An Explicit Time Marching Scheme for Efficient Solution of the Magnetic Field Integral Equation at Low Frequencies

Rui Chen, Member, IEEE, Sadeed B. Sayed, Member, IEEE, H. Arda Ulku, Senior Member, IEEE, and Hakan Bagci, Senior Member, IEEE

Abstract—An explicit marching-on-in-time (MOT) scheme to efficiently solve the time domain magnetic field integral equation (TD-MFIE) with a large time step size (under a low-frequency excitation) is developed. The proposed scheme spatially expands the current using high-order nodal functions defined on curvilinear triangles discretizing the scatterer surface. Applying Nyström discretization, which uses this expansion, to the TD-MFIE, which is written as an ordinary differential equation (ODE) by separating self-term contribution, yields a system of ODEs in unknown time-dependent expansion coefficients. A predictor-corrector method is used to integrate this system for samples of these coefficients. Since the Gram matrix arising from the Nyström discretization is block-diagonal, the resulting MOT scheme replaces the matrix “inversion” required at each time step by a product of the inverse block-diagonal Gram matrix and the right-hand side vector. It is shown that, upon convergence of the corrector updates, this explicit MOT scheme produces the same solution as its implicit counterpart, and is faster for large time step sizes.

Index Terms—Marching-on-in-time (MOT), magnetic field integral equation (MFIE), Nyström method, predictor-corrector scheme.

I. INTRODUCTION

TRANSIENT electromagnetic scattering from perfect electrically conducting (PEC) objects can be analyzed by solving time domain surface integral equations (TD-SIEs) [1]–[18]. A TD-SIE for a PEC scatterer is often constructed by enforcing electric and magnetic field boundary conditions on the scatterer surface. These boundary conditions are expressed in terms of the known incident field and the scattered field; and representing the latter as a space-time integral of the unknown current induced on the scatterer surface. Hence, the electric field integral equation (EFIE) or the magnetic field integral equation (MFIE) (depending on the type of the field used). Also, the combined field integral equation (CFIE) can be obtained by linearly combining the (rotated) EFIE and the MFIE.

One of the prevalent methods developed for solving TD-SIEs is the marching-on-in-time (MOT) scheme [10]–[16]. The classical MOT scheme expands the unknown current induced on the scatterer surface using the Rao-Wilton-Glisson (RWG) basis functions [19] in space and piece-wise Lagrange polynomials [3], [13], [14] in time. This expansion is inserted into the TD-SIE, and the resulting equation is Galerkin-tested in space and point-tested in time yielding a lower triangular matrix system in unknown expansion coefficients.

Applying backward substitution to this matrix system results in a time marching algorithm, where a reduced system of equations known as the MOT system is solved at every time step for the unknown coefficients associated with only that time step. The right-hand side of the MOT system consists of the incident and scattered fields tested at the same time step. After the solution, the coefficients, which are now known, are used for computing the tested scattered field at the next step. This process is repeated until all expansion coefficients at all time steps are computed.

The time step size of the MOT scheme described above is selected as \( \Delta t = 1/(\alpha f_{\text{max}}) \), where \( f_{\text{max}} \) is the maximum frequency of the incident field and \( \alpha \) is an oversampling coefficient. For high-frequency excitations, \( \Delta t \) is small and the MOT system is sparse. Consequently, the computation of the tested scattered field on the right-hand side of the MOT system is more costly than its solution (which is often done using an iterative solver). This computation is often accelerated using (multilevel) plane wave time domain (PWTD) algorithms [10]–[13] or fast Fourier transformation (FFT)-based schemes [14]–[16]. For low-frequency excitations, \( \Delta t \) is large and the MOT system is dense (even full), and the computational cost of solving this system becomes larger than that of computing the tested scattered field on the right-hand side.

In [8], a quasi-explicit MOT scheme, which does not suffer from this drawback, has been developed to solve the TD-MFIE. This scheme expresses the TD-MFIE as an ordinary differential equation (ODE) that relates the current to its temporal derivative. Discretizing this ODE using RWG basis and testing functions [19] yields a system of ODEs in time-dependent expansion coefficients of the RWG basis functions. Integrating this system in time provides the samples of these coefficients. The MOT scheme avoids solving a dense matrix system but still requires solution of a sparse Gram matrix system at each time step.

