

# Communication

## An Explicit Time Marching Scheme for Efficient Solution of the Magnetic Field Integral Equation at Low Frequencies

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**Abstract**—An explicit marching-on-in-time (MOT) scheme to efficiently solve the time domain magnetic field integral equation (TD-MFIE) with a large time step size (under a low-frequency excitation) is developed. The proposed scheme spatially expands the current using high-order nodal functions defined on curvilinear triangles discretizing the scatterer surface. Applying Nyström discretization, which uses this expansion, to the TD-MFIE, which is written as an ordinary differential equation (ODE) by separating self-term contribution, yields a system of ODEs in unknown time-dependent expansion coefficients. A predictor-corrector method is used to integrate this system for samples of these coefficients. Since the Gram matrix arising from the Nyström discretization is block-diagonal, the resulting MOT scheme replaces the matrix “inversion” required at each time step by a product of the inverse block-diagonal Gram matrix and the right-hand side vector. It is shown that, upon convergence of the corrector updates, this explicit MOT scheme produces the same solution as its implicit counterpart, and is faster for large time step sizes.

**Index Terms**—Marching-on-in-time (MOT), magnetic field integral equation (MFIE), Nyström method, predictor-corrector scheme.

### I. INTRODUCTION

TRANSIENT electromagnetic scattering from perfect electrically conducting (PEC) objects can be analyzed by solving time domain surface integral equations (TD-SIEs) [1]–[18]. A TD-SIE for a PEC scatterer is often constructed by enforcing electric and magnetic field boundary conditions on the scatterer surface. These boundary conditions are expressed in terms of the known incident field and the scattered field; and representing the latter as a space-time integral of the unknown current induced on the scatterer surface yields the electric field integral equation (EFIE) or the magnetic field integral equation (MFIE) (depending on the type of the field used). Also, the combined field integral equation (CFIE) can be obtained by linearly combining the (rotated) EFIE and the MFIE.

One of the prevalent methods developed for solving TD-SIEs is the marching-on-in-time (MOT) scheme [10]–[16]. The classical MOT scheme expands the unknown current induced on the scatterer surface using the Rao-Wilton-Glisson (RWG) basis functions [19] in space and piece-wise Lagrange polynomials [3], [13], [14] in time. This expansion is inserted into the TD-SIE, and the resulting equation is Galerkin-tested in space and point-tested in time yielding a lower triangular matrix system in unknown expansion coefficients.

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Applying backward substitution to this matrix system results in a time marching algorithm, where a reduced system of equations known as the MOT system is solved at every time step for the unknown coefficients associated with only that time step. The right-hand side of the MOT system consists of the incident and scattered fields tested at the same time step. After the solution, the coefficients, which are now known, are used for computing the tested scattered field at the next step. This process is repeated until all expansion coefficients at all time steps are computed.

The time step size of the MOT scheme described above is selected as  $\Delta t = 1/(\alpha f_{\max})$ , where  $f_{\max}$  is the maximum frequency of the incident field and  $\alpha$  is an oversampling coefficient. For high-frequency excitations,  $\Delta t$  is small and the MOT system is sparse. Consequently, the computation of the tested scattered field on the right-hand side of the MOT system is more costly than its solution (which is often done using an iterative solver). This computation is often accelerated using (multilevel) plane wave time domain (PWTd) algorithms [10]–[13] or fast Fourier transformation (FFT)-based schemes [14]–[16]. For low-frequency excitations,  $\Delta t$  is large and the MOT system is dense (even full), and the computational cost of solving this system becomes larger than that of computing the tested scattered field on the right-hand side.

In [8], a quasi-explicit MOT scheme, which does not suffer from this drawback, has been developed to solve the TD-MFIE. This scheme expresses the TD-MFIE as an ordinary differential equation (ODE) that relates the current to its temporal derivative. Discretizing this ODE using RWG basis and testing functions [19] yields a system of ODEs in time-dependent expansion coefficients of the RWG basis functions. Integrating this system in time provides the samples of these coefficients. This MOT scheme avoids solving a dense matrix system but still requires solution of a sparse Gram matrix system at each time step.

This sparse matrix inversion required at every step can be eliminated if a spatial discretization scheme, which does not use basis functions defined over two mesh elements, is used. Indeed in [20], an MOT scheme that makes use of this idea is used to solve the time domain integral equations of acoustics. In this work, this scheme is extended to solve the TD-MFIE efficiently when  $\Delta t$  is large. The current is spatially expanded using high-order nodal interpolation functions defined on curvilinear triangular patches. Applying Nyström discretization, which uses this expansion, converts the ODE form of the TD-MFIE into a system of ODEs in time-dependent expansion coefficients. Then, this system is integrated using a predictor-corrector (PE(CE)<sup>m</sup>) scheme to yield the samples of these coefficients. The Gram matrix arising from the Nyström discretization is block-diagonal (with  $2 \times 2$  blocks) and its inverse is constructed very efficiently from the inverse of these blocks at the beginning of time integration. Therefore, the matrix inversion required at every time step is replaced by a block matrix and vector product. This explicit MOT scheme maintains its stability even when using  $\Delta t$  as large as that would be used by its implicit counter-

part (traditional MOT scheme where RWG-based discretization is replaced by Nyström discretization [18]) and is significantly faster for large  $\Delta t$ . Additionally, it is shown that when the orders of the temporal basis function of the implicit solver and the PE(CE)<sup>m</sup> scheme of the explicit solver are the same, both solvers produce the same solution upon convergence of the corrector updates.

