**Large deformation near a crack tip in a fiber-reinforced neo-Hookean sheet**

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**Abstract**

The asymptotic fields at the tip of a crack in a fiber-reinforced neo-Hookean sheet are derived. The investigation is carried out for the case of a strain energy function for a fiber-reinforced hyperelastic material motivated by composite mechanics (Guo et al., 2006, 2007ab), where the fibers are also neo-Hookean. The resulting asymptotic deformation and stress fields depend qualitatively and quantitatively on the degree of fiber reinforcement. For suitable choice of parameters, the strain energy potential for the material reduces to that of a pure neo-Hookean material and the corresponding asymptotic fields to those obtained by Knowles and Sternberg (1983). The result obtained may prove useful in providing a framework for future exploration in modeling and assessing the mechanical behavior near a slit or tear in soft biological tissue reinforced by collagen fibers and in other applications of fiber-reinforced soft materials.

Keywords: Crack tip fields; Fiber-reinforced materials; Asymptotic analysis; Crack tip integrals

**1 Introduction**

Anisotropic soft materials with fiber-like inclusions or specifically oriented microstructures exist widely in nature, e.g., the skeletal muscle (Gennisson et al., 2010), the leaves of the Venus flytrap (Forterre et al., 2005), and the keel of the ice plant (Harrington et al., 2011). Large deformation constitutive models for soft tissue are important for understanding the behavior of muscles, cartilage
and arteries, for example, see the review in Holzapfel et al. (2019). For example, the mechanical anisotropy in these organism materials plays a key role in achieving functions like actuation, seed spreading and predation. Synthesized anisotropic soft materials have also been found to have wide potential use in soft robotics and actuators such as 4D printing fiber reinforced hydrogels (Sydney et al., 2016) and swellable elastomer composites reinforced by magneto-responsive inorganic particles (Erb et al., 2013). In this paper, we focus on theoretical investigation of the crack tip fields of an anisotropic hyperelastic material to promote the understanding of the influence of material anisotropy on the fracture behaviors of soft solids and to provide a basis for guidance on the design of highly tough soft composites through introducing anisotropic microstructures.

Most of the research to date on the analysis of crack tip fields for anisotropic materials has been for the case of small deformation. For example, Sih et al. (1965) investigated the crack-tip stress fields in rectilinear anisotropic bodies using a complex variable approach and found an $r^{-1/2}$ singularity of the stress fields ($r$ is the radial distance from the crack tip). The brittle fracture of unidirectional composites with cracks propagated along the fiber direction was then studied by Sih and Chen (1973). Gdoutos et al. (1989) analyzed the fracture behavior of an anisotropic panel with a crack inclined with respect to the principal axes of material symmetry under in-plane uniform uniaxial stress. More recently, Nobile et al. (2004) investigated an inclined crack in an orthotropic medium under biaxial loading and obtained closed form solutions for the crack tip stress and deformation fields. The asymptotic crack-tip fields in an anisotropic plate under bending, twisting moments and transverse shear loads were recently studied in (Li, 2002; Yuan and Yang, 2000).

The crack tip fields in an initially isotropic hyperelastic solid were first studied by Wong and Shield (1969) for a biaxially stretched neo-Hookean sheet with large stretch throughout the entire domain. Knowles and Sternberg (1973, 1974) investigated the asymptotic crack tip fields in a compressible rubber-like material under plane strain condition within the framework of finite elasticity. A basic analysis method involved in their work is the usage of an asymptotic expansion of the deformed coordinates in the governing equations, which leads to an eigenvalue problem with eigenfunctions characterizing the circumferential distribution of the deformed coordinates. The amplitude of the deformed coordinate field is related to the far-field boundary conditions by the path-independent $J$-integral. Using this asymptotic analysis technique, subsequent work focusing on other type of hyperelastic materials have been emerging until recent years. For example, the asymptotic crack tip fields for a wide range of two-dimensional hyperelastic solids in both homogenous and
interface settings have been reported, e.g., for an incompressible neo-Hookean material under conditions of plane strain (Stephenson, 1982) and its extension to the generalized linear neo-Hookean material (Begley et al., 2015), bimaterial interface crack in neo-Hookean materials under plane stress conditions (Knowles and Sternberg, 1983; Ravichandran and Knauss, 1989), a generalized neo-Hookean material for both homogeneous and interface crack (Geubelle and Knauss, 1994a, 1994b), and a strain hardening material in the plane stress condition or mode III crack (Long et al., 2011; Long and Hui, 2011), to name a few.

In the present paper, we analyze the crack tip fields of an anisotropic hyperelastic sheet using the neo-Hookean fiber-reinforced neo-Hookean matrix material model developed in Guo et al., (2007a, 2007b). For the case of a thin sheet in which plane stress conditions hold, the asymptotic governing equations at the crack tip field for the fiber-reinforced material reduce to those obtained by Knowles and Sternberg (1983) for a pure neo-Hookean material by a suitable scaling of the coordinate system. Furthermore, following the observation by Liu and Moran (2019), where the asymptotic governing equations separate into two independent Laplace equations, we use the asymptotic path-independent integrals derived in that paper to evaluate the crack tip parameters. The analytical deformed coordinates and stress fields agree well with finite element results for different stretch levels, loading modes, and material parameters.

This paper is organized as follows. In Section 2, we introduce the general governing equations of finite elasticity of a fiber-reinforced hyperelastic material. This theory then is specialized to the plane stress case to describe two dimensional crack tip fields in Section 3. In Section 4, we derive the asymptotic governing equations based on the features of the deformation fields at the crack tip. Analytical solutions for the deformed coordinates and stresses are obtained. In Section 5, asymptotic path-independent integrals derived by Liu and Moran (2019) are extended to the present case of anisotropic neo-Hookean materials and used to evaluate the crack tip parameters. In Section 6, a finite element simulation of the crack tip fields of a clamped long strip with an edge crack over half its width. The finite element results and analytical results are compared and analyzed. Concluding remarks are made in Section 7.

2 Preliminary kinematics, stress measures and governing equations

Following Knowles and Sternberg (1983) and using similar notation, we consider a thin sheet of material occupying the three-dimensional undeformed region, $\mathcal{R}$, i.e.,
\[ \mathcal{R} = \{ \mathbf{x} \mid (x_1, x_2) \in \Pi, -\frac{1}{2} t \leq x_3 \leq \frac{1}{2} t \} \]  

where \( \Pi \) is the cross-section of \( \mathcal{R} \) in the plane \( x_3 = 0 \), and \( t \) is the constant initial thickness of the sheet. The mapping from the reference configuration to the current is given by

\[ y(x) = x + \mathbf{u}(x) \]  

The deformation gradient is written as

\[ \mathbf{F} = (\nabla y)^T \]  

where \( \nabla = \mathbf{e}_i \partial / \partial x_i \) is the gradient operator with respect to the undeformed coordinates. For incompressible materials, we have the volume-preserving constraint, \( \det \mathbf{F} = 1 \). The equilibrium equation is written as

\[ \nabla \cdot \mathbf{P}^T = 0 \]  

where \( \mathbf{P} \) is the first Piola-Kirchhoff stress, and the Cauchy (true) stress is given by \( \mathbf{\sigma} = \mathbf{P} \cdot \mathbf{F}^T \).

The right Cauchy Green deformation tensor and its three principal invariants are given by

\[ \mathbf{C} = \mathbf{F}^T \mathbf{F} \quad I_1 = \text{tr} \mathbf{C} \quad I_2 = \frac{1}{2} \left[ (\text{tr} \mathbf{C})^2 - \text{tr} \mathbf{C}^2 \right] \quad I_3 = \det \mathbf{C} = 1 \]  

The invariants may be written in terms of the principal stretches as

\[ I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad I_2 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 \quad I_3 = \lambda_1 \lambda_2 \lambda_3 = 1 \]  

Any isotropic strain energy function can be written as a function of these invariants. For a transversely isotropic material with axis of anisotropy \( \mathbf{a}_o \), Spenser showed that a general strain energy function required the additional invariants (Spencer, 2014)

\[ I_4 = \mathbf{a}_o \cdot \mathbf{C} \cdot \mathbf{a}_o \quad I_5 = \mathbf{a}_o \cdot \mathbf{C}^2 \cdot \mathbf{a}_o \]  

For our purposes, the orientation of the aligned fibers is denoted by \( \mathbf{a}_o \). Let \( \mathbf{a} = \mathbf{F} \cdot \mathbf{a}_o \) and thus

\[ I_4 = \mathbf{a} \cdot \mathbf{a} = \lambda_f^2 \]
where $\lambda_f$ is the fiber stretch.

We will confine our attention to a class of materials for which the strain energy per unit undeformed volume can be written as

$$W = W(I_1, I_4) \quad (9)$$

In particular we will be interested in material which consists of a neo-Hookean matrix reinforced by neo-Hookean fibers and for which the strain energy is given by (Guo et al., 2007a, 2007b, 2006)

$$W = \frac{\mu}{2} [(I_1 - 3) + \kappa (I_4 + 2I_4^{1/2} - 3)] \quad (10)$$

where $\kappa$ is a material constant which represents the relative shear stiffness of the fiber to the matrix. This form of the strain energy function was derived using a composites based approach and a multiplicative of the deformation gradient into a uniaxial stretch along the fiber direction and a subsequent shear deformation. The first term may be thought of as the matrix contribution and the second that of the fibers.

Remark 1: The term involving $I_4^{1/2}$ arises naturally in the composites based approach. It ensures that infinite energy is required to reduce the fiber stretch to zero, compared with a simple spring approach where a term of the form $(I_4 - 1)^2$ requires only a finite energy.

The second Piola-Kirchhoff stress is defined as $S = F^{-1}P$ and is computed from the strain energy function by

$$S = 2 \frac{\partial W}{\partial C} - pC^{-1} \quad (11)$$

where $p = -\frac{1}{3} \text{tr} \sigma$ is the pressure and the second term above is a result of the incompressibility constraint. For the strain energy function (9) this gives

$$S = 2 \frac{\partial W}{\partial I_1} \mathbf{I} + 2 \frac{\partial W}{\partial I_4} \mathbf{a}_o \otimes \mathbf{a}_o - pC^{-1} \quad \mathbf{P} = 2 \frac{\partial W}{\partial I_1} \mathbf{F} + 2 \frac{\partial W}{\partial I_4} \mathbf{F} \mathbf{a}_o \otimes \mathbf{a}_o - p\mathbf{F}^{-T} \quad (12)$$

and for the particular case (10) we obtain
\[ P = \mu F + \mu \kappa (1 - l_4^{-3/2}) F \alpha_\alpha \otimes \alpha_\alpha - p F^{-T} \] 

(13)

3 Plane stress equations

Following Knowles and Sternberg (1983), for a state of plane stress, we assume that the deformation is symmetric about the plane \( x_3 = 0 \). The fibers are assumed to be orthogonal to the \( x_3 \) direction and lie in the \( x_1-x_2 \) plane. Furthermore, \( y_\alpha(x_1,x_2,x_3), \alpha = 1,2 \) is an even function of \( x_3 \), and \( y_3(x_1,x_2,x_3) \) is an odd function of \( x_3 \). Traction free boundary conditions on the plane surfaces of the sheet require \( P_{i3}(x_1,x_2,\pm t) = 0 \) for \( i = 1,2 \). For small \( t \) compared to the characteristic dimension of \( \Pi \), we take \( P_{33} = 0 \) on \( \Pi \) and \( P_{a3} = 0 \) on \( \mathfrak{R} \).

We will henceforth restrict all quantities to the plane \( \Pi \). From the plane stress conditions above, we have

\[ P_{a\beta,\nu} = 0 \] 

(14)

Let \( F_{33} = \lambda \) and \( J = \det [F_{a\beta}] \), then, from the incompressibility constraint,

\[ J = \lambda^{-1}, \ C_{33}^{-1} = \lambda^{-2}, \ F_{a3} = F_{3a} = 0 \] 

(15)

From the expression (13) for the stress and (15), it follows that

\[ P_{a3} = P_{3a} = 0, \quad P_{33} = 0 \] 

(16)

Using (11), the last of these allows for the determination of the following expression for the Lagrangian multiplier

\[ p = \mu \lambda^2 \] 

(17)

where we have used the relation \( \mathbf{e}_3 \alpha_\alpha = 0 \).

