Contact Linearizability of Scalar Ordinary Differential Equations of Arbitrary Order

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Contents

Linearization

We consider ODEs of the form:

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \quad n \geq 3 \tag{1}$$

We want to know if Eq. (1) is linearizable through some contact transformation.

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

$$\downarrow$$

$$u^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\, u^{(k)}(t) = 0$$

Point transformation (PT)

Point transformation (PT) is an analytical diffeomorphism:

$$t = t(x, y)\,, u = u(x, y)\,, t_x u_y - t_y u_x \neq 0.$$

$t = \sqrt{x}\,, u = y\,, x \neq 0.$

$$y''' + 3y''/(2x) = 0$$

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Contact transformation (CT)

Add one more variable $p = y'(x)$ in PT:

$$X = X(x, y, p)\,, Y = Y(x, y, p)\,, P = Y'(X) = Y_p/X_p.$$

The last expression is coming from $Y_p = (Y(X))_p = X_p Y'(X)$.

Another expression for P is from total differentiation

$$P = Y'(X) = D_x Y/D_x X = (Y_x + pY_y + p'Y_p)/(X_x + pX_y + p'X_p).$$

Thus $X_p(Y_x + pY_y) = Y_p(X_x + pX_y)$ is also required in a CT to equate above two expressions.

And the nonsingularity of Jacobian can be simplified as $(PX_y - Y_y)((P_x + pP_y)X_p - (X_x + pX_y)P_p) \neq 0$.

$X = p\,, Y = xp - y\,, P = x.$

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Contact symmetry

Let $(X, Y, P) = T_\alpha(x, y, p) = T(x, y, p, \alpha)$ be a CT with a parameter α, which maps $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$ to itself for all possible values of α. Further, T also satisfies:

$$T_{\alpha+\beta} = T_\alpha \circ T_\beta\,, T_0(x, y, p) = (x, y, p).$$

Definition.

1) Let $(a(x, y, p), b(x, y, p), c(x, y, p)) = \frac{\partial}{\partial\alpha} T(x, y, p, \alpha)|_{\alpha=0}$, we call the vector field $\mathcal{X} := a\,\partial_x + b\,\partial_y + c\,\partial_p$ a contact symmetry generator associated with T;

2) All possible generators form a Lie algebra under Lie bracket $[\mathcal{X}_1, \mathcal{X}_2] = \mathcal{X}_1\mathcal{X}_2 - \mathcal{X}_2\mathcal{X}_1$, we call it the contact symmetry algebra of Eq. (1).

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Main result of the paper

In this paper, we provide a sufficient and necessary condition for the contact linearizability of Eq. (1)

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \quad n \geq 3.$$

Moreover, if Eq. (1) is linearizable through some CT, we algorithmically construct a determining system for all possible CTs (i.e. PDEs for $X(x, y, p)$ and $Y(x, y, p)$).

Main theorem

Let L be the contact symmetry algebra of Eq. (1) and $m = \dim(L)$, our main theorem is:

Main theorem.

Eq. (1) with $n \geq 3$ is linearizable by a contact transformation if and only if one of the following conditions is fulfilled:

1. $n = 3, m = 10$ or $n \geq 4, m = n + 4$ [Lie,1883a/1883b],
2. $n \geq 3$, $m = n + 1$ or $n + 2$ and the derived algebra $DA = [L, L]$ is abelian of dimension n.

Linearization test

Input: q =Eq. (1);
n :=DifferentialOrder (q);
DS := DeterminingSystem (q) (i.e. PDEs of a, b, c);
L := LieSymmetryAlgebra (DS);
$m := \dim(L)$;
if $(n = 3 \wedge m = 10) \vee (n > 3 \wedge m = n + 4)$ then
  return TRUE;
else if $n \geq 3 \wedge (m = n + 1 \vee m = n + 2)$ then
  DA := DerivedAlgebra (L);
  if DA is abelian and $\dim(DA) = n$ then
    return TRUE;
  end if
end if
return FALSE.