## II. FORMULATION

### A. TD-MFIE

Let  $S$  denote the surface of a PEC object residing in an unbounded homogeneous medium with permittivity  $\varepsilon$  and permeability  $\mu$ . An electromagnetic wave with magnetic field  $\mathbf{H}^i(\mathbf{r}, t)$  that is band-limited to  $f_{\max}$  is incident on the object. An electric current  $\mathbf{J}(\mathbf{r}, t)$  is induced on  $S$  and generates a scattered magnetic field  $\mathbf{H}^s(\mathbf{r}, t)$  as

$$\mathbf{H}^s(\mathbf{r}, t) = \int_S \nabla \times \frac{\mathbf{J}(\mathbf{r}', t')}{4\pi R} ds'. \quad (1)$$

Here,  $t' = t - R/c$  is the retarded time,  $R = |\mathbf{r} - \mathbf{r}'|$  is the distance between the points  $\mathbf{r}$  and  $\mathbf{r}'$ , and  $c = 1/\sqrt{\mu\varepsilon}$  is the speed of light. Inserting (1) into the temporal derivative of the magnetic-field boundary condition on  $S$ , i.e.,  $\partial_t \mathbf{J}(\mathbf{r}, t) = \partial_t \hat{\mathbf{n}}(\mathbf{r}) \times [\mathbf{H}^i(\mathbf{r}, t) + \mathbf{H}^s(\mathbf{r}, t)]$ , yields the temporal derivative form of the TD-MFIE as [8]

$$\frac{1}{2} \partial_t \mathbf{J}(\mathbf{r}, t) = \hat{\mathbf{n}}(\mathbf{r}) \times \partial_t \mathbf{H}^i(\mathbf{r}, t) + \hat{\mathbf{n}}(\mathbf{r}) \times \int_S \nabla \times \frac{\partial_t \mathbf{J}(\mathbf{r}', t')}{4\pi R} ds'. \quad (2)$$

Here,  $\hat{\mathbf{n}}(\mathbf{r})$  is the unit normal vector pointing outwards from  $S$  at  $\mathbf{r}$ .

### B. Discretization

To numerically solve (2),  $\mathbf{J}(\mathbf{r}, t)$  is expanded in space using high-order Lagrange polynomial interpolation functions as [21]

$$\mathbf{J}(\mathbf{r}, t) = \sum_{p=1}^{N_p} \sum_{i=1}^{N_n} [\{\mathbf{I}(t)\}_{ip}^u \mathbf{u}(\mathbf{r}) + \{\mathbf{I}(t)\}_{ip}^v \mathbf{v}(\mathbf{r})] \vartheta^{-1}(\mathbf{r}) \ell_{ip}(\mathbf{r}). \quad (3)$$

Here,  $N_p$  and  $N_n$  are numbers of the curvilinear triangular patches discretizing  $S$  and the interpolation nodes on each patch, respectively,  $\ell_{ip}(\mathbf{r})$  represents the Lagrange interpolation function defined at node  $\mathbf{r}_{ip}$  (node  $i$  on patch  $p$ ),  $\vartheta(\mathbf{r})$  is the Jacobian of the transformation that maps the patch description in the Cartesian coordinate system to the right triangle defined by variable pair  $(u, v)$ , vectors  $\mathbf{u}(\mathbf{r}) = \partial_u \mathbf{r}$  and  $\mathbf{v}(\mathbf{r}) = \partial_v \mathbf{r}$  are tangential to  $S$  at  $\mathbf{r}$ , and  $\{\mathbf{I}(t)\}_{ip}^u$  and  $\{\mathbf{I}(t)\}_{ip}^v$  are the time-dependent unknown expansion coefficients of  $\mathbf{J}(\mathbf{r}_{ip}, t)$ 's components along the directions of  $\mathbf{u}(\mathbf{r}_{ip})$  and  $\mathbf{v}(\mathbf{r}_{ip})$ , respectively. To account for the time retardation  $t'$  in (2),  $\{\mathbf{I}(t)\}_{ip}^b$ ,  $b \in \{u, v\}$  are expanded using piece-wise Lagrange polynomial interpolation functions as [3], [13], [14]

$$\{\mathbf{I}(t)\}_{ip}^b = \sum_{l=1}^{N_t} I_l|_{ip}^b T(t - l\Delta t). \quad (4)$$

Here,  $I_l|_{ip}^b = \{\mathbf{I}(l\Delta t)\}_{ip}^b$ ,  $b \in \{u, v\}$ ,  $T(t)$  is constructed using piece-wise Lagrange polynomials, and  $N_t$  is the number of time steps.

### C. Explicit MOT Scheme (E-MOT)

Substituting (3) in (2) and spatially testing with  $\mathbf{u}(\mathbf{r}_{jq})$  and  $\mathbf{v}(\mathbf{r}_{jq})$ ,  $j = 1, \dots, N_n$ ,  $q = 1, \dots, N_p$  yield a time-dependent semi-discrete system of ODEs. This system has to be sampled at times  $t = h\Delta t$  to carry out the time integration using a PE(CE)<sup>m</sup>-type scheme [8], [20], [22]. Consequently one has to use temporal interpolation on  $\{\mathbf{I}(t')\}_{ip}^b$ ,  $b \in \{u, v\}$ . This is done by using (4), which leads to a fully discretized linear system as

$$\mathbf{G}\mathbf{I}_h = \mathbf{V}_h^i - \sum_{l=1}^h \mathbf{Z}_{h-l}^{\text{exp}} \mathbf{I}_l, \quad h = 1, \dots, N_t. \quad (5)$$

Here, the block-diagonal Gram matrix  $\mathbf{G}$  is given by (6) at the top of the next page, where its entries are

$$G_{jq,ip}^{\text{ab}} = \frac{1}{2} \mathbf{a}(\mathbf{r}_{jq}) \cdot \mathbf{b}(\mathbf{r}_{ip}) \vartheta^{-1}(\mathbf{r}_{ip}) \quad (7)$$