This sparse matrix inversion required at every step can be eliminated if a spatial discretization scheme, which does not use basis functions defined over two mesh elements, is used. Indeed in [20], an MOT scheme that makes use of this idea is used to solve the time domain integral equations of acoustics. In this work, this scheme is extended to solve the TD-MFIE efficiently when \( \Delta t \) is large. The current is spatially expanded using high-order nodal interpolation functions defined on curvilinear triangular patches. Applying Nyström discretization, which uses this expansion, converts the ODE form of the TD-MFIE into a system of ODEs in time-dependent expansion coefficients. Then, this system is integrated using a predictor-corrector (PE(CE))\(^\alpha\) scheme to yield the samples of these coefficients. The Gram matrix arising from the Nyström discretization is block-diagonal (with \( 2 \times 2 \) blocks) and its inverse is constructed very efficiently from the inverse of these blocks at the beginning of time integration. Therefore, the matrix inversion required at every time step is replaced by a block matrix and vector product. This explicit MOT scheme maintains its stability even when using \( \Delta t \) as large as that would be used by its implicit counter-
part (traditional MOT scheme where RWG-based discretization is replaced by Nyström discretization [18]) and is significantly faster for large $\Delta t$. Additionally, it is shown that when the orders of the temporal basis function of the implicit solver and the PE(CE)$^m$ scheme of the explicit solver are the same, both solvers produce the same solution upon convergence of the corrector updates.

II. FORMULATION

A. TD-MFIE

Let $S$ denote the surface of a PEC object residing in an unbounded homogeneous medium with permittivity $\varepsilon$ and permeability $\mu$. An electromagnetic wave with magnetic field $H'(r,t)$ that is band-limited to $f_{\text{max}}$ is incident on the object. An electric current $J(r,t)$ is induced on $S$ and generates a scattered magnetic field $H'(r,t)$ as

$$H'(r,t) = \int_S \nabla \times \frac{J(r,t',t)}{4\pi R} \cdot ds'. \quad (1)$$

Here, $t' = t - R/c$ is the retarded time, $R = |r - r'|$ is the distance between the points $r$ and $r'$, and $c = 1/\sqrt{\varepsilon\mu}$ is the speed of light. Inserting (1) into the temporal derivative of the magnetic-field boundary condition on $S$, i.e., $\partial_t H'(r,t) = \partial_0 \mathbf{H}(r) + \mathbf{H}'(r,t)$, yields the temporal derivative form of the TD-MFIE as [8]

$$\frac{1}{2} \partial_t J(r,t) = \hat{\mathbf{n}}(r) \times \partial_0 \mathbf{H}(r,t) + \hat{\mathbf{n}}(r) \times \int_S \nabla \times \frac{\partial_0 J(r',t)}{4\pi R} \cdot ds'. \quad (2)$$

Here, $\hat{\mathbf{n}}(r)$ is the unit normal vector pointing outwards from $S$ at $r$.

B. Discretization

To numerically solve (2), $J(r,t)$ is expanded in space using high-order Lagrange polynomial interpolation functions as [21]

$$J(r,t) = \sum_{p=1}^{N_p} \sum_{i=1}^{N_n} \mathbf{N}_i \mathbf{u}(r) + \mathbf{N}_i \mathbf{v}(r) \mathbf{\partial} \mathbf{r}^{-1} \partial_0 \mathbf{r} \nu(i,p). \quad (3)$$

Here, $N_p$ and $N_n$ are numbers of the curvilinear triangular patches discretizing $S$ and the interpolation nodes on each patch, respectively, $\nu(i,p)$ represents the Lagrange interpolation function defined at node $r_{ip}$ (node $i$ on patch $p$), $\mathbf{\partial} \mathbf{r}$ is the Jacobian of the transformation that maps the patch description in the Cartesian coordinate system to the right triangle defined by variable pair $(u, v)$, vectors $\mathbf{u}(r) = \partial_0 \mathbf{r}$ and $\mathbf{v}(r) = \partial_1 \mathbf{r}$ are tangential to $S$ at $r$, and $\mathbf{N}_i \mathbf{u}(r)$ and $\mathbf{N}_i \mathbf{v}(r)$ are the time-dependent unknown expansion coefficients of $J(r_{ip}, t)$’s components along the directions of $\mathbf{u}(r_{ip})$ and $\mathbf{v}(r_{ip})$, respectively. To account for the time retardation $t'$ in (2), $\mathbf{N}_i \mathbf{u}(t')$, $b \in \{u,v\}$ are expanded using piecewise Lagrange polynomial interpolation functions as [3], [13], [14]

$$\mathbf{N}_i \mathbf{u}(t') = \sum_{l=1}^{N_t} \mathbf{I}_{il} \mathbf{T}(t - l\Delta t). \quad (4)$$

Here, $\mathbf{I}_{il} = \mathbf{N}_i (t\Delta t)$, $b \in \{u,v\}$, $T(t)$ is constructed using piece-wise Lagrange polynomials, and $N_t$ is the number of time steps.

