$\mathcal{L}$]).

The cornerstone is symmetry (group) classification of linear ODE. If two differential equations could be mapped one into another by point transformation, we say that they are equivalent. Definitely, it spans equivalence relations and split all linear ODE's into equivalent classes. By doing so, the classification of scalar linear ODE is as follows [19]):

- ⋄ All 2nd order linear ODEs are equivalent to $u''(t) = 0$ up to a point transformation and their symmetry algebras are isomorphic with dimension 8;
- ⋄ An nth ($n \geq 3$) order linear ODE, which admits an $(n+4)-$dimensional symmetry algebra, is equivalent to $u^{(n)}(t) = 0$ up to a point transformation;
- ⋄ An nth ($n \geq 3$) order linear ODE, which admits an $(n+2)-$dimensional symmetry algebra, is equivalent an linear ODE with constant coefficients up to a point transformation;
- ⋄ An nth ($n \geq 3$) order linear ODE, which belongs to none of the above cases, admits an $(n+1)-$dimensional symmetry algebra (it is the case of linear equation with principally non-constant coefficient).

Our theory involves much analysis on the inner structure of these symmetry algebras such as their generators. One important fact is that any nth order linear ODE always admits an $n-$ dimensional abelian subalgebra which is generated by the fundamental solution set (see Lemma 1 in next section). So here the following theorem naturally comes:

Theorem 3. ([15]) *Eq. (1) with* $n \geq 2$ *is P-linearizable if and only if one of the following conditions is fulfilled:*

1. $n = 2\,, m = 8$ *or* $n \geq 3\,, m = n + 4$ *(maximality of symmetry dimension space which is related to* $u^{(n)}(t) = 0$*);*
2. $n \geq 3\,, m \in \{n+1, n+2\}$ *and dervied algebra* $\mathcal{D}$ *is abelian of dimension* n.

Proof. This is a combination of Theorem 3 and Corollary 4 in [15]. ∎

The theorem gives a foundation for algorithmic test of linearization. Starting with determining system for infinitesimal symmetries, completion to Riquier basis (involution) gives immediately dimension of solution space (number of initial data for Cauchy problem). If it satisfies exact linearization dimension condition, then we have to compute derived algebra and its dimension.

4 Construction of differential system for linearizing mapping

Lemma 1. *Let* $S_n := \{f_i(x)\}_{i=1}^n$ *be a set of linear independent analytical functions defined on some domain* U *and* $\mathcal{S}_n$ *denotes the* $n-$*dimensional abelian Lie algebra generated by* $\{X_i = f_i(x)\frac{\partial}{\partial y}\}_{i=1}^n$. *Then, despite the singular points, there exists a linear ODE in the form of Eq. (1) such that its point symmetry algebra* $\mathcal{L}$ *admits* $\mathcal{S}_n$ *as a subalgebra. Moreover, this ODE is unique if it is required to be homogeneous.*

Proof. Let $W_n(x)$ be the Wronskian of $\{\boldsymbol{y}_i(x)\}_{i=1}^n$, where $\boldsymbol{y}_i(x) = (f_i(x), f_i'(x), ..., f_i^{(n-1)}(x))^{\mathrm{T}}$. First, we prove $W_n(x) \neq 0$ by induction.[6] Since everything is analytical, this is equivalent to say that $\mathrm{Supp}(W_n(x)) = U$. The proof for $n = 1$ is trivial. Now assume that $W_n(x) \neq 0$ but $W_{n+1}(x) = 0$. Thus $f_{n+1}(x)$ is a solution of

$$\sum_{k=0}^{n} A_k(x) y^{(k)}(x) \tag{12}$$

where $A_k(x)$ is the cofactor of $f_{n+1}^{(k)}(x)$ in $W_{n+1}(x)$. Since $A_n(x) = W_n(x) \neq 0$ and $\sum_{k=0}^n A_k(x) f_i^{(k)}(x) = 0$ for all $1 \leq i \leq n$, S_n forms a fundamental solution set for Eq. (12) (despite the singular points of $A_n^{-1}(x)$). This means $f_{n+1}(x)$ is a linear combination of elements in S_n, contradicts to the linear independence of S_{n+1}.

Now back to the lemma. The nth prolongation for each generator of $\mathcal{S}_n$ is obviously $X_i = \sum_{k=0}^n f_i^{(k)}(x) \frac{\partial}{\partial y^{(k)}}$. By Proposition 1 we have

$$X_i(y^{(n)} - F(x, y, y', \ldots, y^{(n-1)}))|_{y^{(n)}=F(x,y,y',\ldots,y^{(n-1)})} = 0.$$

Namely, a linear PDE system for F:

$$f_i^{(n)}(x) = \sum_{k=0}^{n-1} f_i^{(k)}(x) \frac{\partial F}{\partial y^{(k)}}, \quad 1 \leq i \leq n.$$

By our first statement, this is a non-singular system in analytical function field and it follows immediately that F is linear respect to each $y^{(k)}$. Moreover, since the coefficients of $y^{(k)}$ are uniquely determined, F is unique if it is required to be homogeneous. ■

According to Theorem 3 we split our approach into three cases.

4.1 $n = 2\,, m = 8$ or $n \geq 3\,, m = n + 4$

This is the trivial case since Eq. (1) is equivalent to $u^{(n)}(t) = 0$ up to a point transformation. However, the construction algorithm is more complicated than other cases because $\mathcal{D}$ is not abelian ($\mathcal{L}$ still admits an $n-$dimensional abelian subalgebra). Thus we use a quite straight forward approach.