$\mathbf{a}, \mathbf{b} \in \{u, v\}$ ,  $\mathbf{a}, \mathbf{b} \in \{u, v\}$ , the E-MOT matrices  $\mathbf{Z}_{h-l}^{\text{exp}}$  are given by (8) at the top of the next page, where their entries are

$$\mathbf{Z}_{h-l}^{\text{exp}|jq,ip} = \mathbf{a}(\mathbf{r}_{jq}) \cdot \hat{\mathbf{n}}(\mathbf{r}_{jq}) \times \int_{S_p} \vartheta^{-1}(\mathbf{r}') \ell_{ip}(\mathbf{r}') \frac{\mathbf{R}}{4\pi R^3} \left[ \partial_t T(t) + \frac{R}{c} \partial_t^2 T(t) \right]_{t=(h-l)\Delta t - R/c} \times \mathbf{b}(\mathbf{r}') ds' \quad (9)$$

$S_p$  is the support of patch  $p$ ,  $\mathbf{R} = \mathbf{r}_{jq} - \mathbf{r}'$ ,  $R = |\mathbf{r}_{jq} - \mathbf{r}'|$ , the tested excitation vector  $\mathbf{V}_h^i$  are given by

$$\mathbf{V}_h^i = [V_h|_{11}^u, V_h|_{11}^v, V_h|_{21}^u, V_h|_{21}^v, \dots, V_h|_{N_i N_p}^u, V_h|_{N_i N_p}^v]^T$$

with entries  $V_h|_{jq}^i = \mathbf{a}(\mathbf{r}_{jq}) \cdot \hat{\mathbf{n}}(\mathbf{r}_{jq}) \times \partial_t \mathbf{H}^i(\mathbf{r}_{jq}, h\Delta t)$ , unknown coefficient vector  $\mathbf{I}_l$  are given by

$$\mathbf{I}_l = [I_l|_{11}^u, I_l|_{11}^v, I_l|_{21}^u, I_l|_{21}^v, \dots, I_l|_{N_i N_p}^u, I_l|_{N_i N_p}^v]^T$$

and  $\dot{\mathbf{I}}_h$  store samples of the time derivative of the coefficients as

$$\dot{\mathbf{I}}_h = [\dot{I}_h|_{11}^u, \dot{I}_h|_{11}^v, \dot{I}_h|_{21}^u, \dot{I}_h|_{21}^v, \dots, \dot{I}_h|_{N_i N_p}^u, \dot{I}_h|_{N_i N_p}^v]^T$$

with  $\dot{I}_h|_{ip}^b = \{\partial_t \mathbf{I}(h\Delta t)\}_{ip}^b$ . Note that the weak singularity of the integral in (9) is cancelled using the Duffy transformation [23] to permit its numerical evaluation.

A PE(CE)<sup>m</sup> scheme is used to integrate the ODE system (5) to yield  $\mathbf{I}_h$ ,  $h = 1, \dots, N_t$  [8], [20], [22]. Steps of this scheme are briefly summarized as follows:

0) Compute  $\mathbf{G}^{-1}$

Loop over  $h = 1, \dots, N_t$

1) Compute the part of right-hand side of (5) that does not change within time step  $h$ :

$$\mathbf{V}_h^{\text{fix}} = \mathbf{V}_h^i - \sum_{l=1}^{h-1} \mathbf{Z}_{h-l}^{\text{exp}} \mathbf{I}_l. \quad (10)$$

2) Predict  $\mathbf{I}_h$  using  $\mathbf{I}_l$  and  $\dot{\mathbf{I}}_l$ :

$$\mathbf{I}_h = \sum_{k=1}^K [\{\mathbf{p}\}_k \mathbf{I}_{h-1+k-K} + \{\mathbf{p}\}_{K+k} \dot{\mathbf{I}}_{h-1+k-K}]. \quad (11)$$

3) Evaluate  $\dot{\mathbf{I}}_h$  using  $\mathbf{V}_h^{\text{fix}}$  and the predicted  $\mathbf{I}_h$ :

$$\dot{\mathbf{I}}_h = \mathbf{G}^{-1} (\mathbf{V}_h^{\text{fix}} - \mathbf{Z}_0^{\text{exp}} \mathbf{I}_h). \quad (12)$$

4) Set  $\dot{\mathbf{I}}_h^{(0)} = \dot{\mathbf{I}}_h$  and start (CE)<sup>m</sup> updates

Loop over  $n = 1, \dots, m$  (until convergence)

4.1) Correct  $\mathbf{I}_h^{(n)}$  using  $\dot{\mathbf{I}}_h^{(n-1)}$ ,  $\mathbf{I}_l$  and  $\dot{\mathbf{I}}_l$ :

$$\mathbf{I}_h^{(n)} = \sum_{k=1}^K [\{\mathbf{c}\}_k \mathbf{I}_{h-1+k-K} + \{\mathbf{c}\}_{K+k} \dot{\mathbf{I}}_{h-1+k-K} + \{\mathbf{c}\}_{2K+1} \dot{\mathbf{I}}_h^{(n-1)}]. \quad (13)$$

4.2) Evaluate  $\dot{\mathbf{I}}_h^{(n)}$  using  $\mathbf{V}_h^{\text{fix}}$  and the corrected  $\mathbf{I}_h^{(n)}$ :

$$\dot{\mathbf{I}}_h^{(n)} = \mathbf{G}^{-1} (\mathbf{V}_h^{\text{fix}} - \mathbf{Z}_0^{\text{exp}} \mathbf{I}_h^{(n)}). \quad (14)$$

4.3) Check convergence  $\|\mathbf{I}_h^{(n)} - \mathbf{I}_h^{(n-1)}\| < \chi^{\text{PECE}}$ , where  $\chi^{\text{PECE}}$  is a threshold parameter.