With this result, the stresses can be written as

\[ P_{a\beta} = \mu F_{a\beta} + \mu \kappa (1 - \lambda_f^{-3}) F_{av} \alpha_v \alpha_\beta - \mu \lambda^2 F_{\beta\alpha}^{-1} \] 

(18)
Using
\[ J F^{-1} = \boldsymbol{e}_{\alpha\beta} \boldsymbol{e}_{\rho\nu} F \boldsymbol{e}_{\nu\gamma} = \boldsymbol{e}_{\alpha\beta} \boldsymbol{e}_{\rho\nu} y_{\alpha\gamma} \] (19)

where \( \boldsymbol{e}_{\alpha\beta} \) is the two-dimensional alternating symbol (\( \epsilon_{11} = \epsilon_{22} = 0, \epsilon_{12} = -\epsilon_{21} = 1 \)), the stress may be written as
\[ P_{\alpha\beta} = \mu [ y_{\alpha\beta} + \kappa (1 - \lambda_f^{-3}) y_{\alpha\gamma} a^\alpha a^\beta - \lambda^3 \boldsymbol{e}_{\alpha\beta} e_{\alpha\gamma} y_{\alpha\gamma} ] \] (20)

Using this expression in the equilibrium equation (14) and dividing by \( \mu \) we get
\[ y_{\alpha\beta} + \kappa (1 - \lambda_f^{-3}) y_{\alpha\gamma} a^\alpha a^\beta + \kappa (1 - \lambda_f^{-3}) y_{\alpha\beta} a^\alpha a^\gamma - \lambda^3 \boldsymbol{e}_{\beta\alpha} e_{\alpha\gamma} y_{\alpha\gamma} - 3 \lambda^2 \lambda_\beta \boldsymbol{e}_{\beta\alpha} e_{\alpha\gamma} y_{\alpha\gamma} = 0 \] (21)

which is a nonlinear partial differential equation for the \( y_{\alpha\beta} \). The nominal tractions \( t_{\alpha\beta} \) on a surface with outward normal \( n_\beta \) are given by
\[ t_{\alpha\beta} = P_{\alpha\beta} n_\beta = \mu [ y_{\alpha\beta} n_\beta + \kappa (1 - \lambda_f^{-3}) y_{\alpha\gamma} a^\alpha a^\beta n_\beta - \lambda^3 \boldsymbol{e}_{\beta\alpha} e_{\alpha\gamma} y_{\alpha\gamma} n_\beta ] \] (22)

4 Asymptotic crack tip boundary value problem and its solutions

4.1 Asymptotic boundary value problems

We seek a local solution to the equilibrium equation (21) together with traction free boundary conditions on the crack faces. As is customary, the crack tip fields will be determined up to unknown amplitudes, the values of which depend on the far field loading conditions. Referring to Figure 1, a crack is located along the \( x_1 \) axis and symmetrically about the origin. The fiber orientation makes an angle \( \phi \) with the \( x_2 \) axis.
Figure 1 Global and crack-tip coordinate systems in the reference configuration

A polar coordinate system \((r, \theta)\) in the reference configuration is centered at crack tip. We will seek solutions where the in plane stretches are singular, i.e., of the form

\[
\lambda_u = O(r^{-m}) \quad \text{as} \quad r \to 0 \quad (-\pi \leq \theta \leq \pi)
\]

for some real \(m > 0\). Furthermore we will insist that the out of plane stretch exhibits the asymptotic form (Knowles and Sternberg, 1983; Wong and Shield, 1969)

\[
\lambda = J^{-1} = O(r^q) \quad \text{as} \quad r \to 0 \quad (-\pi \leq \theta \leq \pi)
\]

for some real \(q > 0\). This implies that some of the components of the deformation gradient will be unbounded which will simplify the asymptotic forms of the governing equations. In particular, the fiber stretch \(\lambda_f \to \infty\) as \(r \to 0\) and consequently, the out of plane stretch \(\lambda \to 0\) as \(r \to 0\).

**Remark 2:** We will find in Section 5.3 and Appendix C that the assumption for \(\lambda_f \to \infty\) as \(r \to 0\) is valid for fiber orientation angle \(\phi \in [0, \pi/2]\). For the case of \(\phi = \pi/2\) with horizontal fibers, \(\lambda_f\) is bounded on the crack line and a separate analysis for this case is given in Appendix C.

Using these results and noting that the terms involving \(\lambda\) and \(\lambda_f\) in Eqs. (20)~(22) do not come into play in the lowest order analysis, as a result, we can rewrite the stress, the governing equation and the traction free boundary conditions on the crack surfaces, i.e.,
\[ P_{\alpha\beta} = \mu \left( y_{\alpha,\beta} + \kappa y_{\alpha,\alpha} a_\alpha a_\beta \right) \]  
(25)

\[ y_{\alpha,\beta\beta} + \kappa y_{\alpha,\alpha\beta} a_\alpha a_\beta = 0 \]  
(26)

\[ \left( y_{\alpha,\beta} + \kappa y_{\alpha,\alpha} a_\alpha a_\beta \right) n_\beta = 0, \text{ at } x_1 \leq 0, x_2 = 0 \]  
(27)

If two sets of fibers are introduced in the problem with orientation angles \( \phi_a \) and \( \phi_b \) and stiffness ratios \( \kappa_a \) and \( \kappa_b \), the above equations are modified as

\[ P_{\alpha\beta} = \mu \left( y_{\alpha,\beta} + \kappa_a y_{\alpha,\alpha} a_\alpha a_\beta + \kappa_b y_{\alpha,\alpha} b_\alpha b_\beta \right) \]  
(28)

\[ y_{\alpha,\beta\beta} + \kappa_a y_{\alpha,\alpha\beta} a_\alpha a_\beta + \kappa_b y_{\alpha,\alpha\beta} b_\alpha b_\beta = 0 \]  
(29)

\[ y_{\alpha,\beta} + \kappa_a y_{\alpha,\alpha} a_\alpha a_\beta + \kappa_b y_{\alpha,\alpha} b_\alpha b_\beta \quad n_\beta = 0, \text{ at } x_1 \leq 0, x_2 = 0 \]  
(30)

where \( a_\alpha = \sin \phi_a \), \( a_\beta = \cos \phi_a \), \( b_\alpha = \sin \phi_b \), and \( b_\beta = \cos \phi_b \).

It is clear that by a constant rotation and a constant scaling of the coordinate axes, Eq. (26) can be transformed into a Laplace equation for \( y_{\alpha} \). It is convenient to describe the transformation in a manner consistent with the usual description of the same transformation in anisotropic linearly elastic antiplane crack problems (Ting, 1996). Thus noting that \( a_\alpha = \sin \phi \), \( a_\beta = \cos \phi \), equations (25) and (26) above can be written as

\[ c_{55} y_{\alpha,11} + 2c_{45} y_{\alpha,12} + c_{44} y_{\alpha,22} = 0 \]  
(31)

\[ c_{45} y_{\alpha,1} + c_{44} y_{\alpha,2} = 0 \]  
(32)

where

\[ c_{44} = 1 + \kappa \cos^2 \phi, \quad c_{45} = \kappa \sin \phi \cos \phi, \quad c_{55} = 1 + \kappa \sin^2 \phi \]  
(33)

For the sheets with two sets of fibers, from Eqs. (28)–(30), these coefficients are given by
\begin{align}
c_{44} &= 1 + \kappa_a \cos^2 \phi_a + \kappa_b \cos^2 \phi_b, \\
c_{45} &= \kappa_a \sin \phi_a \cos \phi_b + \kappa_b \sin \phi_b \cos \phi_a, \\
c_{55} &= 1 + \kappa_a \sin^2 \phi_a + \kappa_b \sin^2 \phi_b, 
\end{align}
(34)

The boundary value problem is the same as the case for antiplane shear of a linearly anisotropic composite and could be solved using a complex variable approach (Sih et al., 1965). Here, we employ a method based on a linear transformation of the original coordinates \( x_i, i = 1, 2 \) to scaled coordinates \( \eta_i, i = 1, 2 \), that is (Appendix A),

\[ \eta_i = A_{ij} x_j, \quad i, j = 1, 2, \quad A_{11} = \sqrt{\frac{c_{44}}{c_{44}c_{55} - c_{45}^2}}, \quad A_{12} = -\frac{c_{45}}{c_{44}} A_{11}, \quad A_{21} = 0, \quad A_{22} = \frac{1}{\sqrt{c_{44}}} \]
(35)

Using this coordinate transformation, Eqs. (31) and (32) become

\[ \frac{\partial^2 y_a}{\partial \eta_1^2} + \frac{\partial^2 y_a}{\partial \eta_2^2} = 0, \quad \text{as} \quad (\eta_1^2 + \eta_2^2)^{1/2} \to 0 \]
\[ \frac{\partial y_a}{\partial \eta_2} = 0, \quad \text{on} \quad \eta_1 \leq 0, \quad \eta_2 = 0 \]
(36)

where \( y_a \) are now components in the \( \eta \)-coordinate system. The asymptotic governing equation and boundary condition in the \( \eta \)-coordinate system are the same as those for the isotropic neo-Hookean sheet (Knowles and Sternberg, 1983).

Introducing polar coordinates in the scaled coordinate system,

\[ \rho^2 = \eta_1^2 + \eta_2^2, \]
\[ \psi = \arctan \eta_2 / \eta_1 \]
(37)

the governing equations (36) become

\[ \frac{\partial^2 y_a}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial y_a}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 y_a}{\partial \psi^2} = 0, \quad \text{as} \quad \rho \to 0 \]
\[ \frac{\partial y_a}{\partial \psi} = 0, \quad \text{on} \quad \psi = \pm \pi \]
(38)
It can be verified that $\psi = 0, \pi, -\pi$ when $\theta = 0, \pi, -\pi$, respectively, which means that the initial crack surface does not rotate in the scaled coordinate system.

Using Eqs. (35), and $x_1 = r \cos \theta$ and $x_2 = r \sin \theta$, where $r$ and $\theta$ are physical polar coordinates (Figure 1), Eq. (37) becomes

$$
\rho^2 = r^2 \left[ (A_{11} \cos \theta + A_{12} \sin \theta)^2 + (A_{22} \sin \theta)^2 \right],
$$

$$
\psi = \text{atan} \left( \frac{A_{22} \sin \theta}{A_{11} \cos \theta + A_{12} \sin \theta} \right),
$$

\begin{align*}
\rho^2 &= r^2 \left[ (A_{11} \cos \theta + A_{12} \sin \theta)^2 + (A_{22} \sin \theta)^2 \right], \\
\psi &= \text{atan} \left( \frac{A_{22} \sin \theta}{A_{11} \cos \theta + A_{12} \sin \theta} \right),
\end{align*}

(39)

Figure 2 The scaled polar coordinates $\psi$ and $\rho$ for different fiber modulus ratios $\kappa$ and orientation angles $\phi$.

A plot of $\rho$ and $\psi$ with respect to $\theta$ for different $\kappa$ and $\phi$ is shown in Figure 2. It is noted that $\rho$ follows periodic patterns and shows several extreme points. Actually, for the case of one set of fibers, using Eq. (33) and (35), $\rho^2$ can be written as $\rho^2 = [\kappa \cos^2(\phi + \theta) + 1](1 + \kappa)$, and thus $d\rho/d\theta = -\frac{\kappa}{2\rho(1+\kappa)} \sin(2\phi + 2\theta)$, therefore, extreme points for $\rho$, i.e., $\theta^e$, satisfy

$$
\phi + \theta^e = 0, \pm \frac{\pi}{2}, \pm \pi, ..., \quad (40)
$$

In the polar coordinate system for the scaled coordinates, the solution of the asymptotic boundary value problem Eq. (38) can be written in terms of the expression (Knowles and Sternberg, 1983)
\[ y_\alpha \sim \rho^{m/2} v_m(\psi), \text{ as } \rho \to 0 \]  

(41)

Substituting \( y_\alpha \) into (38) leads to an ordinary differential equation

\[ v''_m(\psi) + \frac{m^2}{4} v_m(\psi) = 0 \quad \text{and} \quad v'(\pm \pi) = 0 \]  

(42)

which has a general solution

\[
 v_m = \begin{cases} 
 p_m \sin\left(\frac{m}{2} \psi\right), & m = 1, 3, 5, \ldots \\
 q_m \cos\left(\frac{m}{2} \psi\right), & m = 2, 4, 6, \ldots 
\end{cases}
\]  

(43)

where \( p_m \) and \( q_m \) are arbitrary constants.

4.2 Asymptotic solutions near the crack tip

Taking the two lowest order solutions, namely, \( m = 1 \) and \( 2 \), gives rise to deformed coordinates

\[ y_\alpha \sim p_\alpha \rho^{1/2} \sin\left(\frac{\psi}{2}\right) + q_\alpha \rho \cos\psi, \quad \alpha = 1, 2 \]  

(44)

where \( p_\alpha, q_\alpha, \alpha = 1, 2 \) are constants determined by far-field boundary conditions. Using the expressions (39) in Eq. (44), \( y_1 \) and \( y_2 \) can be written in terms of the physical polar coordinates.

In front of the crack with \( \psi = \theta = 0 \), the deformed coordinates become

\[
 y_1|_{\theta=0} \sim q_1 \rho, \quad y_2|_{\theta=0} \sim q_2 \rho
\]  

(45)

while at the crack faces for \( \psi = \pm \pi \)

\[
 y_1|_{\theta=\pm \pi} \sim \pm p_1 \rho^{1/2} - q_1 \rho, \quad y_2|_{\theta=\pm \pi} \sim \pm p_2 \rho^{1/2} - q_2 \rho
\]  

(46)

In an opened crack with large deformation, the following relation should hold

\[
 y_2|_{\theta=\pi} \gg y_2|_{\theta=0} \Rightarrow p_2 \rho^{1/2} \gg q_2 \rho
\]  

(47)
which means that the influence of $q_2$ can be neglected compared with $p_2 \rho^{1/2}$ at large deformation, and thus for the purposes of simplicity, $q_2$ is assumed to be zero (this approximation was explained in (Liu and Moran, 2019) and will be demonstrated by the subsequent numerical examples). With this in mind, the crack shape can be described by

$$y_1 \sim p_1 \rho^{1/2} \sin \frac{\psi}{2} + q_1 \rho \cos \psi$$

$$y_2 \sim p_2 \rho^{1/2} \sin \frac{\psi}{2}$$  \hspace{1cm} (48)

For isotropic materials under uniaxial tension, the deformed crack is symmetric, thus $y_1$ is an even function, which leads to $p_1 = 0$. This is not the case for anisotropic hyperelastic materials, because, in general, the crack shape is no longer symmetric with respect to the line $\theta = 0$. The parameters $(p_1, q_1, p_2)$ should be determined based on the far-field boundary conditions or local crack tip fields, which will be addressed in Section 5.