First we assume that $n \geq 3$. By Proposition 1, it can be easily verified that the point symmetry algebra of $u^{(n)}(t) = 0$ is generated by $B = \{\frac{\partial}{\partial u}, t\frac{\partial}{\partial u}, ..., t^{n-1}\frac{\partial}{\partial u}, u\frac{\partial}{\partial u}, \frac{\partial}{\partial t}, t\frac{\partial}{\partial t}, t^2\frac{\partial}{\partial t} + (n-1)ut\frac{\partial}{\partial u}\}$. Our task here is to construct a determining system T for (ϕ, ψ) which maps Eq. 1 to $u^{(n)}(t) = 0$. Let L be the determining system of $\mathcal{L}$ with unknown functions (ξ, η). Since $t = \phi(x, y), u = \psi(x, y)$, for convenience we express elements of B in column

[6] Here we mean $W_n(x)$ is not a zero function

vectors $X_1 = (0,1)^{\mathrm{T}}, X_2 = (0,\phi)^{\mathrm{T}}, ..., X_{n+4} = (\phi^2, (n-1)\phi\psi)^{\mathrm{T}}$. By B-K equations:

$$\begin{pmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \sum_{i=1}^{n+4} c_i X_i \tag{13}$$

where c_i are arbitrary constants. This is a linear system and by Crammer Law, (ξ, η) can be expressed in terms of (ϕ, ψ) and their first derivatives. Substitute these expressions back to L, collect the coefficients of $\{c_i\}_{i=1}^{n+4}$ and vanish them all, then we get the desired system T for (ϕ, ψ). Since the expression for (ξ, η) involves high order derivatives of sum of $J^{-1}\phi^k$ $(0 \le k \le n-1)$, there is no doubt that the substitution step would be a mess when n goes large. But fortunately, Eq. 13 could be significantly reduced as shown in the following algorithm.

Algorithm 1 ***P-determining system'***

Input: q, an ODE in the form of Eq. (1) which is equivalent to $u^{(n)}(t) = 0$.
Output: A determining system T for (ϕ, ψ) which maps q to $u^{(n)}(t) = 0$.

```
1: L := DerterminingPDE (q);
2: J := JacobianMatrix (φ, ψ);
3: n :=DifferentialOrder (q);
4: X := (0, 1)^T, Y := (0, ψ)^T, Z := (φ^2, (n − 1)φψ)^T;
5: (ξ, η)^T := J^{-1}(aX + bY + cZ);
6: L' :=Substitutions ((ξ, η), L);
7: C := Coefficients (L', {a, b, c});
8: T := {e = 0 : e ∈ C} ∪ {φ_x ψ_y − φ_y ψ_x ≠ 0}.
```

Remark 2. We only need to justify the 5th step is equivalent to Eq. (13). Let us try a different explanation for step 5: By B-K equations, (ϕ, ψ) maps $\mathcal{L}$ to a isomorphic counterpart $\mathcal{L}'$ which includes generators $X = X_1, Y = X_{n+1}$ and $Z = X_{n+4}$. Since $\mathcal{L}'$ is a Lie algebra, it is closed under Lie bracket. One can easily check that $\{X_1, X_{n+4}\}$ generate $\{X_2, ..., X_n\}$ under Lie bracket. Finally by Lemma 1 and Corollary 1, the subalgebra $\mathcal{S} = \mathrm{Span}(X_1, ..., X_n)$ is admitted by a unique linear homogeneous ODE which is obviously $u^{(n)}(t) = 0$.

According to above discussion, one can easily see that Alg. 1 still works for case $n = 2\,, m = 8$. This is because to determine a linear equation, we do not need all the generators.

4.2 $n \ge 3\,, m = n + 1$

Lemma 2. *Assume that* $X_i = \xi^i(x,y)\frac{\partial}{\partial x} + \eta^i(x,y)\frac{\partial}{\partial y}, i = 1,2,3$ *are linear independent generators and commute to each other. If* X_1 *has the form* $X_1 = f_1(x)\frac{\partial}{\partial y}$*. Then, the rest must have the same form* $X_i = f_i(x)\frac{\partial}{\partial y}, i = 2,3.$

Proof. $[X_i, X_j] = 0$ is equivalent to:

$$\begin{cases} \xi^i \xi^j_x + \eta^i \xi^j_y = \xi^j \xi^i_x + \eta^j \xi^i_y, \\ \xi^i \eta^j_x + \eta^i \eta^j_y = \xi^j \eta^i_x + \eta^j \eta^i_y. \end{cases} \tag{14}$$

Thus $[X_1, X_2] = [X_1, X_3] = 0$ implies

$$X_i = g_i(x)\frac{\partial}{\partial x} + (\frac{y g_i(x) f_1'(x)}{f_1(x)} + f_i(x))\frac{\partial}{\partial y}, i = 2, 3$$

where $g_i(x), f_i(x)$ are undetermined. We are going to prove $g_i(x) = 0$ by contradiction. Assume that $g_2(x) \neq 0$. Since now X_2, X_3 have simpler expressions, the first equation in system (14) can be simplified as $g_2 g_3' = g_3 g_2'$, i.e. $g_3 = C g_2$ for some constant C. Replace X_3 by $X_3 - CX_2$, then we may assume that $g_3 = 0$.[7] From the second equation of (14) (take $i = 2, j = 3$) we have $f_1 f_3' = f_3 f_1'$, i.e. X_1, X_3 are linear dependent. This is a contradiction. ■

Definition 5. *System of differential equations for local diffeomorphism $(\phi(x, y), \psi(x, y))$ is called linearizing differential system of Eq. (1) if every solution of this system maps Eq. (1) into a linear ODE.*

