

SUPPLEMENTARY MATERIAL: Birnbau-Saunders sample selection model

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ARTICLE HISTORY

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Appendix A: Gradient and Hessian Matrix

Let $\boldsymbol{\theta} = (\boldsymbol{\gamma}^\top, \boldsymbol{\beta}^\top, \phi, \rho)$. The log-likelihood function for the Heckman-BS model based on a pair of observations (y, u) is

$$\begin{aligned} \mathcal{L}(\boldsymbol{\theta}) = & u \left\{ -\frac{\phi}{4} \left(\sqrt{\frac{y(\phi+1)}{\phi\boldsymbol{\mu}_1}} - \sqrt{\frac{\phi\boldsymbol{\mu}_1}{y(\phi+1)}} \right)^2 \right. \\ & + \log \Phi \left[\frac{[\boldsymbol{\mu}_2 - 2]}{2\sqrt{\boldsymbol{\mu}_2(1-\rho^2)}} + \frac{\sqrt{\phi}\rho}{\sqrt{2(1-\rho^2)}} \left(\sqrt{\frac{(\phi+1)y}{\phi\boldsymbol{\mu}_1}} - \sqrt{\frac{\phi\boldsymbol{\mu}_1}{y(\phi+1)}} \right) \right] \\ & + \log \left(\frac{\sqrt{\phi+1}}{\sqrt{\phi\boldsymbol{\mu}_1 y}} + \frac{\sqrt{\phi\boldsymbol{\mu}_1}}{\sqrt{y^3(\phi+1)}} \right) + \frac{1}{4} \sqrt{\frac{\phi}{\pi}} \left. \right\} \\ & + (1-u) \log \Phi \left\{ \sqrt{\frac{1}{2}} \left[\sqrt{\frac{2}{\boldsymbol{\mu}_2}} - \sqrt{\frac{\boldsymbol{\mu}_2}{2}} \right] \right\}, \end{aligned} \quad (1)$$

where $\boldsymbol{\mu}_1 = \exp(\boldsymbol{x}^\top \boldsymbol{\beta}) = \exp(\eta_1)$ and $\boldsymbol{\mu}_2 = \exp(\boldsymbol{w}^\top \boldsymbol{\gamma}) = \exp(\eta_2)$. If y is observed, then $u = 1$, otherwise $u = 0$. To simplify the calculation of the Hessian matrix, we

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consider

$$t_0(y, \boldsymbol{\mu}_1, \phi) = \left(\sqrt{\frac{y(\phi+1)}{\phi\boldsymbol{\mu}_1}} - \sqrt{\frac{\phi\boldsymbol{\mu}_1}{y(\phi+1)}} \right), \quad t_1(y, \boldsymbol{\mu}_1, \phi) = -\frac{\phi}{4} [t_0(y, \boldsymbol{\mu}_1, \phi)]^2,$$

$$t_2(y, \boldsymbol{\mu}_1, \phi) = \left(\frac{\sqrt{\phi+1}}{\sqrt{\phi\boldsymbol{\mu}_1 y}} + \frac{\sqrt{\phi\boldsymbol{\mu}_1}}{\sqrt{y^3(\phi+1)}} \right), \quad t_3(\phi) = \frac{1}{4} \sqrt{\frac{\phi}{\pi}},$$

$$t_4(\boldsymbol{\mu}_2, \rho) = \frac{\boldsymbol{\mu}_2 - 2}{2\sqrt{\boldsymbol{\mu}_2(1-\rho^2)}}, \quad t_5(\phi, \rho) = \frac{\rho\sqrt{\phi}}{\sqrt{2(1-\rho^2)}}, \quad t_6(\boldsymbol{\mu}_2) = \sqrt{\frac{1}{2}} \left[\sqrt{\frac{2}{\boldsymbol{\mu}_2}} - \sqrt{\frac{\boldsymbol{\mu}_2}{2}} \right],$$

and for $k = 1, \dots, p$, and $i = 1, \dots, q$,

$$\frac{\partial t_0}{\partial \boldsymbol{\gamma}_i} = \frac{\partial t_1}{\partial \boldsymbol{\gamma}_i} = \frac{\partial t_2}{\partial \boldsymbol{\gamma}_i} = \frac{\partial t_3}{\partial \boldsymbol{\gamma}_i} = \frac{\partial t_5}{\partial \boldsymbol{\gamma}_i} = 0, \quad \frac{\partial t_4}{\partial \boldsymbol{\gamma}_i} = \frac{\partial t_4}{\partial \boldsymbol{\mu}_2} \frac{\partial \boldsymbol{\mu}_2}{\partial \eta_2} \frac{\partial \eta_2}{\partial \boldsymbol{\gamma}_i} = \frac{\boldsymbol{\mu}_2 + 2}{4\sqrt{\boldsymbol{\mu}_2(1-\rho^2)}} \mathbf{w}_i,$$

$$\frac{\partial t_6}{\partial \boldsymbol{\gamma}_i} = \frac{\partial t_6}{\partial \boldsymbol{\mu}_2} \frac{\partial \boldsymbol{\mu}_2}{\partial \eta_2} \frac{\partial \eta_2}{\partial \boldsymbol{\gamma}_i} = -\frac{1}{2\sqrt{2}} \left(\sqrt{\frac{2}{\boldsymbol{\mu}_2}} + \sqrt{\frac{\boldsymbol{\mu}_2}{2}} \right) \frac{\partial t_3}{\partial \boldsymbol{\beta}_k} \mathbf{w}_i, \quad \frac{\partial t_4}{\partial \boldsymbol{\beta}_k} = \frac{\partial t_5}{\partial \boldsymbol{\beta}_k} = \frac{\partial t_6}{\partial \boldsymbol{\beta}_k} = 0,$$

$$\frac{\partial t_0}{\partial \boldsymbol{\beta}_k} = \frac{\partial t_0}{\partial \boldsymbol{\mu}_1} \frac{\partial \boldsymbol{\mu}_1}{\partial \eta_1} \frac{\partial \eta_1}{\partial \boldsymbol{\beta}_k} = -\frac{1}{2} \left(\sqrt{\frac{y(\phi+1)}{\phi\boldsymbol{\mu}_1}} + \sqrt{\frac{\phi\boldsymbol{\mu}_1}{y(\phi+1)}} \right) \mathbf{x}_{ki}, \quad \frac{\partial t_4}{\partial \phi} = \frac{\partial t_6}{\partial \phi} = 0,$$

$$\frac{\partial t_1}{\partial \boldsymbol{\beta}_k} = \frac{\partial t_1}{\partial \boldsymbol{\mu}_1} \frac{\partial \boldsymbol{\mu}_1}{\partial \eta_1} \frac{\partial \eta_1}{\partial \boldsymbol{\beta}_k} = -\frac{\phi}{2} t_0 \frac{\partial t_0}{\partial \boldsymbol{\beta}_k}, \quad \frac{\partial t_0}{\partial \rho} = \frac{\partial t_1}{\partial \rho} = \frac{\partial t_2}{\partial \rho} = \frac{\partial t_3}{\partial \rho} = \frac{\partial t_6}{\partial \rho} = 0,$$

$$\frac{\partial t_2}{\partial \boldsymbol{\beta}_k} = \frac{\partial t_2}{\partial \boldsymbol{\mu}_1} \frac{\partial \boldsymbol{\mu}_1}{\partial \eta_1} \frac{\partial \eta_1}{\partial \boldsymbol{\beta}_k} = \frac{1}{2} \left(\sqrt{\frac{\phi\boldsymbol{\mu}_1}{y^3(\phi+1)}} - \sqrt{\frac{\phi+1}{\phi\boldsymbol{\mu}_1 y}} \right) \mathbf{x}_k, \quad \frac{\partial t_3}{\partial \phi} = \frac{1}{8\sqrt{\pi\phi}},$$