C. Explicit MOT Scheme (E-MOT)

Substituting (3) in (2) and spatially testing with $\mathbf{u}(r_{jq})$ and $\mathbf{v}(r_{jq})$, $j = 1, \ldots, N_q$, $q = 1, \ldots, N_p$ yield a time-dependent semi-discrete system of ODEs. This system has to be sampled at times $t_h = h\Delta t$ to carry out the time integration using a PE(CE)$^m$-type scheme [8], [20], [22]. Consequently one has to use temporal interpolation on $\{\mathbf{T}(t')\}_{i,p}$, $b \in \{u,v\}$. This is done by using (4), which leads to a fully discretized linear system as

$$\mathbf{G} \mathbf{i}_h = \mathbf{v}_h - \sum_{l=1}^{h} \mathbf{Z}_{h\rightarrow l} \mathbf{i}_l, \quad h = 1, \ldots, N_t. \quad (5)$$

Here, the block-diagonal Gram matrix $\mathbf{G}$ is given by (6) at the top of the next page, where its entries are

$$G_{ip,jq}^{ab} = \frac{1}{2} a(r_{jq}) \cdot b(r_{ip}) \nabla^{-1} (r_{ip}) \quad \text{at } a, b \in \{u,v\}, \mathbf{Z}_{h\rightarrow l} \text{ are given by (8) at the top of the next page, where their entries are }$$

$$Z_{h\rightarrow l}^{ab} = a(r_{jq}) \cdot \hat{n}(r_{jq}) \int_{S_p} \nabla^{-1} (r') \delta_{lp} \delta(r') \quad \text{where } S_p \text{ is the support of patch } p. \quad (9)$$

Note that the weak singularity of the integral in (9) is cancelled using the Duffy transformation [23] to permit its numerical evaluation.

A PE(CE)$^m$ scheme is used to integrate the ODE system (5) to yield $\mathbf{i}_h$, $h = 1, \ldots, N_t$ [8], [20], [22]. Steps of this scheme are briefly summarized as follows:

0. Compute $\mathbf{G}^{-1}$

1) Compute the part of right-hand side of (5) that does not change within time step $h$:

$$\mathbf{V}_{h\rightarrow l}^{\text{fix}} = \mathbf{v}_h - \sum_{l=1}^{h-1} \mathbf{Z}_{h\rightarrow l} \mathbf{i}_l. \quad (10)$$

2) Predict $\mathbf{i}_h$ using $\mathbf{i}_l$ and $\mathbf{G}^{-1}$:

$$\mathbf{i}_h = \sum_{k=1}^{K} \left[ (\mathbf{p})_k \mathbf{i}_{h-k-1} + (\mathbf{p})_{K+k} \hat{\mathbf{i}}_{h-k-1} \right]. \quad (11)$$

3) Evaluate $\hat{\mathbf{i}}_h$ using $\mathbf{V}_{h\rightarrow l}^{\text{fix}}$ and the predicted $\mathbf{i}_h$:

$$\hat{\mathbf{i}}_h = \mathbf{G}^{-1} \left( \mathbf{V}_{h\rightarrow l}^{\text{fix}} - \mathbf{Z}_{h\rightarrow l} \mathbf{i}_h \right). \quad (12)$$

4) Set $\hat{\mathbf{i}}_h = \hat{\mathbf{i}}_h$ and start (CE)$^m$ updates

Loop over $n = 1, \ldots, m$ (until convergence)

4.1) Correct $\mathbf{i}_h^{(n)}$ using $\mathbf{i}_h^{(n-1)}$, $\mathbf{i}_l$, and $\mathbf{G}^{-1}$:

$$\mathbf{i}_h^{(n)} = \sum_{k=1}^{K} \left[ (\mathbf{c})_k \mathbf{i}_{h-k-1} + (\mathbf{c})_{K+k} \hat{\mathbf{i}}_{h-k-1} \right] + (\mathbf{c})_{2K+1} \hat{\mathbf{i}}_h^{(n-1)}. \quad (13)$$

4.2) Evaluate $\hat{\mathbf{i}}_h^{(n)}$ using $\mathbf{V}_{h\rightarrow l}^{\text{fix}}$ and the corrected $\mathbf{i}_h^{(n)}$:

$$\hat{\mathbf{i}}_h^{(n)} = \mathbf{G}^{-1} \left( \mathbf{V}_{h\rightarrow l}^{\text{fix}} - \mathbf{Z}_{h\rightarrow l} \mathbf{i}_h^{(n)} \right). \quad (14)$$

4.3) Check convergence $\| \mathbf{i}_h^{(n)} - \mathbf{i}_h^{(n-1)} \| < \chi^{\text{PECIE}}$, where $\chi^{\text{PECIE}}$ is a threshold parameter.