Based on the transformation (35), the deformation and stress can be conveniently expressed by the matrix forms

$$\mathbf{F} = \tilde{\mathbf{F}} \mathbf{A}, \quad \mathbf{C} = \mathbf{A}^T \tilde{\mathbf{F}}^T \tilde{\mathbf{F}} \mathbf{A}, \quad \lambda_j^2 = \mathbf{a}_j \cdot (\mathbf{Ca}_j)$$  \hspace{1cm} (49)

$$\mathbf{P}/\mu \sim \mathbf{FAD}, \quad r \to 0$$  \hspace{1cm} (50)

$$\mathbf{\sigma}/\mu \sim \tilde{\mathbf{FADA}}^T \tilde{\mathbf{F}}^T, \quad r \to 0$$  \hspace{1cm} (51)

$$W/\mu \sim \frac{1}{2} \left[ \text{tr}(\tilde{\mathbf{FADA}}^T \tilde{\mathbf{F}}^T) - 3 \right] = \frac{1}{2}(\text{tr}\mathbf{\sigma} - 3), \quad \text{as} \quad r \to 0$$  \hspace{1cm} (52)

where $\tilde{\mathbf{F}}$ is the deformation gradient matrix with respect to the scaled coordinates, satisfying

$$\tilde{\mathbf{F}} = \begin{bmatrix} y_{1,n_1} & y_{1,n_2} \\ y_{2,n_1} & y_{2,n_2} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} c_{55} & c_{45} \\ c_{45} & c_{44} \end{bmatrix}$$  \hspace{1cm} (53)

Substituting $\mathbf{A}_j$ in Eq. (35) into Eqs. (50)-(52), it can be shown that $\mathbf{ADA}^T = \mathbf{I}$, and

$$\mathbf{P}/\mu \sim \tilde{\mathbf{F}} \mathbf{D}, \quad \mathbf{\sigma}/\mu \sim \tilde{\mathbf{F}} \tilde{\mathbf{F}}^T, \quad W/\mu \sim \frac{1}{2} \text{tr}\mathbf{\sigma}, \quad \text{as} \quad r \to 0$$  \hspace{1cm} (54)
where \( \mathbf{D} = [A_{11}^{-1} 0; c_{45}A_{22} A_{22}^{-1}] \). It can be easily verified that for an isotropic neo-Hookean material with \( \kappa = 0 \), Eqs. (49)–(52) reduce to those shown in Knowles and Sternberg (1983) since \( \mathbf{A} = \mathbf{I} \) and \( \mathbf{D} = \mathbf{I} \).

Using Eqs. (48), (49) and (53), the components of \( \mathbf{F} \) can be explicitly derived as

\[
F_{11} = y_{1,1} \sim A_{11} y_{1,1} = A_{11} \left( -\frac{1}{2} p_1 \rho^{-1/2} \sin \frac{\nu}{2} + q_1 \right),
\]
\[
F_{21} = y_{2,1} \sim A_{11} y_{2,1} = -\frac{1}{2} A_{11} p_2 \rho^{-1/2} \sin \frac{\nu}{2}
\]
\[
F_{12} = y_{1,2} \sim A_{11} y_{1,2} + A_{22} y_{1,2} = \frac{1}{2} p_1 \rho^{-1/2} \left( -A_{12} \sin \frac{\nu}{2} + A_{22} \cos \frac{\nu}{2} \right) + A_{12} q_1
\]
\[
F_{22} = y_{2,2} \sim A_{11} y_{2,2} + A_{22} y_{2,2} = \frac{1}{2} p_2 \rho^{-1/2} \left( -A_{12} \sin \frac{\nu}{2} + A_{22} \cos \frac{\nu}{2} \right)
\]

Using a similar procedure and Eqs. (50)–(52), the components of the first Piola-Kirchhoff stress, the Jacobian, the Cauchy stress and strain energy function are given by

\[
P_{11}/\mu \sim -\frac{1}{2} p_1 \rho^{-1/2} A_{11}^{-1} \sin \frac{\nu}{2} + A_{11}^{-1} q_1
\]
\[
P_{22}/\mu \sim \frac{1}{2} A_{22}^{-1} p_2 \rho^{-1/2} \cos \frac{\nu}{2}
\]
\[
P_{21}/\mu \sim \frac{1}{2} \rho^{-1} \frac{1}{2} A_{22}^{-1} p_2 \rho^{-1/2} \cos \frac{\nu}{2}
\]
\[
J = A_{11} A_{22} J_\eta \sim \frac{1}{2} A_{11} A_{22} \rho^{-1/2} p_2 q_1 \cos \frac{\nu}{2}
\]
\[
\sigma_{11}/\mu \sim -\frac{1}{4} \rho^{-1} p_1^2 - p_1 q_1 \rho^{-1/2} \sin \frac{\nu}{2} + q_1^2
\]
\[
\sigma_{22}/\mu \sim \frac{1}{4} p_2 \rho^{-1}
\]
\[
\sigma_{12}/\mu \sim -\frac{1}{4} p_1 p_2 \rho^{-1} - \frac{1}{2} \rho^{-1/2} p_2 q_1 \sin \frac{\nu}{2}
\]
\[
W/\mu \sim \frac{1}{8} \rho^{-1} \left( p_1^2 + p_2^2 \right) - \frac{1}{2} p_1 q_1 \rho^{-1/2} \sin \frac{\nu}{2} + \frac{1}{2} q_1^2 - \frac{3}{2}
\]

It can be seen that, as for the isotropic solution, the Cauchy stress has a \( \rho^{-1} \) (or \( r^{-1} \), Eq. (39)) singularity rather than a \( \rho^{-1/2} \) as in the case of linear elasticity. Using Eq. (39), the expressions above in the scaled coordinates can be transformed into the original coordinates \( (r, \theta) \).
In the following section, we show how the crack tip parameters can be evaluated using asymptotic path independent integrals introduced in Liu and Moran (2019) for the isotropic case which are extended here to the fiber-reinforced anisotropic case.

5 Crack tip integrals and evaluation of the crack tip parameters

5.1 Crack tip integrals

A globally path-independent J-integral. For a homogeneous hyperelastic material with traction free crack surface, free body forces and quasistatic loading, the J-integral is path-independent (Rice, 1968). For purposes of the present description, we will refer to this type of path-independence as global path-independence. Thus,

\[ J = \int_{\Gamma} \left( W \delta_{ij} - P_{q, y_i} \right) n_j \, d\Gamma \]  (60)

where the integrand \( W \delta_{ij} - P_{q, y_i} \) is divergence free, \( \Gamma \) is a contour with arbitrary shape and location. It can be shown that the J-integral in Eq. (60) is related to the asymptotic J-integral in the scaled coordinate system \( \eta_i, i = 1, 2 \) by (see proof in Appendix B),

\[ J = \sqrt{c_{44}} \tilde{J} \]  (61)

where \( \tilde{J} \) is defined as \( \tilde{J} = \lim_{\Gamma_{\eta} \to 0} \int_{\Gamma_{\eta}} \left( \tilde{W} \delta_{ij} - \tilde{P}_{q, y_i} \right) \tilde{n}_j \, d\Gamma_{\eta} \), with \( \tilde{W} = \frac{1}{2} (\text{tr} (\tilde{F}^T \tilde{F}) - 3) \), \( \tilde{P} = \mu \tilde{F} \), and \( \tilde{n} = \{ \cos \psi, \sin \psi \}^T \), representing the locally path-independent J-integral derived from the asymptotic governing equations (36) in the scaled coordinate system. It has been shown that (Liu and Moran, 2019; Ravichandran and Knauss, 1989), by choosing a circular contour near the crack tip with \( d\Gamma_{\eta} = \rho d\psi \), \( \tilde{J} \) can be calculated as \( \tilde{J} = \frac{\mu \pi}{4} \left( p_1^2 + p_2^2 \right) \), and thus the J-integral for the fiber-reinforced materials is given by

\[ J = \frac{\mu \pi \sqrt{c_{44}}}{4} \left( p_1^2 + p_2^2 \right) \]  (62)
where $c_{44} = 1 + \kappa \cos^2 \phi$ is a coefficient in Eq. (33). Note that $J$ is expressed in terms of the square of two crack parameters $p_1$ and $p_2$. Knowles and Sternberg (1983) demonstrated that by rotating the coordinate system by an angle $\varepsilon$, to be determined, the two parameters $p_1$ and $p_2$ can be replaced by the pair $p_2$ and $\varepsilon$. This leads to simplification of expressions in discussing an asymptotic solution, but may not be practical in numerical calculations and a straightforward calculation of $p_1$ and $p_2$ may be preferable.

A plot of the coefficient $\sqrt{c_{44}}$ with respect to $\kappa$ and $\phi$ in Figure 3 shows that $c_{44} = 1$ when $\kappa = 0$ or $\phi = \frac{\pi}{2}$, indicating that $J = \frac{\mu \pi}{4} \left( p_1^2 + p_2^2 \right)$ in these special cases. The $J$-integral is maximum when $\phi = 0$, when the fibers are perpendicular to the crack surfaces, i.e., the largest energy release rate.

![Figure 3](image)

**Figure 3** Evolution of $\sqrt{c_{44}}$ with respect to the orientation angle $\phi$ and the modulus ratio $\kappa$

*Asymptotically path-independent $J$-integrals.* Liu and Moran (2019) noted that for a neo-Hookean material with no fibers, the asymptotic boundary value problem (38) separates into two independent boundary value problems for each degree of freedom $y_1$ and $y_2$ and that asymptotic path-independent integrals and additional crack tip interaction energy integrals can be derived for each boundary value problem, separately. Thus, for the current problem, two $J$-integrals can be written down for each asymptotic boundary value problem separately as $r \to 0$. 
\[ J_\gamma^\alpha = \lim_{{r \to 0}} \int_{\Gamma} \left( W_\gamma^\alpha \delta_{ij} - P_j^\gamma y_{\alpha,i} \right) n_j \, d\Gamma, \quad \alpha = 1, 2, \text{ no summation on } \alpha \]  \hspace{1cm} (63)

where the strain energy functions and the stresses are given as

\[ W_\gamma^\alpha = \frac{1}{2} \mu y_{\alpha,i} D_{ij} y_{\alpha,j} = \frac{1}{2} \mu (y_{\alpha,1}^2 + y_{\alpha,2}^2) \quad \text{and} \quad P_j^\gamma = P_{\alpha j} \]  \hspace{1cm} (64)

Note that \( D_{ij} \) and \( P_{\alpha j} \) are given in Eq. (53) and Eq. (50), respectively, and Eqs. (52) and (54) are used to derive \( W_\gamma^\alpha \). The integrals (63) are asymptotically path-independent, i.e., the integrands are divergence free, vanishes on the crack faces and they are independent of contour shape and location as long as the contour is located in the region of dominance of the singular fields. The terminology asymptotic is used because the relations in Eq. (64) which hold asymptotically are used to develop the integral (63). The integral \( J_\gamma^\alpha \) can also be expressed as \( J_\gamma^\alpha = \sqrt{c_{44}} \tilde{J}_\gamma^\alpha \), where \( \tilde{J}_\gamma^\alpha \) is the asymptotically path-independent \( J \)-integral in the scaled coordinate system (Eq. (B.9) in Appendix B) and its integration over a circular contour was shown to be \( \tilde{J}_\gamma^\alpha = \frac{\mu \pi}{4} p_{\alpha}^2 \), \( \alpha = 1, 2 \) (Liu and Moran, 2019). Thus, \( J_\gamma^\alpha \) is given by

\[ J_\gamma^\alpha = \sqrt{c_{44}} \tilde{J}_\gamma^\alpha = \frac{\mu \pi \sqrt{c_{44}}}{4} p_{\alpha}^2, \quad \alpha = 1, 2 \]  \hspace{1cm} (65)

The integrals \( J_\gamma^\alpha \) can be used to extract the crack tip parameters \( p_1 \) and \( p_2 \). The asymptotic solution (48) does not decompose into mode I (opening) and mode II (shear) as it does in linear elasticity. Nonetheless, the integrals, \( J_\gamma^1 \) and \( J_\gamma^2 \), permit us to identify the two crack tip parameters, \( p_1 \) and \( p_2 \), separately. We note also that \( J = J_\gamma^1 + J_\gamma^2 \).