$$\frac{\partial t_0}{\partial \phi} = -\frac{1}{2} \left(\sqrt{\frac{y}{(\phi+1)\phi^3\boldsymbol{\mu}_1}} + \sqrt{\frac{\boldsymbol{\mu}_1}{y(\phi+1)^3\phi}} \right), \quad \frac{\partial t_5}{\partial \phi} = \frac{\rho}{2\sqrt{2\phi(1-\rho^2)}},$$

$$\frac{\partial t_1}{\partial \phi} = -\frac{t_0^2}{4} + \frac{t_0}{4} \left(\sqrt{\frac{y}{(\phi+1)\phi\boldsymbol{\mu}_1}} + \sqrt{\frac{\phi\boldsymbol{\mu}_1}{y(\phi+1)^3}} \right), \quad \frac{\partial t_4}{\partial \rho} = \frac{\rho t_4}{(1-\rho^2)},$$

$$\frac{\partial t_2}{\partial \phi} = \frac{1}{2} \left(\sqrt{\frac{\boldsymbol{\mu}_1}{y^3(\phi+1)^3\phi}} - \frac{1}{\sqrt{y\boldsymbol{\mu}_1(\phi+1)\phi^3}} \right), \quad \frac{\partial t_5}{\partial \rho} = \frac{t_5}{\rho(1-\rho^2)}.$$

Define

$$\begin{aligned} f_1(y, \boldsymbol{\mu}_1, \phi) &= t_1(y, \boldsymbol{\mu}_1, \phi), & f_2(y, \boldsymbol{\mu}_1, \phi) &= \log [t_2(y, \boldsymbol{\mu}_1, \phi)], & f_3(\phi) &= \log [t_3(\phi)], \\ f_4(y, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \phi, \rho) &= \log \Phi(t_4 + t_5 t_0), & f_5(\boldsymbol{\mu}_2) &= \log \Phi[t_6(\boldsymbol{\mu}_2)]. \end{aligned}$$

The first-order derivatives of the above functions are

$$\frac{\partial f_1}{\partial \gamma_i} = \frac{\partial f_2}{\partial \gamma_i} = \frac{\partial f_3}{\partial \gamma_i} = 0, \quad \frac{\partial f_4}{\partial \gamma_i} = \frac{\phi(t_4 + t_5 t_0)}{\Phi(t_4 + t_5 t_0)} \frac{\partial t_4}{\partial \gamma_i}, \quad \frac{\partial f_5}{\partial \gamma_i} = \frac{\phi(t_6)}{\Phi(t_6)} \frac{\partial t_6}{\partial \gamma_i},$$

$$\frac{\partial f_1}{\partial \beta_k} = -\frac{\phi}{2} t_0 \frac{\partial t_0}{\partial \beta_k}, \quad \frac{\partial f_2}{\partial \beta_k} = \frac{1}{t_2} \frac{\partial t_2}{\partial \beta_k}, \quad \frac{\partial f_3}{\partial \beta_k} = \frac{\partial f_5}{\partial \beta_k} = 0,$$

$$\frac{\partial f_4}{\partial \beta_k} = \frac{\phi(t_4 + t_5 t_0)}{\Phi(t_4 + t_5 t_0)} t_5 \frac{\partial t_0}{\partial \beta_k}, \quad \frac{\partial f_1}{\partial \phi} = \frac{\partial t_1}{\partial \phi}, \quad \frac{\partial f_2}{\partial \phi} = \frac{1}{t_2} \frac{\partial t_2}{\partial \phi},$$

$$\frac{\partial f_3}{\partial \phi} = \frac{1}{2\phi}, \quad \frac{\partial f_4}{\partial \phi} = \frac{\phi(t_4 + t_5 t_0)}{\Phi(t_4 + t_5 t_0)} \left(\frac{\partial t_5}{\partial \phi} t_0 + t_5 \frac{\partial t_0}{\partial \phi} \right), \quad \frac{\partial f_5}{\partial \phi} = 0,$$

$$\frac{\partial f_1}{\partial \rho} = \frac{\partial f_2}{\partial \rho} = \frac{\partial f_3}{\partial \rho} = \frac{\partial f_5}{\partial \rho} = 0, \quad \frac{\partial f_4}{\partial \rho} = \frac{\phi(t_4 + t_5 t_0)}{\Phi(t_4 + t_5 t_0)} \left(\frac{\partial t_4}{\partial \rho} + \frac{\partial t_5}{\partial \rho} t_0 \right).$$

Thus, we have that the maximum likelihood estimators are obtained by the solution of the system of equations

$$\left\{ \begin{aligned} \frac{\partial \mathcal{L}(\boldsymbol{\theta})}{\partial \gamma_j} &= \sum_{i=1}^n u_i \left(\frac{\partial f_{4i}}{\partial \gamma_j} \right) + \sum_{i=1}^n (1 - u_i) \left(\frac{\partial f_{5i}}{\partial \gamma_j} \right) = 0, \quad j = 1, \dots, q, \\ \frac{\partial \mathcal{L}(\boldsymbol{\theta})}{\partial \beta_k} &= \sum_{i=1}^n u_i \left(\frac{\partial f_{1i}}{\partial \beta_k} + \frac{\partial f_{2i}}{\partial \beta_k} + \frac{\partial f_3}{\partial \beta_k} + \frac{\partial f_{4i}}{\partial \beta_k} \right) = 0, \quad k = 1, \dots, p, \\ \frac{\partial \mathcal{L}(\boldsymbol{\theta})}{\partial \phi} &= \sum_{i=1}^n u_i \left(\frac{\partial f_{1i}}{\partial \phi} + \frac{\partial f_{2i}}{\partial \phi} + \frac{\partial f_3}{\partial \phi} + \frac{\partial f_{4i}}{\partial \phi} \right) = 0, \\ \frac{\partial \mathcal{L}(\boldsymbol{\theta})}{\partial \rho} &= \sum_{i=1}^n u_i \frac{\partial f_{4i}}{\partial \rho} = 0. \end{aligned} \right.$$

Let $M(x) = \frac{\phi(x)}{\Phi(x)}$, $x \in \mathbb{R}$, follow that $\frac{\partial M(x)}{\partial a} = -\frac{\partial x}{\partial a} M(x) \left(x + M(x) \right)$, $a \in (\boldsymbol{\gamma}^\top, \boldsymbol{\beta}^\top, \phi, \rho)$. The second-order derivatives of the above functions are

$$\frac{\partial^2 f_4}{\partial \gamma_i \partial \gamma_j} = \frac{\partial}{\partial \gamma_j} M(t_4 + t_5 t_0) \frac{\partial t_4}{\partial \gamma_i} + M(t_4 + t_5 t_0) \frac{\partial^2 t_4}{\partial \gamma_i \partial \gamma_j}$$