End loop over $n$

5) Once convergence is achieved, e.g., at iteration $m$, set $\mathbf{i}_h = \mathbf{i}_h^{(m)}$ and $\hat{\mathbf{i}}_h = \hat{\mathbf{i}}_h^{(m)}$.

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In this section, it is shown that, when both I-MOT and E-MOT use the Gram matrix $G$ does not depend on $\Delta t$ and is block diagonal with $2 \times 2$ blocks. Therefore, $G^{-1}$ is constructed very efficiently using the inverses of $G$'s blocks and is stored before the time marching starts. Since $G^{-1}$ is also block diagonal, the solutions of matrix systems (12) and (14) are done by simply multiplying $G^{-1}$ with the right-hand side vector. At the start of time marching, it is assumed that $I_0 = 0$ and $\tilde{I}_0 = 0$, for $l = 1 - K, \ldots, 0$. This assumption does not introduce any significant errors in the solution since $H'(r, t)$ is vanishingly small for $t \leq 0$, $\forall r$ in $S$ and bandlimited to $f_{max}$.

### D. Implicit MOT Scheme (I-MOT)

Substituting (3) and (4) in (2) and testing with $u(r, q)$ and $v(r, q)$, $j = 1, \ldots, N_u$, $q = 1, \ldots, N_p$ at times $t = h \Delta t$, $h = 1, \ldots, N_t$ yield the I-MOT system as [18]

$$Z_{0}^{imp}I_0 = V_{h}^{imp} - \sum_{l=1}^{h-1}Z_{l}^{imp}I_l, \ h = 1, \ldots, N_t. \quad (15)$$

Here, $I_0$ and $V_{h}^{imp}$ are same as those in (5) and the implicit MOT matrices $Z_{l}^{imp}$ are given by

$$Z_{l}^{imp} = G \partial T(t)|_{l=(h-l)\Delta t} + Z_{l}^{exp} \quad (16)$$

The unknown vectors $I_0$, $h = 1, \ldots, N_t$ are computed recursively using a time marching scheme. First, at $t = \Delta t$, $I_0$ is obtained by solving $Z_{0}^{imp}I_0 = V_{h}^{imp} \ [h = 1 \text{ in } (15) \text{. Then, at } t = 2\Delta t, I_2 \text{ is found by solving } Z_{2}^{imp}I_2 = V_{2}^{imp} - Z_{1}^{imp}I_1 \ [h = 2 \text{ in } (15) \text{. Next, at } t = 3\Delta t, I_3 \text{ is computed by solving } Z_{3}^{imp}I_3 = V_{3}^{imp} - Z_{2}^{imp}I_2 - Z_{1}^{imp}I_1 \ [h = 3 \text{ in } (15) \text{ and so on. During this time marching scheme, the solution of the MOT system (15) required at each time step is obtained using the transpose-free quasi-minimal residual (TFQMR) scheme [25]. This iterative solver is stopped when $\|I^{(n)}_0 - I^{(n-1)}_0\| < \chi_{TFQMR}$, where $I^{(n)}_0$ and $\chi_{TFQMR}$ represent the solution vector at iteration $n$ and a threshold parameter, respectively.}

### E. Comparison of Implicit and Explicit Solutions

In this section, it is shown that, when both I-MOT and E-MOT use the same $\Delta t$, corrector updates of E-MOT produce the same result as the iterative solution of the I-MOT system upon convergence, if $T(t)$ (consisting of piece-wise Lagrange polynomials) has the same order as the Lagrange polynomial interpolation used to derive the corrector coefficients $c$. This is best demonstrated by an example. Assume that $c$ is obtained using a third-order backward difference method [24], which leads to

$$I_h = \begin{bmatrix} I_{h-3} & 0 & 0 \\ 0 & I_{h-2} & -2I_{h-3} & \varepsilon \frac{6}{\Delta t}I_{h-3} \end{bmatrix}$$

(17)

Assuming that the corrector updates have converged, inserting (17) into (5) yields the matrix system

$$\frac{11}{6\Delta t}GI_h = 3 + \frac{3}{2\Delta t}GI_{h-2} - \frac{1}{3\Delta t}GI_{h-3} = V_h - \sum_{l=1}^h Z_{l}^{exp}I_l, \quad h = 1, \ldots, N_t \quad (18)$$

whose solution for $I_h$ should be equal to the result of the corrector updates. Merging the “past/history” terms with $I_{h-3}, I_{h-2}, \text{ and } I_{h-1}$ to the right-hand side of (18) and collecting the “instantaneous/current” terms with $I_0$ on the left-hand side yield:

$$\tilde{Z}_{0}^{exp}I_h = V_{h}^{imp} - \sum_{l=1}^{h-1}Z_{l}^{exp}I_l, \quad h = 1, \ldots, N_t \quad (19)$$

where $\tilde{Z}_{0}^{exp} = 11/(6\Delta t)G + Z_{0}^{exp}$, $\tilde{Z}_{1}^{exp} = -3/(\Delta t)G + Z_{1}^{exp}$, $\tilde{Z}_{2}^{exp} = 3/(2\Delta t)G + Z_{2}^{exp}$, $\tilde{Z}_{3}^{exp} = -1/(3\Delta t)G + Z_{3}^{exp}$, and $\tilde{Z}_{h}^{exp} = Z_{h}^{exp}, \quad l = 1, \ldots, h - 4.$