End loop over  $n$

5) Once convergence is achieved, e.g., at iteration  $m$ , set  $\mathbf{I}_h = \mathbf{I}_h^{(m)}$  and  $\dot{\mathbf{I}}_h = \dot{\mathbf{I}}_h^{(m)}$ .

End loop over  $h$ .



Consequently, the number of  $\mathbf{Z}_{h-l}^{\text{exp}}$ , which are not completely zero, decreases with increasing  $\Delta t$  (for lower frequency excitations) while at the same time they become denser matrices. For example, for  $D_{\text{max}}/(c\Delta t) < 1$ , every non-zero  $\mathbf{Z}_{h-l}^{\text{exp}}$  is a fully dense matrix. For smaller  $\Delta t$  (for higher frequency excitations), the opposite happens: the number of  $\mathbf{Z}_{h-l}^{\text{exp}}$ , which are not completely zero, increases while at the same time they become sparser matrices.

(ii)  $\mathbf{Z}_{h-l}^{\text{exp}}$  and  $\mathbf{Z}_{h-l}^{\text{imp}}$  have the same sparsity structure since the entries of  $\mathbf{G}$  contribute to entries of  $\mathbf{Z}_{h-l}^{\text{imp}}$  when  $\partial_t T(t)|_{t=(h-l)\Delta t} \neq 0$  and  $\mathbf{r}_{jq} = \mathbf{r}_{ip}$  (test and source interpolation nodes are the same), and the entries of  $\mathbf{Z}_{h-l}^{\text{exp}}$  are already non-zero for these cases.

(iii) Let  $C_{\text{fix}}^{\text{exp}}$  and  $C_{\text{fix}}^{\text{imp}}$  represent the cost of computing  $\mathbf{V}_h^{\text{fix}} = \mathbf{V}_h^i - \sum_{l=1}^{h-1} \mathbf{Z}_{h-l}^{\text{exp}} \mathbf{I}_l$  [see (10)] by E-MOT and  $\mathbf{V}_h^i - \sum_{l=1}^{h-1} \mathbf{Z}_{h-l}^{\text{imp}} \mathbf{I}_l$  [see the right-hand side of (15)] by I-MOT, respectively, at a given time step. Since the sparseness levels of  $\mathbf{Z}_{h-l}^{\text{exp}}$  and  $\mathbf{Z}_{h-l}^{\text{imp}}$  are the same (even though not all entries of these matrices are the same—see comment (ii) above), the cost of computing the summations  $\sum_{l=1}^{h-1} \mathbf{Z}_{h-l}^{\text{exp}} \mathbf{I}_l$  and  $\sum_{l=1}^{h-1} \mathbf{Z}_{h-l}^{\text{imp}} \mathbf{I}_l$  is the same, which means that  $C_{\text{fix}}^{\text{exp}} = C_{\text{fix}}^{\text{imp}}$ . Note that, regardless of  $\Delta t$ , the computation of  $\sum_{l=1}^{h-1} \mathbf{Z}_{h-l}^{\text{exp}} \mathbf{I}_l$  or  $\sum_{l=1}^{h-1} \mathbf{Z}_{h-l}^{\text{imp}} \mathbf{I}_l$  at any time step  $h > \lfloor D_{\text{max}}/(c\Delta t) \rfloor + T_{\text{max}}$  requires  $O(N_s^2)$  operations. Here,  $N_s = 2N_p N_n$  is the number of unknowns. However, depending on  $\Delta t$ , cost of this computation relative to overall cost of E-MOT or I-MOT changes.

Let  $C_{\text{tot}}^{\text{imp}} = C_{\text{sol}}^{\text{imp}} + C_{\text{fix}}^{\text{imp}}$  and  $C_{\text{tot}}^{\text{exp}} = C_{\text{sol}}^{\text{exp}} + C_{\text{fix}}^{\text{exp}}$  denote the total computational cost per time step for I-MOT and E-MOT, respectively. Here,  $C_{\text{sol}}^{\text{exp}}$  is the computational cost for Steps 2-4 of the PE(CE)<sup>m</sup> method used by E-MOT and is obtained as

$$C_{\text{sol}}^{\text{exp}} \sim O(\{m[2K+1]+2K\}N_s) + O([m+1]\gamma N_s) + O([m+1]2N_s)$$

where  $m$  is the number of corrector updates and  $\gamma$  is defined as the average number of non-zero entries in rows of  $\mathbf{Z}_0^{\text{exp}}$ . The first term is the total cost of summations in Step 2 (once) and Step 4.1 ( $m$  times), respectively. The second term is the total cost of matrix-vector products  $\mathbf{Z}_0^{\text{exp}} \mathbf{I}_h$  in Step 3 (once) and  $\mathbf{Z}_0^{\text{exp}} \mathbf{I}_h^{(n)}$  in Step 4.2 ( $m$  times). The last term is total cost of matrix-vector products involving  $\mathbf{G}^{-1}$  in Step 3 (once) and Step 4.2 ( $m$  times).

I-MOT uses an iterative method to solve the matrix system (15). Let  $C_{\text{sol}}^{\text{imp}}$  represent the cost of this solution:

$$C_{\text{sol}}^{\text{imp}} \sim O(N_{\text{iter}}^{\text{imp}} F_{\text{iter}}^{\text{imp}} \gamma N_s)$$

where  $N_{\text{iter}}^{\text{imp}}$  and  $F_{\text{iter}}^{\text{imp}}$  are the numbers of iterations and matrix-vector products  $\mathbf{Z}_0^{\text{imp}} \mathbf{I}_h^{(n)}$  carried out at every iteration, respectively. Note that  $\mathbf{Z}_0^{\text{imp}}$  and  $\mathbf{Z}_0^{\text{exp}}$  have the same  $\gamma$  (see comment (ii) above).