$$\begin{aligned}\frac{\partial^2 t_4}{\partial \gamma_i \partial \gamma_j} &= \frac{t_4}{4} w_i w_j, & \frac{\partial^2 f_5}{\partial \gamma_i \partial \gamma_j} &= \frac{\partial}{\partial \gamma_j} M(t_6) \frac{\partial t_6}{\partial \gamma_i} + M(t_6) \frac{\partial^2 t_6}{\partial \gamma_i \partial \gamma_j}, & \frac{\partial^2 t_6}{\partial \gamma_i \partial \gamma_j} &= \frac{1}{4} t_6 w_i w_j, \\ \frac{\partial^2 f_5}{\partial \gamma_i \partial \phi} &= \frac{\partial}{\partial \phi} M(t_6) \frac{\partial t_6}{\partial \gamma_i}, & \frac{\partial^2 f_4}{\partial \gamma_i \partial \beta_l} &= \frac{\partial}{\partial \beta_l} M(t_4 + t_5 t_0) \frac{\partial t_4}{\partial \gamma_i}, & \frac{\partial^2 f_5}{\partial \gamma_i \partial \beta_l} &= \frac{\partial^2 f_5}{\partial \gamma_i \partial \rho} = 0, \\ \frac{\partial^2 f_4}{\partial \gamma_i \partial \phi} &= \frac{\partial}{\partial \phi} M(t_4 + t_5 t_0) \frac{\partial t_4}{\partial \gamma_i}, & \frac{\partial^2 f_4}{\partial \beta_k \partial \gamma_j} &= \frac{\partial}{\partial \gamma_j} M(t_4 + t_5 t_0) t_5 \frac{\partial t_0}{\partial \beta_k}, & \frac{\partial^2 t_4}{\partial \gamma_i \partial \rho} &= \frac{\rho}{(1 - \rho^2)} \frac{\partial t_4}{\partial \gamma_i}, \\ \frac{\partial^2 t_0}{\partial \beta_k \partial \beta_l} &= \frac{1}{4} t_0 x_k x_l, & \frac{\partial^2 f_2}{\partial \beta_k \partial \beta_l} &= \frac{\partial t_2^{-1}}{\partial \beta_l} \frac{\partial t_2}{\partial \beta_k} + \frac{1}{t_2} \frac{\partial^2 t_2}{\partial \beta_k \partial \beta_l}, & \frac{\partial t_2^{-1}}{\partial \beta_l} &= -\frac{1}{t_2^2} \frac{\partial t_2}{\partial \beta_l},\end{aligned}$$

$$\frac{\partial^2 t_2}{\partial \beta_k \partial \beta_l} = \frac{1}{4} t_2 x_k x_l, \quad \frac{\partial^2 f_3}{\partial \beta_k \partial \beta_l} = 0, \quad \frac{\partial^2 t_0}{\partial \phi \partial \beta_l} = \frac{1}{4} \left(\sqrt{\frac{\mu_1}{y(\phi+1)^3 \phi}} - \sqrt{\frac{y}{(\phi+1)\phi^3 \mu_1}} \right) x_l,$$

$$\frac{\partial^2 f_3}{\partial \beta_k \partial \phi} = 0, \quad \frac{\partial^2 f_1}{\partial \beta_k \partial \rho} = 0, \quad \frac{\partial^2 f_1}{\partial \phi \partial \gamma_j} = \frac{\partial^2 f_2}{\partial \phi \partial \gamma_j} = \frac{\partial^2 f_3}{\partial \phi \partial \gamma_j} = 0,$$

$$\frac{\partial^2 f_3}{\partial \beta_k \partial \rho} = 0, \quad \frac{\partial t_2^{-1}}{\partial \phi} = -\frac{1}{t_2^2} \frac{\partial t_2}{\partial \phi}, \quad \frac{\partial^2 f_4}{\partial \phi \partial \gamma_j} = \frac{\partial}{\partial \gamma_j} M(t_4 + t_5 t_0) \left(\frac{\partial t_5}{\partial \phi} t_0 + t_5 \frac{\partial t_0}{\partial \phi} \right),$$

$$\frac{\partial^2 f_2}{\partial \beta_k \partial \rho} = 0, \quad \frac{\partial^2 t_5}{\partial \phi \partial \rho} = \frac{1}{2\sqrt{2\phi(1-\rho^2)^3}}, \quad \frac{\partial^2 f_2}{\partial \phi^2} = -\frac{1}{t_2^2} \left(\frac{\partial t_2}{\partial \phi} \right)^2 + \frac{1}{t_2} \frac{\partial^2 t_2}{\partial \phi^2},$$

$$\frac{\partial^2 f_3}{\partial \phi^2} = -\frac{1}{2\phi^2}, \quad \frac{\partial^2 t_5}{\partial \phi^2} = -\frac{\rho}{4\sqrt{2\phi^3(1-\rho^2)}}, \quad \frac{\partial^2 f_2}{\partial \phi \partial \beta_l} = -\frac{1}{t_2^2} \frac{\partial t_2}{\partial \beta_l} + \frac{1}{t_2} \frac{\partial^2 t_2}{\partial \phi \partial \beta_l},$$

$$\frac{\partial^2 f_4}{\partial \gamma_i \partial \rho} = \frac{\partial}{\partial \rho} M(t_4 + t_5 t_0) \frac{\partial t_4}{\partial \gamma_i} + M(t_4 + t_5 t_0) \frac{\partial^2 t_4}{\partial \gamma_i \partial \rho}, \quad \frac{\partial^2 f_1}{\partial \beta_k \partial \gamma_j} = \frac{\partial^2 f_2}{\partial \beta_k \partial \gamma_j} = \frac{\partial^2 f_3}{\partial \beta_k \partial \gamma_j} = 0,$$

$$\frac{\partial^2 f_1}{\partial \beta_k \partial \beta_l} = -\frac{\phi}{2} \left(\frac{\partial t_0}{\partial \beta_l} \frac{\partial t_0}{\partial \beta_k} + t_0 \frac{\partial^2 t_0}{\partial \beta_k \partial \beta_l} \right), \quad \frac{\partial^2 f_2}{\partial \beta_k \partial \phi} = \frac{\partial t_2^{-1}}{\partial \phi} \frac{\partial t_2}{\partial \beta_k} + \frac{1}{t_2} \frac{\partial^2 t_2}{\partial \beta_k \partial \phi},$$

$$\frac{\partial^2 f_1}{\partial \beta_k \partial \phi} = -\frac{1}{2} t_0 \frac{\partial t_0}{\partial \beta_k} - \frac{\phi}{2} \left(\frac{\partial t_0}{\partial \phi} \frac{\partial t_0}{\partial \beta_k} + t_0 \frac{\partial^2 t_0}{\partial \beta_k \partial \phi} \right), \quad \frac{\partial^2 f_1}{\partial \phi \partial \rho} = \frac{\partial^2 f_2}{\partial \phi \partial \rho} = \frac{\partial^2 f_3}{\partial \phi \partial \rho} = 0,$$

$$\frac{\partial^2 f_4}{\partial \beta_k \partial \beta_l} = \frac{\partial}{\partial \beta_l} M(t_4 + t_5 t_0) t_5 \frac{\partial t_0}{\partial \beta_k} + M(t_4 + t_5 t_0) \left(t_5 \frac{\partial^2 t_0}{\partial \beta_k \partial \beta_l} \right),$$

$$\frac{\partial^2 t_0}{\partial \beta_k \partial \phi} = \frac{1}{4} \left(\sqrt{\frac{y}{(\phi+1)\phi^3 \mu_1}} + \sqrt{\frac{\mu_1}{y(\phi+1)^3 \phi}} \right) x_k,$$

$$\frac{\partial^2 t_2}{\partial \beta_k \partial \phi} = \frac{1}{4} \left(\sqrt{\frac{\mu_1}{y^3 \phi(\phi+1)^3}} + \frac{1}{\sqrt{\mu_1 y \phi^3 (\phi+1)}} \right) x_k,$$

$$\frac{\partial^2 f_4}{\partial \beta_k \partial \rho} = \frac{\partial}{\partial \rho} M(t_4 + t_5 t_0) t_5 \frac{\partial t_0}{\partial \beta_k} + M(t_4 + t_5 t_0) \frac{\partial t_5}{\partial \rho} \frac{\partial t_0}{\partial \beta_k},$$

$$\frac{\partial^2 t_2}{\partial \phi \partial \beta_l} = \frac{1}{4} \left(\sqrt{\frac{\mu_1}{y^3 \phi(\phi+1)^3}} + \frac{1}{\sqrt{\mu_1 y \phi^3 (\phi+1)}} \right) x_l,$$

