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A PSEUDOCOMPRESSIBILITY METHOD FOR THE INCOMPRESSIBLE BRINKMAN-FORCHHEIMER EQUATIONS

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Abstract. In this work, we study the Brinkman-Forchheimer equations for unsteady flows. We prove the continuous dependence of the solution on the Brinkman's and Forchheimer's coefficients as well as the initial data and external forces. Next, we propose and study a perturbed compressible system that approximate the Brinkman-Forchheimer equations. Finally, we propose a time discretization of the perturbed system by a semi-implicit Euler scheme and next a lowest-order Raviart-Thomas element is applied for spatial discretization. Some numerical results are given.

1. Introduction

Transport phenomena in porous media are ubiquitous to the extent that it is hardly difficult to find applications without some sort of porous material included. It also spans wide range of scales from small scale applications in confine spaces like bio-organic tissues to large scale applications in subsurface oil and gas reservoirs. With this large number of scales involved, the governing equations that describe conservation laws are, in a sense, simpler than those governing fluid flows. Some remarks, however, are worth mentioning. Probably most important is the fact that the governing laws in porous media have been adapted based on the assumption of the validity of the continuum hypothesis.

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This implies that field variables represent continuous functions of space and time and, hence, enable the conservation laws to be written in the form of partial differential equations [14, 15]. These conservation laws have been postulated based on upscaling the conservation laws at the fluid continuum level using theory of volume averaging, method of homogenization, theory of mixtures, etc. Unfortunately, the governing equations based on these theories are usually difficult to solve and they are generally unclosed because they contain terms at the pore scale which are hard to implement. Therefore, to obtain field equations that are workable, researchers suggested terms to the governing equations in addition to some properties of the porous material. In other words, some *ad hoc* terms are introduced to extend the governing equations to encounter more applications. Then, it remains to experimentalists and theoreticians to validate these terms. As an example, the simplest Darcy law [4] suggests that the mass flux and pressure gradient are proportional and the relationship between them is linear. This linear relationship has later been found to be valid for values of Reynolds number less than one. As Reynolds number becomes greater than one, this linear relationship is no longer valid. To account for such nonlinearities, Forchheimer [5, 6] suggested a quadratic term of the velocity to be included when considering momentum balance. Furthermore, Brinkman [1, 2] considers another term to account for the possible no slip condition once a confining wall exists. This produced the widely used Darcy-Brinkman-Forchheimer equations. The main feature of this equation is that it is nonlinear which apparently poses great difficulty to find analytical solution.

The Brinkman model is believed accurate when the flow velocity is too large for Darcy's law to be valid, and additionally the porosity is not too small. In this article, we are concerned with structural stability for the following Brinkman-Forchheimer (BF in short) equations:

$$\begin{cases} \partial_t \boldsymbol{u} - \gamma \Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|^\alpha \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega_T, \\ \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega_T, \\ \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Sigma_T, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega_T = \Omega \times (0,T)$, $\Sigma_T = \Gamma \times (0,T)$. $\boldsymbol{u}$ and p represent respectively fluid velocity and pressure. The constant $\gamma > 0$ is the Brinkman coefficient, $a > 0$ is the Darcy coefficient, b is the Forchheimer coefficient and $\alpha \in [1,2]$ is a given number. Ω is a bounded domain of $\mathbb{R}^3$ with sufficiently smooth boundary Γ. Δ is the Laplace operator, $\|\cdot\|$ and $\langle u, v\rangle$ denote respectively the norm and inner product on $L^2(\Omega)$.

2. The existence of solutions for the (BF) equations

The aim of this subsection is to give a variational formulation of problem (1.1) and to prove the existence of weak solutions. We introduce the spaces $\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \quad \mathrm{div}\boldsymbol{v} = 0\}$ and $\boldsymbol{H} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \quad \mathrm{div}\boldsymbol{v} = 0\}$. The variational formulation of (1.1) can be written as: Given $\boldsymbol{f} \in L^2(0, T, \boldsymbol{L}^2(\Omega))$ and $\boldsymbol{u}_0 \in \boldsymbol{V}$, find $\boldsymbol{u} \in \boldsymbol{L}^2(0, T, \boldsymbol{V}) \cap L^\infty(0, T, \boldsymbol{H}_0^1(\Omega))$, $\partial_t \boldsymbol{u} \in L^2(0, T, \boldsymbol{L}^2(\Omega))$ satisfying (1.1) such that for almost all t and $\boldsymbol{v} \in \boldsymbol{V}$,

$$\frac{d}{dt}\langle \boldsymbol{u}(t), \boldsymbol{v}\rangle + \gamma\langle \nabla \boldsymbol{u}(t), \nabla \boldsymbol{v}\rangle + a\langle \boldsymbol{u}(t), \boldsymbol{v}\rangle + b\langle |\boldsymbol{u}(t)|^\alpha \boldsymbol{u}(t), \boldsymbol{v}\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle. \quad (2.1)$$

Next, we discuss the solvability of the variational problem (2.1) by means of Faedo-Galerkin's method. This will be achieved in several steps that we describe below.

Step 1: *Construction of approximating solutions.* Consider an orthonormal basis of $\boldsymbol{V}$ constituted of $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m$. Using this basis, we introduce the space $\boldsymbol{V}_m = \langle \boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\rangle$ and define the approximate solution $\boldsymbol{u}_m$ of (2.1) as

$$\boldsymbol{u}_m(t) = \sum_{i=1}^{m} g_{im}(t)\boldsymbol{w}_i \quad (2.2)$$

such that

$$\begin{aligned} \frac{d}{dt}\langle \boldsymbol{u}_m(t), \boldsymbol{w}_i\rangle + \gamma\langle \nabla \boldsymbol{u}_m(t), \nabla \boldsymbol{w}_i\rangle + a\langle \boldsymbol{u}_m(t), \boldsymbol{w}_i\rangle + b\langle |\boldsymbol{u}_m(t)|^\alpha \boldsymbol{u}_m(t), \boldsymbol{w}_i\rangle& \\ = \langle \boldsymbol{f}, \boldsymbol{w}_i\rangle, \quad t \in [0, T], \ \ i = 1, \ldots, m,& \quad (2.3) \\ \boldsymbol{u}_m(0) = \boldsymbol{u}_{0m} \to \boldsymbol{u}_0 \in \boldsymbol{V}_m.& \quad (2.4) \end{aligned}$$

Using the theory of ordinary differential equations, it follows that the problem (2.3) has a solution $\boldsymbol{u}_m$ defined on $[0, t_m]$ with $t_m < T$.

Step 2: *a priori estimates.* First, we multiply (2.3) by $g_{im}(t)$ and sum up the obtained set of equations corresponding to $i = 1, \ldots, m$. We obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m(t)\|^2 + \gamma\|\nabla \boldsymbol{u}_m(t)\|^2 + a\|\boldsymbol{u}_m(t)\|^2 + b\|\boldsymbol{u}_m(t)\|_{\alpha+2}^{\alpha+2} \\ &= \langle \boldsymbol{f}(t), \boldsymbol{u}_m(t)\rangle, \leq \|\boldsymbol{f}(t)\|\|\boldsymbol{u}_m(t)\| \leq \frac{1}{2}\Big(\frac{1}{a}\boldsymbol{f}(t)\|^2 + a\|\boldsymbol{u}_m(t)\|^2\Big). \end{aligned}$$

Thus, we have

$$\frac{d}{dt}\|\boldsymbol{u}_m(t)\|^2 + 2\gamma\|\nabla \boldsymbol{u}_m(t)\|^2 + a\|\boldsymbol{u}_m(t)\|^2 + 2b\|\boldsymbol{u}_m(t)\|_{\alpha+2}^{\alpha+2} \leq \frac{1}{a}\|\boldsymbol{f}(t)\|^2.$$

Integrating this inequality from 0 to T with ($T \leq t_m$) leads to

$$\begin{aligned}
&\sup_{0\leq t\leq T}\|\boldsymbol{u}_m(t)\|^2 + 2\gamma\int_0^T \|\nabla \boldsymbol{u}_m(t)\|^2 dt \\
&+ a\int_0^T \|\boldsymbol{u}_m(t)\|^2 dt + 2b\int_0^T \|\boldsymbol{u}_m(t)\|_{\alpha+2}^{\alpha+2} dt \\
&\leq \frac{1}{a}\int_0^T \|\boldsymbol{f}(t)\|^2 dt + \|\boldsymbol{u}_0\|^2 < \infty.
\end{aligned} \tag{2.5}$$