Theorem 4. *Assume that there exists a maximal linearizing differential system T of Eq. (1). Then, (ϕ, ψ) is a solution of T if and only if (ϕ, ψ) maps one nonzero element $\xi_0(x, y)\frac{\partial}{\partial x} + \eta_0(x, y)\frac{\partial}{\partial y} \in \mathcal{D}$ into the form $f_0(t)\frac{\partial}{\partial u}$ where $f_0 \neq 0$.*

Proof. By the footnote of Theorem 3, we only need to prove the inverse. First, by B-K equations, the point transformation (ϕ, ψ) induces an isomorphism $(\xi, \eta) \mapsto (\xi\phi_x + \eta\phi_y, \xi\psi_x + \eta\psi_y)$ between Lie algebras. If (ϕ, ψ) maps one nonzero element $(\xi_0, \eta_0) \in \mathcal{D}$ to the form $(0, f_0(t))$. Since $\mathcal{D}$ is abelian and its dimension is at least 3, by Lemma 2 we know (ϕ, ψ) maps all elements $(\xi, \eta) \in \mathcal{D}$ into the form $(0, f(t))$ simultaneously. Then by Lemma 1, these new generators $f(t)\frac{\partial}{\partial u}$ are admitted by a linear ODE which is the image of Eq. (1) under (ϕ, ψ). ■

Remark 3. From the proof we know that (ϕ, ψ) maps one nonzero element $(\xi_0, \eta_0) \in \mathcal{D}$ to the form $(0, f_0(t))$ is equivalent to say that (ϕ, ψ) maps all elements $(\xi, \eta) \in \mathcal{D}$ into the form $(0, f(t))$, in other words, $\xi\phi_x + \eta\phi_y = 0$ and the Jacobian of $(\xi\psi_x + \eta\psi_y)$ and ϕ vanishes.

For any given ODE in the form of Eq. (1), the system for (ξ, η) in symmetry algebra can be constructed directly by command **> *DerterminingPDE***(Eq. (1)) in Maple™ Software and the system for derived algebra by commands in LAVF package. By Proposition 1 and Section 3.1, these systems are always linear. And since the derived algebras $\mathcal{D}$ for case $m = n + 2$ is still abelian with dimension n, the following algorithm is valid for both cases.

[7] This assumption does not change any commutative condition.

Algorithm 2 ***P-determining system***

Input: q, a P-linearizable ODE in the form of Eq. (1) with $m \in \{n+1, n+2\}$.
Output: A determining system T for (ϕ, ψ) which linearizes q.

1: $L :=$ ***DerterminingPDE*** (q);
2: $D :=$ ***DeterminingSystem*** $([L, L])$;
3: $P :=$ ***ParametricDerivatives*** (D);
4: $S := \{\xi\phi_x + \eta\phi_y\, , (\xi\psi_x + \eta\psi_y)_x\phi_y - (\xi\psi_x + \eta\psi_y)_y\phi_x\} \mod D$;
5: $C :=$ ***Coefficients*** (S, P);
6: $T := \{e = 0 : e \in C\} \cup \{\phi_x\psi_y - \phi_y\psi_x \neq 0\}$;
7: $J :=$ ***JacobianMatrix*** (ϕ, ψ);
8: $X := (0, \psi)^{\mathrm{T}}$;
9: $(\xi, \eta)^{\mathrm{T}} := J^{-1}aX$;
10: $L' :=$***Substitutions*** $((\xi, \eta), L)$;
11: $C' :=$ ***Coefficients*** $(L', \{a\})$;
12: $T := \{e = 0 : e \in C'\} \cup T$.

Remark 4. Since D is a linear system, S is obviously linear respect to P. Then in step 5 we collect all the coefficients of P in S and vanish them all to get a determining system. One can actually stop at here because (ϕ, ψ) from step 6 is already capable to linearize q but not to a homogeneous ODE. Thus the following steps is to make sure that $(\phi, \psi)(q)$ admits the generator $y\partial_y$, which guarantees the homogeneous of linear equation under mapping (Corollary 1).

4.3 $n \geq 3\, , m = n + 2$

From [9] we know that, when $m = n + 2$, Eq. (1) is equivalent to a linear ODE with constant coefficients up to a point transformation. In this regard we would like to make a refinement on Alg. 2 which includes more equations in T such that $(\phi, \psi)(q)$ is not only linear but also with constant coefficients. This is just another simple application of Corollary 1.

Algorithm 3 ***P-determining system***$^{+}$

Input: q, a P-linearizable ODE in the form of Eq. (1) with $m = n + 2$.
Output: A determining system T for (ϕ, ψ) which maps q to a linear ODE with constant coefficients.

1: $T :=$ ***P-DeterminingSystem*** (q); (Alg. 2)
2: $J :=$ ***JacobianMatrix*** (ϕ, ψ);
3: $(\xi, \eta)^{\mathrm{T}} := J^{-1}(a, 0)^{\mathrm{T}}$;(Corollary 1)
4: $L' :=$***Substitutions*** $((\xi, \eta), L)$;
5: $C :=$ ***Coefficients*** $(L', \{a\})$;
6: $T := \{e = 0 : e \in C\} \cup T$.

It is well known that a linear PDE system (e.g. T from Alg. 2 or Alg. 3) is generally impossible to solve algorithmically by computational technique. But our next algorithm is about to find a linear ODE with constant coefficients, which is equivalent to Eq. (1), without any information about point transformations.

The following Lemma is simple but critical.