$$\frac{\partial^2 f_4}{\partial \phi \partial \beta_l} = \frac{\partial}{\partial \beta_l} M(t_4 + t_5 t_0) \left(\frac{\partial t_5}{\partial \phi} t_0 + t_5 \frac{\partial t_0}{\partial \phi} \right) + M(t_4 + t_5 t_0) \left(\frac{\partial t_5}{\partial \phi} \frac{\partial t_0}{\partial \beta_l} + t_5 \frac{\partial^2 t_0}{\partial \phi \partial \beta_l} \right),$$

$$\frac{\partial^2 t_2}{\partial \phi^2} = \frac{1}{4} \left(\frac{4\phi+3}{\phi^2 \sqrt{\mu_1 y \phi(\phi+1)^3}} - \frac{4\phi+1}{(\phi+1)^2} \sqrt{\frac{\mu_1}{y^3 \phi^3 (\phi+1)}} \right),$$

$$\frac{\partial^2 t_0}{\partial \phi^2} = \frac{1}{4} \left[\frac{4\phi+3}{\phi^4} \sqrt{\frac{y}{\mu_1 \phi(\phi+1)^3}} - \frac{4\phi+1}{(\phi+1)^4} \sqrt{\frac{\mu_1}{y \phi^3 (\phi+1)}} \right],$$

$$\frac{\partial^2 f_4}{\partial \beta_k \partial \phi} = \frac{\partial}{\partial \phi} M(t_4 + t_5 t_0) t_5 \frac{\partial t_0}{\partial \beta_k} + M(t_4 + t_5 t_0) \left(\frac{\partial t_5}{\partial \phi} \frac{\partial t_0}{\partial \beta_k} + t_5 \frac{\partial^2 t_0}{\partial \beta_k \partial \phi} \right),$$

$$\frac{\partial^2 f_4}{\partial \phi^2} = \frac{\partial}{\partial \phi} M(t_4 + t_5 t_0) \left(\frac{\partial t_5}{\partial \phi} t_0 + t_5 \frac{\partial t_0}{\partial \phi} \right) + M(t_4 + t_5 t_0) \left(\frac{\partial^2 t_5}{\partial \phi^2} t_0 + 2 \frac{\partial t_5}{\partial \phi} \frac{\partial t_0}{\partial \phi} + t_5 \frac{\partial^2 t_0}{\partial \phi^2} \right),$$

$$\frac{\partial^2 f_4}{\partial \phi \partial \rho} = \frac{\partial}{\partial \rho} M(t_4 + t_5 t_0) \left(\frac{\partial t_5}{\partial \phi} t_0 + t_5 \frac{\partial t_0}{\partial \phi} \right) + M(t_4 + t_5 t_0) \left(\frac{\partial^2 t_5}{\partial \phi \partial \rho} t_0 + \frac{\partial t_5}{\partial \rho} \frac{\partial t_0}{\partial \phi} \right),$$

$$\frac{\partial^2 f_4}{\partial \rho \partial \gamma_j} = \frac{\partial}{\partial \gamma_j} M(t_4 + t_5 t_0) \left(\frac{\partial t_4}{\partial \rho} + \frac{\partial t_5}{\partial \rho} t_0 \right) + M(t_4 + t_5 t_0) \frac{\partial^2 t_4}{\partial \rho \partial \gamma_j},$$

$$\frac{\partial^2 f_4}{\partial \rho \partial \phi} = \frac{\partial}{\partial \phi} M(t_4 + t_5 t_0) \left(\frac{\partial t_4}{\partial \rho} + \frac{\partial t_5}{\partial \rho} t_0 \right) + M(t_4 + t_5 t_0) \left(\frac{\partial^2 t_5}{\partial \rho \partial \phi} t_0 + \frac{\partial t_5}{\partial \rho} \frac{\partial t_0}{\partial \phi} \right),$$

$$\frac{\partial^2 f_4}{\partial \rho^2} = \frac{\partial}{\partial \rho} M(t_4 + t_5 t_0) \left(\frac{\partial t_4}{\partial \rho} + \frac{\partial t_5}{\partial \rho} t_0 \right) + M(t_4 + t_5 t_0) \left(\frac{\partial^2 t_4}{\partial \rho^2} + \frac{\partial^2 t_5}{\partial \rho^2} t_0 + \frac{\partial t_5}{\partial \rho} \frac{\partial t_0}{\partial \rho} \right),$$

$$\begin{aligned} \frac{\partial^2 f_1}{\partial \phi \partial \beta_l} &= \frac{\partial^2 t_1}{\partial \phi \partial \beta_l}, \\ &= -\frac{1}{2} t_0 \frac{\partial t_0}{\partial \beta_l} + \frac{1}{4} \frac{\partial t_0}{\partial \beta_l} \left(\sqrt{\frac{y}{(\phi+1)\phi\boldsymbol{\mu}_1}} + \sqrt{\frac{\phi\boldsymbol{\mu}_1}{y(\phi+1)^3}} \right), \\ &\quad + \frac{t_0}{8} \left(\sqrt{\frac{\phi\boldsymbol{\mu}_1}{y(\phi+1)^3}} - \sqrt{\frac{y}{(\phi+1)\phi\boldsymbol{\mu}_1}} \right) x_l, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 f_1}{\partial \phi^2} &= \frac{\partial^2 t_1}{\partial \phi^2}, \\ &= -\frac{t_0}{2} \frac{\partial t_0}{\partial \phi} + \frac{1}{4} \frac{\partial t_0}{\partial \phi} \left(\sqrt{\frac{y}{(\phi+1)\phi\boldsymbol{\mu}_1}} + \sqrt{\frac{\phi\boldsymbol{\mu}_1}{y(\phi+1)^3}} \right), \\ &\quad - \frac{t_0}{8} \left((2\phi-1) \sqrt{\frac{\boldsymbol{\mu}_1}{y(\phi+1)^5\phi}} + (2\phi+1) \sqrt{\frac{y}{(\phi+1)^3\phi\boldsymbol{\mu}_1}} \right), \end{aligned}$$

$$\frac{\partial^2 t_4}{\partial \rho \partial \gamma_j} = \frac{\rho}{(1-\rho^2)} \frac{\partial t_4}{\partial \gamma_j}, \quad \frac{\partial^2 f_4}{\partial \rho \partial \beta_l} = \frac{\partial}{\partial \beta_l} M(t_4 + t_5 t_0) \left(\frac{\partial t_4}{\partial \rho} + \frac{\partial t_5}{\partial \rho} t_0 \right), \quad \frac{\partial^2 t_5}{\partial \rho^2} = \frac{3t_5}{(1-\rho^2)^2},$$

$$\frac{\partial^2 t_5}{\partial \rho \partial \phi} = \frac{1}{2\sqrt{2\phi(1-\rho^2)^3}}, \quad \frac{\partial^2 t_4}{\partial \rho^2} = \frac{1+\rho^2}{(1-\rho^2)^2} t_4 + \frac{\rho}{(1-\rho^2)} \frac{\partial t_4}{\partial \rho}.$$

Let $S_{ab} = \frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial a \partial b}$, with $a, b \in (\boldsymbol{\gamma}^\top, \boldsymbol{\beta}^\top, \phi, \rho)$. Therefore, for $i, j = 1, \dots, q$ and