Now, we multiply (2.3) by $g'_{im}(t)$ and add the obtained equations for $i = 1, \ldots, m$. We obtain

$$\begin{aligned}
\|\partial_t \boldsymbol{u}_m(t)\|^2 + \frac{1}{2}\gamma\frac{d}{dt}\|\nabla \boldsymbol{u}_m(t)\|^2 + \frac{1}{2}a\frac{d}{dt}\|\boldsymbol{u}_m(t)\|^2 + \frac{b}{\alpha+2}\frac{d}{dt}\|\boldsymbol{u}_m(t)\|_{\alpha+2}^{\alpha+2} \\
= \langle \boldsymbol{f}(t), \partial_t \boldsymbol{u}_m(t)\rangle \leq \frac{1}{2}\|\boldsymbol{f}(t)\|^2 + \frac{1}{2}\|\partial_t \boldsymbol{u}_m(t)\|^2.
\end{aligned} \tag{2.6}$$

Next, we integrate (2.6) from 0 to T and get

$$\begin{aligned}
&\int_0^T \|\partial_t \boldsymbol{u}_m(t)\|^2 dt + \gamma\|\nabla \boldsymbol{u}_m(T)\|^2 + a\|\boldsymbol{u}_m(T)\|^2 + \frac{2b}{\alpha+2}\|\boldsymbol{u}_m(t)\|_{\alpha+2}^{\alpha+2} \\
&\leq \frac{1}{2}\int_0^T \|\boldsymbol{f}(t)\|^2 dt + \gamma\|\nabla \boldsymbol{u}_{m0}\|^2 + a\|\boldsymbol{u}_{m0}\|^2 + \frac{2b}{\alpha+2}\|\boldsymbol{u}_{m0}\|_{\alpha+2}^{\alpha+2}.
\end{aligned}$$

In particular, we obtain

$$\int_0^T \|\partial_t \boldsymbol{u}_m(t)\|^2 dt \leq \frac{1}{2}\int_0^T \|\boldsymbol{f}(t)\|^2 dt + c < \infty, \tag{2.7}$$

where $c = \gamma\|\nabla \boldsymbol{u}_0\|^2 + a\|\boldsymbol{u}_0\|^2 + \frac{2b}{\alpha+2}\|\boldsymbol{u}_0\|_{\alpha+2}^{\alpha+2}$. Eventually, integrating (2.6) from 0 to t, we get

$$\begin{aligned}
\int_0^t \|\partial_t \boldsymbol{u}_m(s)\|^2 ds + \gamma\|\nabla \boldsymbol{u}_m(t)\|^2 + a\|\boldsymbol{u}_m(t)\|^2 + \frac{2b}{\alpha+2}\|\boldsymbol{u}_m(t)\|_{\alpha+2}^{\alpha+2} \\
\leq \frac{1}{2}\int_0^T \|\boldsymbol{f}(s)\|^2 ds + c.
\end{aligned} \tag{2.8}$$

In particular, we infer

$$\sup_{0\leq t\leq T}\|\nabla \boldsymbol{u}_m(t)\|^2 \leq \frac{1}{2\gamma}\int_0^T \|\boldsymbol{f}(s)\|^2 ds + c < \infty. \tag{2.9}$$

Before going further, let us mention that the above *a priori* estimates show that in fact $t_m = T$.

Step 3: *Passage to the limit.* We need to pass to the limit when m approaches infinity. It follows from (2.5), (2.7) and (2.9) that there is a subsequence of $(\boldsymbol{u}_m)_m$ that we still denote $(\boldsymbol{u}_m)_m$ by abuse of notation and some $\boldsymbol{u}$ such that

$$\boldsymbol{u}_m \longrightarrow \boldsymbol{u} \quad \text{weak star} \quad \text{in } L^\infty(0,T,\boldsymbol{H}_0^1(\Omega)), \tag{2.10}$$

$$\boldsymbol{u}_m \longrightarrow \boldsymbol{u} \quad \text{weak} \quad \text{in } L^2(0,T,\boldsymbol{V}), \tag{2.11}$$

$$|\boldsymbol{u}_m|^\alpha \boldsymbol{u}_m \longrightarrow \boldsymbol{w} \quad \text{weak} \quad \text{in } L^{\frac{\alpha+2}{\alpha+1}}(0,T,\boldsymbol{L}^{\frac{\alpha+2}{\alpha+1}}(\Omega)), \tag{2.12}$$

$$\partial_t \boldsymbol{u}_m \longrightarrow \partial_t \boldsymbol{u} \quad \text{weak} \quad \text{in } L^2(0,T,\boldsymbol{L}^2(\Omega)). \tag{2.13}$$

From (2.11) and (2.13), we deduce that $\boldsymbol{u}$ is bounded in $\boldsymbol{H}^1(Q)$. Since the imbedding of $\boldsymbol{H}^1(Q)$ in $\boldsymbol{L}^2(Q)$ is compact, we can extract a subsequence denoted again $\boldsymbol{u}_m$ such that

$$\boldsymbol{u}_m \longrightarrow \boldsymbol{u} \quad \text{strongly} \quad \text{in } L^2(0,T,\boldsymbol{L}^2(\Omega)) \quad \text{and} \quad \text{a.e. in } Q. \tag{2.14}$$

Therefore, it remains to prove that $\boldsymbol{w} = |\boldsymbol{u}|^\alpha \boldsymbol{u}$. For this purpose, we apply the same argument used in [8, Lemma 1.3]. Indeed, using (2.14), it follows that

$$|\boldsymbol{u}_m|^\alpha \boldsymbol{u}_m \longrightarrow |\boldsymbol{u}|^\alpha \boldsymbol{u} \quad \text{weak} \quad \text{in } L^{\frac{\alpha+2}{\alpha+1}}(0,T,\boldsymbol{L}^{\frac{\alpha+2}{\alpha+1}}(\Omega)).$$

Thanks to (2.12), one obtains $\boldsymbol{w} = |\boldsymbol{u}|^\alpha \boldsymbol{u}$. Now, It is easy to pass to the limit in ((2.3)-(2.4)) and conclude that $\boldsymbol{u}$ is solution of (2.1).

Therefore, we have

Theorem 2.1. *For any $\boldsymbol{u}_0 \in \boldsymbol{V}$ and $\boldsymbol{f} \in L^2(0,T,\boldsymbol{L}^2(\Omega))$, problem* (1.1) *admits at least one solution u satisfying $\boldsymbol{u} \in L^\infty(0,T,\boldsymbol{V}) \cap L^2(0,T,\boldsymbol{H}^2(\Omega))$, $\partial_t \boldsymbol{u} \in L^2(0,T,\boldsymbol{L}^2(\Omega))$, and $p \in L^2(0,T,L^2(\Omega))$. Moreover, for any $T > 0$, we have*

$$\sup_{0\le t\le T} \|\nabla \boldsymbol{u}(t)\| \le C \qquad \textit{and} \qquad \int_0^T \|\partial_t \boldsymbol{u}(t)\|^2 \, dt \le C, \tag{2.15}$$

where C denotes a positive constant depending on $\boldsymbol{u}_0$ and the parameters of problem (1.1).