*Asymptotically path-independent interaction energy integrals.* In addition to the \( J \)-integrals in Eq. (63), we introduced asymptotically path-independent interaction energy integrals for the isotropic case to evaluate all the crack tip parameters, separately (Liu and Moran, 2019). These integrals can be extended to the crack problem for an anisotropic material as follows

\[ I_\gamma^\alpha = \lim_{{r \to 0}} \int_{\Gamma} \left( P_j^\gamma y_{\alpha,i}^\text{max} n_i - P_j^\gamma y_{\alpha,i}^\text{min} n_i - P_j^\text{max} y_{\alpha,i} n_i \right) \, d\Gamma \quad \text{for} \quad \alpha = 1, 2 \]  \hspace{1cm} (66)
where \( P_j^{\gamma} = P_{a_j} \), \( y_{aux} \) is an auxiliary field that satisfies the asymptotic governing equations (36), and the auxiliary stress \( P_j^{aux} = y_{aux} D_{ij} \) with \( D \) given in Eq. (53). It can be shown that the integrand of (66) is divergence free and vanishes on the crack faces as long as the contour is located in the region of dominance of the singular fields. Specifically, using Eq. (20) and \( P_{j,j}^{aux} = 0 \), and \( P_{j,j}^{\gamma} = 0 \), we obtain

\[
\int_{\Gamma^{out} + \Gamma^{in} + \Gamma^{in}} \left( P_j^{\gamma} y_{j}^{aux} n_i - P_j^{\gamma} y_{j}^{aux} n_i - P_j^{aux} y_{\alpha,j} n_j \right) d\Gamma \\
= \int_{A} \left( P_j^{\gamma} y_{j}^{aux} \delta_{ij} - P_j^{\gamma} y_{j}^{aux} - P_j^{aux} y_{\alpha,j} \right) dA \\
= \int_{A} \mu(\lambda\varepsilon_{r\theta}^{r\theta} y_{r\beta}) y_{\beta}^{aux} r dr d\theta 
\]

(67)

where the contours \( \Gamma^{out}, \Gamma^{+}, \Gamma^{in} \) and \( \Gamma^{-} \) enclose a simply connected region \( A \) (Figure 4). Recalling the relations \( \lambda = \frac{1}{2} - \rho^{1/2} \sim r^{1/2} \) and \( y_{r\theta} \sim r^{-1/2} \), we obtain that \( \mu(\lambda\varepsilon_{r\theta}^{r\theta} y_{r\beta}) \sim \rho^{1/2} \sim r^{-1/2} \) for \( y_{aux} = \rho^{1/2} \sin \frac{\theta}{2} \) and \( \mu(\lambda\varepsilon_{r\theta}^{r\theta} y_{r\beta}) \sim r^{-1} \) for \( y_{aux} = \ln \rho \). Using these results, it can be shown that the area integral in Eq. (67) behaves as

\[
\lim_{r \to 0} \int \mu(\lambda\varepsilon_{r\theta}^{r\theta} y_{r\beta}) r dr d\theta \sim \begin{cases} 
 r^{3/2}, & \text{for } y_{aux} = \rho^{1/2} \sin \frac{\theta}{2} \\
 r, & \text{for } y_{aux} = \ln \rho 
\end{cases} \\
\sim 0, \text{ as } r \to 0 
\]

(68)

for the two auxiliary fields of interest. Thus, the integral (66) is asymptotically path independent.

![Figure 4 Integration domain near the crack tip. The boundary of the region A is composed of the outer and inner contours, \( \Gamma^{out} \) and \( \Gamma^{in} \), and the two crack surfaces \( \Gamma^{+} \) and \( \Gamma^{-} \).]
Similar to $J^{y_a}$, for the purpose of calculation, $I^{y_a}$ can be expressed as $I^{y_a} = \sqrt{c_{44}} \bar{T}_{y_a}$ (see proof in Appendix B), where the integral $\bar{T}_{y_a}$ (see Eq. (B11)) is asymptotically path-independent in the scaled coordinate system. The integrals $\bar{T}_{y_a}$ on a circular contour are shown to be $\bar{T}_{i_a}^y = \frac{i}{2} \pi \mu p_a$, $\alpha = 1, 2$ when $y^{\text{aux}} = \rho^{1/2} \sin \frac{\psi}{2}$ for the first order crack tip parameters $p_a$, and $\bar{T}_{i_a}^y = -2 \pi \mu q_a$, $\alpha = 1, 2$ when $y^{\text{aux}} = \ln \rho$ for the second order parameters $q_a$ (Liu and Moran, 2019). Thus,

$$I_{1}^{y_a} = \sqrt{c_{44}} \bar{T}_{1}^{y_a} = \frac{\sqrt{c_{44}} \pi \mu p_a}{2}, \alpha = 1, 2, \text{ for } y^{\text{aux}} = \rho^{1/2} \sin \frac{\psi}{2}$$  \hspace{1cm} (69)

$$I_{2}^{y_a} = \sqrt{c_{44}} \bar{T}_{2}^{y_a} = -2 \sqrt{c_{44}} \pi \mu q_a, \alpha = 1, 2, \text{ for } y^{\text{aux}} = \ln \rho$$  \hspace{1cm} (70)

where $\rho$ and $\psi$ are given in Eq. (39).

### 5.2 Numerical evaluation of the crack tip parameters

**Domain integration methods.** In a finite element simulation, the $J$-integral is often calculated based on an domain integral formulation, which can be given as (Li et al., 1985; Moran and Shih, 1987)

$$J = \int_{\Gamma^{\text{out}} \cup \Gamma^{\text{in}} \cup \Gamma^{+} \cup \Gamma^{-}} \left( P_{y} y_{i,j} - W \delta_{i,j} \right) g n_{j} d\Gamma$$

$$= \int_{A} \left( P_{y} y_{i,j} - W \delta_{i,j} \right) \frac{\partial g}{\partial x_{j}} dA + \int_{A} \left( P_{y} y_{i,j} - W \delta_{i,j} \right) g dA$$  \hspace{1cm} (71)

where the contours $\Gamma^{\text{in}}$, $\Gamma^{\text{out}}$, $\Gamma^{+}$ and $\Gamma^{-}$, are shown in Figure 4, $g$ is a sufficiently smooth function in $A$ that vanishes on the outer contour $\Gamma^{\text{out}}$ and is unity on the inner contour $\Gamma^{\text{in}}$, and other quantities, $P_{y}$, $y_{i,j}$ and $W$, are available in the finite element analysis. It can be shown that the second term in Eq. (71) is divergence free, i.e., $(P_{y} y_{i,j} - W \delta_{i,j})_{,j} = 0$, thus,

$$J = \int_{A} \left( P_{y} y_{i,j} - W \delta_{i,j} \right) \frac{\partial g}{\partial x_{j}} dA = \frac{\mu \pi \sqrt{c_{44}}}{4} \left( p_{1}^{2} + p_{2}^{2} \right)$$  \hspace{1cm} (72)
Considering the asymptotic divergence-free property of the integrand $P_{j}^{y_{a,l}} - W_{j}^{y_{a,i}}\delta_{ij}$ in $J^{y_{a}}$ (Eq. (63)) in the region dominated by the asymptotic fields, $J^{y_{a}}$ can also be calculated in a similar way, i.e.,

$$J^{y_{a}} = \int_{\Gamma_{an,1} + \Gamma_{an,2}} (P_{j}^{y_{a,l}} - W_{j}^{y_{a,i}}\delta_{ij}) g n_{j} d\Gamma$$

$$= \int_{A} (P_{j}^{y_{a,l}} - W_{j}^{y_{a,i}}\delta_{ij}) \frac{\partial g}{\partial x_{j}} dA + \int_{A} (P_{j}^{y_{a,l}} - W_{j}^{y_{a,i}}\delta_{ij}) g dA$$

$$= \int_{A} (P_{j}^{y_{a,l}} - W_{j}^{y_{a,i}}\delta_{ij}) \frac{\partial g}{\partial x_{j}} dA$$

$$= \frac{\mu\pi\sqrt{c_{44}}}{4} p_{\alpha}^{2}, \alpha = 1, 2, \text{ no summation on } \alpha$$

from which the parameters $p_{\alpha}, \alpha = 1, 2$ can be obtained.

Remark 3: In this expression, we have used the asymptotic divergence free property of the integrand. This means that, not just the inner contour, but the outer area needs to reside in the region dominated by the asymptotic crack tip fields. Our numerical results in the next sections show this is readily achieved. If the area extends beyond the region of asymptotic dominance, the integral $\int_{A}(\bullet), g dA$ needs to be included.

Using the domain integration method and considering the asymptotic divergence-free property of the integrand in Eq. (66), the asymptotic interaction energy integrals $I^{y_{a}}$ can be written as

$$I^{y_{a}} = \int_{A} \left( -P_{j}^{y_{a,l}} y_{aux}^{j} \frac{\partial g}{\partial x_{i}} + (P_{j}^{y_{aux}} y_{aux}^{j} n_{j} + P_{j}^{aux} y_{aux}^{j} n_{j}) \frac{\partial g}{\partial x_{j}} \right) dA$$

$$= \begin{cases} -2\sqrt{c_{44}}\pi q_{\alpha}, \alpha = 1, 2, & \text{if } y^{aux} = \ln \rho \\ \sqrt{c_{44}}\pi \mu p_{\alpha}, \alpha = 1, 2, & \text{if } y^{aux} = \rho^{1/2} \sin \frac{\theta}{2} \end{cases}$$

(74)

where the same caveats on the integration area $A$ as those in Remark 3 hold.

A coordinate-based method. If the finite element mesh near the crack tip is well organized, a coordinate-based approach can be used to evaluate the crack tip parameters, e.g., a $P_{i}$ contour
integral as given in Chang and Li (2004). This approach is simply extended to the present case using the relation in Eqs. (39), (45) and (46), which gives,

\[
p_1 = \left[ y_1^n(r_0, \pi) - y_1^n(r_0, -\pi) \right]/(4A_1r_1^{1/2}), \quad q_1 = y_1^n(r_0, 0)/(A_1r_0),
\]

\[
p_2 = \left[ y_2^n(r_0, \pi) - y_2^n(r_0, -\pi) \right]/(4A_1r_0^{1/2}),
\]

where \( y_i^n(r_0, \theta), i = 1, 2 \) are the deformed coordinates at the point \((r_0, \theta)\) in the finite element analysis and \( A_{11} \) is defined in (35).

6 Numerical results and analysis

6.1 Finite element model

To verify the theoretical results and compute the crack tip parameters, we use the finite element method to model the crack tip fields of a long cracked strip under combined stretch and shear. The finite element simulation is implemented in an in-house finite element code where a total Lagrange formulation is used to solve the nonlinear governing equations (21) and (22) under the boundary conditions shown in Figure 5. The resulting nonlinear finite element equations are solved by the Newton-Raphson iteration scheme with fixed step size. The norm of the residual force is used as the error indicator to determine the convergence of the program in a load step, and its tolerance is taken as \( 5 \times 10^{-7} \). Since the plane-stress case is considered here, no issue regarding volumetric locking due to incompressibility are encountered and specific techniques, such as the reduced integration or the B-bar method, are not required. The domain is discretized by eight-node quadrilateral elements with four Gauss integration points. The mesh discretization inside two contours near the crack tip at \( r_1 = 1 \times 10^{-4} H_0 \) and \( r_2 = 1 \times 10^{-3} H_0 \) are shown and the smallest size of the element near to the crack tip is about \( 10^{-5} H_0 \). The tangent modulus of the material is given by

\[
\mathbb{C}_{ijkl} = \frac{\partial P_{ik}}{\partial F_{kl}} = \mu \delta_{ik} \delta_{jl} + 3\mu \kappa I_4^{5/2} a_i a_j a_k a_l + \mu \kappa (1 - I_4^{3/2}) \delta_{ik} a_j a_l + \mu \lambda^2 (2F^{-1}_{jk} F^{-1}_{ik} + F^{-1}_{jk} F^{-1}_{ki})
\]

Other details of finite element implementation are given in (Belytschko et al., 2013).
6.2 The crack tip parameters for a clamped long strip

As shown in Figure 5, the size of the strip is \( L_0 \times H_0 \) and is imposed with uniform \( x_2 \) displacements with opposite directions on the top and bottom surfaces, while the \( x_1 \)-directional displacements are constrained to be zero on these surfaces. The domain can be divided into four regions \( Q_1 - Q_4 \) (separated by the dashed lines) with different deformation features (Rivlin and Thomas, 1953). Far from the crack in region \( Q_3 \), the deformation is uniform due to long enough length of the strip, while no deformation is present in region \( Q_4 \). Thus, for the \( J \)-integral (60) evaluation along the contour shown in red and green in Figure 5, only the integral on the green path survives and gives (Long and Hui, 2015; Rice, 1968; Rivlin and Thomas, 1953)

\[
J_{\text{far}} = W(F_s)H_0
\]  

(77)

where \( W \) is defined in Eq. (10), the deformation gradient is \( F_s = \text{diag}(1, \lambda_s, \lambda_s^{-1}) \), and \( \lambda_s \) is the applied stretch in the \( x_2 \) direction. It is noted that \( F_{11}^s = 1 \) and \( F_{22}^s = \lambda_s \) are constant deformation in region \( Q_3 \), which is remote from the crack tip and left and right edges.

![Figure 5 Sketch of a long cracked strip and the finite element mesh near the crack tip.](image)

To verify the three methods for calculating \( p_1 \) and \( p_2 \), i.e., the domain integration methods Eqs. (73) and (74), and the coordinate-based method Eq. (75), we first simulate three cases for fiber orientation angles \( \phi = 0, \frac{\pi}{4}, \frac{3\pi}{8} \) and \( \kappa = 1 \). The results of the coordinate-based method are calculated at \( r_0 = 1 \times 10^{-4} H_0 \). For the computation of \( J^{\gamma_s} \) in Eq. (73), the integration domain \( A \) is a circle with
\[ r_0 = 1 \times 10^{-4} H_0 \] and the weight function is taken as \[ g = (r_0 - r)/r_0 \], where \( r \) is the distance to the crack tip. The weight function for \( g \) for \( I^{\gamma_x} \) is set to zero on the outer contour and to unity in the interior domain. The variations of \( p_1 \) and \( p_2 \) with respect to \( \lambda_1 \) (Figure 6b and c) show good agreement between the results obtained by the domain integration methods and coordinate-based method. The parameter \( p_1 \) vanishes when \( \phi = 0 \), indicating the deformed crack is symmetric with respect to \( x_2 = 0 \). The parameter \( p_2 \) is one order larger than \( p_1 \), and increases with \( \lambda_1 \). For a given stretch, \( p_2 \) is largest when \( \phi = 0 \) i.e., the fiber orientation is perpendicular to the initial crack line. The \( J \)-integral in Figure 6a also shows good agreement between the results by different calculation methods for several combinations of \( \phi \) and \( \kappa \). It can be seen that the \( J \)-integrals, like \( W(\mathbf{F}) \), scale positively with \( \kappa \).