$k, l = 1, \dots, p$, we have that the elements of the Hessian matrix are

$$\begin{aligned}
S_{\gamma_i \gamma_j} &= u \frac{\partial^2 f_4}{\partial \gamma_i \partial \gamma_j} + (1-u) \frac{\partial^2 f_5}{\partial \gamma_i \partial \gamma_j}, \\
S_{\gamma_i \beta_l} &= u \frac{\partial^2 f_4}{\partial \gamma_i \partial \beta_l} + (1-u) \frac{\partial^2 f_5}{\partial \gamma_i \partial \beta_l}, \\
S_{\gamma_i \phi} &= u \frac{\partial^2 f_4}{\partial \gamma_i \partial \phi} + (1-u) \frac{\partial^2 f_5}{\partial \gamma_i \partial \phi}, \\
S_{\gamma_i \rho} &= u \frac{\partial^2 f_4}{\partial \gamma_i \partial \rho} + (1-u) \frac{\partial^2 f_5}{\partial \gamma_i \partial \rho}, \\
S_{\beta_k \gamma_j} &= u \left(\frac{\partial^2 f_1}{\partial \beta_k \partial \gamma_j} + \frac{\partial^2 f_2}{\partial \beta_k \partial \gamma_j} + \frac{\partial^2 f_3}{\partial \beta_k \partial \gamma_j} + \frac{\partial^2 f_4}{\partial \beta_k \partial \gamma_j} \right), \\
S_{\beta_k \beta_l} &= u \left(\frac{\partial^2 f_1}{\partial \beta_k \partial \beta_l} + \frac{\partial^2 f_2}{\partial \beta_k \partial \beta_l} + \frac{\partial^2 f_3}{\partial \beta_k \partial \beta_l} + \frac{\partial^2 f_4}{\partial \beta_k \partial \beta_l} \right), \\
S_{\beta_k \phi} &= u \left(\frac{\partial^2 f_1}{\partial \beta_k \partial \phi} + \frac{\partial^2 f_2}{\partial \beta_k \partial \phi} + \frac{\partial^2 f_3}{\partial \beta_k \partial \phi} + \frac{\partial^2 f_4}{\partial \beta_k \partial \phi} \right), \\
S_{\beta_k \rho} &= u \left(\frac{\partial^2 f_1}{\partial \beta_k \partial \rho} + \frac{\partial^2 f_2}{\partial \beta_k \partial \rho} + \frac{\partial^2 f_3}{\partial \beta_k \partial \rho} + \frac{\partial^2 f_4}{\partial \beta_k \partial \rho} \right), \\
S_{\phi \gamma_j} &= u \left(\frac{\partial^2 f_1}{\partial \phi \partial \gamma_j} + \frac{\partial^2 f_2}{\partial \phi \partial \gamma_j} + \frac{\partial^2 f_3}{\partial \phi \partial \gamma_j} + \frac{\partial^2 f_4}{\partial \phi \partial \gamma_j} \right), \\
S_{\phi \beta_l} &= u \left(\frac{\partial^2 f_1}{\partial \phi \partial \beta_l} + \frac{\partial^2 f_2}{\partial \phi \partial \beta_l} + \frac{\partial^2 f_3}{\partial \phi \partial \beta_l} + \frac{\partial^2 f_4}{\partial \phi \partial \beta_l} \right), \\
S_{\phi \phi} &= u \left(\frac{\partial^2 f_1}{\partial \phi^2} + \frac{\partial^2 f_2}{\partial \phi^2} + \frac{\partial^2 f_3}{\partial \phi^2} + \frac{\partial^2 f_4}{\partial \phi^2} \right), \\
S_{\phi \rho} &= u \left(\frac{\partial^2 f_1}{\partial \phi \partial \rho} + \frac{\partial^2 f_2}{\partial \phi \partial \rho} + \frac{\partial^2 f_3}{\partial \phi \partial \rho} + \frac{\partial^2 f_4}{\partial \phi \partial \rho} \right), \\
S_{\rho \gamma_j} &= u \frac{\partial^2 f_4}{\partial \rho \partial \gamma_j}, \\
S_{\rho \beta_l} &= u \frac{\partial^2 f_4}{\partial \rho \partial \beta_l}, \\
S_{\rho \phi} &= u \frac{\partial^2 f_4}{\partial \rho \partial \phi}, \\
S_{\rho \rho} &= u \frac{\partial^2 f_4}{\partial \rho^2}.
\end{aligned}$$

Appendix B: Additional Simulations

We present additional simulations of the fit of the Heckman-BS selection model. These simulations aim to evaluate the behavior of the maximum likelihood estimators in finite samples under the Heckman-BS model with different settings for the covariates and different values for the correlation parameter. The models have been defined according to Scenarios 1 and 2 of the paper. In Scenario 1, the model is fitted in the presence of exclusion restriction. In Scenario 2, the model considers the absence of exclusion restriction. In both scenarios, we maintained average censorship of approximately 30%.

• Scenario 1

Table 1. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 1 with $\rho = -0.7$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters	n	Configuration 1		Configuration 2		Configuration 3	
		Mean	RMSE	Mean	RMSE	Mean	RMSE
$\gamma_1 = 1.600$	500	1.609	0.086	1.611	0.212	1.610	0.090
	1000	1.604	0.060	1.603	0.151	1.605	0.060
	2000	1.601	0.042	1.604	0.105	1.601	0.042
$\gamma_2 = 0.800$	500	0.816	0.096	0.822	0.200	0.811	0.120
	1000	0.803	0.067	0.808	0.144	0.808	0.079
	2000	0.803	0.042	0.801	0.097	0.805	0.062
$\gamma_3 = 0.200$	500	0.205	0.079	0.212	0.337	0.209	0.118
	1000	0.202	0.061	0.203	0.240	0.201	0.075
	2000	0.199	0.042	0.203	0.162	0.197	0.058
$\gamma_4 = 0.700$	500	0.711	0.093	0.716	0.058	0.715	0.091
	1000	0.705	0.061	0.708	0.040	0.705	0.063
	2000	0.706	0.044	0.704	0.027	0.704	0.040
$\beta_1 = 1.000$	500	0.993	0.140	0.994	0.136	0.989	0.131
	1000	0.996	0.091	0.994	0.099	0.995	0.090
	2000	0.998	0.059	0.998	0.067	0.998	0.061
$\beta_2 = 0.700$	500	0.701	0.071	0.694	0.116	0.709	0.091
	1000	0.703	0.047	0.703	0.079	0.699	0.058
	2000	0.700	0.031	0.701	0.054	0.701	0.039
$\beta_3 = 1.100$	500	1.100	0.055	1.108	0.192	1.096	0.079
	1000	1.100	0.041	1.100	0.139	1.101	0.053
	2000	1.099	0.029	1.099	0.092	1.099	0.038
$\phi = 1.200$	500	1.221	0.141	1.219	0.102	1.225	0.135
	1000	1.208	0.094	1.214	0.072	1.210	0.097
	2000	1.205	0.064	1.206	0.049	1.205	0.064
$\rho = -0.700$	500	-0.684	0.133	-0.707	0.085	-0.684	0.126
	1000	-0.696	0.083	-0.697	0.064	-0.696	0.078
	2000	-0.698	0.053	-0.700	0.045	-0.698	0.052

Table 2. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 1 with $\rho = -0.5$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters	n	Configuration 1		Configuration 2		Configuration 3	
		Mean	RMSE	Mean	RMSE	Mean	RMSE
$\gamma_1 = 1.600$	500	1.610	0.086	1.608	0.220	1.610	0.089
	1000	1.604	0.061	1.604	0.152	1.605	0.061
	2000	1.601	0.042	1.605	0.109	1.601	0.042
$\gamma_2 = 0.800$	500	0.817	0.097	0.826	0.209	0.812	0.126
	1000	0.803	0.067	0.806	0.151	0.810	0.082
	2000	0.802	0.043	0.801	0.102	0.806	0.064
$\gamma_3 = 0.200$	500	0.204	0.081	0.221	0.346	0.209	0.119
	1000	0.203	0.062	0.200	0.244	0.201	0.076
	2000	0.199	0.043	0.202	0.168	0.196	0.060
$\gamma_4 = 0.700$	500	0.713	0.094	0.718	0.060	0.715	0.095
	1000	0.705	0.064	0.707	0.041	0.706	0.068
	2000	0.706	0.045	0.704	0.027	0.704	0.042
$\beta_1 = 1.000$	500	0.999	0.141	0.995	0.136	0.994	0.134
	1000	0.997	0.096	0.993	0.100	0.996	0.094
	2000	1.001	0.064	0.997	0.068	0.999	0.066
$\beta_2 = 0.700$	500	0.699	0.072	0.695	0.116	0.710	0.095
	1000	0.703	0.051	0.704	0.081	0.700	0.061
	2000	0.700	0.034	0.701	0.055	0.701	0.042
$\beta_3 = 1.100$	500	1.100	0.057	1.108	0.194	1.095	0.080
	1000	1.100	0.041	1.099	0.141	1.101	0.054
	2000	1.099	0.030	1.099	0.094	1.099	0.039
$\phi = 1.200$	500	1.211	0.124	1.220	0.102	1.214	0.119
	1000	1.204	0.088	1.214	0.070	1.206	0.088
	2000	1.202	0.060	1.206	0.048	1.203	0.060
$\rho = -0.500$	500	-0.486	0.165	-0.509	0.113	-0.482	0.163
	1000	-0.494	0.122	-0.493	0.087	-0.495	0.114
	2000	-0.500	0.080	-0.498	0.060	-0.497	0.079