3. Continuous dependence on data

In this section, we show that the solution depends continuously on the initial velocity and the external forces. For that purpose, let $(\boldsymbol{u}_1, p_1)$ and $(\boldsymbol{u}_2, p_2)$ such that

$$\begin{aligned} &\partial_t \boldsymbol{u}_1 - \gamma\Delta \boldsymbol{u}_1 + a\boldsymbol{u}_1 + b|\boldsymbol{u}_1|^\alpha \boldsymbol{u}_1 + \nabla p_1 = \boldsymbol{f}_1 \quad \text{in } \Omega_T,\\ &\operatorname{div} \boldsymbol{u}_1 = 0 \quad \text{in } \Omega_T, \end{aligned} \tag{3.1}$$

$$\begin{aligned}&\boldsymbol{u}_1=\mathbf{0}\quad \text{on } \Sigma_T,\\&\boldsymbol{u}_1(0)=\boldsymbol{u}_0^1\quad \text{in } \Omega,\end{aligned}$$

and

$$\begin{aligned}&\partial_t\boldsymbol{u}_2-\gamma\Delta\boldsymbol{u}_2+a\boldsymbol{u}_2+b|\boldsymbol{u}_2|^\alpha\boldsymbol{u}_2+\nabla p_2=\boldsymbol{f}_2\quad \text{in } \Omega_T,\\&\operatorname{div}\boldsymbol{u}_2=0\quad \text{in } \Omega_T,\\&\boldsymbol{u}_2=\mathbf{0}\quad \text{on } \Sigma_T,\\&\boldsymbol{u}_2(0)=\boldsymbol{u}_0^2\quad \text{in } \Omega.\end{aligned}\tag{3.2}$$

Now, we are able to state the main result of this section.

Theorem 3.1. *Let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ as in* (3.1) *and* (3.2)*. Then, there exists a positive constant K depending on the parameters γ, a and Ω such that*

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2\le e^{-Kt}\|\boldsymbol{u}_1(0)-\boldsymbol{u}_2(0)\|^2+\int_0^t e^{K(s-t)}\|\boldsymbol{f}_1(s)-\boldsymbol{f}_2(s)\|^2ds.\tag{3.3}$$

Proof. We know that both $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ satisfy the variational formulation (2.1) with respect to $\boldsymbol{u}_0^1$, $\boldsymbol{f}_1$ and $\boldsymbol{u}_0^2$, $\boldsymbol{f}_2$ respectively. Rewriting (2.1) for $\boldsymbol{u}_1$ with a test function $\boldsymbol{v}-\boldsymbol{u}_1$, we obtain

$$\begin{aligned}\frac{d}{dt}\langle\boldsymbol{u}_1(t),\boldsymbol{v}-\boldsymbol{u}_1(t)\rangle+\gamma\langle\nabla\boldsymbol{u}_1(t),\nabla\big(\boldsymbol{v}-\boldsymbol{u}_1(t)\big)\rangle+a\langle\boldsymbol{u}_1(t),\boldsymbol{v}-\boldsymbol{u}_1(t)\rangle\\+b\langle|\boldsymbol{u}_1(t)|^\alpha\boldsymbol{u}_1(t),\boldsymbol{v}-\boldsymbol{u}_1(t)\rangle=\langle\boldsymbol{f}_1,\boldsymbol{v}-\boldsymbol{u}_1(t)\rangle.\end{aligned}\tag{3.4}$$

Equivalently, for $\boldsymbol{u}_2$, we take $\boldsymbol{v}-\boldsymbol{u}_2$ as a test function and obtain

$$\begin{aligned}\frac{d}{dt}\langle\boldsymbol{u}_2(t),\boldsymbol{v}-\boldsymbol{u}_2(t)\rangle+\gamma\langle\nabla\boldsymbol{u}_2(t),\nabla\big(\boldsymbol{v}-\boldsymbol{u}_2(t)\big)\rangle+a\langle\boldsymbol{u}_2(t),\boldsymbol{v}-\boldsymbol{u}_2(t)\rangle\\+b\langle|\boldsymbol{u}_2(t)|^\alpha\boldsymbol{u}_2(t),\boldsymbol{v}-\boldsymbol{u}_2(t)\rangle=\langle\boldsymbol{f}_2,\boldsymbol{v}-\boldsymbol{u}_2(t)\rangle.\end{aligned}\tag{3.5}$$

Now, let $\boldsymbol{w}(t)=\boldsymbol{u}_2(t)-\boldsymbol{u}_1(t)$ and $\boldsymbol{w}(0)=\boldsymbol{u}_2(0)-\boldsymbol{u}_1(0)$. Choosing $\boldsymbol{v}=\boldsymbol{u}_2$ in (3.4) and $\boldsymbol{v}=\boldsymbol{u}_1$ in (3.5), summing up both equations, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}(t)\|^2+\gamma\|\nabla\boldsymbol{w}(t)\|^2+a\|\boldsymbol{w}(t)\|^2+b\langle|\boldsymbol{u}_2|^\alpha\boldsymbol{u}_2-|\boldsymbol{u}_1|^\alpha\boldsymbol{u}_1,\boldsymbol{w}(t)\rangle\\=\langle\boldsymbol{f}_2(t)-\boldsymbol{f}_1(t),\boldsymbol{w}(t)\rangle.\end{aligned}$$

Since the operator $T(\xi)=|\xi|^\alpha\xi$ is monotone, we have obviously

$$\langle|\boldsymbol{u}_1|^\alpha\boldsymbol{u}_1-|\boldsymbol{u}_2|^\alpha\boldsymbol{u}_2,\boldsymbol{u}(t)\rangle\ge 0.$$

Therefore,

$$\frac{d}{dt}\|\boldsymbol{w}(t)\|^2+2\gamma\|\nabla\boldsymbol{w}(t)\|^2+a\|\boldsymbol{w}(t)\|^2\le\frac{1}{a}\|\boldsymbol{f}_2(t)-\boldsymbol{f}_1(t)\|^2.$$

Thanks to (4.6), we obtain

$$\frac{d}{dt}\|\boldsymbol{w}(t)\|^2 + c\|\boldsymbol{w}(t)\|^2 \leq \frac{1}{a}\|\boldsymbol{f}_2(t) - \boldsymbol{f}_1(t)\|^2,$$

where the constant c depends on D(the constant in (4.6)), γ and a. Eventually, using Gronwall's lemma, we obtain the estimate (3.3). □

Remark 3.2. Obviously, this last result implies in particular the uniqueness of solutions of problem (1.1).

4. Continuous dependence on the Brinkman's and Forchheimer's coefficients

In this section, we show that the solutions of the BF problem (1.1) depend continuously on the Brinkman's and Forchheimer's coefficients, thereby we extend the result of [3]. Let $(\boldsymbol{u}_1, p_1)$ and $(\boldsymbol{u}_2, p_2)$ such that

$$\begin{aligned}
&\partial_t \boldsymbol{u}_1 - \gamma_1 \Delta \boldsymbol{u}_1 + a\boldsymbol{u}_1 + b_1|\boldsymbol{u}_1|^\alpha \boldsymbol{u}_1 + \nabla p_1 = 0 \quad \text{in } \Omega_T,\\
&\operatorname{div}\boldsymbol{u}_1 = 0 \quad \text{in } \Omega_T,\\
&\boldsymbol{u}_1 = \mathbf{0} \quad \text{on } \Sigma_T,\\
&\boldsymbol{u}_1(0) = \boldsymbol{u}_0 \quad \text{in } \Omega,
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\partial_t \boldsymbol{u}_2 - \gamma_2 \Delta \boldsymbol{u}_2 + a\boldsymbol{u}_2 + b_2|\boldsymbol{u}_2|^\alpha \boldsymbol{u}_2 + \nabla p_2 = 0 \quad \text{in } \Omega_T,\\
&\operatorname{div}\boldsymbol{u}_2 = 0 \quad \text{in } \Omega_T,\\
&\boldsymbol{u}_2 = \mathbf{0} \quad \text{on } \Sigma_T,\\
&\boldsymbol{u}_2(0) = \boldsymbol{u}_0 \quad \text{in } \Omega.
\end{aligned} \tag{4.2}$$

We set $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $p = p_1 - p_2$. Then, $(\boldsymbol{u}, p)$ satisfies

$$\begin{aligned}
&\partial_t \boldsymbol{u} - \gamma_1 \Delta \boldsymbol{u} - \gamma \Delta \boldsymbol{u}_2 + a\boldsymbol{u} + \nabla p\\
&= -b|\boldsymbol{u}_1|^\alpha \boldsymbol{u}_1 - b_2\big(|\boldsymbol{u}_1|^\alpha \boldsymbol{u}_1 - |\boldsymbol{u}_2|^\alpha \boldsymbol{u}_2\big) \quad \text{in } \Omega_T,\\
&\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega_T,\\
&\boldsymbol{u} = \mathbf{0} \quad \text{on } \Sigma_T,\\
&\boldsymbol{u}(0) = \mathbf{0} \quad \text{in } \Omega,
\end{aligned} \tag{4.3}$$

where $\gamma = \gamma_1 - \gamma_2$ and $b = b_1 - b_2$. Now, we claim the following