Lemma 3. *Let E_1, E_2 be two nth order ($n \geq 3$) linear ODEs with constant coefficients and they admit characteristic polynomials $f_1(x), f_2(x)$ respectively. Then E_1, E_2 are equivalent up to a point transformation if $f_2(x) = k^n f_1(\frac{x-b}{k})$ for some constants k, b ($k \neq 0$).*[8]

Proof. The transformation is given by $\bar{x} = \frac{x}{k}, \bar{y} = ye^{\frac{bx}{k}}$. In fact, assume that $f_1(x) = \prod_{i=1}^{n}(x-\lambda_i)$, then $f_2(x) = \prod_{i=1}^{n}(x-k\lambda_i-b)$. The given transformation maps each solution $y = x^m \mathrm{e}^{\lambda_i x}$ to $\bar{y} = k^m \bar{x}^m \mathrm{e}^{(k\lambda_i+b)\bar{x}}$. ∎

We will show the algorithm first and prove its correctness later:

Algorithm 4 ***Linear ODE Recovery for*** $m = n+2$ ***(P)***

Input: q, a P-linearizable ODE in the form of Eq. (1) with $m = n+2$.
Output: E, a linear ODE with constant coefficients which is equivalent to q.

1: $L :=$ ***DerterminingPDE*** (q);
2: $SC :=$ ***StructureCoefficients*** $(\mathcal{L}) = (c_{ij}^l)$;
3: $r := 0$;
4: **repeat**
   $r := r+1, f(x) :=$***CharacteristicPolynomial*** (c_{rj}^l) , $g(x) := f(x)/x^2$
5: **until** $g'(x) \nmid g(x)$;
6: Output $g(x)$ (or output E whose characteristic polynomial is $g(x)$).

Remark 5. In the 3rd step, $SC = (c_{ij}^l)$ is an $(n+2)\times(n+2)\times(n+2)$ cuboidal matrix such that $[X_i, X_j] = \sum_{l=1}^{n+2} c_{ij}^l X_l$ where $\{X_i\}_{i=1}^{n+2}$ are some bases for $\mathcal{L}$. As mentioned in Section 3.1, c_{ij}^l may vary as a function of the base point. So here SC should be evaluated at any non-singular point (x_0, y_0) (respect to the PDEs L). In step 4 we compute the characteristic polynomial of the rth slice of SC. The later proof will show that our repeat loop does terminate within $(n+2)$ rounds.

Theorem 5. *Alg. 4 terminates and the output E is equivalent to q up to a point transformation.*

Proof. Assume that q is equivalent to a linear ODE E' with constant coefficients. There is a clear description for the point symmetry algebra $\mathcal{L}$ of E'. $\mathcal{L}$ is generated by $n+2$ linear independent symmetries

$$\{y_1(x)\frac{\partial}{\partial y}, ..., y_n(x)\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\}$$

[8] This is not a necessary condition. Take $y''' = 0$ and $y''' - y' = 0$ as a counter example.

where $\{y_i(x)\}_{i=1}^n$ form a fundamental solution set of E'. Moreover, $\{y_i(x)\frac{\partial}{\partial y}\}_{i=1}^n$ generate the derived algebra $\mathcal{D}$ of $\mathcal{L}$ and $\mathcal{D}$ is abelian. Thus $\mathcal{L} = \mathcal{D} \oplus$ [9] $\mathcal{M}$ where $\mathcal{M}$ is the subalgebra generated by $X = \frac{\partial}{\partial x}$ and $Y = y\frac{\partial}{\partial y}$. Assume that the characteristic polynomial of E' is $f_1(x) = \prod_{i=1}^s (x - \lambda_i)^{n_i}$ where $\sum_{i=1}^s n_i = n$ and λ_i's are pairwise distinct. There is a standard basis for $\mathcal{D}$ and $\mathcal{M}$ (and thus for $\mathcal{L}$):

$$\{\frac{x^m}{m!}\mathrm{e}^{\lambda_i x}\frac{\partial}{\partial y} \mid 1 \le i \le s, 0 \le m \le n_i - 1\}, \{\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\}$$

The matrix of the adjoint action $\mathrm{ad}_X : \mathcal{D} \to \mathcal{D}$ under this basis is a Jordan canonical form whose diagonal entries are all the λ_i's (include the multipel roots), and its characteristic polynomial is obviously $f_1(x)$. For the matrix of $-\mathrm{ad}_Y$ on $\mathcal{D}$, it is always identity. Now take any $A \in \mathcal{L}$, $A = kX - bY + D, D \in \mathcal{D}$. Since $\mathcal{D}$ is abelian, $\mathrm{ad}_A = k \cdot \mathrm{ad}_X - b \cdot \mathrm{ad}_Y$ on $\mathcal{D}$. Thus, under the standard basis, it can be represented as an upper triangular matrix with diagonal entries $k\lambda_i + b$, and the characteristic polynomial is no one but $g(x) = \prod_{i=1}^s (x - k\lambda_i - b)^{n_i} = k^n f_1(\frac{x-b}{k})$. On the other hand, since $\mathrm{ad}_A(\mathcal{M}) \subset [\mathcal{L}, \mathcal{L}] = \mathcal{D}$, the matrix of the adjoint action $\mathrm{ad}_A : \mathcal{L} \to \mathcal{L}$ under the standard basis is an upper triangular matrix with two zeros in the last two diagonal entries. The characteristic polynomial is just $x^2 g(x)$. As a matter of fact, if the characteristic polynomial of ad_A is all we want, the basis of $\mathcal{L}$ is no need to be considered because similar matrices share the same characteristic polynomial.

By Lemma 3, it is clear that once we get $g(x) = k^n f_1(\frac{x-b}{k})$ for some $k \ne 0$, then it comes the desired equation E. This could be done within $(n+2)$ rounds by the **repeat** loop because:

1. Since (c_{ij}^l) are constructed by some basis of $\mathcal{L}$, such nonzero k does exist;
2. If $k = 0$, then $g(x) = (x - b)^n$, which implies $g' \mid g$,[10] so we exclude all the 'wrong' polynomials;
3. When $k \ne 0$, since E' is equivalent to E but not to $u^{(n)} = 0$, g has at least 2 distinct roots, otherwise it contradicts to Lemma 3. Thus it has to be $g' \nmid g$. So we do not miss the 'right' polynomial.