When  $\Delta t \ll D_{\text{max}}/c$  (high-frequency excitation),  $\mathbf{Z}_0^{\text{exp}}$  and  $\mathbf{Z}_0^{\text{imp}}$  are both very sparse, i.e.,  $\gamma \ll N_s$  (see comment (i) above). Therefore,  $C_{\text{sol}}^{\text{imp}}$  and  $C_{\text{sol}}^{\text{exp}}$  scale as  $C_{\text{sol}}^{\text{imp}} \sim O(N_{\text{iter}}^{\text{imp}} F_{\text{iter}}^{\text{imp}} \gamma N_s)$  and  $C_{\text{sol}}^{\text{exp}} \sim O(mKN_s) + O(m\gamma N_s)$ , respectively. Comparing these estimates with  $C_{\text{fix}}^{\text{exp}}$  and  $C_{\text{fix}}^{\text{imp}}$  (see comment (iii) above), one can see that  $C_{\text{sol}}^{\text{imp}} \ll C_{\text{fix}}^{\text{imp}}$  and  $C_{\text{sol}}^{\text{exp}} \ll C_{\text{fix}}^{\text{exp}}$ , which consequently results in  $C_{\text{tot}}^{\text{imp}} \approx C_{\text{fix}}^{\text{imp}}$  and  $C_{\text{tot}}^{\text{exp}} \approx C_{\text{fix}}^{\text{exp}}$ , respectively. This means that both solvers have similar total execution times under high-frequency excitations as also shown by results presented in Section III.

As the frequency of excitation decreases (for large  $\Delta t$ ,  $\Delta t \approx D_{\text{max}}/c$ ),  $\mathbf{Z}_0^{\text{exp}}$  and  $\mathbf{Z}_0^{\text{imp}}$  become denser (and even full) resulting in  $\gamma \approx N_s$  (see comments (i) above). Consequently,  $C_{\text{sol}}^{\text{imp}} \sim O(N_{\text{iter}}^{\text{imp}} F_{\text{iter}}^{\text{imp}} N_s^2)$  and  $C_{\text{sol}}^{\text{exp}} \sim O(mN_s^2)$  resulting in  $C_{\text{fix}}^{\text{imp}} \ll C_{\text{sol}}^{\text{imp}}$  and  $C_{\text{fix}}^{\text{exp}} \ll C_{\text{sol}}^{\text{exp}}$ , respectively (see comment (iii) above). In summary, under low-frequency excitations,  $C_{\text{tot}}^{\text{imp}} \approx C_{\text{sol}}^{\text{imp}}$  and  $C_{\text{tot}}^{\text{exp}} \approx C_{\text{sol}}^{\text{exp}}$  and E-MOT is faster than I-MOT for  $m < N_{\text{iter}}^{\text{imp}} F_{\text{iter}}^{\text{imp}}$ . In Section III, it is shown by numerical results that this condition is satisfied.

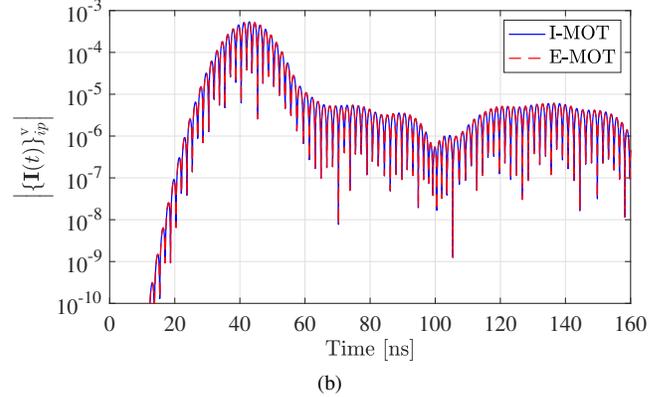
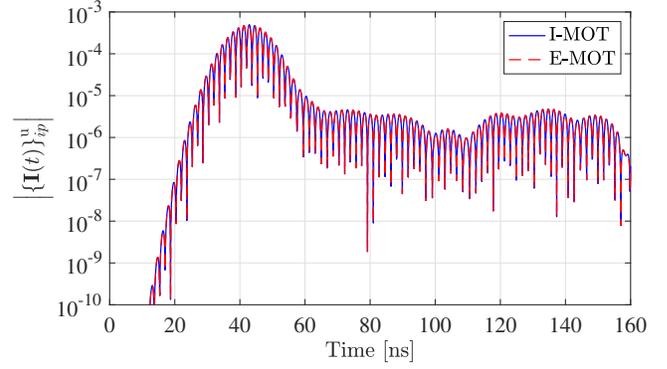


Fig. 1. (a)  $|\{\mathbf{I}(t)\}_{ip}^u|$  and (b)  $|\{\mathbf{I}(t)\}_{ip}^v|$ ,  $p = 1014$ ,  $i = 6$  calculated by the MOT schemes at  $\mathbf{r}_{ip} = (0.97, -0.245, 0.005)$  m.

It is well-known that  $C_{\text{fix}}^{\text{imp}}$  can be reduced using PWT [10]–[13] or FFT-based schemes [14]–[16]. But the same algorithms can be applied to  $C_{\text{fix}}^{\text{exp}}$  without any modifications, ensuring that  $C_{\text{fix}}^{\text{exp}} = C_{\text{fix}}^{\text{imp}}$  would still hold. Similarly, FFT-based methods have been developed to reduce the cost of matrix-vector product  $\mathbf{Z}_0^{\text{imp}} \mathbf{I}_h^{(n)}$  required by I-MOT [17]. E-MOT can utilize this method in the same way to compute the matrix-vector products  $\mathbf{Z}_0^{\text{exp}} \mathbf{I}_h$  and  $\mathbf{Z}_0^{\text{exp}} \mathbf{I}_h^{(n)}$ .