Theorem 4.1. *Let $\boldsymbol{u}$ be the solution of* (4.3)*. Then, there exist positive constants M and Q, depending on the parameters of* (4.3) *such that*

$$\|\nabla \boldsymbol{u}(t)\|^2 + \|\boldsymbol{u}(t)\|^2 \leq M\gamma^2 + Qb^2. \tag{4.4}$$

Proof. Multiplying (4.3) by $\boldsymbol{u}$ and integrating over Ω leads to

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|^2 + \gamma_1\|\nabla\boldsymbol{u}(t)\|^2 + a\|\boldsymbol{u}(t)\|^2 = -b_2\langle|\boldsymbol{u}_1|^\alpha\boldsymbol{u}_1 - |\boldsymbol{u}_2|^\alpha\boldsymbol{u}_2, \boldsymbol{u}(t)\rangle - b\langle|\boldsymbol{u}_1(t)|^\alpha\boldsymbol{u}_1(t), \boldsymbol{u}(t)\rangle - \gamma\langle\nabla\boldsymbol{u}_2(t), \nabla\boldsymbol{u}(t)\rangle.$$

With the monotonicity of the operator $T(\xi) = |\xi|^\alpha\xi$, we get

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|^2 + \gamma_1\|\nabla\boldsymbol{u}(t)\|^2 + a\|\boldsymbol{u}(t)\|^2 \leq \Big|b\langle|\boldsymbol{u}_1(t)|^\alpha\boldsymbol{u}_1(t), \boldsymbol{u}(t)\rangle\Big| + \Big|\gamma\langle\nabla\boldsymbol{u}_2(t), \nabla\boldsymbol{u}(t)\rangle\Big|. \tag{4.5}$$

Next, we define the Sobolev constant D as

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \quad \|\boldsymbol{v}\|_p \leq D\|\nabla\boldsymbol{v}\| \qquad \text{for any } 1 \leq p \leq 6. \tag{4.6}$$

Thus, thanks to Hölder's inequality and the Sobolev inequality (4.6), we get

$$\Big|b\langle|\boldsymbol{u}_1(t)|^\alpha\boldsymbol{u}_1(t), \boldsymbol{u}(t)\rangle\Big| \leq |b|D^{\alpha+2}\|\nabla\boldsymbol{u}_1(t)\|^{\alpha+1}\|\nabla\boldsymbol{u}(t)\| \leq \frac{b^2D^{2\alpha+4}}{\gamma_1}\|\nabla\boldsymbol{u}_1(t)\|^{2(\alpha+1)} + \frac{\gamma_1}{4}\|\nabla\boldsymbol{u}(t)\|^2. \tag{4.7}$$

Using (4.7) and the fact that

$$\Big|\gamma\langle\nabla\boldsymbol{u}_2(t), \nabla\boldsymbol{u}(t)\rangle\Big| \leq \frac{|\gamma|^2}{\gamma_1}\|\nabla\boldsymbol{u}_2\|^2 + \frac{\gamma_1}{4}\|\nabla\boldsymbol{u}\|^2, \tag{4.8}$$

we obtain thanks to (4.5)

$$\frac{d}{dt}\|\boldsymbol{u}(t)\|^2 + \gamma_1\|\nabla\boldsymbol{u}(t)\|^2 + 2a\|\boldsymbol{u}(t)\|^2 \leq 2b^2\frac{D^{2\alpha+4}}{\gamma_1}\|\nabla\boldsymbol{u}_1(t)\|^{2\alpha+1} + \frac{2\gamma^2}{\gamma_1}\|\nabla\boldsymbol{u}_2(t)\|^2 \leq \quad Q_0b^2 + \frac{2\gamma^2}{\gamma_1}\|\nabla\boldsymbol{u}_2(t)\|^2, \tag{4.9}$$

where $Q_0 = 2D^{2\alpha+4}C^{2\alpha+1}\gamma_1^{-1}$ and C is the constant in (2.15). Now, multiplying (4.3) by $\partial_t\boldsymbol{u}$ and integrating over Ω leads to

$$\|\partial_t\boldsymbol{u}(t)\|^2 + \frac{1}{2}\frac{d}{dt}\Big(\gamma_1\|\nabla\boldsymbol{u}(t)\|^2 + a\|\boldsymbol{u}(t)\|^2\Big) \tag{4.10}$$

$$\leq \Big|b_2\langle|\boldsymbol{u}_1(t)|^\alpha\boldsymbol{u}_1(t) - |\boldsymbol{u}_2(t)|^\alpha\boldsymbol{u}_2(t), \partial_t\boldsymbol{u}(t)\rangle\Big| + \Big|\gamma\langle\Delta\boldsymbol{u}_2(t), \partial_t\boldsymbol{u}(t)\rangle\Big| + \Big|b\langle|\boldsymbol{u}_1(t)|^\alpha\boldsymbol{u}_1(t), \partial_t\boldsymbol{u}(t)\rangle\Big|. \tag{4.11}$$

Using the mean value theorem and Hölder's inequality and (4.6) and (2.15), it is easy to show that

$$\Big|b_2\langle|\boldsymbol{u}_1(t)|^\alpha\boldsymbol{u}_1(t) - |\boldsymbol{u}_2(t)|^\alpha\boldsymbol{u}_2(t), \partial_t\boldsymbol{u}(t)\rangle\Big| \tag{4.12}$$

$$\leq b_2 D^{\alpha+1} C^{\alpha} \|\nabla \boldsymbol{u}(t)\| \|\partial_t \boldsymbol{u}(t)\| \leq b_2^2 D^{2\alpha+2} C^{2\alpha} \|\nabla \boldsymbol{u}(t)\|^2 + \frac{1}{3} \|\partial_t \boldsymbol{u}(t)\|^2.$$

Using the first equation in (4.2) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\Big|\gamma\langle \Delta \boldsymbol{u}_2(t), \partial_t \boldsymbol{u}(t)\rangle\Big| &\leq \frac{\gamma}{\gamma_2}\Big|\langle \partial_t \boldsymbol{u}_2(t) + a\boldsymbol{u}_2(t) + b_2|\boldsymbol{u}_2(t)|^{\alpha}\boldsymbol{u}_2(t), \partial_t \boldsymbol{u}(t)\rangle\Big|, \\
&\leq \frac{1}{3}\|\partial_t \boldsymbol{u}\|^2 + \frac{3\gamma^2}{4\gamma_2^2}\Big\|\partial_t \boldsymbol{u}_2 + a\boldsymbol{u}_2 + b_2|\boldsymbol{u}_2|^{\alpha}\boldsymbol{u}_2\Big\|^2, \\
&\leq \frac{1}{3}\|\partial_t \boldsymbol{u}\|^2 + \frac{3\gamma^2}{4\gamma_2^2}\Big(\|\partial_t \boldsymbol{u}_2\|^2 + a^2\|\boldsymbol{u}_2\|^2 + b_2^2\|\boldsymbol{u}_2\|_{2\alpha+2}^{2\alpha+2}\Big), \\
&\leq \frac{1}{3}\|\partial_t \boldsymbol{u}\|^2 + \frac{3\gamma^2}{4\gamma_2^2}\Big(\| \ \partial_t \boldsymbol{u}_2\|^2 + a^2 D^2 C^2 + b_2^2 (DC)^{2\alpha+2}\Big). \qquad (4.13)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\Big|b\langle |\boldsymbol{u}_1(t)|^{\alpha}\boldsymbol{u}_1(t), \partial_t \boldsymbol{u}(t)\rangle\Big| &\leq |b| D^{\alpha+1}\|\nabla \boldsymbol{u}_1\|^{\alpha+1}\|\partial_t \boldsymbol{u}\| \qquad (4.14) \\
&\leq b^2 \frac{3}{4} D^{2(\alpha+1)}\|\nabla \boldsymbol{u}_1\|^{2(\alpha+1)} + \frac{1}{3}\|\partial_t \boldsymbol{u}\|^2 \\
&\leq b^2 \frac{3}{4} D^{2(\alpha+1)} C^{2(\alpha+1)} + \frac{1}{3}\|\partial_t \boldsymbol{u}\|^2.
\end{aligned}$$