■

At the end of this section, we would like to point out that only the determining system from Alg. 2 is not finite-dimensional (i.e. the general solution consists of arbitrary functions). But the following theorem guarantees that, after adding some restrictions, the determining system can always be finite-dimensional, and it also demonstrates why the systems from Alg. 1 and Alg. 3 are in finite dimension.

Theorem 6. *Linearizing differential system admits finite-dimensional construction.*

[9] This is a direct product for linear space, not for Lie algebra.
[10] Actually it is not hard to prove that $g' \mid g$ is equivalent to $g(x) = (x - b)^n$ for some b.

Proof. Let consider the system of differential equations which is constructed by Alg. 1. Two possible transformation (ϕ_1, ψ_1) and (ϕ_2, ψ_2) satisfy

$$(\phi_1, \psi_1) \cdot (\phi_2, \psi_2)^{-1} = T,$$

where T – symmetry transformation of $u^{(n)} = 0$. The space of symmetry could depend only on finite number of constants, and thus solution space of the determining system is finite-dimensional.

If $m = n + 2$, then mapping two operators into

$$\left\{ \frac{\partial}{\partial x}, y\frac{\partial}{\partial y} \right\}$$

and one operator from derived algebra into

$$X = \frac{\partial}{\partial y}$$

ensures[11] that image will be homogeneous linear ODE with constant coefficients. Equivalence group of constant coefficient linear ODE is already finite-dimensional.

General case (homogeneous one) with principally non-costant coefficients $(m = n + 1)$ admits equivalence transformation

$$(t, u) = (\lambda(x), \mu(x)y).$$

Obviously by choosing $\mu(x) = 1/f_1(x)$ and $\lambda(x) = f_2(x)/f_1(x)$, where f_1, f_2 are linear independent elements from the fundamental solution set, this transformation maps $\{f_1 \frac{\partial}{\partial y}, f_2 \frac{\partial}{\partial y}\}$ to $\{\frac{\partial}{\partial u}, t\frac{\partial}{\partial u}\}$, which implies the existence of the form

$$u^{(n)}(t) = \sum_{k=2}^{n-1} a_k(x)\, u^{(k)}(t).$$

This is an analog of Lagueerre-Forsyth form which was extensively used in **LinearizationTestII** [15]. The descriptive property of it is the admittance (in derived algebra) of three operators

$$\left\{ \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y} \right\}.$$

The linearizing differential system which maps Eq. (1) to such forms is obviously finite-dimensional because $\mu(x)$ and $\lambda(x)$ are related to the fundamental solution set of some linear ODE. ■

[11] It is always possible to do because of Lemma 3, as one of eigenvalues could be shifted to 0.

5 Differential Thomas Decomposition

From practical point of view, it is required to construct at least one solution to linearizing differential system, but algorithms 2,3 and 4 yield large overdetermined PDE system which is hard to analyze or construct particular solution. Our approach here is to complete the system to standard basis by means of differential Thomas decomposition.

The differential Thomas decomposition was suggested in the papers of [33,34] as a generalization of the Riquier-Janet theory of *passive* linear and *orthonomic* PDE systems (for modern description see [29]) to polynomially-nonlinear systems of general form. The Thomas decomposition provides a universal algorithmic technique ([2,27]) for symbolic analysis of *differential systems*, which is defined as follows.

Definition 6. *([2,27,35]) A* differential system *is a system $S := \{S^{=}, S^{\neq}\}$ of differential equations and (possibly) inequations of the form*

$$S^{=} := \{g_1 = 0, \ldots, g_s = 0\},\ S^{\neq} := \{h_1 \neq 0, \ldots, h_t \neq 0\},$$

where s is a positive integer as well as t if $S^{\neq} \neq \emptyset$, and g_i, h_j are elements in the differential polynomial ring in finitely many differential indeterminates (dependent variables) over the differential field of characteristics zero.

The Thomas decomposition applied to a differential system S yields a finite set of passive (involutive) and *differentially triangular* differential systems called *simple* (see [27]) that partition the solution set of the input differential system[12]. Algebraically, this provides a characterizable decomposition ([6]) of the radical differential ideal $\sqrt{\mathcal{I}}$ where $\mathcal{I}$ is the differential ideal generated by the polynomials in $S^{=}$.

Example 4. We consider a serial example from [18]

$$(y^2)^{(n)} + y^2 = 0\,, \quad n \geq 3 \tag{15}$$

which is obviously equivalent to $u^{(n)} + u = 0$. It definitely belongs to the linearizable case with $m = n+2$. For different values of n we make a comparison of timings with **TestLinearizationII** in [18] (OOM means out of memory). The new procedure requires some preprocessing based on substitution of explicit expressions into generators for determining system, but the final step for Thomas Decomposition is significantly improved with fixed number of independent and dependent variables $(2, 2)$. It is a bottle neck of exact linearization algorithm, for more details on upper bounds of algorithms in differential algebra we reference to classical paper ([4]).

[12] Like with Riquier basis, every simple system from partition admits well-posed Cauchy problem.

Table 1: CPU times (sec.)