Bluman-Kumei equations

Assume that a CT $(X, Y, P) = T(x, y, p)$ maps Eq. (1) to:

$$Y^{(n)} = F(X, Y, Y', \ldots, Y^{(n-1)}) \tag{2}$$

Then T induces an isomorphism between their contact symmetry algebras,

$$T : a\,\partial_x + b\,\partial_y + c\,\partial_p \mapsto A\,\partial_X + B\,\partial_Y + C\,\partial_P,$$

which is expressed as B-K equations:

$$\begin{aligned} a(x,y,p)X_x + b(x,y,p)X_y + c(x,y,p)X_p &= A(x,y,p)\,, \\ a(x,y,p)Y_x + b(x,y,p)Y_y + c(x,y,p)Y_p &= B(x,y,p)\,, \\ a(x,y,p)P_x + b(x,y,p)P_y + c(x,y,p)P_p &= C(x,y,p)\,. \end{aligned}$$

Remark: In B-K equations, new symmetry (A, B, C) is expressed in old variables.

Algorithm for determining system of CT

Assume that Eq. (1) belongs to the 2nd case in our main theorem and Eq. (2) is linear.

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The derived algebra of Eq. (2) has a very simple structure:

$$DA_2 = \{(A, B, C) = (0, f(X), f'(X)) : f(X) \text{ is a solution of Eq. (2)}\}.$$

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Since $B(x, y, p)$ is a function of X in DA_2, combined with B-K equations we have the following algorithm:

1. Input: $q =$Eq. (1);
2. $DS :=$ DeterminingSystem (q);
3. $L :=$ LieSymmetryAlgebra (DS);
4. $DA :=$ DerivedAlgebra (L);
5. In B-K equations set $S := \{A = 0\, , \frac{B_x}{X_x} = \frac{B_y}{X_y} = \frac{B_p}{X_p}\}$;
6. Reduce S by the system of DA;
7. Vanish all the coefficients of parametric derivatives in S, denoted by Sys;
8. Output $Sys = Sys \cup \{X_p(Y_x + pY_y) = Y_p(X_x + pX_y)\}$.

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Example

Let us consider

$$-16y'^2y''y^{(4)} + 48y'^2y'''^2 + y'y''^5x - 48y'y''^2y''' - y''^5y + 12y''^4 = 0 \quad (3)$$

This example passes our linearization test with dimension $m = 6$. It requires also the computation of the derived algebra which is 4-dimensional and abelian. Our second algorithm gives the system of differential equations and inequations

$$\{X_x = 0, X_y = 0, Y_{xx} = 0, Y_{xy} = 0, Y_x + pY_y = 0\}, \{X_p \neq 0, Y_p \neq 0, Y_x \neq 0\}$$

which forms basis of linearizing mappings of Eq. (3).

Example

Illustration.

$$DA = \{b_{ppp} = \frac{4b_p - 16pb_{pp} + pb - p^2 a}{16p^2}, a_p = \frac{b_p}{p}, a_x = b_x = a_y = b_y = c = 0\}$$

$$ParaDriv = \{a, b, b_p, b_{pp}\}$$

$$S = \{aX_x + bX_y = 0, \frac{aY_{xx} + bY_{xy}}{X_x} = \frac{aY_{xy} + bY_{yy}}{X_y} = \frac{B_p}{X_p}\} \mod DA$$

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Remark: Here we write $\frac{B_x}{X_x} = \frac{B_y}{X_y} = \frac{B_p}{X_p}$ only for conciseness, but in our code they are 3 determinents without any denominators.

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Conclusions

- We constructed a new algebraic linearization test for ODEs by contact transformation.
- Moreover, we find a way to algorithmically construct the determining system for linearizing CTs of linearizable ODEs.
- For cases $m = n + 1, n + 2$, our algorithm works efficiently.
- For cases $n = 3, m = 10$ or $n \geq 4, m = n + 4$, we still have an algorithm for linearizing CTs. But it is not as practical as the 2nd algorithm since there are fewer useful properties in their derived algebras.

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