Detailed results for \( p_1 \), \( p_2 \) and \( q_1 \) for various cases are shown in Table 1. Note that, considering the relation in Eq. (47), i.e., \( p_2 \rho^{1/2} \gg q_2 \rho \), \( q_2 \) is negligible for the problem considered in this paper and we do not list the value of \( q_2 \) in Table 1. It can be seen that parameters obtained from different methods show good agreement with one another. Without additional comments, the parameters obtained by the interaction energy integral, Eq. (74), will be used to produce the analytical results in Sections 6.3–6.6 to study the influence of material anisotropy on the crack fields.

Figure 6 (a) Comparison of the \( J \)-integrals by the analytical and numerical methods for different \( \phi \) and \( \kappa \). (b)-(c) Comparison of \( p_1 \) and \( p_2 \) obtained by different methods for different orientation angles \( \phi \) and stretches \( \lambda_1 \).
Table 1 The coefficients \((p_1, q_1, p_2)\) for several representative cases, where \(\Xi^I: (p_1, q_1, p_2)\) are obtained by the interaction energy integrals in Eq. (74), \(\Xi^J: (p_1, p_2)\) by the \(J\)-integrals in Eq. (73), and \(\Xi^C: (p_1, q_1, p_2)\) by the coordinate-based method in Eq. (75).

| Case 1: different stretch ratios, \(\kappa = 1\), \(\phi = \pi/4\) |
|-----------------|-------|-------|
| \(\lambda_s = 1.4\) | 1.8   | 2.0   |
| \(\Xi^I: (0.026, 1.476, 0.276)\) | (0.047, 1.474, 0.511) | (0.055, 1.461, 0.621) |
| \(\Xi^J: (0.026, 0.277)\) | (0.047, 0.512) | (0.055, 0.622) |
| \(\Xi^C: (0.025, 1.517, 0.276)\) | (0.046, 1.497, 0.510) | (0.055, 1.483, 0.620) |

| Case 2: different modulus ratios, \(\lambda_s = 2\), \(\phi = \pi/4\) |
|-----------------|-------|-------|
| \(\kappa = 0\) | 5     | 20    |
| \(\Xi^I: (0.000, 1.091, 0.598)\) | (0.170, 1.859, 0.702) | (0.322, 1.987, 0.868) |
| \(\Xi^J: (0.000, 0.597)\) | (0.170, 0.700) | (0.321, 0.867) |
| \(\Xi^C: (0.000, 1.086, 0.595)\) | (0.168, 1.951, 0.696) | (0.317, 1.985, 0.856) |

| Case 3: different orientation angles, \(\lambda_s = 2\), \(\kappa = 5\) |
|-----------------|-------|-------|
| \(\phi = 0\) | \(\pi/8\) | \(3\pi/8\) |
| \(\Xi^I: (0.000, 1.143, 0.899)\) | (0.115, 1.325, 0.846) | (0.078, 2.048, 0.561) |
| \(\Xi^J: (0.000, 0.897)\) | (0.115, 0.844) | (0.078, 0.559) |
| \(\Xi^C: (0.000, 1.133, 0.895)\) | (0.114, 1.358, 0.841) | (0.078, 2.144, 0.557) |

| Case 4: mixed modes, \(\lambda_s = 2\), \(\phi = \pi/4\), \(\kappa = 1\) |
|-----------------|-------|-------|
| \(\lambda_{sx} = -2.0\) | -1.4  | 1.4   | 2.0   |
| \(\Xi^I: (-0.349, 1.605, 0.591)\) | (-0.112, 1.522, 0.608) | (0.228, 1.414, 0.640) | (0.492, 1.368, 0.661) |
| \(\Xi^J: (-0.348, 0.590)\) | (-0.111, 0.607) | (0.227, 0.638) | (0.491, 0.659) |
| \(\Xi^C: (-0.347, 1.435, 0.587)\) | (-0.111, 1.463, 0.604) | (0.226, 1.518, 0.636) | (0.489, 1.596, 0.657) |

| Case 5: Two sets of fibers, \(\lambda_s = 2\), \(\kappa_1 = \kappa_2 = 5\), \(\phi_b = \phi_a + \pi/4\) |
|-----------------|-------|-------|
| \(\phi_a = 0\) | \(\pi/8\) | \(\pi/4\) |
| \(\Xi^I: (0.095, 2.479, 0.938)\) | (0.145, 1.541, 0.830) | (0.165, 1.147, 0.695) |
| \(\Xi^J: (0.095, 0.936)\) | (0.145, 0.829) | (0.165, 0.692) |
| \(\Xi^C: (0.095, 2.519, 0.933)\) | (0.145, 1.748, 0.825) | (0.164, 1.207, 0.689) |
6.3 Crack tip fields for different stretch ratios

Using the values of $\Xi^i:(p_i, q_i, p_2)$ shown in Table 1, we can compute the analytical expressions for $y_1$ and $y_2$ and other quantities in Eqs. (55)~(59) as functions of $r$ and $\theta$. We first study the case with the material parameters $\kappa=1$ and $\phi=\pi/4$. The deformed coordinates $y_1$ and $y_2$ for different loading levels $\lambda_\kappa=1.4, 1.8, 2.0$ are shown in Figure 7. The analytical and finite element results agree with each other for different parameters at both $r_1=1\times10^{-2}H_0$ and $r_2=1\times10^{-3}H_0$, indicating the accuracy of the asymptotic solution (48) for characterizing the deformed shape of the crack. The amplitude of both $y_1$ and $y_2$ increases with the stretch ratio, indicating that the crack opens more as $\lambda_\kappa$ increases. Unlike the crack shape of isotropic materials under remote uniaxial tension, neither $y_1$ nor $y_2$ is symmetric or anti-symmetric in general with respect to $\theta=0$ because of the presence of fibers. The change of $y_1$ and $y_2$ along the radius $r$ at $\theta=0$ and $\pm\frac{\pi}{2}$ in Figure 8 also shows good agreement between the theoretical and finite element results. It can be shown from Eq. (48), that $\log_{10}|y_1| \sim \frac{1}{2} \log_{10} r$ at $\theta=\pm\frac{\pi}{2}$ and $\log_{10}|y_1| \sim \log_{10} r$ at $\theta=0$, and the slopes of the lines in Figure 8 are verified to be $\frac{1}{2}$ for the results at $\theta=\pm\frac{\pi}{2}$ and 1 at $\theta=0$. As $\lambda_\kappa$ increases, $y_1$ and $y_2$ at $\theta=\pm\frac{\pi}{2}$ increase, while they remain almost unchanged for $\theta=0$. This can also be seen in Figure 7, where both $y_1$ and $y_2$ exhibit fixed points for different stretch ratios at $\theta=0$. These observations indicate that when the stretch becomes larger, the $y_1$ locations of the material points in the front of the crack tip do not change significantly.

The variations of $y_1$ and $y_2$ are consistent with the change of coefficients, that is, $p_1$ and $p_2$ increase with $\lambda_\kappa$, while $q_i$ is nearly constant 1.5 (Figure 7). Though $p_i$ is much less than $q_i$ (e.g., $p_1=0.025$, $q_1=1.517$ for $\lambda_\kappa=1.4$), $p_ir^{\frac{1}{2}}$ and $q_ir$ have the same order in magnitude for the considered range of $r$, which means that both first order and second order terms are essential to describe the variation of $y_1$. From Figure 8a and b, the asymptotic fields agree with the finite element results up to the range $r/H_0<10^{-2}$ for $y_1$ at the crack line $\theta=0$ and $y_2$. The valid ranges of the asymptotic solution for $y_1$, at the polar angles $\theta=\frac{\pi}{2}$, or $-\frac{\pi}{2}$, become $r/H_0<10^{-3}$ for $\lambda_\kappa=1.4$, and
This indicates that the larger the stretch, the larger is the zone of dominance of the $y_1$ crack tip fields. Overall, at a moderate stretch ($\lambda_s \sim 1.4$), the asymptotic solutions for both $y_1$ and $y_2$ hold up to $r/H_0 \sim 10^{-3}$.

Using Eq. (46) and the coefficients $(p_1, q_1, p_2)$ in Figure 7, the crack shapes are shown in Figure 9. The crack shapes in the remainder of the paper are also produced in this way. Good agreement between the finite element and theoretical results can be seen for different stretch levels $\lambda_s$. As expected, the crack opens to large degree when $\lambda_s$ increases and the crack shapes are not symmetric with respect to the initial crack line. The crack shape tends to a straight line with a slope angle $\zeta$ (in the $x_1-x_2$ coordinate system) when $r$ is small enough. In such a case, $\zeta$ should satisfy

$$\tan \zeta \sim \frac{y_2(r, \pi) - y_2(r, -\pi)}{y_1(r, \pi) - y_1(r, -\pi)} \sim \frac{p_2}{p_1}, \quad r \to 0$$

(78)

For isotropic materials, it can be shown that $p_1 = 0$ and $\zeta = \frac{\pi}{2}$, which mean that the crack line is perpendicular to the initial horizontal crack line, while for the anisotropic material, $p_1$ does not vanish, and an inclination angle less than $\frac{\pi}{2}$ can be observed. For the case $\kappa = 1$ and $\phi = \pi/4$, $p_1$ is much smaller than $p_2$ (Table 1), and thus $\zeta$ is near to $\frac{\pi}{2}$, leading to a crack nearly perpendicular to the horizontal crack line (inset of Figure 9). Interestingly, it can be verified that, using $p_1$ and $p_2$ in Table 1, $\tan \zeta \approx 11$ for different $\lambda_s$, which means that the slope of the crack surface near its tip remains relatively unchanged during the stretching process.
Figure 7 Variation of $y_1$ and $y_2$ with respect to $\theta$ at $r_1 = 1 \times 10^{-4} H_0$ (a and b) and $1 \times 10^{-3} H_0$ (c and d) for three different stretches, i.e., $\lambda = 1.4, 1.8, 2.0$. Other material parameters are $\kappa = 1$ and $\phi = \pi/4$. The parameters $\Xi^i : (p_1, q_1, p_2)$ are given in Table 1.

Figure 8 Variation of (a) $|y_1|$ and (b) $|y_2|$ with respect to the radius $r$ at fixed polar angles $\theta = \frac{\pi}{2}, 0, -\frac{\pi}{2}$. The solid and dash lines are the analytical results (Eq. (48)) for the stretches $\lambda = 1.4$ and 2.0, respectively, where $\Xi^i : (p_1, q_1, p_2)$ are given in Table 1. The circles and rectangles are the finite element results. Other material parameters are $\kappa = 1$ and $\phi = \pi/4$. 

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Figure 9 The crack shapes at different stretches $\lambda_s = 1.4, 1.8, 2.0$ within the range $r \in [0, 4 \times 10^{-3} H_0]$ . Other material parameters are $\kappa = 1$ and $\phi = \pi/4$ . Note that $y_1$ and $y_2$ axes are scaled differently in the main plot, but are the same in the inset plot to show the physical crack shapes.
Stress distributions as a function of $\theta$ are shown in Figure 10, where good agreement between the finite element and theoretical results can be found. The finite element stress results are extracted at the Gauss points on a circle located at $r = 3.29 \times 10^{-5} L_0$, and the theoretical results are obtained from Eq. (58) with the parameters $\Xi^1: (p_1, q_1, p_2)$ given in Table 1. The stress results in the remainder of the paper are also produced in this way. Comparing the three stress components, $\sigma_{22}$ is found to be the largest. It can be understood that, from Eq. (58), $\sigma_{22}/\sigma_{11} \sim p_2^2/p_1^2$ and $\sigma_{22}/\sigma_{12} \sim p_2/p_1$ if only the leading order is considered, and $p_2$ is one order higher than $p_1$ according to Table 1. Therefore, the magnitude of $\sigma_{22}$ is approximately two orders higher than that of $\sigma_{11}$ and one order higher than that of $\sigma_{12}$. It is noted that since $\sigma_{22}$ is proportional to $\rho^{-1}$, it has the same extreme points as a function of the angle $\theta$ as those of $\rho^{-1}$ (Figure 2) for different stretch levels. For the case $\phi = \frac{\pi}{4}$, using Eq. (40), the locations of extreme points for $\sigma_{22}$ satisfy

$$\theta^* = \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}, \ldots,$$  (79)

This condition is well verified by the results in Figure 10b, indicating that $\sigma_{22}$ achieves a maximum at $\theta = \frac{\pi}{4}$ and $-\frac{3\pi}{4}$ which exactly coincide with the initial fiber directions. Both $\sigma_{11}$ and $\sigma_{12}$ decrease with $\theta$ due to the influence of the higher order terms shown in Eq. (58), meaning that $\sigma_{11}$ and $\sigma_{12}$ are larger below the crack surface ($\theta < 0$) than those above it ($\theta > 0$). The $\theta$ values of extreme points for $\sigma_{11}$ and $\sigma_{12}$ deviate from those in Eq. (79) due to the influence of the higher order terms in Eq. (58).

In Figure 11, we can see that the Cauchy stress component $\sigma_{22}$ transitions from its $r^{-1}$ behavior in the region dominated by the asymptotic field to a uniform far-field value, which has also been observed in (Geubelle and Knauss, 1994b; Krishnan et al., 2008; Ravichandran and Knauss, 1989) for crack problems in isotropic hyperelastic sheets. The far-field value is given by
\( \sigma_{22}^i/\mu = \lambda_\kappa^2 - (1 - \lambda_i^3) \cos^2 \phi \lambda_\kappa - 1/\lambda_\kappa^2 \), where \( \lambda_i^2 = I_4 = \cos^2 \phi \lambda_\kappa^2 + \sin^2 \phi \) and \( \lambda_\kappa \) is the far-field stretch. In each region, the finite element results agree well with the corresponding analytical results. When the stretch ratio increases from \( \lambda_\kappa = 1.4 \) to 2.0, \( \sigma_{22} \) increases as expected, and the size of the region dominated by the asymptotic solution also increases.