Test	Order n of the ODE (15)										
	3	4	5	6	7	8	9	10	11	12	13
Alg. 2	0.6	1.0	1.8	4.8	8.6	22.4	49.1	137.7	412.7	1001	3268
II in [15]	0.5	1.8	9.4	30.1	209.9	789.8	2011	8200	25217	OOM	OOM

Example 5. (Tremblay-Turbiner-Winternitz system) These equations come from the theory of super-integrable Hamiltonian systems, which recently have found the application in quantum mechanics as new exactly solvable models [5]. The main equation is given by expression

$$y(x)\, y'''(x) + y'(x)\left(16\,\omega^2\, y(x) + 3\, y''(x)\right) = 0\,, \tag{16}$$

which depends on one parameter ω. Assuming it as function

$$\omega := \omega(x),\ \omega'(x) = 0,$$

the classical group analysis is applicable. We will search for generators of the form

$$\xi_x := C_1 x,\ \eta_y := C_2 y,\ \eta_\omega := C_3 \omega,$$

it leads to

$$\omega(C_1 + C_3) = 0.$$

When $\omega \neq 0$, point transformation $x = \omega\bar{x}$ maps (16) to parameterless form

$$y(x)\, y'''(x) + y'(x)\,(16\, y(x) + 3\, y''(x)) = 0\,,$$

and

$$y(x)\, y'''(x) + 3y'(x)\, y''(x) = 0\,, \omega = 0.$$

The linearization test shows both cases admit the maximum dimension of symmetry algebra and thus are equivalent to $u''' = 0$. The linearizing differential system for $\omega \neq 0$ is outputted by Thomas Decomposition:

$$f_{xx} = \frac{g_{xy} f_x}{g_y},\ f_y = 0,$$

$$g_{xxx} = -\frac{16 g_y^2 g_x + 6 g_y g_{xy} g_{xx} - 3 g_x g_{xy}^2}{2 g_y^2}, g_{xxy} = \frac{16 g_y^2 + 3 g_{xy}^2}{2 g_y}, g_{yy} = \frac{g_y}{y},$$

$$g_y \neq 0,\ g_x \neq 0,\ f_x \neq 0,\ g \neq 0,\ 2 f g_x + f_x g \neq 0,\ f \neq 0.$$

The system[13] could be easily solved by heuristical methods because of simple structure of equations - starting from g, we reduce it to system of ODE and then finally concluding with f.

[13] We remind here that in practice we need only one solution of linearizing determining system.

Remark 6. Simplification of initial differential system by Thomas Decomposition is obtained by adding integrability conditions and reduction steps. According to Algorithm 1 and Remark 2 it is enough to take only 3 generators in B-K equation, but from practical point of view acceleration of computations could be achieved by adding more generators. Some of these new equations will contain integrability conditions and thus it reduces number of reduction steps in Thomas Decomposition.

6 Generalizations to Contact Transformation

Contact transformation (or symmetry) is the generalized version of point transformation (or symmetry). The transformation involves three variables x, y and y'. For conciseness we only introduce several definitions, more detailed introduction could be found in [17].

Definition 7. *A contact transformation is defined by the formulas*

$$t = \phi(x, y, p)\,,\ u = \psi(x, y, p)\,,\ q = \theta(x, y, p)$$

with some restrictions

$$\psi_p = \theta\phi_p\,,\ (\psi_x + p\psi_y) = \theta(\phi_x + p\phi_y).$$

Here, we use the notation $p = y'(x)$ and $q = u'(t)$. And the non-singularity of the Jacobian matrix could be simplified as

$$(\theta\phi_y - \psi_y)((\theta_x + p\theta_y)\phi_p - (\phi_x + p\phi_y)\theta_p) \neq 0\,.$$

Also, a contact symmetry generator is an operator of the form

$$\mathcal{X} := \xi(x, y, p)\,\partial_x + \eta(x, y, p)\,\partial_y + \eta^{[1]}(x, y, p)\partial_p\,,$$

where

$$\eta_p - p\xi_p = 0,\ \eta^{[1]} = \eta_x + p(\eta_y - \xi_x) - p^2\eta_y\,.$$

Thus $\xi_p = 0$ implies $\mathcal{X}$ is the 1st prolongation of a point symmetry generator.[14] *Similarly, all contact generators admitted by Eq. (1) still form a Lie algebra.*

In this section, we always assume that Eq. (1) is linearizable by contact transformation (or C-linearizable for short), $\mathcal{L}$ is the contact symmetry algebra of Eq. (1), $m = \dim(\mathcal{L})$ and $\mathcal{D} = [\mathcal{L}, \mathcal{L}]$ is the derived algebra of $\mathcal{L}$. The algorithms in Section 4 can be easily generalized to contact transformation. Our previous result in [17] could be quite helpful:

Theorem 7. *(Theorem 8 in [17]) Eq.* (1) *with* $n \geq 3$ *is C-linearizable if and only if one of the following conditions is fulfilled:*

[14] Contact symmetry generators can be also defined in the same way like Definition 1

1. $m \in \{n+1, n+2\}$ *and* $\mathcal{D}$ *is abelian of dimension* n*;*
2. $n \geq 4\,, m = n+4$*;*
3. $n = 3\,, m = 10$*. (In this case Eq. (1) is equivalent to* $u'''(t) = 0$ *up to a contact transformation)*

Remark 7. The first two cases in Theorem 7 is exactly the same like Theorem 3. This is because for linear ODEs, except for those who are equivalent to $u''' = 0$, they do not have nontrivial contact symmetry algebras (i.e. $\mathcal{L}$ is merely the first prolongation of point symmetry algebra).

The generalized algorithms can be given after a few adjustments, here we only show one of them because they are quite similar with point transformation counterparts.

Algorithm 5 ***C-determining system***

Input: q, a C-linearizable ODE in the form of Eq. (1) with $n \geq 3, m \in \{n+1, n+2\}$.
Output: A determining system T for (ϕ, ψ) which linearizes q.