Assume that $T(t)$ is constructed using third-order Lagrange polynomials [13] [same order as the one used for backward difference in (17)], then $\partial T(t)|_{l=0} = 11/(6\Delta t)$, $\partial T(t)|_{l=\Delta t} = -3/\Delta t$, $\partial T(t)|_{l=2\Delta t} = 3/(2\Delta t)$, and $\partial T(t)|_{l=3\Delta t} = -1/(3\Delta t).$ Inserting these expressions in (16) yields $Z_{0}^{exp} = 11/(6\Delta t)G + Z_{0}^{exp}$, $Z_{1}^{exp} = -3/(\Delta t)G + Z_{1}^{exp}$, $Z_{2}^{exp} = 3/(2\Delta t)G + Z_{2}^{exp}$, $Z_{3}^{exp} = -1/(3\Delta t)G + Z_{3}^{exp}$, and $Z_{h}^{exp} = Z_{h}^{exp}, \quad l = 1, \ldots, h - 4$, which demonstrates that the system in (19) is same as the I-MOT system. Then, one can argue that, upon convergence, the corrector updates produce the same result as the iterative solution of the I-MOT system.

This statement is verified by the numerical examples in Section III. Note that the numerical results also show that E-MOT can use the same $\Delta t$ as I-MOT without sacrificing the stability of the solution.

### F. Computational Complexity Analysis

Before comparing the computational complexity of I-MOT and E-MOT, three observations are noted:

(i) The E-MOT matrices $Z_{h-1}^{exp} = 0$ for $h - l > |D_{max}/(c\Delta t)| + T_{max}$. Here, $T_{max}$ is the order of Lagrange polynomials used for constructing $T(t)$ and $D_{max}$ is the largest distance possible on $S.$
Consequently, the number of $Z_{\text{exp}}^{\exp}$, which are not completely zero, decreases with increasing $\Delta t$ (for lower frequency excitations) while at the same time they become denser matrices. For example, when $D_{\text{max}}/(c\Delta t) < 1$, every non-zero $Z_{\text{exp}}^{\exp}$ is a fully dense matrix. For smaller $\Delta t$ (for higher frequency excitations), the opposite happens: the number of $Z_{\text{exp}}^{\exp}$, which are not completely zero, increases while at the same time they become sparser matrices.

(ii) $Z_{\text{exp}}^{\exp}$ and $Z_{\text{imp}}^{\exp}$ have the same sparsity structure since the entries of $G$ contribute to entries of $Z_{\text{imp}}^{\exp}$ when $\partial_t T(t)|_{t=(h-1)\Delta t} \neq 0$ and $r_{ij} = r_{ip}$ (test and source interpolation nodes are the same), and the entries of $Z_{\text{imp}}^{\exp}$ are already non-zero for these cases.