![Graph showing Cauchy stress component \( \sigma_{22} \) along the crack line \( \theta = 0 \) for different stretch ratios \( \lambda_\kappa = 1.4 \) and 2.0. The material parameters are \( \phi = \frac{\pi}{4} \) and \( \kappa = 1 \).](image)

**Remark 4: fiber size and the region of dominance of the asymptotic fields.** In present homogenized constitutive relation, there is no explicit fiber dimension. As shown above, the asymptotic fields hold up to a range of \( r/H_0 < 10^{-3} \) for both \( y_1 \) and \( y_2 \). For a moderate load level \( \lambda_\kappa = 1.4 \), and a specimen with a characteristic size \( H_0 \) of \( \sim 10^{-1} \) m, the region dominated by the asymptotic fields is in a range \( r < 10^{-4} \) m. This is the case for some typical biological soft tissues, for which the smallest characteristic diameter of fiber is in the order of nanometers. For example, the fundamental fiber-like constituents, actin and myosin, in muscle have a size about 10-30 nm (AL-Khayat, 2013). The diameter of collagen fibrils in arteries and tendons is about 10 to 300 nm (Fascia, 2012). For these soft tissues, the effect of the fiber can be homogenized over the scale \( r \sim 10^{-4} \) m where the asymptotic fields still dominate. In a recent work by (Ben-Or Frank et al., 2019), the fiber size used
for a micromechanically-motivated analysis for fibrous tissue is \( \sim 1 \) micron, which can still be homogenized over the range \( r \sim 10^{-4} \) m. Thus, the asymptotic fields have the potential to characterize the deformation in the crack-tip region over physically meaningful scales. A full assessment that accounts for micro-mechanical fracture and damage processes remains to be carried out.

### 6.4 Crack tip fields for different modulus ratios and fiber orientation angles

The effects of fiber-matrix modulus ratio \( \kappa \). Figure 12 shows the distributions of \( y_1 \) and \( y_2 \) in the circumferential and radial directions for \( \phi = \frac{\pi}{4} \) and \( \lambda_s = 2 \). Good agreement between the finite element results and the analytical results can be observed. Both \( y_1 \) and \( y_2 \) at \( 1 \times 10^{-4} H_0 \) increase significantly with \( \kappa \) (Figure 12a and b), indicating that the crack opens more for a larger \( \kappa \) even if the stretch is the same. This is also consistent with the radial change of \( y_1 \) and \( y_2 \) in Figure 12c and d, and the change of the crack shapes for different \( \kappa \) in Figure 13. For the isotropic case with \( \kappa = 0 \), the deformed coordinate \( y_1 \) is an even function of \( \theta \) and \( y_2 \) an odd function (Figure 12a and b), leading to a symmetric crack with respect to \( \theta = 0 \) (Figure 13). When \( \kappa \) is large enough (e.g., \( \kappa = 20 \)), \( y_1 \) is the same order of magnitude as that of \( y_2 \) and the crack shape tends to be a straight line. The slope angle \( \zeta \) can be obtained by Eq. (78), and its can be verified that \( \zeta \) decreases from 90° for \( \kappa = 0 \) to about 70° for \( \kappa = 20 \). In Figure 12c and d, the slope of \( \log_{10} |y_1| \) with respect to \( \log_{10} r \) is still 0.5 at \( \theta = \pm \frac{\pi}{2} \) and 1 at \( \theta = 0 \), because the relation \( \log_{10} |y_1| \sim \frac{1}{2} \log_{10} r \) still holds at \( \theta = \pm \frac{\pi}{2} \) and \( \log_{10} |y_1| \sim \log_{10} r \) at \( \theta = 0 \). The region of validity for the asymptotic fields for \( y_1 \) is up to \( r / H_0 \sim 10^{-3} \) for different modulus ratios for a stretch \( \lambda_s = 2 \) (Figure 12c), while for \( y_2 \), it remains over a range \( r/H_0 < 10^{-2} \) for different modulus ratios (Figure 12d). This indicates that the fiber modulus ratio does not significantly affect the valid region of the asymptotic fields. In addition, \( y_1 \) at \( \theta = 0 \) does not change too much for \( \kappa = 1 \) and 20, which means that material points in the front of the crack tip show little change for different value of \( \kappa \), similar to the case in Figure 8a.
Figure 12 (a and b) Circumferential variation of $y_1$ and $y_2$ with respect to $\theta$ at $r_1=1\times10^{-4}L_0$ for different modulus ratios $\kappa=0, 1, 5, 20$ and (c and d) radial variation of $y_1$ and $y_2$ at $\theta=-\pi/2, 0$ and $\pi/2$. The applied stretch is $\lambda_s=2.0$ and the fiber orientation angle is $\phi=\pi/4$. The parameters $\Xi^1: (p_1, q_1, p_2)$ for each case are given in Table 1.

As shown in Figure 14 for the stress distribution, good agreement between the analytical and finite element results can be seen for all the stress components. The stresses increase with $\kappa$ for a given stretch ratio $\lambda_s$. For the isotropic material with $\kappa=0$, all the stress components approximately maintains an constant for different $\theta$, in which, $\sigma_{11}$ is almost vanishing and $\sigma_{22}$ is the major component. This is because the material near the crack tip is nearly in a state of uniaxial stretch in the $x_2$ direction. The extreme points of $\sigma_{22}$ are the same as those in Figure 10c, since $\theta^c$ (Eq. (40)) is only dependent on the orientation angle $\phi$. It is noted that, unlike the case of $\kappa=1$ in Figure 10, $\sigma_{11}$ and $\sigma_{12}$ follows the same periodic pattern as $\sigma_{22}$ for $\kappa=20$, since $p_1$ has the same order in magnitude as $p_2$ and only the first term $p_1^2\rho^{-1}$ and $p_1p_2\rho^{-1}$ dominate in $\sigma_{11}$ and $\sigma_{12}$ (Eq. (58)).
respectively. The above analysis indicates that the presence of fiber shows significant influence on the crack tip shape and on the stress distribution in the vicinity of the crack tip.

Figure 13 The crack shapes for different modulus coefficients $\kappa = 0, 1, 5, 20$. Other parameters are $\lambda_0 = 2.0$ and $\phi = \pi/4$. The parameters $\Xi^l: (p_1, q_1, p_2)$ are given in Table 1.
Figure 14 Stress distribution for different modulus ratios $\kappa = 0, 5, 20$. Other parameters are $\lambda_s = 2.0$ and $\phi = \pi/4$. The parameters $\Xi^l : (p_1, q_1, p_2)$ are given in Table 1.

The effects of the fiber orientation angle $\phi$. The deformed coordinates $y_1$ and $y_2$ for different fiber orientation angle $\phi = 0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}$ are given in Figure 15, where other parameters are $\kappa = 5$ and $\lambda_s = 2.0$. Good agreement between the finite element and analytical results can be observed. For the case with the fibers perpendicular to the crack line, i.e., $\phi = 0$, $y_1$ is much smaller than $y_2$ due to $p_1 = 0$ and the crack shape is symmetric with respect to $\theta = 0$ (Figure 16). Among the four cases, the case for $\phi = \frac{\pi}{4}$ shows the best agreement between the analytical and finite element results for the crack shape even far from the crack tip (Figure 16).
From Figure 17, the analytical stress distribution also agrees with the finite element results very well. The orientation angle has significant influence on both stress magnitude and distribution. In the case \( \phi = 0 \), \( \sigma_{11} \) is nearly zero, and much smaller compared with the results of \( \phi = \frac{\pi}{8}, \frac{3\pi}{8} \) while \( \sigma_{22} \) has a maximum peak value for the case \( \phi = 0 \). It can be seen that the locations of extreme points of \( \sigma_{22} \) for different \( \phi \) agree with the relation in Eq. (40), i.e., \( \theta^e = -\phi, \phi \pm \frac{\pi}{4}, \phi \pm \pi, \ldots \). For example, when \( \phi = 0 \), \( \sigma_{22} \) has maximum value when \( \theta = \pm \frac{\pi}{2} \) (i.e., the fiber direction) and minimum when \( \theta = 0 \) (i.e., the direction perpendicular to fiber direction).

![Figure 15](image)

Figure 15 Variation of \( y_1 \) and \( y_2 \) with respect to \( \theta \) at \( r_1 = 1 \times 10^{-4} H_0 \) for different fiber orientation angles, i.e., \( \phi = 0, \frac{\pi}{8}, \frac{3\pi}{8} \). The applied stretch is \( \lambda_s = 2.0 \) and the fiber modulus ratio is \( \kappa = 5 \). The parameters \( \Xi^i : (p_1, q_1, p_2) \) for each case are given in Table 1.
Figure 16 The crack shapes for different orientation angles $\phi = 0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}$. Other parameters are $\lambda_c = 2.0$ and $\kappa = 5$. The parameters $\Xi^I(p_1, q_1, p_2)$ are given in Table 1.

**Remark 5:** The case of horizontal fibers $\phi = \frac{\pi}{2}$. In this case, we observed the beginning of a cusp-like behavior in $y_1$ at $\theta = 0$ in the finite element result for large $\kappa$ (approximately, $\kappa \geq 5$) and the finite element simulation diverges subsequently due to severe mesh distortion before the applied stretch reaches the maximum value $\lambda_c = 2$. This cusp-like behavior of $y_1$ is induced by the compressive stress in the front the crack tip at $\theta = 0$ (Figure C1b in Appendix C), leading to shortening of fibers. This compressive stress should come from the constrained lateral contraction by the stiff horizontal fibers along the crack line. The shortening of fibers at $\theta = 0$ gives rise to a fact that the asymptotic approximation $\lambda_f^{-1} \to 0$ as $r \to 0$ does not hold and the asymptotic analysis needs to be modified in this special case. A detailed analysis of this case is shown in Appendix C, which quantitatively explains the emergence of the cusp in $y_1$ at the crack tip front. Though a decrease of $\lambda_f$ at $\theta = 0$ leads to a modification of the solution for $y_1$, it has no influence on the major deformation component $y_2$, as shown in Figure C1d, see also a theoretical explanation in Appendix C.
Figure 17 Stress distribution for different fiber orientation angles $\phi = 0, \frac{\pi}{8}, \frac{3\pi}{8}$. Other parameters are $\kappa = 5$ and $\lambda_s = 2$.

Remark 6: damage initiation at the crack tip. The asymptotic crack tip fields may be useful in developing criteria for damage initiation at the crack tip. For example, a critical value of the fiber stretch at a critical distance to the crack tip may have the potential to characterize the initiation of damage. To achieve this, we need to calculate the fiber stretch, using Eq. (8), as

$$I_4 = \lambda_s^2 \sim \frac{p_1^2 + p_2^2}{4\rho} \left( A_{11} \frac{\sin \phi \sin \psi}{2} - A_{22} \frac{\cos \phi \cos \frac{\psi}{2}}{2} + A_{12} \frac{\cos \phi \sin \frac{\psi}{2}}{2} \right)^2 ,$$

(80)

where $\rho$ and $\psi$ are the polar coordinates in the scaled coordinate system given in Eq. (37). The variation of $I_4$ with respect to the physical polar angle $\theta$ is shown in Figure 18, where good
agreement can be found between the finite element results and the analytical results. For each fiber orientation angle, a maximum stretch $\lambda_{f,\text{max}}$ can be identified at a specific polar angle $\theta_m$. Interestingly, the largest fiber stretch is located at an inclined angle $\theta_m$ rather than along a horizontal line $\theta=0$.

Figure 18 Fiber stretch with respect to $\theta$ for different fiber orientation angles.

### 6.5 Crack tip fields for mixed crack modes

In this section, we consider the crack tip fields for mixed crack modes, where an $x_1$ directional shear, $k$, is applying on the top and bottom surfaces of the strip (Figure 5) in addition to an $x_2$ directional stretch $\lambda_s=2$. In this case, the $x_1$-directional displacement on the top surfaces of the strip (Figure 5) is specified by $y_1 = x_1 + kH_0/2$. It can be seen from Figure 19 that a change in $k$ shows a significant influence on $y_1$, but little on $y_2$ in the crack tip region. As $k$ increases from -1 to 1, $y_1$ inverts from a shape with negative slope to one with positive slope (Figure 19a and c). This can be verified by the values of $p_1$ in Table 1, which are negative for $k=-1.0$ and -0.4 and positive for $k=0.4$ and 1.0. In addition, $p_1$ and $p_2$ are of the same order of magnitude when $|k|$ is 0.4 or larger, which means that the first order asymptotic solution $\rho^{1/2} \sin \frac{\nu}{2}$ is important in $y_1$ even if the materials
is isotropic. It is noted that $y_1$ for $k = -1$ and 1 is not symmetric with respect to the initial crack line because of the existence of fibers. The $J$-integrals obtained by different methods are in good agreement with one another for different shear levels (Figure 19d), further verifying the accuracy of the asymptotic solutions. Here, it should be noted that $F_s$ in (77) is given by

$$F_s = \begin{bmatrix} 1 & k & 0; 0 & \lambda_s & 0; 0 & 0 \end{bmatrix},$$

where $k$ varies and $\lambda_s$ is fixed at $\lambda_s = 2$.

The stress distributions for different value of $k$ are shown in Figure 20 with good agreement between the analytical and finite element results. It can be seen that $\sigma_{11}$ and $\sigma_{12}$ change significantly when varying $k$ over the range 1, 0.4, and -1, in which, $\sigma_{11}$ is always positive while $\sigma_{12}$ changes its sign for $k = 1$ and -1. The sign change in $\sigma_{12}$ is induced by the reverse of direction of the applied shear $k$ and mathematically characterized by the change of sign of $p_1$ (Table 1). The extreme points for all stress components do not change since both $\kappa$ and $\phi$ are the same for all cases.