Thus, combining ((4.12)-(4.14)), we obtain from (4.10)

$$\begin{aligned}
&\frac{d}{dt}\big(\gamma_1\|\nabla \boldsymbol{u}(t)\|^2 + a\|\boldsymbol{u}(t)\|^2\big) \qquad (4.15) \\
&\leq M_0\|\nabla \boldsymbol{u}(t)\|^2 + Q_1 b^2 + \frac{3}{2\gamma_2^2}\gamma^2\|\partial_t \boldsymbol{u}_2(t)\|^2 + M_1\gamma^2,
\end{aligned}$$

where

$$\begin{aligned}
&M_0 = 27 b_2^2 D^{2\alpha+2} C^{2\alpha}, \qquad Q_1 = \frac{3}{2} D^{2\alpha+2} C^{2\alpha+1}, \\
&M_1 = \frac{3}{2}\gamma_2^{-2}(a(DC)^2 + b_2^2 (DC)^{2\alpha+2}).
\end{aligned}$$

Multiplying (4.9) by $\beta = 2\frac{M_0}{\gamma_1}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\beta\|\boldsymbol{u}(t)\|^2\Big) + 2M_0\|\nabla \boldsymbol{u}(t)\|^2 + 2a\beta\|\boldsymbol{u}(t)\|^2 \qquad (4.16) \\
&\leq \beta Q_0 b^2 + 2\beta\frac{\gamma^2}{\gamma_1}\|\nabla \boldsymbol{u}_2(t)\|^2.
\end{aligned}$$

Next, we add the bounds (4.15) and (4.16) to get

$$\frac{d}{dt}\Big(\gamma_1\|\nabla\boldsymbol{u}(t)\|^2+(a+\beta)\|\boldsymbol{u}(t)\|^2\Big)+M_0\|\nabla\boldsymbol{u}(t)\|^2+2a\beta\|\boldsymbol{u}(t)\|^2$$
$$\leq\gamma^2\Big(\frac{2\beta}{\gamma_1}\|\nabla\boldsymbol{u}_2(t)\|^2+\frac{9}{2\gamma_2^2}\|\partial_t\boldsymbol{u}_2(t)\|^2+M_1\Big)+b^2\big(Q_1+\beta Q_0\big).$$

Now, we set $F(t)=\gamma_1\|\nabla\boldsymbol{u}(t)\|^2+(a+\beta)\|\boldsymbol{u}(t)\|^2$. It is easy to show that

$$F'(t)+Q_2F(t)\leq\gamma^2\Big(\frac{2\beta}{\gamma_1}\|\nabla\boldsymbol{u}_2(t)\|^2+\frac{9}{2\gamma_2^2}\|\partial_t\boldsymbol{u}_2(t)\|^2+M_1\Big)+b^2\big(Q_1+\beta Q_0\big),$$

where $Q_2=\min\big(\frac{M_0}{\gamma_1},\frac{2a\beta}{a+\beta}\big)$. Therefore, integrating from 0 to t, applying Gronwall's inequality and using (2.15), we obtain

$$F(t)\leq\gamma^2\Big(2\beta\gamma_1^{-1}C+\frac{9}{2}\gamma_2^{-2}C+\frac{M_1}{Q_2}\Big)+b^2(Q_1+\beta Q_0).$$

Furthermore, we can derive (4.4) with

$$Q=\frac{Q_1+\beta Q_0}{\min(\gamma_1,(a+\beta))}\quad\text{and}\quad M=\frac{2\beta\gamma_1^{-1}C+\frac{9}{2\gamma_2^2}C+\frac{M_1}{Q_1}}{\min(\gamma_1,(a+\beta))}.$$

This achieves the proof of the continuous dependence on the Brinkman's and Forchheimer coefficients. □

5. Approximation by the artificial compressibility method

The importance of the treatment of the incompressibility constraint has been recognized since a long time. Towards incompressible flows, density variation is not linked to the pressure. A possible method is to introduce an artificial compressibility, i.e. adding the time-derivation of pressure to the continuity equation, first proposed by Chorin (1967), named pseudo-compressibility. This method introduced waves of finite speed into the incompressible flow, in which the artificial pressure waves would otherwise be infinite. In this section, we study a perturbed system that approximates, in the limit, the Brinkman-Forchheimer equations via the artificial compressibility method. We prove the existence of weak solution and show how the solution of the perturbed problem converges to the solution of the Brinkman-Forchheimer problem when $\varepsilon\to 0$ where $\epsilon>0$ is arbitrary. The pseudo-compressibility method that we propose to study is to approximate the solution $(\boldsymbol{u},p)$ of the incompressible Brinkman-Forchheimer equations by $(\boldsymbol{u}^\varepsilon,p^\varepsilon)$ satisfying the following perturbed system:

$$\partial_t\boldsymbol{u}_\epsilon-\gamma\Delta\boldsymbol{u}_\epsilon+a\boldsymbol{u}_\epsilon+b|\boldsymbol{u}_\epsilon|^\alpha\boldsymbol{u}_\epsilon+\nabla p_\epsilon=\boldsymbol{f}\qquad\text{in}\Omega_T,$$

$$\begin{aligned}
&\operatorname{div}\boldsymbol{u}_\epsilon + \epsilon\partial_t p_\epsilon = 0 \quad \text{in}\Omega_T,\\
&\boldsymbol{u}_\epsilon = \mathbf{0} \quad \text{on } \Sigma_T,\\
&\boldsymbol{u}_\epsilon(0) = \boldsymbol{u}_0 \quad \text{in } \Omega,\\
&p_\epsilon(0) = p_0 \quad \text{in } \Omega,
\end{aligned} \tag{5.1}$$

where $p_0 \in L^2(\Omega)$ is arbitrary and independent of ϵ.

5.1. **Existence of solutions of the perturbed problem.** We can easily verify that for a given $\boldsymbol{f} \in L^2(0,T,\boldsymbol{L}^2(\Omega))$ and an $\boldsymbol{u}_0 \in \boldsymbol{L}^2(\Omega)$, the problem (5.1) is equivalent to the variational formulation: Find $\boldsymbol{u}_\epsilon \in L^2(0,T,\boldsymbol{H}_0^1(\Omega)) \cap L^\infty(0,T,\boldsymbol{L}^2(\Omega))$ and $p_\epsilon \in L^2(0,T,L^2(\Omega))$ such that

$$\begin{aligned}
&\frac{d}{dt}\langle \boldsymbol{u}^\varepsilon(t), \boldsymbol{v}\rangle + \gamma\langle \nabla\boldsymbol{u}^\varepsilon(t), \nabla\boldsymbol{v}\rangle + a\langle \boldsymbol{u}^\varepsilon(t), \boldsymbol{v}\rangle + b\langle |\boldsymbol{u}^\varepsilon(t)|^\alpha \boldsymbol{u}^\varepsilon(t), \boldsymbol{v}\rangle\\
&- \langle p^\varepsilon(t),\ \operatorname{div}\boldsymbol{v}\rangle = \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^{\alpha+2}(\Omega),
\end{aligned} \tag{5.2}$$

$$\langle\ \operatorname{div}\boldsymbol{u}^\varepsilon(t), q\rangle = -\varepsilon\langle \partial_t p^\varepsilon(t), q\rangle, \qquad \forall q \in L^2(\Omega). \tag{5.3}$$

Now, we state the existence result as follows

Theorem 5.1. *For $\varepsilon > 0$ fixed, given $\boldsymbol{f} \in L^2(0,T,\boldsymbol{L}^2(\Omega))$, $\boldsymbol{u}_0 \in \boldsymbol{H}$ and $p_0 \in L^2(\Omega)$, there exists at least one solution $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ of the perturbed problems (5.1). Moreover,*

$$(\boldsymbol{u}^\varepsilon, p^\varepsilon) \in L^2(0,T,\boldsymbol{H}_0^1(\Omega)) \cap L^\infty(0,T,\boldsymbol{L}^2(\Omega)) \times L^2(0,T,L^2(\Omega)).$$