1: $L :=$ ***DerterminingPDE*** (q);
2: $L := L \cup \{\eta^{[1]} = \eta_x + p(\eta_y - \xi_x) - p^2\xi_y\}$;
3: $D :=$ ***DeterminingSystem*** $([L, L])$;
4: $P :=$ ***ParametricDerivatives*** (D);
5: $\theta := \psi_p/\phi_p\,, \eta^{[1]} := \eta_x + p(\eta_y - \xi_x) - p^2\xi_y, \bar{\eta} := \xi\psi_x + \eta\psi_y + \eta^{[1]}\psi_p$;
6: $T := \{\phi_p(\psi_x + p\psi_y) = \psi_p(\phi_x + p\phi_y)\} \cup \{(\theta\phi_y - \psi_y)((\theta_x + p\theta_y)\phi_p - (\phi_x + p\phi_y)\theta_p) \neq 0\,, \phi_p \neq 0\}$;
7: $S := \{\xi\phi_x + \eta\phi_y + \eta^{[1]}\phi_p\,, \phi_x\bar{\eta}_y - \phi_y\bar{\eta}_x\,, \phi_y\bar{\eta}_p - \phi_p\bar{\eta}_y\,, \phi_p\bar{\eta}_x - \phi_x\bar{\eta}_p\} \mod D$;
8: $C :=$ ***Coefficients*** (S, P);
9: $T := \{e = 0 : e \in C\} \cup T$.
10: $J :=$ ***JacobianMatrix*** (ϕ, ψ, θ);
11: $(\xi, \eta, \eta^{[1]})^{\mathrm{T}} := J^{-1}(0, a\psi, a\theta)^{\mathrm{T}}$;
12: $L' :=$***Substitutions*** $((\xi, \eta, \eta^{[1]}), L)$;
13: $C' :=$ ***Coefficients*** $(L', \{a\})$;
14: $T := \{e = 0 : e \in C'\} \cup T$.

Remark 8. In contact symmetry algebra, the third coordinate $\eta^{[1]}$ is uniquely determined by (ξ, η), so in the first step, Maple™ constructs L in terms of (ξ, η). But generally, (ξ, η) does not form a Lie algebra without $\eta^{[1]}$, thus a prolongation of L in step 2 is necessary. As to the last three expressions in step 5, they are equivalent to say that rank$\begin{pmatrix} \bar{\eta}_x & \bar{\eta}_y & \bar{\eta}_p \\ \phi_x & \phi_y & \phi_p \end{pmatrix} = 1$ (i.e. $\bar{\eta}$ is a function of ϕ). And in step 6 $\phi_p \neq 0$ is added to exclude point transformations.

Example 6. *([31])* Let us consider

$$-16y'^2y''y^{(4)} + 48y'^2y'''^2 + y'y''^5x - 48y'y''^2y''' - y''^5y + 12y''^4 = 0 \qquad (17)$$

This example also passes our linearization test with dimension $m = 6$. It requires also the computation of the derived algebra which is 4-dimensional and abelian. From step 6 of Alg. 5 we have $\xi_x = \xi_y = 0$ in the derived algebra, which means none of its generators is prolonged from a point symmetry generator (i.e. Eq. (17) is C-linearizable but not P-linearizable). The determining system for all possible linearizing contact transformations is given by

$$\{\phi_x = 0, \phi_y = 0, \psi_{xx} = 0, \psi_{xy} = 0, \psi_x + p\psi_y = 0\}, \{\phi_p \neq 0, \psi_p \neq 0, \psi_x \neq 0\}.$$

It includes also solutions with non-homogeneous right-hand side. We do not write here explicitly system with finite-dimensional space of solutions because of its brevity.

All implementations of algorithms (both for contact and point cases) could be found on GitHub([16]).

7 Conclusion

Exact linearization theory based on Lie algebras admits quite natural generalization to wide classes of differential equations and systems. The key point here is symmetry (group) classification of linear equations in equivalent class. Future work in this regard should rely on algorithmic classification of branches under the action of equivalence group ([12]).

The generalizations is straightforward for differential equations with finite-dimensional symmetry group, but becomes more complicated when address to partial differential equations. It suffers from obvious obstacle of existence of infinite-dimensional abelian subalgebra of superposition symmetry. Any method based on structure constants as finite-dimensional description of abstract Lie algebra cannot really work with it.

Infinitesimal symmetries for PDE have natural description on the level of Lie pseudogroups. The transitive Lie pseudogroups admit algorithmic construction of its Cartan structure ([13]). It would be interesting part of future work to extract linearizability property on this basis.

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References

1. Vladimir Arnold, Ordinary Differential Equations, Springer, Berlin, 1992.
2. Thomas Bächler, Vladimir Gerdt, Markus Lange-Hegermann, and Daniel Robertz. *Algorithmic Thomas decomposition of algebraic and differential systems.* Journal of Symbolic Computation 47, 10 (2012): 1233–1266.