### III. NUMERICAL RESULTS

In this section, E-MOT and I-MOT are used to analyze transient electromagnetic scattering from a unit sphere centered at the origin. The sphere is excited by a planewave with magnetic field

$$\mathbf{H}^i(\mathbf{r}, t) = \hat{\mathbf{y}} \sqrt{\varepsilon/\mu} G(t - \hat{\mathbf{k}} \cdot \mathbf{r}/c) \quad (20)$$

where  $\hat{\mathbf{k}} = \hat{\mathbf{z}}$  is the direction of propagation and  $G(t) = \cos[2\pi f_0(t - t_0)]e^{-(t-t_0)^2/(2w^2)}$  is a Gaussian pulse, where  $w = 7/(2\pi f_{\text{bw}})$ ,  $f_{\text{bw}}$ ,  $f_0$ , and  $t_0 = 3.5w + 10/(f_0 + f_{\text{bw}})$  are the pulse duration, bandwidth, modulation frequency, and delay, respectively. Note that  $f_{\text{max}} = f_0 + f_{\text{bw}}$  is the effective maximum frequency. In all examples, the order of  $\ell_{ip}(\mathbf{r})$  used by the Nyström method is two ( $N_n = 6$ ) [21] and the order of Lagrange polynomials used for constructing  $T(t)$  is three ( $T_{\text{max}} = 3$ ). The predictor coefficients  $\mathbf{p}$  and the corrector coefficients  $\mathbf{c}$  are obtained using the Adams-Bashforth and backward difference methods, respectively, resulting in  $K = 4$  for the PE(CE)<sup>m</sup> scheme [24]. Convergence thresholds for corrector updates and the TFQMR scheme are  $\chi^{\text{PECE}} = \chi^{\text{TFQMR}} = 10^{-13}$ .

For the first set of simulations,  $N_p = 1126$  ( $N_s = 13512$ ),  $f_0 = 300$  MHz,  $f_{\text{bw}} = 200$  MHz,  $N_t = 1600$ , and  $\Delta t = 0.1$  ns  $[1/(20f_{\text{max}})]$ . Figs. 1(a) and (b) compare  $|\{\mathbf{I}(t)\}_{ip}^u|$  and  $|\{\mathbf{I}(t)\}_{ip}^v|$  computed by I-MOT and E-MOT at  $\mathbf{r}_{ip} = (0.97, -0.245, 0.005)$  m ( $p = 1014$ ,  $i = 6$ ), respectively. The results agree very well.

TABLE I  
EXECUTION TIMES OF THE MOT SCHEMES' DIFFERENT STAGES.

$N_s$	$f_0$	I-MOT						E-MOT					$\frac{T_{\text{TFQMR}}^{\text{imp}}}{T_{\text{PECE}}^{\text{exp}}}$	$\frac{T_{\text{tot}}^{\text{imp}}}{T_{\text{tot}}^{\text{exp}}}$
		$\sigma_{\text{err}}^{\text{imp}}$	$T_{\text{fix}}^{\text{imp}} [\text{s}]$	$T_{\text{TFQMR}}^{\text{imp}} [\text{s}]$	$T_{\text{tot}}^{\text{imp}} [\text{s}]$	$\{N_{\text{iter}}^{\text{imp}} F_{\text{iter}}^{\text{imp}}\}_{\text{avg}}$	$\sigma_{\text{err}}^{\text{exp}}$	$T_{\text{fix}}^{\text{exp}} [\text{s}]$	$T_{\text{PECE}}^{\text{exp}} [\text{s}]$	$T_{\text{tot}}^{\text{exp}} [\text{s}]$	$m_{\text{avg}}$			
2424	5	0.011	61.81	763.61	825.53	66.66	0.011	65.43	129.02	194.56	9.78	5.92	4.24	
2424	10	0.007	67.67	567.24	635.00	68.02	0.007	68.05	67.68	135.83	7.19	8.38	4.68	
2424	25	0.004	83.87	100.29	184.21	70.15	0.004	83.65	7.40	91.14	4.19	13.55	2.02	
2424	50	0.025	89.84	55.09	144.99	80.26	0.025	90.02	3.46	93.57	3.92	15.92	1.55	
2424	100	0.047	91.70	28.56	120.31	92.53	0.047	91.71	2.07	93.86	5.20	13.80	1.28	
2424	200	0.046	93.85	19.52	113.41	96.40	0.046	93.99	1.33	95.42	4.77	14.68	1.19	
2424	400	0.068	99.47	8.61	108.13	99.80	0.068	99.62	0.63	100.33	4.32	13.67	1.08	
7464	5	0.010	583.40	6266.59	6850.92	57.62	0.010	585.11	1155.56	1741.13	9.79	5.42	3.93	
7464	10	0.006	638.02	4650.56	5289.34	58.56	0.006	637.91	640.10	1278.45	7.20	7.27	4.14	
7464	25	0.004	779.17	784.12	1563.65	60.78	0.004	780.93	64.87	846.22	4.06	12.09	1.85	
7464	50	0.025	828.96	349.04	1178.32	70.56	0.025	829.76	22.90	853.07	3.56	15.24	1.38	
7464	100	0.047	850.05	136.80	987.14	79.08	0.047	851.62	10.80	862.83	4.85	12.67	1.14	
7464	200	0.046	857.49	69.68	927.46	81.46	0.046	851.57	5.59	857.57	4.68	12.47	1.08	
7464	400	0.066	882.09	44.40	926.77	82.91	0.066	884.64	3.70	888.75	4.46	12.00	1.04	