(iii) Let $C_{\text{fix}}^{\exp}$ and $C_{\text{fix}}^{\exp}$ represent the cost of computing $V_h^{\text{fix}} = V_h - \sum_{l=1}^{h-1} Z_{\text{exp}}^{\exp} I_l$ [see (10)] by E-MOT and $V_h = \sum_{l=1}^{h-1} Z_{\text{exp}}^{\exp} I_l$ [see the right-hand side of (15)] by I-MOT, respectively, at a given time step. Since the sparseness levels of $Z_{\text{exp}}^{\exp}$ and $Z_{\text{imp}}^{\exp}$ are the same (even though not all entries of these matrices are the same—see comment (ii) above), the cost of computing the summations $\sum_{l=1}^{h-1} Z_{\text{exp}}^{\exp} I_l$ and $\sum_{l=1}^{h-1} Z_{\text{exp}}^{\exp} I_l$ is the same, which means that $C_{\text{fix}}^{\exp} = C_{\text{fix}}^{\exp}$. Note that, regardless of $\Delta t$, the computation of $\sum_{l=1}^{h-1} Z_{\text{exp}}^{\exp} I_l$ or $\sum_{l=1}^{h-1} Z_{\text{exp}}^{\exp} I_l$ at any time step $h > |D_{\text{max}}/(c\Delta t)| + T_{\text{max}}$ requires $O(N_h^2)$ operations. Here, $N_h = 2N_h N_h$ is the number of unknowns. However, depending on $\Delta t$, cost of this computation relative to overall cost of E-MOT or I-MOT changes. Let $C_{\text{imp}}^{\exp} = C_{\text{imp}}^{\exp} + C_{\text{imp}}^{\exp} + C_{\text{exp}}^{\exp}$. $C_{\text{imp}}^{\exp}$ denotes the total computational cost per time step for I-MOT and E-MOT, respectively. Here, $C_{\text{exp}}^{\exp}$ is the computational cost for Steps 2-4 of the PE(C) scheme used by E-MOT and is obtained as $C_{\text{exp}}^{\exp} \sim C\left(m(2K + 1 + 2K)N_h\right) + O\left([m + 1] \gamma N_h\right) + O\left([m + 1] 2N_h\right)$ where $m$ is the number of corrector updates and $\gamma$ is defined as the average number of non-zero entries in rows of $Z_{\text{imp}}^{\exp}$. The first term is the total cost of summations in Step 2 (once) and Step 4.1 ($m$ times), respectively. The second term is the total cost of matrix-vector products $Z_{\text{imp}}^{\exp} I_l$ in Step 3 (once) and $Z_{\text{imp}}^{\exp} I_l$ in Step 4.2 ($m$ times). The last term is the total cost of matrix-vector products involving $G^{-1}$ in Step 3 (once) and Step 4.2 ($m$ times).

I-MOT uses an iterative method to solve the matrix system (15). Let $C_{\text{imp}}^{\exp}$ represent the cost of this solution:

$$C_{\text{imp}}^{\exp} \sim O(N_{\text{iter}} F_{\text{iter}} \gamma N_h)$$

where $N_{\text{iter}}$ and $F_{\text{iter}}$ are the numbers of iterations and matrix-vector products $Z_{\text{imp}}^{\exp} I_l$ carried out at every iteration, respectively. Note that $Z_{\text{imp}}^{\exp}$ and $Z_{\text{imp}}^{\exp}$ have the same $\gamma$ (see comment (ii) above).

When $\Delta t \ll D_{\text{max}}/c$ (high-frequency excitation), $Z_{\text{imp}}^{\exp}$ and $Z_{\text{imp}}^{\exp}$ are both very sparse, i.e., $\gamma \ll N_h$ (see comment (i) above). Therefore, $C_{\text{imp}}^{\exp}$ and $C_{\text{imp}}^{\exp}$ scale as $C_{\text{imp}}^{\exp} \sim O(N_{\text{iter}} F_{\text{iter}} N_h)$ and $C_{\text{imp}}^{\exp} \sim O(mK N_h) + O(mr N_h)$, respectively. Comparing these estimates with $C_{\text{fix}}^{\exp}$ and $C_{\text{exp}}^{\exp}$ (see comment (iii) above), one can see that $C_{\text{fix}}^{\exp} \ll C_{\text{fix}}^{\exp}$ and $C_{\text{exp}}^{\exp} \ll C_{\text{exp}}^{\exp}$, which consequently results in $C_{\text{tot}}^{\exp} \approx C_{\text{fix}}^{\exp}$ and $C_{\text{tot}}^{\exp} \approx C_{\text{exp}}^{\exp}$, respectively. This means that both solvers have similar total execution times under high-frequency excitations as also shown by results presented in Section III.

As the frequency of excitation decreases (for large $\Delta t$, $\Delta t \approx D_{\text{max}}/c$), $Z_{\text{imp}}^{\exp}$ and $Z_{\text{imp}}^{\exp}$ become denser (and even full) resulting in $\gamma \approx N_h$ (see comments (i) above). Consequently, $C_{\text{imp}}^{\exp} \sim O(N_{\text{iter}} F_{\text{iter}} N_h)$ and $C_{\text{imp}}^{\exp} \sim O(mK N_h)$ resulting in $C_{\text{imp}}^{\exp} \approx C_{\text{sol}}^{\exp}$ and $C_{\text{exp}}^{\exp} \approx C_{\text{sol}}^{\exp}$, respectively (see comment (iii) above). In summary, under low-frequency excitations, $C_{\text{tot}}^{\exp} \approx C_{\text{tot}}^{\exp}$ and $C_{\text{tot}}^{\exp} \approx C_{\text{exp}}^{\exp}$, and E-MOT is faster than I-MOT for $m < N_{\text{iter}} F_{\text{iter}}$.

In Section III, it is shown by numerical results that this condition is satisfied.