Proof. To prove the well-posedness of the problem ((5.2)-(5.3)), we will use the Faedo-Galerkin method. We find the approximate solution of $\boldsymbol{u}^\varepsilon(t)$ and $p_\varepsilon(t)$ respectively in the forms

$$\boldsymbol{u}_m^\varepsilon(t) = \sum_{i=1}^m g_{im}(t)\boldsymbol{w}_i \qquad \text{and} \qquad p_m^\varepsilon(t) = \sum_{j=1}^m \xi_{jm}(t) r_j, \tag{5.4}$$

where the coefficients $\boldsymbol{w}_i$ and r_j satisfy the system of ODE's:

$$\begin{aligned}
&\frac{d}{dt}\langle \boldsymbol{u}_m^\varepsilon(t), \boldsymbol{w}_k\rangle + \gamma\langle \nabla\boldsymbol{u}_m^\varepsilon(t), \nabla\boldsymbol{w}_k\rangle + a\langle \boldsymbol{u}_m^\varepsilon(t), \boldsymbol{w}_k\rangle + b\langle |\boldsymbol{u}_m^\varepsilon(t)|^\alpha \boldsymbol{u}_m^\varepsilon(t), \boldsymbol{w}_k\rangle\\
&- \langle p_m^\varepsilon(t),\ \operatorname{div}\boldsymbol{w}_k\rangle = \langle \boldsymbol{f}(t), \boldsymbol{w}_k\rangle, \qquad k = 1, \dots, m
\end{aligned} \tag{5.5}$$

$$\langle\ \operatorname{div}\boldsymbol{u}_m^\varepsilon(t), r_l\rangle = -\varepsilon\langle \partial_t p_m^\varepsilon(t), r_l\rangle, \qquad l = 1, \dots, m. \tag{5.6}$$

Moreover, we have the following initial conditions

$$\boldsymbol{u}_m^\varepsilon(0) = \boldsymbol{u}_{0m} \qquad \text{and} \qquad p_m^\varepsilon(0) = p_{0m}, \tag{5.7}$$

where $\boldsymbol{u}_{0m}$ and p_{0m} are the orthogonal projections of $\boldsymbol{u}_0$ and p_0 onto the subspaces spanned by $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m$ and $r_1, \ldots, r_m$ in $\boldsymbol{L}^2(\Omega)$ and $L^2(\Omega)$, respectively.

It is clear that for each m, there exist solutions $\boldsymbol{u}_m^\varepsilon(t)$ and $p_m^\varepsilon(t)$ in the form (5.4) which satisfy ((5.5)-(5.7)) almost everywhere on $0 \le t \le T_m$ for some T_m, $0 < T_m \le T$. The following estimates allow us to take $T_m = T$ for all m.

a) *First estimates:* We multiply the k^{th} equation of (5.5) by $g_{km}(t)$ and sum up over k. Next, we multiply the l^{th} equation of (5.6) by $\xi_{lm}(t)$ and sum up over l. We get

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m^\varepsilon(t)\|^2 + \gamma\|\nabla \boldsymbol{u}_m^\varepsilon(t)\|^2 + a\|\boldsymbol{u}_m^\varepsilon(t)\|^2 + b\|\boldsymbol{u}_m^\varepsilon(t)\|_{\alpha+2}^{\alpha+2} + \frac{\varepsilon}{2}\frac{d}{dt}\|p_m^\varepsilon(t)\|^2$$
$$= \langle \boldsymbol{f}(t), \boldsymbol{u}_m^\varepsilon(t)\rangle \le \frac{a}{2}\|\boldsymbol{u}_m^\varepsilon(t)\|^2 + \frac{1}{2a}\|\boldsymbol{f}(t)\|^2,$$

so that

$$\frac{d}{dt}\Big(\|\boldsymbol{u}_m^\varepsilon(t)\|^2 + \varepsilon\|p_m^\varepsilon(t)\|^2\Big) + 2\gamma\|\nabla \boldsymbol{u}_m^\varepsilon(t)\|^2 + a\|\boldsymbol{u}_m^\varepsilon(t)\|^2 + 2b\|\boldsymbol{u}_m^\varepsilon(t)\|_{\alpha+2}^{\alpha+2}$$
$$\le \frac{1}{2a}\|\boldsymbol{f}(t)\|^2. \tag{5.8}$$

Integrating both sides of (5.8) over the interval $(0,t)$ we get

$$\|\boldsymbol{u}_m^\varepsilon(t)\|^2 + \varepsilon\|p_m^\varepsilon(t)\|^2 + 2\gamma\int_0^t \|\nabla \boldsymbol{u}_m^\varepsilon(s)\|^2 ds + a\int_0^t \|\boldsymbol{u}_m^\varepsilon(s)\|^2 ds$$
$$+2b\int_0^t \|\boldsymbol{u}_m^\varepsilon(s)\|_{\alpha+2}^{\alpha+2} ds \le \frac{1}{2a}\int_0^t \|\boldsymbol{f}(s)\|^2 ds + \|\boldsymbol{u}_m^\varepsilon(0)\|^2 + \varepsilon\|p_m^\varepsilon(0)\|^2.$$

From this inequality we infer

$$\sup_{t\in[0,T]}\Big(\|\boldsymbol{u}_m^\varepsilon(t)\|^2 + \varepsilon\|p_m^\varepsilon(t)\|^2\Big) \le d_1, \tag{5.9}$$

$$\int_0^T \|\nabla \boldsymbol{u}_m^\varepsilon(t)\|^2 dt \le \frac{d_1}{2\gamma}, \tag{5.10}$$

$$\int_0^T \|\boldsymbol{u}_m^\varepsilon(t)\|^2 dt \le \frac{d_1}{a}, \tag{5.11}$$

$$\int_0^T \|\boldsymbol{u}_m^\varepsilon(t)\|_{\alpha+2}^{\alpha+2} dt \le \frac{d_1}{2b}, \tag{5.12}$$

where

$$d_1 = \|\boldsymbol{u}_0\|^2 + \varepsilon\|p_0\|^2 + \frac{1}{2a}\int_0^T \|\boldsymbol{f}(s)\|^2 ds.$$

b) *Second estimates:* In addition to the previous estimates, we want to prove an estimate of the fractional derivative with respect to time of $\boldsymbol{u}_m^\varepsilon(t)$ in order to pass to the limit in the nonlinear term. We let

$$\phi_m(t) = \boldsymbol{f}(t) - \gamma\Delta \boldsymbol{u}_m^\varepsilon(t) - a\boldsymbol{u}_m^\varepsilon(t) - b|\boldsymbol{u}_m^\varepsilon(t)|^\alpha \boldsymbol{u}_m^\varepsilon(t).$$

Therefore,

$$\|\phi_m(t)\|_{\boldsymbol{H}^{-1}(\Omega)} \le \|\boldsymbol{f}(t)\| + a\|\boldsymbol{u}_m^\varepsilon(t)\| + \gamma\|\nabla \boldsymbol{u}_m^\varepsilon(t)\| + b\|\boldsymbol{u}_m^\varepsilon(t)\|_{\alpha+2}^{\alpha+1}. \quad (5.13)$$

Hence, ((5.5)-(5.6)) reads now

$$\frac{d}{dt}\langle \boldsymbol{u}_m^\varepsilon(t), \boldsymbol{w}_k\rangle - \langle p_m^\varepsilon(t),\ \operatorname{div}\boldsymbol{w}_k\rangle = \langle \phi_m(t), \boldsymbol{w}_k\rangle, \quad k = 1,\dots,m, \quad (5.14)$$

$$\langle\ \operatorname{div}\boldsymbol{u}_m^\varepsilon(t), r_l\rangle = -\varepsilon\langle \partial_t p_m^\varepsilon(t), r_l\rangle, \quad l = 1,\dots,m. \quad (5.15)$$