Table 3. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 1 with $\rho = -0.2$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters	n	Configuration 1		Configuration 2		Configuration 3	
		Mean	RMSE	Mean	RMSE	Mean	RMSE
		$\gamma_1 = 1.600$	500	1.611	0.086	1.607	0.223
	1000	1.605	0.059	1.607	0.156	1.607	0.061
	2000	1.600	0.042	1.604	0.106	1.601	0.043
$\gamma_2 = 0.800$	500	0.818	0.098	0.821	0.206	0.810	0.127
	1000	0.803	0.067	0.805	0.154	0.811	0.087
	2000	0.803	0.044	0.802	0.103	0.806	0.064
$\gamma_3 = 0.200$	500	0.203	0.082	0.229	0.362	0.210	0.121
	1000	0.203	0.061	0.191	0.246	0.199	0.079
	2000	0.200	0.044	0.202	0.167	0.198	0.059
$\gamma_4 = 0.700$	500	0.716	0.094	0.718	0.061	0.714	0.097
	1000	0.704	0.063	0.706	0.041	0.705	0.070
	2000	0.705	0.045	0.704	0.028	0.704	0.042
$\beta_1 = 1.000$	500	1.005	0.128	0.996	0.135	1.001	0.131
	1000	0.999	0.090	0.995	0.099	0.999	0.090
	2000	1.002	0.062	0.998	0.068	1.000	0.063
$\beta_2 = 0.700$	500	0.697	0.075	0.697	0.118	0.709	0.098
	1000	0.703	0.054	0.704	0.082	0.700	0.066
	2000	0.699	0.036	0.701	0.056	0.701	0.044
$\beta_3 = 1.100$	500	1.099	0.057	1.105	0.192	1.096	0.080
	1000	1.101	0.043	1.098	0.141	1.100	0.055
	2000	1.099	0.030	1.099	0.096	1.100	0.040
$\phi = 1.200$	500	1.200	0.103	1.221	0.101	1.202	0.101
	1000	1.199	0.069	1.211	0.068	1.201	0.071
	2000	1.200	0.048	1.205	0.046	1.201	0.047
$\rho = -0.200$	500	-0.196	0.196	-0.211	0.140	-0.192	0.205
	1000	-0.197	0.151	-0.195	0.105	-0.200	0.147
	2000	-0.202	0.103	-0.198	0.076	-0.198	0.100

Table 4. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 1 with $\rho = 0$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters	n	Configuration 1		Configuration 2		Configuration 3	
		Mean	RMSE	Mean	RMSE	Mean	RMSE
		$\gamma_1 = 1.600$	500	1.612	0.085	1.608	0.219
	1000	1.605	0.059	1.606	0.154	1.607	0.062
	2000	1.600	0.042	1.602	0.109	1.601	0.043
$\gamma_2 = 0.800$	500	0.815	0.097	0.824	0.209	0.809	0.127
	1000	0.804	0.066	0.804	0.152	0.810	0.087
	2000	0.802	0.044	0.803	0.105	0.805	0.063
$\gamma_3 = 0.200$	500	0.205	0.081	0.223	0.360	0.209	0.121
	1000	0.203	0.060	0.188	0.243	0.199	0.079
	2000	0.200	0.044	0.201	0.172	0.199	0.058
$\gamma_4 = 0.700$	500	0.714	0.094	0.717	0.060	0.713	0.098
	1000	0.705	0.064	0.705	0.039	0.705	0.069
	2000	0.705	0.045	0.703	0.028	0.704	0.043
$\beta_1 = 1.000$	500	1.005	0.116	0.996	0.132	1.006	0.118
	1000	1.000	0.082	0.995	0.098	1.003	0.082
	2000	1.003	0.056	0.998	0.067	1.000	0.058
$\beta_2 = 0.700$	500	0.698	0.075	0.698	0.117	0.706	0.098
	1000	0.702	0.055	0.703	0.083	0.697	0.066
	2000	0.699	0.037	0.701	0.057	0.700	0.044
$\beta_3 = 1.100$	500	1.099	0.056	1.103	0.192	1.096	0.080
	1000	1.101	0.043	1.099	0.140	1.100	0.055
	2000	1.099	0.030	1.099	0.097	1.100	0.040
$\phi = 1.200$	500	1.199	0.098	1.222	0.101	1.199	0.098
	1000	1.198	0.064	1.211	0.068	1.199	0.066
	2000	1.200	0.044	1.205	0.046	1.200	0.044
$\rho = 0.000$	500	0.000	0.205	-0.008	0.147	-0.003	0.210
	1000	-0.001	0.155	0.003	0.109	-0.010	0.153
	2000	-0.004	0.107	0.003	0.079	-0.001	0.104

Table 5. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 1 with $\rho = 0.2$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters	n	Configuration 1		Configuration 2		Configuration 3	
		Mean	RMSE	Mean	RMSE	Mean	RMSE
$\gamma_1 = 1.600$	500	1.611	0.084	1.610	0.222	1.609	0.090
	1000	1.605	0.060	1.607	0.156	1.607	0.062
	2000	1.599	0.042	1.603	0.108	1.602	0.043
$\gamma_2 = 0.800$	500	0.814	0.098	0.825	0.207	0.808	0.126
	1000	0.804	0.066	0.803	0.152	0.810	0.086
	2000	0.803	0.044	0.805	0.105	0.805	0.063
$\gamma_3 = 0.200$	500	0.205	0.081	0.220	0.366	0.210	0.121
	1000	0.204	0.062	0.188	0.247	0.200	0.080
	2000	0.200	0.043	0.202	0.172	0.198	0.058
$\gamma_4 = 0.700$	500	0.713	0.094	0.717	0.059	0.712	0.098
	1000	0.705	0.064	0.705	0.039	0.706	0.068
	2000	0.705	0.044	0.703	0.027	0.704	0.044
$\beta_1 = 1.000$	500	1.007	0.101	0.996	0.130	1.005	0.104
	1000	1.000	0.072	0.995	0.097	1.004	0.071
	2000	1.002	0.050	0.998	0.067	1.000	0.051
$\beta_2 = 0.700$	500	0.696	0.074	0.698	0.117	0.705	0.096
	1000	0.702	0.055	0.704	0.084	0.697	0.066
	2000	0.699	0.037	0.701	0.057	0.701	0.044
$\beta_3 = 1.100$	500	1.099	0.057	1.103	0.189	1.097	0.081
	1000	1.101	0.042	1.100	0.139	1.100	0.055
	2000	1.099	0.030	1.099	0.096	1.100	0.040
$\phi = 1.200$	500	1.202	0.104	1.222	0.102	1.201	0.107
	1000	1.199	0.069	1.210	0.068	1.202	0.070
	2000	1.201	0.047	1.204	0.046	1.200	0.048
$\rho = 0.200$	500	0.192	0.197	0.193	0.147	0.192	0.199
	1000	0.195	0.148	0.202	0.105	0.186	0.149
	2000	0.195	0.104	0.200	0.076	0.199	0.100