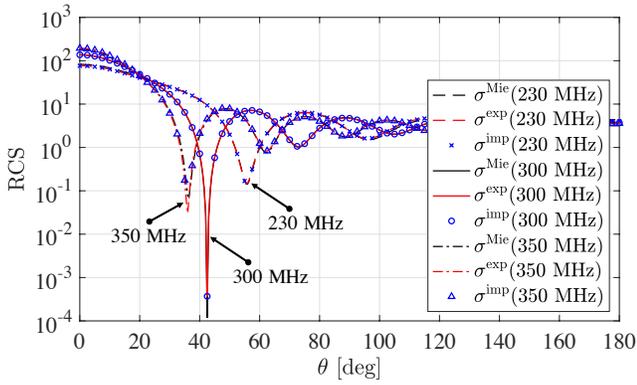


Fig. 2.  $\sigma^{\text{Mie}}(\theta, \varphi, f)$ ,  $\sigma^{\text{imp}}(\theta, \varphi, f)$ , and  $\sigma^{\text{exp}}(\theta, \varphi, f)$  versus  $\theta$  for  $\varphi = 0^\circ$  and  $f \in \{230, 300, 350\}$  MHz.

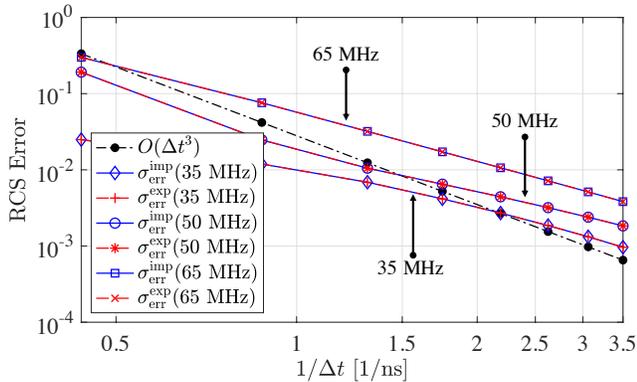


Fig. 3.  $L_2$ -norm error in RCS obtained from the solutions computed by the MOT schemes versus  $1/\Delta t$ .

Let  $\sigma^{\text{Mie}}(\theta, \varphi, f)$ ,  $\sigma^{\text{imp}}(\theta, \varphi, f)$ , and  $\sigma^{\text{exp}}(\theta, \varphi, f)$  denote the radar cross section (RCS) values computed using the Mie series solution and the Fourier transform of the solutions obtained by I-MOT and E-MOT, respectively. Fig. 2 plots  $\sigma^{\text{Mie}}(\theta, \varphi, f)$ ,  $\sigma^{\text{imp}}(\theta, \varphi, f)$ , and  $\sigma^{\text{exp}}(\theta, \varphi, f)$  versus  $\theta$  for  $\varphi = 0^\circ$  and  $f \in \{230, 300, 350\}$  MHz. The figure clearly shows that the solutions obtained by I-MOT and E-MOT are accurate within the band of the excitation.

For the second set of simulations, two spatial discretizations are

considered:  $N_p = 202$  ( $N_s = 2424$ ) and  $N_p = 622$  ( $N_s = 7464$ ). For each discretization, seven sets of excitations are considered:  $f_0 = \{5, 10, 25, 50, 100, 200, 400\}$  MHz and  $f_{\text{bw}} = 0.75f_0$ . E-MOT and I-MOT are both executed for  $N_t = 500$  time steps with  $\Delta t = 1/(17.5f_0) = 1/(10f_{\text{max}})$ . The ‘‘denseness’’ level in  $\mathbf{Z}_0^{\text{exp}}$  and  $\mathbf{Z}_0^{\text{imp}}$  for these values of  $\Delta t$  and for the two different discretizations are:  $\gamma = \{2424, 1777, 303, 143, 61, 38, 13\}$  for  $N_p = 202$  and  $\gamma = \{7464, 5471, 881, 340, 116, 54, 32\}$  for  $N_p = 622$ , respectively. For each excitation and discretization, RCS values  $\sigma^{\text{imp}}(\theta, \varphi, f)$  and  $\sigma^{\text{exp}}(\theta, \varphi, f)$  are computed for  $\theta = [0^\circ, 180^\circ]$ ,  $\varphi = 0^\circ$ , and  $f = f_0$  using the Fourier transform of the solutions obtained by I-MOT and E-MOT, respectively. The  $L_2$ -norm error in RCS is defined as

$$\sigma_{\text{err}}^{\text{type}} = \sqrt{\frac{\sum_{n=0}^{360} |\sigma^{\text{type}}(n\Delta\theta, \varphi, f) - \sigma^{\text{Mie}}(n\Delta\theta, \varphi, f)|^2}{\sum_{n=0}^{360} |\sigma^{\text{Mie}}(n\Delta\theta, \varphi, f)|^2}} \quad (21)$$

where  $\text{type} \in \{\text{imp}, \text{exp}\}$ ,  $f = f_0$ ,  $\Delta\theta = 0.5^\circ$ , and  $\varphi = 0^\circ$ . Table I lists  $\sigma_{\text{err}}^{\text{imp}}$  and  $\sigma_{\text{err}}^{\text{exp}}$  and shows that the error level in the solutions obtained by I-MOT and E-MOT are the same.