III. NUMERICAL RESULTS

In this section, E-MOT and I-MOT are used to analyze transient electromagnetic scattering from a unit sphere centered at the origin. The sphere is excited by a plane wave with magnetic field

$$H^\perp(x, t) = \sqrt{\epsilon} \mu G(t - \mathbf{k} \cdot \mathbf{r}/c)$$

where $\mathbf{k} = \mathbf{z}$ is the direction of propagation and $G(t) = \cos[2\pi f_0(t - t_0)/(t - t_0)]$ is a Gaussian pulse, where $w = 7/(2\pi f_{\text{low}})$. $f_{\text{low}}$, $f_{\text{low}}$, and $f_0 = 3.5\mu$ are the pulse duration, bandwidth, modulation frequency, and delay, respectively. Note that $f_{\text{max}} = f_0 + f_{\text{low}}$ is the effective maximum frequency. In all examples, the order of $f_{\text{p}}(\mathbf{r})$ used by the Nystrom method is two ($N_h = 6$) [21] and the order of Lagrange polynomials used for constructing $T(\mathbf{r})$ is three ($T_{\text{max}} = 3$). The predictor coefficients $p$ and the corrector coefficients $\gamma$ are obtained using the Adams-Bashforth and backward difference methods, respectively, resulting in $K = 4$ for the PE(C) scheme [24]. Convergence thresholds for corrector updates and the TFQMR scheme are $\chi_{\text{PECE}}^{\text{TFQMR}} = 10^{-13}$.

For the first set of simulations, $N_h = 1126$ ($N_h = 13512$), $f_0 = 300\text{ MHz}$, $f_{\text{low}} = 200\text{ MHz}$, $N_h = 1600$, and $\Delta t = 0.1\text{ ns}[1/(20f_{\text{max}})]$. Figs. 1(a) and (b) compare $||\{I(t)\}^{(p)}||_p$ and $||\{I(t)\}^{(i)}||_p$ computed by I-MOT and E-MOT at $t_{\text{p}} = (0.97, -0.245, 0.005)$ m ($p = 1014, i = 6$), respectively. The results agree very well.
Table I

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Considered: N<sub>p</sub> = 202 (N<sub>e</sub> = 2424) and N<sub>p</sub> = 622 (N<sub>e</sub> = 7464). For each discretization, seven sets of excitations are considered: f<sub>0</sub> = {5, 10, 25, 50, 100, 200, 400} MHz and f<sub>exp</sub> = 0.75 f<sub>0</sub>. The “denseness” level in Z<sub>0</sub> and Z<sub>0</sub> for these values of Δt and for the two different discretizations are: γ = {2424, 1777, 303, 143, 61, 38, 13} for N<sub>p</sub> = 202 and γ = {7464, 5471, 881, 340, 116, 54, 32} for N<sub>p</sub> = 622, respectively. For each excitation and discretization, RCS values σ<sup>imp</sup>(θ, ϕ, f) and σ<sup>exp</sup>(θ, ϕ, f) are computed for θ = [0<sup>°</sup>, 180<sup>°</sup>], ϕ = 0<sup>°</sup>, and f = f<sub>0</sub> using the Fourier transform of the solutions obtained by I-MOT and E-MOT, respectively. The L<sub>2</sub>-norm error in RCS is defined as

<sup>σ</sup><sub>err</sub> = \sqrt{\frac{\sum_{n=0}^{360} |σ<sup>type</sup>(n Δθ, ϕ, f) - σ<sup>Mie</sup>(n Δθ, ϕ, f)|^2}{\sum_{n=0}^{360} |σ<sup>Mie</sup>(n Δθ, ϕ, f)|^2}}

where type ∈ {imp, exp}, f = f<sub>0</sub>, Δθ = 0.5<sup>°</sup>, and ϕ = 0<sup>°</sup>. Table I lists σ<sub>err</sub><sup>imp</sup> and σ<sub>err</sub><sup>exp</sup> and shows that the error level in the solutions obtained by I-MOT and E-MOT are the same.

Execution times of MOT schemes’ different stages are compared in Table I. Here, T<sub>fix</sub><sup>imp</sup> and T<sub>fix</sub><sup>exp</sup> are the total times required for computing V<sub>h</sub> = \sum_{l=1}^{N<sub>e</sub>-1} Z<sub>0</sub><sup>imp</sup> I<sub>l</sub> and V<sub>h</sub> = \sum_{l=1}^{N<sub>e</sub>-1} Z<sub>0</sub><sup>exp</sup> I<sub>l</sub> for h = 1, ..., N<sub>e</sub>, by I-MOT and E-MOT, respectively. T<sub>TFQMR</sub><sup>imp</sup> is the total time required to iteratively solve the I-MOT system in (15) by the TFQMR method for h = 1, ..., N<sub>e</sub>, T<sub>PECE</sub> is the total time required by the PECE method for h = 1, ..., N<sub>e</sub>, T<sub>fix</sub><sup>imp</sup> and T<sub>fix</sub><sup>exp</sup> are the total execution times of I-MOT and E-MOT, respectively. Finally, {N<sub>imp</sub><sup>fix</sup> T<sub>fix</sub><sup>imp</sup> T<sub>PECE</sub> T<sub>TFQMR</sub> m<sub>avg</sub>} are the average values of N<sub>imp</sub><sup>fix</sup> T<sub>fix</sub><sup>imp</sup> T<sub>PECE</sub> T<sub>TFQMR</sub> m<sub>avg</sub> over N<sub>e</sub> time steps, respectively.