We extend the functions $\boldsymbol{u}_m^\varepsilon(t)$, $p_m^\varepsilon(t)$, $\boldsymbol{f}(t)$ $g_{km}(t)$, $\xi_{lm}(t)$ and $\phi_m^\varepsilon(t)$ by 0 outside the interval $[0, T]$. Then we obtain in the sense of distributions on $\mathbb{R}$:

$$\begin{aligned}
&\frac{d}{dt}\langle \widetilde{\boldsymbol{u}}_m^\varepsilon(t), \boldsymbol{w}_k\rangle - \langle \widetilde{p}_m^\varepsilon(t),\ \operatorname{div}\boldsymbol{w}_k\rangle \\
&= \langle \widetilde{\phi}_m(t), \boldsymbol{w}_k\rangle + \langle \boldsymbol{u}_{0m}, \boldsymbol{w}_k\rangle\delta(0) - \langle \boldsymbol{u}_m^\varepsilon(T), \boldsymbol{w}_k\rangle\delta(T), \quad k = 1,\dots,m, \\
&\langle\ \operatorname{div}\widetilde{\boldsymbol{u}}_m^\varepsilon(t), r_l\rangle + \varepsilon\langle \partial_t \widetilde{p}_m^\varepsilon(t), r_l\rangle \\
&= \langle p_{0m}, r_l\rangle\delta(0) - \langle p_m^\varepsilon(T), r_l\rangle\delta(T), \qquad l = 1,\dots,m,
\end{aligned}$$

where $\delta(0)$ and $\delta(T)$ are respectively the Dirac distributions at 0 and T. Next, by taking the Fourier transforms, we get

$$\begin{aligned}
&2\mathrm{i}\pi\tau\langle \widehat{\boldsymbol{u}}_m^\varepsilon(\tau), \boldsymbol{w}_k\rangle - \langle \widehat{p}_m^\varepsilon(\tau),\ \operatorname{div}\boldsymbol{w}_k\rangle \\
&= \langle \widehat{\phi}_m(\tau), \boldsymbol{w}_k\rangle + \langle \boldsymbol{u}_{0m}, \boldsymbol{w}_k\rangle - \langle \boldsymbol{u}_m^\varepsilon(T), \boldsymbol{w}_k\rangle \exp(-2\mathrm{i}\pi\tau T), k = 1,\dots,m, \\
&\langle\ \operatorname{div}\widehat{\boldsymbol{u}}_m^\varepsilon(\tau), r_l\rangle + 2\mathrm{i}\pi\tau\varepsilon\langle \widehat{p}_m^\varepsilon(\tau), r_l\rangle \\
&= \varepsilon\langle p_{0m}, r_l\rangle - \varepsilon\langle p_m^\varepsilon(T), r_l\rangle \exp(-2\mathrm{i}\pi\tau T), \qquad l = 1,\dots,m.
\end{aligned}$$

Let $\widehat{g}_{km}(\tau)$ and $\widehat{\xi}_{lm}(\tau)$ be respectively the Fourier transform of $g_{km}(t)$ and $\xi_{lm}(t)$. Now, we multiply both last equations by $\widehat{g}_{km}(\tau)$ (the former) and the $\widehat{\xi}_{lm}(\tau)$ (the latter) and add the results for $k = 1,\dots,m$ and $l = 1,\dots,m$. The result of this calculation reads

$$\begin{aligned}
&2\mathrm{i}\pi\tau\Big(\|\widehat{\boldsymbol{u}}_m^\varepsilon(\tau)\|^2 + \varepsilon\|\widehat{p}_m^\varepsilon(\tau)\|^2\Big) \\
&= \langle \widehat{\phi}_m(\tau), \widehat{\boldsymbol{u}}_m^\varepsilon(\tau)\rangle + \langle \boldsymbol{u}_{0m}, \widehat{\boldsymbol{u}}_m^\varepsilon(\tau)\rangle + \varepsilon\langle p_{0m}, \widehat{p}_m^\varepsilon(\tau)\rangle \\
&- \Big(\langle \boldsymbol{u}_m^\varepsilon(T), \widehat{\boldsymbol{u}}_m^\varepsilon(\tau)\rangle + \varepsilon\langle p_m^\varepsilon(T), \widehat{p}_m^\varepsilon(\tau)\rangle\Big) \exp(-2\mathrm{i}\pi\tau T).
\end{aligned}$$

Using estimate (5.9) we get

$$\begin{aligned}&2\pi|\tau|\Big(\|\widehat{\boldsymbol{u}}_m^\varepsilon(\tau)\|^2+\varepsilon\|\widehat{p}_m^\varepsilon(\tau)\|^2\Big)\\&\le\|\widehat{\phi}_m(\tau)\|_{\boldsymbol{H}^{-1}(\Omega)}\|\widehat{\boldsymbol{u}}_m^\varepsilon(\tau)\|+2\sqrt{d_1}\|\widehat{\boldsymbol{u}}_m^\varepsilon(\tau)\|+2\sqrt{d_1}\varepsilon\|\widehat{p}_m^\varepsilon(\tau)\|.\end{aligned}$$

But

$$\|\widehat{\phi}_m(\tau)\|_{\boldsymbol{H}^{-1}(\Omega)}\le\int_0^T\|\phi_m(t)\|_{\boldsymbol{H}^{-1}(\Omega)}\,\mathrm{d}t. \tag{5.16}$$

Using estimates ((5.10)-(5.13)) and applying the compactness result of [16, Theorem 2.2 page 274], we can show that

$$\int_{-\infty}^{+\infty}|\tau|^{2\theta}\|\widehat{\boldsymbol{u}}_m^\varepsilon(\tau)\|^2d\tau\le\text{Const, for some }\theta>0. \tag{5.17}$$

c) *Passage to the limit:* We need to pass to the limit when $m\to\infty$ and $\varepsilon\to 0$. A t this step, by fixing ε, we are only concerned with the passage to the limit as $m\to\infty$. Based on ((5.9)-(5.12)), it is clear that there exist subsequences still denoted $\boldsymbol{u}_m^\varepsilon$ and p_m^ε satisfying

$$\boldsymbol{u}_m^\varepsilon\longrightarrow\boldsymbol{u}^\varepsilon\quad\text{in }L^2(0,T;\boldsymbol{H}_0^1(\Omega))\quad\text{weakly,} \tag{5.18}$$

$$\boldsymbol{u}_m^\varepsilon\longrightarrow\boldsymbol{u}^\varepsilon\quad\text{in }L^\infty(0,T;\boldsymbol{L}^2(\Omega))\quad\text{weak-star,} \tag{5.19}$$

$$p_m^\varepsilon\longrightarrow p^\varepsilon\quad\text{in }L^\infty(0,T;L^2(\Omega))\quad\text{weak-star.} \tag{5.20}$$

$$\boldsymbol{u}_m^\varepsilon\longrightarrow\boldsymbol{u}^\varepsilon\quad\text{in }L^{\alpha+2}(0,T;\boldsymbol{L}^{\alpha+2}(\Omega))\quad\text{weakly.} \tag{5.21}$$

Moreover, due to (5.18), (5.17) and [16, Theorem 2.3 page 276], we have

$$\boldsymbol{u}_m^\varepsilon\quad\longrightarrow\quad\boldsymbol{u}^\varepsilon\quad\text{in }L^2(0,T;\boldsymbol{L}^2(\Omega))\quad\text{strongly.} \tag{5.22}$$

In order to prove that $(\boldsymbol{u}^\varepsilon,p^\varepsilon)$ satisfies ((5.2)-(5.3)), we shall use the same arguments of [16]. Multiplying both sides of ((5.5)-(5.6)) by $\psi(t)\in\mathcal{C}^\infty(0,T)$ such that $\psi(T)=0$ and integrating over $(0,T)$, we get

$$\begin{aligned}&-\int_0^T\langle\boldsymbol{u}_m^\varepsilon(t),\boldsymbol{w}_k\rangle\psi'(t)dt+\gamma\int_0^T\langle\nabla\boldsymbol{u}_m^\varepsilon(t),\psi(t)\nabla\boldsymbol{w}_k\rangle dt\\&+a\int_0^T\langle\boldsymbol{u}_m^\varepsilon(t),\boldsymbol{w}_k\psi(t)\rangle dt+b\int_0^T\langle|\boldsymbol{u}_m^\varepsilon(t)|^\alpha\boldsymbol{u}_m^\varepsilon(t),\boldsymbol{w}_k\psi(t)\rangle dt\\&-\int_0^T\langle\nabla p_m^\varepsilon(t),\boldsymbol{w}_k\psi(t)\rangle dt=\langle\boldsymbol{u}_{0m},\boldsymbol{w}_k\rangle\psi(0)+\int_0^T\langle\boldsymbol{f}(t),\boldsymbol{w}_k\psi(t)\rangle dt,\end{aligned} \tag{5.23}$$

$$\int_0^T\langle\,\mathrm{div}\boldsymbol{u}_m^\varepsilon(t),r_l\psi(t)\rangle dt=\varepsilon\langle p_{0m},r_l\rangle\psi(0)+\int_0^T\varepsilon\langle p_m^\varepsilon(t),r_l\rangle\psi'(t)dt. \tag{5.24}$$

Using the previous convergence properties, it is easy to pass to the limit $m \to \infty$ in the linear terms. For the nonlinear term, we apply the following Lemma that we postpone the proof to the end of this paragraph.