![Figure 19 Distribution of (a) $y_1$ and (b) $y_2$ for different shear ratios $k = -1.0, -0.4, 0.4, 1.0$. Other parameters are $\kappa = 1.0$, $\phi = \pi/4$ and $\lambda_s = 2$. (c) and (d) are the crack shapes and comparison of the](image-url)
results of $J$-integrals, respectively. The parameters $\Xi^i: (p_1, q_1, p_2)$ for each case are given in Table 1.

![Image of graphs](image)

Figure 20 Stress distributions with respect to $\theta$ for different mixed crack modes with $k = -1.0, 0.4, 1.0$. Other parameters are $\kappa = 1.0$, $\phi = \pi/4$ and $\lambda_s = 2$.

### 6.6 Crack tip fields for two sets of fibers

In this section, the crack tip fields for the case of two sets of fibers are considered, where the asymptotic solutions still hold after modifying the anisotropy parameters $c_{44}$, $c_{45}$, and $c_{55}$ according to Eq. (34). For the finite element simulation, slight modifications of the expression for the stress and tangent modulus are made based on Eqs. (20) and (76), which are straightforward and not given here. In such cases, the locations of extreme points for $\rho$ (Eq. (39)) satisfy

$$\kappa_a \sin[2(\phi_a + \theta^c)] + \kappa_b \sin[2(\phi_b + \theta^c)] = 0,$$

(81)
Considering a special case for $\kappa_a = \kappa_b = 5$, it can be shown that the nontrivial solution of $\theta^e$ is $\theta^e = -(\phi_a + \phi_b) / 2 + \frac{\pi}{2} \chi$, where $\chi$ is an integer. Let $\phi_b = \phi_a + \frac{\pi}{4}$, then $\theta^e$ becomes $\theta^e = -\phi_a - \frac{\pi}{8} + \frac{\pi}{2} \chi$.

Taking $\phi_a = 0, \frac{\pi}{8}$ and $\frac{\pi}{4}$, and $\phi_b = \phi_a + \frac{\pi}{4}$, and fixing the $x_2$ directional stretch as $\lambda_s = 2$ similar to the cases in Section 5.3, the deformed coordinates and stresses can be obtained as shown in Figure 21 and Figure 22. The deformed coordinates show similar variation with $\phi_a$ with those in Figure 15 and so do the crack shapes. Also, variation of $\phi_a$ has significant influence on both the magnitude and distribution of each stress component. The stress $\sigma_{22}$ is still the dominant component compared with the other two, $\sigma_{11}$ and $\sigma_{12}$, and it reaches a maximum when $\phi_a = 0$. It can be verified that the extreme points of $\sigma_{22}$ are described by $\theta^e = -\phi_a - \frac{\pi}{8} + \frac{\pi}{2} \chi$ for different $\phi_a$ as shown in Figure 21b. Actually, in such a case, $\sigma_{22}$ reaches its largest and smallest values in the directions along and perpendicular to the angular bisector of the two sets of fibers, respectively. It can be shown from Eq. (62) that the $J$-integral has maximum value when the initial crack line is perpendicular to the angular bisector of the two sets of fibers. When the orientation of the two sets of fibers is changed from $\phi_a = 0$ to $\frac{\pi}{4}$, $\sigma_{11}$ increases and the extreme points of $\sigma_{11}$ and $\sigma_{12}$ are quite near to those of $\sigma_{22}$.

![Figure 21](image1.png)

Figure 21 The deformed coordinates $y_1$ and $y_2$ for the cases with two sets of fibers with $\phi_a = 0, \frac{\pi}{8}$ and $\frac{\pi}{4}$. Other parameters are $\kappa_a = \kappa_b = 5$, $\phi_b = \phi_a + \frac{\pi}{4}$ and $\lambda_s = 2$. The coefficients $\Xi^1 : (p_1, q_1, p_2)$ are given in Table 1.
Figure 22 Stress distributions for two sets fibers for different orientation angles $\phi_a = 0, \frac{\pi}{8}$ and $\frac{\pi}{4}$, and $\phi_b = \phi_a + \frac{\pi}{4}$. Other parameters are $\kappa_a = \kappa_b = 5$ and $\lambda_v = 2$.

6.7 Application to a square sheet with an inclined crack

In this section, we explore the asymptotic fields for a more general geometry consisting of a square sheet, side $L_0$, with a crack inclined angle $\phi_c$ to the horizontal (Figure 23a). The sheet is reinforced by fibers with orientation angle $\phi_f$ subject to overall stretch $\lambda_v$ in the vertical direction. The mesh scheme near the crack tip is similar to that of the long strip example in Figure 5. For the case $\phi_c = \frac{\pi}{6}$, $\phi_f = \frac{\pi}{4}$ and $\lambda_v = 1.5$, the initial and the deformed shapes of the sheet are shown in Figure 23b. Good agreement can be observed between the finite element results and the asymptotic solutions for both $y_1$ and $y_2$ in the local $x_1-x_2$ coordinate system (Figure 23c and d).
Figure 23 (a) The initial geometry of a stretched square sheet with an initial inclined crack. (b) The initial and deformed shapes of the model. (c) and (d) are the distribution of \( y_1 \) and \( y_2 \) with respect to \( \theta \) in the local \( x_1-x_2 \) coordinate system. The coefficients are \((p_1, q_1, p_2) = (0.136, 1.359, 0.313)\).

7 Conclusion

In this paper, we investigate the crack tip fields for a neo-Hookean sheet reinforced by neo-Hookean fibers using an asymptotic analysis of the governing equations. An anisotropic constitutive model proposed in (Guo et al., 2007a, 2007b, 2006) is employed to characterize the large deformation behavior of the material. Using a linear transformation of the original coordinates which involves the anisotropic parameters, the asymptotic governing equations are transformed into Laplace equations, exhibiting the same form as those obtained by Knowles and Sternberg (1983) for the isotropic case. The analytical expressions for the crack tip fields, such as deformed coordinates, stresses and \( J \)-integrals, are obtained. A set of asymptotically path-independent \( J \)-integrals and interaction energy integrals is introduced to extract the crack tip parameters, extending those developed by Liu and Moran (2019) for an isotropic neo-Hookean material to the present anisotropic case. A finite element simulation of the crack tip fields for a long cracked strip is conducted, and good agreement between
the analytical and finite element results, e.g., deformed coordinates, stresses and crack shape, are demonstrated for different stretch levels, materials parameters and loading modes. The crack tip fields are found to be significantly affected by the anisotropic constitutive parameters, e.g., the fiber orientation angle and the fiber-matrix modulus ratio. For example, introducing fibers in an isotropic matrix makes the crack shape unsymmetrical and the angular variation of the stress components follow a periodic pattern with peaks related to the fiber orientations.

The asymptotic solutions, crack tip integrals, and numerical results obtained may lay foundation for further exploration of fracture and tearing in fiber-reinforced soft materials, such as biological tissue. For that purpose, the case of nonlinear fibers needs to be considered and is the subject of our ongoing work. Indeed, the case of horizontal fibers in the present paper invokes some elements of nonlinearity that do not vanish asymptotically and, as seen in the Appendix C, special treatment was required. The asymptotic fields have the potential to be used as enrichment fields in Extended Finite Element Methods (XFEM) and thus enable more efficient modelling of crack propagation in anisotropic hyperplastic sheets. The zone of dominance of the asymptotic fields derived here may be sufficiently large to characterize fracture initiation and damage processes in some biological tissues. Micromechanical studies along these line would be worth pursuing.

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Appendix A Coordinate transformation of the Laplace equation

We introduce a linear transformation,

\[ \eta_i = A_{ij} x_j \]  

(A.1)

where \( d\eta_i \) and \( dx_j \) are line elements in the scaled and original coordinate systems, and \( A_{ij} \) are undetermined constant coefficients. Using Eq. (A1) in the governing equations (31) and (32), we obtain
\[ g_{11} \frac{\partial^2 y_\alpha}{\partial \eta_1^2} + g_{12} \frac{\partial^2 y_\alpha}{\partial \eta_1 \partial \eta_2} + g_{22} \frac{\partial^2 y_\alpha}{\partial \eta_2^2} = 0 \]  
(A.2)

\[ g_1 \frac{\partial y_\alpha}{\partial \eta_1} + g_2 \frac{\partial y_\alpha}{\partial \eta_2} = 0 \]  
(A.3)

where

\[ g_{11} = c_{55} A_{11}^2 + 2c_{45} A_{11} A_{12} + c_{44} A_{12}^2 \]  
(A.4)

\[ g_{12} = 2c_{55} A_{11} A_{21} + 2c_{45} (A_{11} A_{22} + A_{12} A_{21}) + 2c_{44} A_{12} A_{22} \]  
(A.5)

\[ g_{22} = c_{55} A_{21}^2 + 2c_{45} A_{21} A_{22} + c_{44} A_{22}^2 \]  
(A.6)

\[ g_1 = c_{45} A_{11} + c_{44} A_{12} \]  
(A.7)

\[ g_2 = c_{45} A_{21} + c_{44} A_{22} \]  
(A.8)

To obtain a Laplace equation from Eq. (A.2) and leave the original crack line not rotated, we let

\[ g_{11} = g_{22} = 1, \quad g_{12} = 0 \quad \text{and} \quad g_1 = 0 \]  
(A.9)

Then the governing equation in the scaled coordinate system and the coefficients can be shown as follows

\[ \frac{\partial^2 y_\alpha}{\partial \eta_1^2} + \frac{\partial^2 y_\alpha}{\partial \eta_2^2} = 0 \]  
(A.10)

\[ \frac{\partial y_\alpha}{\partial \eta_2} = 0, \quad \text{on} \ \eta_1 \leq 0, \ \eta_2 = 0 \]  
(A.11)

\[ A_{11} = \frac{c_{44}}{\sqrt{c_{44} c_{55} - c_{45}^2}}, \quad A_{12} = -\frac{c_{45}}{c_{44}} A_{11}, \quad A_{21} = 0, \quad A_{22} = \frac{1}{c_{44}} \]  
(A.12)
Appendix B Proof of Eq. (61)

To prove Eq. (61), we first give the definitions of the strain energy function, and the first Piola-Kirchhoff stress in the scaled coordinate system \( \eta_i, i=1,2 \) as

\[
\tilde{W} = \frac{1}{2} (\text{tr}(\tilde{F}^T \tilde{F}) - 3), \quad \tilde{\Pi} = \mu \tilde{F}
\]  

(B.1)

From Eqs. (50) and (54), it can be seen that those quantities are related to \( W \) and \( P \) by

\[
\tilde{W} = W, \quad P = \tilde{\Pi} A D
\]  

(B.2)

For the purpose of subsequent use, the polar coordinates and their derivatives in the original and scaled coordinate systems are related by

\[
\rho \tilde{n} = rA_n, \quad \tilde{n}d\rho + \rho d\psi \tilde{n}_d = A(n dr + rd\theta n_d)
\]  

(B.3)

where \( \tilde{n} = \{\cos \psi, \sin \psi\}^T \), \( \tilde{n}_d = \{-\sin \psi, \cos \psi\}^T \), \( n = \{\cos \theta, \sin \theta\}^T \), \( n_d = \{-\sin \theta, \cos \theta\}^T \), and the matrix \( A = [A_y]_{2 \times 2} \) is given in Eq. (35). On a circular contour \( \Gamma \) in the coordinate system \((r, \theta)\), \( dr = 0 \), thus,

\[
rd\theta n_d = A^{-1}(\tilde{n}d\rho + \rho d\psi \tilde{n}_d) \implies rd\theta n = TA^{-1}\tilde{n}d\rho + \frac{\rho}{A_{11}A_{22}} A^T \tilde{n}d\psi
\]  

(B.4)

where \( T = [01; -10] \). Thus, the J-integral on \( \Gamma \) satisfies

\[
J = \int_{\Gamma} (W \delta_{ij} - P_{y, i, 1}) n_j rd\theta
\]

\[
= \int_{\Gamma_y} (W \delta_{ij} - P_{y, i, 1}) T \sigma A_{ij} \tilde{n}_i d\psi + \int_{\Gamma_y} (W \delta_{ij} - P_{y, i, 1}) \frac{\rho}{A_{11}A_{22}} A_{ij} \tilde{n}_j d\psi
\]  

(B.5)

The first part of the above integration must vanish because, on a circular contour \( \Gamma_y \), the scaled radius \( \rho \) is equal at the crack surfaces, i.e., \( \rho(\theta = -\pi) = \rho(\theta = \pi) = rA_{11} \) (Eq. (39)). Considering Eq. (B.2) and \( ADA^T = I \), \( y_{i, 1} = A_{11} y_{i, 1} \), the following relations can be shown

\[
\frac{1}{A_{11}A_{22}} W \delta_{ij} A_{ij} \tilde{n}_k = \frac{1}{A_{11}A_{22}} W A_{ij} \tilde{n}_i = \frac{1}{A_{22}} \tilde{\Pi} \delta_{ij} \tilde{n}_j
\]  

(B.6)

\[
\frac{1}{A_{11}A_{22}} P_{y, i, 1} A_{ij} \tilde{n}_k = \frac{1}{A_{11}A_{22}} A_{11} y_{i, 1} \tilde{P}_k A_{ij} A_{yk} \tilde{n}_s = \frac{1}{A_{22}} y_{i, 1} \tilde{P}_k \tilde{n}_k
\]  

(B.7)
Substituting Eqs. (B.6) and (B.7) into Eq. (B.5) and using $A_{22} = 1/\sqrt{c_{44}}$, it can be shown that

$$J = \sqrt{c_{44}} \int_{\Gamma_y} (\tilde{W} \delta_{ij} - \tilde{P}_{ij} y_{\alpha,\eta_j}) \bar{n}_j \rho \, d\psi = \sqrt{c_{44}} \tilde{J}$$