Table 6. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 1 with $\rho = 0.5$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters	n	Configuration 1		Configuration 2		Configuration 3	
		Mean	RMSE	Mean	RMSE	Mean	RMSE
$\gamma_1 = 1.600$	500	1.612	0.083	1.608	0.220	1.606	0.088
	1000	1.605	0.060	1.609	0.157	1.606	0.061
	2000	1.599	0.042	1.605	0.107	1.602	0.044
$\gamma_2 = 0.800$	500	0.815	0.096	0.822	0.207	0.807	0.128
	1000	0.805	0.066	0.801	0.151	0.810	0.086
	2000	0.802	0.043	0.804	0.102	0.805	0.063
$\gamma_3 = 0.200$	500	0.208	0.081	0.214	0.355	0.210	0.120
	1000	0.204	0.061	0.190	0.245	0.199	0.079
	2000	0.200	0.044	0.202	0.169	0.199	0.057
$\gamma_4 = 0.700$	500	0.712	0.093	0.713	0.056	0.710	0.093
	1000	0.707	0.061	0.706	0.040	0.706	0.066
	2000	0.705	0.043	0.704	0.028	0.704	0.042
$\beta_1 = 1.000$	500	1.005	0.079	0.995	0.125	1.005	0.086
	1000	1.000	0.057	0.998	0.093	1.003	0.056
	2000	1.001	0.039	0.999	0.065	1.000	0.040
$\beta_2 = 0.700$	500	0.696	0.070	0.699	0.115	0.704	0.093
	1000	0.701	0.052	0.703	0.083	0.697	0.064
	2000	0.699	0.035	0.701	0.057	0.700	0.043
$\beta_3 = 1.100$	500	1.100	0.056	1.102	0.185	1.097	0.080
	1000	1.101	0.042	1.098	0.136	1.100	0.054
	2000	1.099	0.030	1.097	0.096	1.100	0.039
$\phi = 1.200$	500	1.210	0.130	1.226	0.105	1.211	0.131
	1000	1.205	0.086	1.210	0.070	1.211	0.088
	2000	1.204	0.062	1.205	0.047	1.203	0.060
$\rho = 0.500$	500	0.490	0.158	0.496	0.124	0.487	0.168
	1000	0.493	0.118	0.500	0.086	0.483	0.122
	2000	0.495	0.081	0.502	0.062	0.496	0.078

• **Scenario 2**

We conducted Monte Carlo simulations to evaluate and compare the performance of the maximum likelihood estimators of the parameters of the Heckman-BS model for finite samples and under the absence of exclusion constraint. We set $\beta_1 = 1, \beta_2 = 0.7, \beta_3 = 1.1, \gamma_1 = 1.6, \gamma_2 = 0.8, \gamma_3 = 0.2, \phi = 1.2$ and consider some values for the correlation parameter ρ .

Table 7. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 2 with $\rho = -0.7$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters	n	Configuration 1		Configuration 2		Configuration 3	
		Mean	RMSE	Mean	RMSE	Mean	RMSE
$\gamma_1 = 1.600$	500	1.607	0.086	1.622	0.190	1.604	0.087
	1000	1.603	0.060	1.612	0.134	1.604	0.059
	2000	1.601	0.042	1.602	0.088	1.601	0.040
$\gamma_2 = 0.800$	500	0.813	0.089	0.793	0.328	0.804	0.117
	1000	0.808	0.065	0.800	0.232	0.801	0.081
	2000	0.804	0.046	0.804	0.160	0.802	0.064
$\gamma_3 = 0.200$	500	0.203	0.023	0.203	0.016	0.206	0.110
	1000	0.202	0.016	0.202	0.011	0.196	0.074
	2000	0.201	0.011	0.201	0.007	0.198	0.054
$\beta_1 = 1.000$	500	0.914	0.281	0.987	0.160	0.856	0.339
	1000	0.975	0.147	0.990	0.109	0.794	0.390
	2000	0.996	0.073	0.997	0.078	0.963	0.176
$\beta_2 = 0.700$	500	0.742	0.141	0.699	0.192	0.779	0.195
	1000	0.710	0.071	0.710	0.139	0.809	0.219
	2000	0.701	0.035	0.698	0.095	0.720	0.095
$\beta_3 = 1.100$	500	1.111	0.035	1.101	0.009	1.113	0.087
	1000	1.102	0.018	1.101	0.006	1.126	0.072
	2000	1.100	0.008	1.100	0.004	1.104	0.044
$\phi = 1.200$	500	1.227	0.161	1.225	0.117	1.231	0.172
	1000	1.214	0.107	1.210	0.082	1.211	0.129
	2000	1.207	0.075	1.207	0.057	1.205	0.084
$\rho = -0.700$	500	-0.541	0.459	-0.683	0.133	-0.435	0.595
	1000	-0.662	0.209	-0.697	0.079	-0.322	0.727
	2000	-0.695	0.060	-0.698	0.056	-0.629	0.312

Table 8. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 2 with $\rho = -0.5$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters	n	Configuration 1		Configuration 2		Configuration 3	
		Mean	RMSE	Mean	RMSE	Mean	RMSE
		$\gamma_1 = 1.600$	500	1.609	0.087	1.624	0.195
	1000	1.605	0.060	1.611	0.136	1.604	0.059
	2000	1.601	0.042	1.601	0.090	1.601	0.040
$\gamma_2 = 0.800$	500	0.816	0.091	0.796	0.339	0.805	0.115
	1000	0.807	0.066	0.799	0.238	0.806	0.081
	2000	0.804	0.047	0.804	0.160	0.805	0.063
$\gamma_3 = 0.200$	500	0.203	0.023	0.204	0.016	0.207	0.113
	1000	0.202	0.016	0.201	0.011	0.197	0.075
	2000	0.201	0.011	0.201	0.008	0.199	0.056
$\beta_1 = 1.000$	500	0.959	0.220	0.986	0.172	0.937	0.236
	1000	0.983	0.137	0.990	0.119	0.942	0.201
	2000	0.998	0.087	0.999	0.084	0.996	0.100
$\beta_2 = 0.700$	500	0.723	0.110	0.698	0.198	0.742	0.145
	1000	0.708	0.067	0.709	0.142	0.731	0.116
	2000	0.701	0.041	0.698	0.098	0.704	0.052
$\beta_3 = 1.100$	500	1.106	0.028	1.101	0.011	1.104	0.083
	1000	1.102	0.016	1.101	0.007	1.108	0.059
	2000	1.100	0.010	1.100	0.005	1.100	0.040
$\phi = 1.200$	500	1.206	0.143	1.219	0.110	1.207	0.150
	1000	1.206	0.096	1.208	0.078	1.204	0.116
	2000	1.203	0.073	1.205	0.054	1.202	0.080
$\rho = -0.500$	500	-0.404	0.342	-0.472	0.195	-0.361	0.402
	1000	-0.465	0.194	-0.493	0.121	-0.384	0.356
	2000	-0.492	0.106	-0.499	0.084	-0.485	0.129