3. François Boulier, Daniel Lazard, François Ollivier and Michel Petitot *Representation for the radical of a finitely generated differential ideal*. Proceedings of the International Symposium on Symbolic and Algebraic Computation (ISSAC'95), ACM New York (1995): 158–166.
4. James Freitag and Omar Leon Sanchez *Effective uniform bounding in partial differential fields*. Advances in Mathematics 288, Elsevier (2016): 308–336.
5. Giorgio Gubbiotti and Maria Clara Nucci *Are all classical superintegrable systems in two-dimensional space linearizable?*, Journal of Mathematical Physics 58 (2017): 012902.
6. Evelyne Hubert, Notes on triangular sets and triangulation-decomposition algorithms. II Differential systems, In: Franz Winkler, and Ulrich Langer (Editors), Symbolic and Numerical Scientific Computation, Hagenberg, 2001, Lecture Notes in Computer Science, Vol. 2630, Springer, Berlin, 2003, pp. 40–87.
7. Nail Ibragimov and Sergey Meleshko. *Linearization of third-order ordinary differential equations by point and contact transformations.*. Journal of Mathematical Analysis and Applications, 308 (2005): 266–289.
8. Nail Ibragimov, A Practical Course in Differential Equations and Mathematical Modelling. Classical and New Methods. Nonlinear Mathematical Models. Symmetry and Invariance Principles, Higher Education Press, World Scientific, New Jersey, 2009.
9. Krause, J., Michel, L., Classification of the Symmetries of Ordinary Differential Equations, Proceedings of the Summer School in Group Theoretical Methods in Physics, Moscow, 1990, Springer LNP 382, 1991.
10. Sukeyuki Kumei and George Bluman. *When nonlinear differential equations are equivalent to linear differential equations?*. SIAM Journal on Applied Mathematics, 42 (1982): 1157–1173.
11. Markus Lange-Hegermann. *The Differential Counting Polynomial*. Foundations of Computational Mathematics, 18 (2018): 291–308.
12. Ian Lisle and Tracy Huang. *Algorithmic symmetry classification with invariance*. Journal of Engineering Mathematics (2010): 201–216.
13. Ian Lisle and Greg Reid. *Geometry and Structure of Lie Pseudogroups from Infinitesimal Defining Systems*. Journal of Symbolic Computation (1998): 355–379
14. Ian Lisle and Tracy Huang. *Algorithmic calculus for Lie determining systems*. Journal of Symbolic Computation 79 (2017): 482–498.
15. Dmitry A. Lyakhov, Vladimir P. Gerdt and Dominik L. Michels. *Algorithmic Verification of Linearizability for Ordinary Differential Equations*. Proceedings of the 42nd International Symposium on Symbolic and Algebraic Computation (ISSAC'17), Michael Burr (Ed.), ACM New York (2017): 285–292.
16. https://github.com/444083220/ODE-linearization-scripts
17. Yang Liu, Dmitry Lyakhov and Dominik Michels. *Contact Linearizability of Scalar Ordinary Differential Equations of Arbitrary Order.*. Proceedings of the International Workshop on Computer Algebra in Scientific Computing CASC'2020), Springer (2020): 421–430.
18. Dmitry A. Lyakhov, Vladimir P. Gerdt and Dominik L. Michels. *On the algorithmic linearizability of nonlinear ordinary differential equations*. Journal of Symbolic Computation 98, Elsevier (2020): 3–22.
19. Fazal Mahomed and Peter Leach. *Symmetry Lie Algebra of nth Order Ordinary Differential Equations*. Journal of Mathematical Analysis and Applications 151, no. 1 (1990): 80–107.

20. Zahra Mohammadi, Gregory Reid and Tracy Huang. *Introduction of the MapDE algorithm for determination of mappings relating differential equations.*. Proceedings of the 2019 on International Symposium on Symbolic and Algebraic Computation (2019): 331–338.
21. Zahra Mohammadi, Gregory Reid and Tracy Huang. *Extensions of the MapDE algorithm for mappings relating differential equations.*. arXiv preprint arXiv:1903.03727 (2019).
22. Zahra Mohammadi, Gregory Reid and Tracy Huang. *Symmetry-based algorithms for invertible mappings of polynomially nonlinear PDE to linear PDE.*. Mathematics in Computer Science (2020): 1–24.
23. Zahra Mohammadi, Gregory Reid and Tracy Huang. *The Lie Algebra of Vector Fields Package with Applications to Mappings of Differential Equations.*. Communications in Computer and Information Science. Volume 1125 CCIS (2020): 337–340.
24. Peter Olver. *Equivalence, Invariance and Symmetry.* Cambridge University Press, 1995.
25. Lev Ovsyannikov. *Group Analysis of Differential Equations.* Academis Press, New York, 1982.
26. Greg Reid. *Finding abstract Lie symmetry algebras of differential equations without integrating determining equations.* European Journal of Applied Mathematics 2, no. 04 (1991): 319–340.
27. Daniel Robertz. *Formal Algorithmic Elimination for PDEs.* Lecture Notes in Mathematics, Vol. 2121, Springer, 2014.
28. Colin Rust, Gregory Reid and Alan Wittkopf. *Existence and uniqueness theorems for formal power series solutions of analytic differential systems.* Proceedings of the International Symposium on Symbolic and Algebraic Computation (1999): 105–112.
29. Werner Seiler. *Involution: The Formal Theory of Differential Equations and its Applications in Computer Algebra.* Algorithms and Computation in Mathematics 24. Springer, Heidelberg, 2010.
30. Celestin Soh and Fazal Mahomed. *Contact Symmetry Algebras of Scalar Ordinary Differential Equations.* Nonlinear Dynamics 28 (2002): 213–230.
31. Supaporn Suksern, Nail Ibragimov and Sergey Meleshko. *Criteria for fourth-order ordinary differential equations to be linearizable by contact transformations.*. Communications in Nonlinear Science and Numerical Simulation, 14 (2009): 266–289.
32. Sergei Svirshchevskii. *Lie-Bäcklund symmetries of linear ODEs and generalized separation of variables in nonlinear equations.* Physics Letter A 199 (1995): 344-348.
33. Joseph Thomas, Differential Systems, AMS Colloquium Publications, Vol. XXI, AMS, New York, 1937.
34. Joseph Thomas, Systems and Roots, William Byrd Press, Richmond, 1962.
35. Dongming Wang, Elimination Methods, Springer, Wien, 2001.
36. Valeriy Yumaguzhin. *Classification of 3rd order linear ODE up to equivalence.* Differential Geometry and its Applications 6, no. 4 (1996): 343-350.