Table I shows that, as expected, T<sub>fix</sub><sup>imp</sup> and T<sub>TFQMR</sub><sup>imp</sup> are almost the same. Also, for all values of Δt, T<sub>PECE</sub> < T<sub>TFQMR</sub> since m<sub>avg</sub> < {N<sub>imp</sub><sup>fix</sup> T<sub>exp</sup> T<sub>PECE</sub> T<sub>TFQMR</sub>}. This has different consequences for simulations with large and small Δt. For large Δt (low-frequency excitation), T<sub>TFQMR</sub> ≫ T<sub>fix</sub>, T<sub>PECE</sub> ≫ T<sub>fix</sub>, and T<sub>PECE</sub> ≈ T<sub>fix</sub>, therefore T<sub>tot</sub> > T<sub>fix</sub>. Indeed the results show that the E-MOT is roughly four times faster than I-MOT. As Δt gets smaller (frequency gets higher), T<sub>fix</sub> and T<sub>PECE</sub> become larger than T<sub>TFQMR</sub> and T<sub>PECE</sub> and the fact that T<sub>PECE</sub> < T<sub>TFQMR</sub> does not reflect on the total MOT times T<sub>tot</sub> and T<sub>exp</sub> anymore. Indeed, the results show that both schemes require almost the same time to complete the simulations as Δt gets smaller. The CPU times presented in Table I confirm computational complexity analysis in Section II-F.
For the last set of simulations, \(N_p = 622\) (\(N_s = 7464\)), \(f_0 = 50\) MHz and \(f_{\text{bw}} = 37.5\) MHz; and I-MOT and E-MOT are executed for different values of \(\Delta t\) changed between \(2.29\) ns \((1/\sqrt{\chi_{\max}})\) and \(0.29\) ns \((1/\sqrt{40\chi_{\max}})\). RCS L2-norm error values \(\sigma_{\text{imp}}\) and \(\sigma_{\exp}\) are computed for \(f \in \{35, 50, 65\}\) MHz, \(\Delta t = 0.5^\circ\), and \(\varphi = 0^\circ\) using (21). Fig. 3 plots \(\sigma_{\text{imp}}\) and \(\sigma_{\exp}\) versus \(1/\Delta t\) for \(f \in \{35, 50, 65\}\) MHz. This figure clearly shows that E-MOT has the same accuracy as I-MOT (as discussed in Section II-E). Additionally, Fig. 3 shows that accuracy of the schemes follows the convergence curve of \(O(\Delta t^3)\) for larger values of \(\Delta t\) (note that the order of Lagrange polynomials constructing \(T(t)\) is \(T_{\max} = 3\) but, for smaller values of \(\Delta t\), the accuracy is limited by the second-order spatial discretization and as a result the convergence with respect to \(1/\Delta t\) becomes slightly slower than \(O(\Delta t^3)\).

IV. CONCLUSION

An explicit MOT scheme to efficiently solve the TD-MFIE for large time step sizes (under low-frequency excitations) is developed. The current is spatially expanded using high-order nodal functions defined on curvilinear triangles discretizing the scatterer surface. Applying Nyström discretization, which uses this expansion, to the TD-MFIE yields a system of ODEs in time-dependent expansion coefficients. This system is integrated in time using a PE(CE)m method to compute the samples of these coefficients. The Gram matrix arising from the Nyström discretization is block-diagonal with \(2 \times 2\) blocks and its inverse is constructed from the inverse of the blocks before the time marching. Therefore, the matrix inversion needed at each time step is replaced by the product of the inverse block-diagonal Gram matrix and the right-hand side vector. Numerical results show that the resulting MOT scheme can use the time step sizes as large as those would be used by its implicit counterpart without sacrificing the stability, has the same level of accuracy, and is more than four times faster for low-frequency excitations.

The explicit MOT scheme can be extended to solve the TD-CFIE obtained by combining the TD-MFIE with the Calderón-preconditioned TD-EFIE. The Calderón-preconditioning ensures that the TD-CFIE is a second-kind surface integral equation [26]-[29].

REFERENCES