Table 9. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 2 with $\rho = -0.2$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters	n	Configuration 1		Configuration 2		Configuration 3	
		Mean	RMSE	Mean	RMSE	Mean	RMSE
		$\gamma_1 = 1.600$	500	1.611	0.086	1.626	0.202
	1000	1.605	0.061	1.609	0.137	1.604	0.059
	2000	1.602	0.042	1.602	0.092	1.601	0.040
$\gamma_2 = 0.800$	500	0.816	0.093	0.795	0.352	0.805	0.119
	1000	0.808	0.067	0.800	0.242	0.806	0.083
	2000	0.803	0.048	0.803	0.166	0.805	0.063
$\gamma_3 = 0.200$	500	0.203	0.023	0.203	0.016	0.208	0.118
	1000	0.202	0.016	0.201	0.011	0.199	0.075
	2000	0.201	0.011	0.200	0.008	0.199	0.056
$\beta_1 = 1.000$	500	0.996	0.185	0.993	0.173	0.992	0.192
	1000	0.997	0.128	0.992	0.125	0.996	0.143
	2000	1.003	0.091	1.000	0.086	1.001	0.102
$\beta_2 = 0.700$	500	0.710	0.101	0.695	0.201	0.719	0.123
	1000	0.704	0.068	0.710	0.146	0.705	0.087
	2000	0.700	0.046	0.698	0.099	0.704	0.059
$\beta_3 = 1.100$	500	1.103	0.026	1.101	0.011	1.099	0.081
	1000	1.101	0.016	1.100	0.008	1.102	0.057
	2000	1.100	0.011	1.100	0.005	1.100	0.040
$\phi = 1.200$	500	1.176	0.118	1.209	0.096	1.176	0.119
	1000	1.190	0.074	1.202	0.066	1.184	0.084
	2000	1.194	0.054	1.203	0.047	1.191	0.059
$\rho = -0.200$	500	-0.154	0.312	-0.178	0.233	-0.145	0.323
	1000	-0.183	0.209	-0.195	0.160	-0.175	0.240
	2000	-0.197	0.149	-0.201	0.114	-0.189	0.169

Table 10. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 2 with $\rho = 0$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters	n	Configuration 1		Configuration 2		Configuration 3	
		Mean	RMSE	Mean	RMSE	Mean	RMSE
		$\gamma_1 = 1.600$	500	1.608	0.085	1.629	0.201
	1000	1.605	0.062	1.609	0.138	1.605	0.060
	2000	1.601	0.043	1.602	0.095	1.601	0.041
$\gamma_2 = 0.800$	500	0.812	0.093	0.790	0.345	0.806	0.122
	1000	0.809	0.067	0.801	0.242	0.807	0.084
	2000	0.803	0.047	0.801	0.170	0.804	0.063
$\gamma_3 = 0.200$	500	0.203	0.023	0.204	0.016	0.207	0.121
	1000	0.202	0.017	0.201	0.011	0.198	0.076
	2000	0.201	0.011	0.200	0.008	0.199	0.056
$\beta_1 = 1.000$	500	1.011	0.175	1.001	0.167	1.016	0.180
	1000	1.002	0.119	0.995	0.122	1.012	0.134
	2000	1.007	0.086	1.001	0.084	1.005	0.092
$\beta_2 = 0.700$	500	0.704	0.098	0.692	0.201	0.708	0.123
	1000	0.702	0.067	0.709	0.146	0.697	0.087
	2000	0.698	0.048	0.697	0.099	0.702	0.060
$\beta_3 = 1.100$	500	1.101	0.026	1.100	0.012	1.097	0.083
	1000	1.100	0.017	1.100	0.008	1.100	0.057
	2000	1.100	0.012	1.100	0.006	1.099	0.040
$\phi = 1.200$	500	1.169	0.118	1.205	0.093	1.165	0.119
	1000	1.186	0.069	1.201	0.065	1.178	0.079
	2000	1.192	0.049	1.202	0.045	1.188	0.051
$\rho = 0.000$	500	0.012	0.305	0.011	0.239	0.007	0.320
	1000	0.006	0.215	-0.003	0.168	-0.008	0.241
	2000	-0.007	0.163	-0.002	0.121	-0.000	0.175

Table 11. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 2 with $\rho = 0.2$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters	n	Configuration 1		Configuration 2		Configuration 3	
		Mean	RMSE	Mean	RMSE	Mean	RMSE
		$\gamma_1 = 1.600$	500	1.608	0.085	1.632	0.206
	1000	1.604	0.061	1.611	0.138	1.604	0.061
	2000	1.601	0.042	1.602	0.095	1.600	0.041
$\gamma_2 = 0.800$	500	0.814	0.093	0.790	0.349	0.808	0.121
	1000	0.809	0.068	0.798	0.243	0.807	0.084
	2000	0.804	0.046	0.801	0.170	0.803	0.062
$\gamma_3 = 0.200$	500	0.202	0.023	0.204	0.017	0.206	0.123
	1000	0.202	0.017	0.201	0.011	0.198	0.077
	2000	0.201	0.011	0.200	0.008	0.200	0.055
$\beta_1 = 1.000$	500	1.030	0.171	1.006	0.161	1.027	0.169
	1000	1.008	0.109	0.998	0.116	1.016	0.120
	2000	1.008	0.077	1.002	0.079	1.008	0.082
$\beta_2 = 0.700$	500	0.692	0.099	0.692	0.204	0.700	0.122
	1000	0.699	0.067	0.708	0.145	0.693	0.087
	2000	0.697	0.049	0.696	0.099	0.699	0.061
$\beta_3 = 1.100$	500	1.098	0.026	1.100	0.011	1.095	0.083
	1000	1.100	0.017	1.100	0.008	1.099	0.056
	2000	1.099	0.012	1.100	0.005	1.099	0.039
$\phi = 1.200$	500	1.175	0.121	1.205	0.096	1.169	0.125
	1000	1.188	0.075	1.203	0.068	1.184	0.083
	2000	1.195	0.056	1.203	0.046	1.191	0.058
$\rho = 0.200$	500	0.165	0.307	0.198	0.234	0.171	0.322
	1000	0.189	0.213	0.188	0.162	0.173	0.238
	2000	0.187	0.160	0.194	0.116	0.189	0.173

Table 12. Empirical mean of the maximum likelihood estimates with their respective root mean square error (RMSE) for the Birnbaum-Saunders sample selection model under Scenario 2 with $\rho = 0.5$. Sample size $n = 500, n = 1000$ and $n = 2000$ with $N = 1000$ Monte Carlo replicates.

Parameters	n	Configuration 1		Configuration 2		Configuration 3	
		Mean	RMSE	Mean	RMSE	Mean	RMSE
		$\gamma_1 = 1.600$	500	1.607	0.086	1.624	0.203
	1000	1.604	0.062	1.609	0.134	1.604	0.060
	2000	1.601	0.042	1.602	0.093	1.599	0.042
$\gamma_2 = 0.800$	500	0.816	0.091	0.803	0.346	0.808	0.123
	1000	0.809	0.067	0.799	0.234	0.807	0.082
	2000	0.804	0.047	0.802	0.161	0.803	0.062
$\gamma_3 = 0.200$	500	0.203	0.023	0.204	0.017	0.205	0.118
	1000	0.202	0.016	0.201	0.011	0.198	0.076
	2000	0.201	0.011	0.201	0.008	0.200	0.056
$\beta_1 = 1.000$	500	1.043	0.169	1.008	0.141	1.067	0.217
	1000	1.012	0.099	0.998	0.099	1.014	0.101
	2000	1.004	0.053	1.001	0.067	1.009	0.075
$\beta_2 = 0.700$	500	0.680	0.104	0.691	0.198	0.672	0.138
	1000	0.694	0.065	0.704	0.140	0.692	0.083
	2000	0.697	0.043	0.697	0.098	0.696	0.059
$\beta_3 = 1.100$	500	1.096	0.026	1.099	0.010	1.090	0.084
	1000	1.099	0.016	1.100	0.007	1.099	0.054
	2000	1.099	0.011	1.100	0.005	1.099	0.039
$\phi = 1.200$	500	1.207	0.146	1.215	0.107	1.199	0.151
	1000	1.206	0.096	1.209	0.076	1.204	0.111
	2000	1.204	0.076	1.205	0.051	1.203	0.082
$\rho = 0.500$	500	0.415	0.317	0.485	0.184	0.375	0.396
	1000	0.469	0.196	0.490	0.119	0.462	0.209
	2000	0.488	0.118	0.496	0.083	0.476	0.158