(B.8)

Following a similar procedure, it can be shown that the asymptotic path-independent integrals in Eqs. (63) and (66) satisfy

$$J^{\gamma_x} = \sqrt{c_{44}} \tilde{J}^{\gamma_x}, \quad I^{\gamma_x} = \sqrt{c_{44}} \tilde{I}^{\gamma_x}$$

(B.9)

where $\tilde{J}^{\gamma_x}$ and $\tilde{I}^{\gamma_x}$ are asymptotic path-independent in the scaled coordinate system $\eta, i = 1, 2$,

$$\tilde{J}^{\gamma_x} = \lim_{\alpha_{\eta} \to 0} \int_{\Gamma_{\eta}} (\tilde{W}^{\gamma_x} \delta_{ij} - \tilde{P}_{ij}^{\gamma_x} y_{\alpha,\eta_j}) \bar{n}_j \, d\Gamma_{\eta}, \quad \alpha = 1, 2, \text{ no summation on } \alpha$$

(B.10)

$$\tilde{I}^{\gamma_x} = \lim_{\alpha_{\eta} \to 0} \int_{\Gamma_{\eta}} (\tilde{P}_{ij}^{\gamma_x} y_{\alpha,\eta_j} \bar{n}_i - \tilde{P}_{ij}^{\gamma_x} y_{\alpha,\eta_j} \tilde{n}_j - \tilde{P}_{ij}^{\gamma_x} y_{\alpha,\eta_j} \bar{n}_j) \, d\Gamma_{\eta}, \quad \alpha = 1, 2, \text{ no summation on } \alpha$$

(B.11)

where $\tilde{W}^{\gamma_x} = W^{\gamma_x} = \frac{1}{2} \mu(y_{\alpha,\eta_j}^2 + y_{\alpha,\eta_i}^2)$, $\tilde{P}_{ij}^{\gamma_x} = \mu y_{\alpha,\eta_j} = \mu \tilde{F}_{\alpha,j}$, $\tilde{P}_{ij}^{\gamma_x} = \mu y_{\alpha,\eta_i}$.

Appendix C Analysis of the case $\phi = \frac{\pi}{2}$

In this section, we consider the crack tip fields for the special case of horizontal fibers ($\phi = \frac{\pi}{2}$). As noted in Section 2, in this case, and for $\kappa \geq 5$, the deformed coordinate $y_1$ exhibits a beginning of cusp-like behavior at $\theta = 0$ (see Fig. C1c). Also, the numerical simulation diverges for this case at a moderate macroscopic stretch of $\lambda = 1.12$. Here, we carry out additional numerical and theoretical analysis to explain these results.

From Figure C1b, the $x_1$ directional stress, $\sigma_{11}$, is seen to differ significantly for the cases with or without fibers. A high $x_1$ directional compressive stress is found for the case with stiff fibers for $\kappa = 5$, while the stress is tensile for the case $\kappa = 0$. This is explained by observing that the lateral contraction of the material along the crack line is resisted by the horizontal fibers, especially for $\kappa \geq 5$. For the other two cases with small $\kappa$, neither a compressive stress nor the cusp is observed ahead of the crack tip and the asymptotic solution works well for predicting the deformed shape. Thus, we need to modify the previous asymptotic analysis for $y_1$ for this special case of stiff horizontal fibers.
An interesting observation is that the deformation component \( y_2 \) shows no cusp like behavior and there is good agreement between the finite element results and the asymptotic results in Eq. (48) for both small and large \( \kappa \) (Figure C1d). To understand these results, we first focus the attention on \( y_2 \). Due to the symmetry of deformation, it can be noticed that \( y_2 = 0 \), \( y_{2,1} = 0 \) and \( y_{2,11} = 0 \) along the crack line \( \theta = 0 \). Because of this, the governing equation for \( y_2 \) becomes \( y_{2,22} = 0 \) at \( \theta = 0 \). Therefore, in the horizontal fiber case, the asymptotic governing equation for \( y_2 \), i.e., \( (1 + \kappa)y_{2,11} + y_{2,22} = 0 \) (Eq. (26)), holds for different \( \theta \). The corresponding deformation (Eq. (48)) show good agreement with the finite element results for this special case.

We now consider the asymptotic governing equation for \( y_1 \). In Section 4.1, the asymptotic governing equation is obtained by using the asymptotic relation \( \lambda_f \to \infty \) and \( \lambda \to 0 \) as \( r \to 0 \). It can be inferred from Eq. (55) that this relation generally holds except for the case of horizontal fibers, with \( \phi = \frac{\pi}{2} \). In this case, the fiber stretch, \( \lambda_f^2 = y_{1,1}^2 + y_{2,1}^2 \), does not approach infinity as \( r \to 0 \) at \( \theta = 0 \) (see explanation of Eq. ((C.2))). The condition that \( \lambda \to 0 \) as \( r \to 0 \) still holds because of the \( r^{-1/2} \) singularity of \( y_{2,2} \) and \( y_{2,1} \) (Eq. (55)). Thus, using \( a_n = \{1, 0\}^T \), \( \lambda_f^2 = y_{1,1}^2 + y_{2,1}^2 \) and \( \lambda_f \lambda_{f,1} = y_{1,1}y_{1,11} + y_{2,1}y_{2,11} \) in Eq. (21), the asymptotic governing equation for \( y_1 \) becomes

\[
\left[ 1 + \kappa \left( 1 - \lambda_f^{-3} + 3\lambda_f^{-3} y_{1,11}^2 \right) \right] y_{1,11} + y_{1,22} + 3\lambda_f^{-3} y_{1,1}y_{2,1}y_{2,11} = 0
\]  

(C.1)

It is a nontrivial task to solve the above nonlinear differential equation and we seek an approximate analysis to qualitatively understand the case of horizontal fibers. For convenience, we divide the domain surrounding the crack tip into region I and II, with \( |\theta| \leq \theta_0 \) and \( |\theta| > \theta_0 \) (Figure C1a), where \( \theta_0 \) is a yet undetermined small angle. This division simplifies the transition of mechanical fields between the region far away from the line \( \theta = 0 \) and that at \( \theta = 0 \). From Eq. (55), \( y_{2,1} \) satisfies the asymptotic relation \( y_{2,1} \to 0 \) for \( |\theta| \leq \theta_0 \) and \( y_{2,1} \sim r^{-1/2} \) for \( |\theta| > \theta_0 \). It can be seen that \( \lambda_f \) satisfies

\[
\lambda_f = \sqrt{y_{1,1}^2 + y_{2,1}^2} \sim \begin{cases} r^{-1/2}, & \text{when } |\theta| > \theta_0, r \to 0 \\ y_{1,1}, & \text{when } |\theta| \leq \theta_0 \end{cases}
\]  

(C.2)
The last term on the left hand side of Eq. (C.1) satisfies 
\[3\lambda_f^5 y_{1,1} y_{2,1} y_{2,11} \sim 0 \quad \text{in} \quad |\theta|<\theta_0\] 
for \(\theta=0\) and 
\[3\lambda_f^5 y_{1,1} y_{2,1} y_{2,11} r^{3/2-1/2-3/2} \sim r^{1/2} \quad \text{in} \quad |\theta|>\theta_0.\] 
Thus, this term can be neglected in Eq. (C.1). Combining Eqs. (C.1) and (C.2), the asymptotic governing equation follows the reduced form

\[(1+\kappa) y_{1,11} + y_{1,22} = 0, \quad \text{in region II with} \quad |\theta|>\theta_0\]  
(C.3)

\[
\left[1+\kappa(1+2y_{1,1}^{-3})\right] y_{1,11} + y_{1,22} = 0, \quad \text{in region I with} \quad |\theta|<\theta_0
\] (C.4)

These two equations can be expressed in a unified form as

\[
c_{ss} y_{1,11} + y_{1,22} = 0, \quad \text{with} \quad c_{ss} = \begin{cases} 1+\kappa(1+2y_{1,1}^{-3}), & |\theta|\leq\theta_0 \\ 1+\kappa, & |\theta|>\theta_0 \end{cases} 
\] (C.5)

The coefficient \(c_{ss}\) is a function of \(\theta\), which has a larger value in the region \(|\theta|\leq\theta_0\) than that in \(|\theta|>\theta_0\). This motivates us to introduce an assumed unified distribution of \(c_{ss}\) as follows

\[
c_{ss} = 1+\kappa(1+2y_{1,1}^{-3}e^{-\theta^2/\Lambda}) 
\] (C.6)

where \(y_{1,1}\) is now taken to be constant. It can be shown that, for a suitable parameter \(\Lambda=0.1\), \(c_{ss} \approx 1+\kappa\) when \(\theta=\pm\pi\) and \(c_{ss} = 1+\kappa(1+2y_{1,1}^{-3})\) when \(\theta=0\). This treatment simplifies the nonlinear differential equation (C.1) into a solvable linear one (C.5) while maintaining its essential feature. The solution is assumed to satisfy an asymptotic form \(y_{1} \sim r^m v(\theta)\) where \(m\) is a constant and \(v(\theta)\) is a continuous function of \(\theta\). Then, Eq. (C.5) along with boundary conditions become

\[
\frac{\partial^2 v}{\partial \theta^2} + s^2 v + (c_{ss} - 1) \left((1-s) \sin 2\theta \frac{\partial v}{\partial \theta} + \sin^2 \theta \frac{\partial^2 v}{\partial \theta^2} - \left(s \cos 2\theta + s^2 \cos^2 \theta \right)v \right) = 0 \quad \text{C.7}
\]

\[
\frac{\partial v}{\partial \theta} = 0, \quad \theta = \pm\pi
\] (C.8)
Equation (C.7) reduces to that of the crack tip problem for an isotropic neo-Hookean material when \( \kappa = 0 \) and \( c_{55} = 1 \). For the general case, we solve this equation using the finite element method with one dimensional 2-node linear elements, which leads to a polynomial eigenvalue problem

\[
(k_0 + mk_1 + m^2 k_2) \mathbf{v} = \mathbf{0}
\]

(C.9)

where \( k_0, k_1 \) and \( k_2 \) are the stiffness matrices, and \( \mathbf{v} \) is the discretized eigenvector. In this problem, we focus on the even eigenmodes in the range \( \theta \in [-\pi, \pi] \), since \( y_1 \) is an even function with respect to \( \theta \). The first and the third even eigenmodes, \( v_1(\theta) \) and \( v_3(\theta) \), and corresponding eigenvalues for different \( \bar{y}_{1,1} \) are shown in Figure C1e and f. The first eigenmode arises from the assumed distribution of \( c_{55} \) in Eq. (C.6), because no such eigenmode is found from (C.9) if we set \( c_{55} = 1 \) (the lowest even eigenmode is \( r \cos \theta \)). The second even eigenmode follows \( v_2(\theta) = \cos \theta \) with eigenvalue \( m = 1 \).

From Figures C1e and f, \( \bar{y}_{1,1} \) has significant influence only on the value of \( v_1(\theta) \) at \( \theta = 0 \), and hence should play a role in adjusting the amplitude of the cusp. Here, we prescribe \( \bar{y}_{1,1} \) as 0.8 which is near to the simulated compressive stretch at \( r/H_0 = 1 \times 10^{-4} \). The corresponding first and third eigenvalues are \( m_1 = 0.175 \) and \( m_3 = 1.823 \). Thus, \( y_1 \) can be expressed as

\[
y_1 = q \cdot r^{0.175} v_1(\theta) + s \cdot r \cos(\theta) + t \cdot r^{1.823} v_3(\theta)
\]

(C.10)

where \( q, s \) and \( t \) are undetermined coefficients. After doing a mathematical fitting based on the finite element results of \( y_1 \) at \( r_0/L = 2.5 \times 10^{-5} \), we determine the coefficients \( q = 1.501 \times 10^{-5} \), \( s = 0.560 \) and \( t = 2.820 \times 10^2 \). Figure C1c indicates good agreement between the finite element results and Eq. (C.10) for the case \( \kappa = 5 \). This provides justification for the approximate analysis for \( y_1 \).
Figure C1 The crack tip results for the cases of the horizontal fibers with orientation angle $\phi = \pi / 2$ and the uniaxial stretch $\lambda_s = 1.112$. (a) Qualitative division of the domains with different fiber stretch ratios $\lambda_f$. (b) Distribution of the $\sigma_{11}$ directional stress, at the crack tip on the reference configuration for modulus ratios $\kappa = 0, 1$ and 5. The radius of the circle contour is $r_0/H_0 = 1 \times 10^{-4}$. The location of the crack tip is marked by a black dot. (c-d) Comparison of the finite element results and theoretical analysis for $y_1$ and $y_2$ at $r_0/H_0 = 1 \times 10^{-4}$. The theoretical result of $y_1$ for $\kappa = 5$ are obtained from Eq. (C10) and all other theoretical results are obtained from Eq. (48). The non-vanishing coefficients for $y_1$ are $q_1 = 1.131$ and 1.189 for $\kappa = 0$ and 1, respectively, and these for $y_2$ are $p_2 = 0.084$, 0.084 and 0.083 for $\kappa = 0, 1$ and 5, respectively. (e-f) the first and the third even eigenmodes $v_1(\theta)$ and $v_3(\theta)$ for the eigenvalue problem in Eq. (C.9) for different $\gamma_{1,1}$. The
corresponding eigenvalues $m_i$ are 0.175, 0.237, 0.213 for $\bar{y}_{1,i} = 0.8$, 0.5 and 0.2, respectively, and $m_3$ are 1.823, 1.763, 1.782, respectively.

References