Execution times of MOT schemes’ different stages are compared in Table I. Here,  $T_{\text{fix}}^{\text{imp}}$  and  $T_{\text{fix}}^{\text{exp}}$  are the total times required for computing  $\mathbf{V}_h^{\text{imp}} - \sum_{l=1}^{h-1} \mathbf{Z}_{h-l}^{\text{imp}} \mathbf{I}_l$  and  $\mathbf{V}_h^{\text{exp}} - \sum_{l=1}^{h-1} \mathbf{Z}_{h-l}^{\text{exp}} \mathbf{I}_l$  for  $h = 1, \dots, N_t$ , by I-MOT and E-MOT, respectively.  $T_{\text{TFQMR}}^{\text{imp}}$  is the total time required to iteratively solve the I-MOT system in (15) by the TFQMR method for  $h = 1, \dots, N_t$ .  $T_{\text{PECE}}^{\text{exp}}$  is the total time required by the PE(CE)<sup>m</sup> method for  $h = 1, \dots, N_t$ .  $T_{\text{tot}}^{\text{imp}}$  and  $T_{\text{tot}}^{\text{exp}}$  are the total execution times of I-MOT and E-MOT, respectively. Finally,  $\{N_{\text{iter}}^{\text{imp}} F_{\text{iter}}^{\text{imp}}\}_{\text{avg}}$  and  $m_{\text{avg}}$  are the average values of  $N_{\text{iter}}^{\text{imp}} F_{\text{iter}}^{\text{imp}}$  and  $m$  over  $N_t$  time steps, respectively.

Table I shows that, as expected,  $T_{\text{fix}}^{\text{imp}}$  and  $T_{\text{fix}}^{\text{exp}}$  are almost the same. Also, for all values of  $\Delta t$ ,  $T_{\text{PECE}}^{\text{exp}} < T_{\text{TFQMR}}^{\text{imp}}$  since  $m_{\text{avg}} \ll \{N_{\text{iter}}^{\text{imp}} F_{\text{iter}}^{\text{imp}}\}_{\text{avg}}$ . This has different consequences for simulations with large and small  $\Delta t$ . For large  $\Delta t$  (low-frequency excitation),  $T_{\text{TFQMR}}^{\text{imp}} \gg T_{\text{fix}}^{\text{imp}}$ ,  $T_{\text{PECE}}^{\text{exp}} \gg T_{\text{fix}}^{\text{exp}}$ , and  $T_{\text{tot}}^{\text{imp}} \approx T_{\text{fix}}^{\text{imp}}$ , therefore  $T_{\text{tot}}^{\text{imp}} > T_{\text{tot}}^{\text{exp}}$ . Indeed the results show that the E-MOT is roughly four times faster than I-MOT. As  $\Delta t$  gets smaller (frequency gets higher),  $T_{\text{fix}}^{\text{imp}}$  and  $T_{\text{fix}}^{\text{exp}}$  become larger than  $T_{\text{TFQMR}}^{\text{imp}}$  and  $T_{\text{PECE}}^{\text{exp}}$ , and the fact that  $T_{\text{PECE}}^{\text{exp}} < T_{\text{TFQMR}}^{\text{imp}}$  does not reflect on the total MOT times  $T_{\text{tot}}^{\text{imp}}$  and  $T_{\text{tot}}^{\text{exp}}$  anymore. Indeed, the results show that both schemes require almost the same time to complete the simulations as  $\Delta t$  gets smaller. The CPU times presented in Table I confirm computational complexity analysis in Section II-F.

For the last set of simulations,  $N_p = 622$  ( $N_s = 7464$ ),  $f_0 = 50$  MHz and  $f_{bw} = 37.5$  MHz; and I-MOT and E-MOT are executed for different values of  $\Delta t$  changed between 2.29 ns  $[1/(5f_{max})]$  and 0.29 ns  $[1/(40f_{max})]$ . RCS  $L_2$ -norm error values  $\sigma_{err}^{imp}$  and  $\sigma_{err}^{exp}$  are computed for  $f \in \{35, 50, 65\}$  MHz,  $\Delta\theta = 0.5^\circ$ , and  $\varphi = 0^\circ$  using (21). Fig. 3 plots  $\sigma_{err}^{imp}$  and  $\sigma_{err}^{exp}$  versus  $1/\Delta t$  for  $f \in \{35, 50, 65\}$  MHz. This figure clearly shows that E-MOT has the same accuracy as I-MOT (as discussed in Section II-E). Additionally, Fig. 3 shows that accuracy of the schemes follows the convergence curve of  $O(\Delta t^3)$  for larger values of  $\Delta t$  (note that the order of Lagrange polynomials constructing  $T(t)$  is  $T_{max} = 3$ ) but, for smaller values of  $\Delta t$ , the accuracy is limited by the second-order spatial discretization and as a result the convergence with respect to  $1/\Delta t$  becomes slightly slower than  $O(\Delta t^3)$ .

#### IV. CONCLUSION

An explicit MOT scheme to efficiently solve the TD-MFIE for large time step sizes (under low-frequency excitations) is developed. The current is spatially expanded using high-order nodal functions defined on curvilinear triangles discretizing the scatterer surface. Applying Nyström discretization, which uses this expansion, to the TD-MFIE yields a system of ODEs in time-dependent expansion coefficients. This system is integrated in time using a PE(CE)<sup>m</sup> method to compute the samples of these coefficients. The Gram matrix arising from the Nyström discretization is block-diagonal with  $2 \times 2$  blocks and its inverse is constructed from the inverse of the blocks before the time marching. Therefore, the matrix inversion needed at each time step is replaced by the product of the inverse block-diagonal Gram matrix and the right-hand side vector. Numerical results show that the resulting MOT scheme can use the time step sizes as large as those would be used by its implicit counterpart without sacrificing the stability, has the same level of accuracy, and is more than four times faster for low-frequency excitations.

The explicit MOT scheme can be extended to solve the TD-CFIE obtained by combining the TD-MFIE with the Calderón-preconditioned TD-EFIE. The Calderón-preconditioning ensures that the TD-CFIE is a second-kind surface integral equation [26]–[29].

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