Lemma 5.2. *We have the following property as* $m \to \infty$

$$\int_0^T \langle |\boldsymbol{u}_m^\varepsilon(t)|^\alpha \boldsymbol{u}_m^\varepsilon(t), \boldsymbol{w}_k \psi(t)\rangle dt \longrightarrow \int_0^T \langle |\boldsymbol{u}^\varepsilon(t)|^\alpha \boldsymbol{u}^\varepsilon(t), \boldsymbol{w}_k \psi(t)\rangle dt. \quad (5.25)$$

Using lemma (5.2), we can pass to the limit in ((5.23)-(5.24)) and obtain for $k = 1, \dots, m$ and $l = 1, \dots, m$

$$\begin{aligned}
&-\int_0^T \langle \boldsymbol{u}^\varepsilon(t), \boldsymbol{w}_k\rangle \psi'(t) dt + \gamma \int_0^T \langle \nabla \boldsymbol{u}^\varepsilon(t), \psi(t) \nabla \boldsymbol{w}_k\rangle dt \\
&+ a \int_0^T \langle \boldsymbol{u}^\varepsilon(t), \boldsymbol{w}_k \psi(t)\rangle dt \\
&+ b \int_0^T \langle |\boldsymbol{u}^\varepsilon(t)|^\alpha \boldsymbol{u}^\varepsilon(t), \boldsymbol{w}_k \psi(t)\rangle dt - \int_0^T \langle \nabla p^\varepsilon(t), \boldsymbol{w}_k \psi(t)\rangle dt \\
&= \langle \boldsymbol{u}_0, \boldsymbol{w}_k\rangle \psi(0) + \int_0^T \langle \boldsymbol{f}(t), \boldsymbol{w}_k \psi(t)\rangle dt, \qquad (5.26)
\end{aligned}$$

$$\int_0^T \langle \operatorname{div} \boldsymbol{u}^\varepsilon(t), r_l \psi(t)\rangle dt = \varepsilon \langle p_0, r_l\rangle \psi(0) + \int_0^T \varepsilon \langle p^\varepsilon(t), r_l\rangle \psi'(t) dt. \quad (5.27)$$

Now, observe that the nonlinear term is continuous with respect to $\boldsymbol{w}_k$. Indeed, for any $\boldsymbol{w}_k \in \boldsymbol{H}_0^1(\Omega)$, we have

$$\begin{aligned}
\Big| \int_0^T \langle |\boldsymbol{u}^\varepsilon(t)|^\alpha \boldsymbol{u}^\varepsilon(t), \boldsymbol{w}_k \psi(t)\rangle dt \Big| &\le \sup_{t\in[0,T]} |\psi(t)| \int_0^T \int_\Omega |\boldsymbol{u}^\varepsilon(t)|^{\alpha+1} |\boldsymbol{w}_k| dx\, dt, \\
&\le C \|\boldsymbol{w}_k\|_{\boldsymbol{L}^6(\Omega)} \int_0^T \Big(\int_\Omega |\boldsymbol{u}^\varepsilon(t)|^{\frac{6(\alpha+1)}{5}} dx \Big) dt.
\end{aligned}$$

Thanks to the embeddings $\boldsymbol{L}^{\alpha+2}(\Omega) \hookrightarrow \boldsymbol{L}^{\frac{6(\alpha+1)}{5}}(\Omega)$ with $\alpha \in [1,2]$ and $\boldsymbol{H}_0^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, we get

$$\begin{aligned}
\Big| \int_0^T \langle |\boldsymbol{u}^\varepsilon(t)|^\alpha \boldsymbol{u}^\varepsilon(t), \boldsymbol{w}_k \psi(t)\rangle dt \Big| &\le C \|\boldsymbol{u}^\varepsilon\|^{\alpha+1}_{L^{\alpha+2}(0,T;\boldsymbol{L}^{\alpha+2}(\Omega))} \|\boldsymbol{w}_k\|_{\boldsymbol{L}^6(\Omega)}, \\
&\le C \|\boldsymbol{w}_k\|_{\boldsymbol{H}_0^1(\Omega)}. \qquad (5.28)
\end{aligned}$$

By linearity, the equation (5.26) still holds for any $\boldsymbol{w}$ that is a linear combination of functions $\boldsymbol{w}_k$ and by continuity argument this equation is still true for any $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$. Similarly, the equation (5.27) still holds true for any $r \in L^2(\Omega)$. Then, we can write

$$- \int_0^T \langle \boldsymbol{u}^\varepsilon(t), \boldsymbol{w}\rangle \psi'(t) dt + \gamma \int_0^T \langle \nabla \boldsymbol{u}^\varepsilon(t), \psi(t) \nabla \boldsymbol{w}\rangle dt + a \int_0^T \langle \boldsymbol{u}^\varepsilon(t), \boldsymbol{w}\psi(t)\rangle dt$$
$$+ b \int_0^T \langle |\boldsymbol{u}^\varepsilon(t)|^\alpha \boldsymbol{u}^\varepsilon(t), \boldsymbol{w}\psi(t)\rangle dt - \int_0^T \langle \nabla p^\varepsilon(t), \boldsymbol{w}\psi(t)\rangle dt$$
$$= \langle \boldsymbol{u}_0, \boldsymbol{w}\rangle \psi(0) + \int_0^T \langle \boldsymbol{f}(t), \boldsymbol{w}\psi(t)\rangle dt, \tag{5.29}$$

$$\int_0^T \langle \operatorname{div} \boldsymbol{u}^\varepsilon(t), r\psi(t)\rangle dt = \varepsilon \langle p_0, r\rangle \psi(0) + \int_0^T \varepsilon \langle p^\varepsilon(t), r\rangle \psi'(t) dt. \tag{5.30}$$

In particular, choosing in ((5.29)-(5.30)) $\psi = \varphi \in \mathcal{D}(0,T)$, we see that $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ satisfies ((5.2)-(5.3)) in the sense of distributions.

It remains to prove that $\boldsymbol{u}^\varepsilon$ and p^ε satisfy the initial conditions in (5.1). For this purpose, we take again, $\psi \in \mathcal{C}^\infty(0,T)$ with $\psi(T) = 0$, multiply ((5.2)-(5.3)) by ψ and integrate over $(0,T)$. This yields

$$- \int_0^T \langle \boldsymbol{u}^\varepsilon(t), \boldsymbol{w}\rangle \psi'(t) dt + \gamma \int_0^T \langle \nabla \boldsymbol{u}^\varepsilon(t), \psi(t) \nabla \boldsymbol{w}\rangle dt + a \int_0^T \langle \boldsymbol{u}^\varepsilon(t), \boldsymbol{w}\psi(t)\rangle dt$$
$$+ b \int_0^T \langle |\boldsymbol{u}^\varepsilon(t)|^\alpha \boldsymbol{u}^\varepsilon(t), \boldsymbol{w}\psi(t)\rangle dt - \int_0^T \langle \nabla p^\varepsilon(t), \boldsymbol{w}\psi(t)\rangle dt$$
$$= \langle \boldsymbol{u}^\varepsilon(0), \boldsymbol{w}\rangle \psi(0) + \int_0^T \langle \boldsymbol{f}(t), \boldsymbol{w}\psi(t)\rangle dt, \tag{5.31}$$

$$\int_0^T \langle \operatorname{div} \boldsymbol{u}^\varepsilon(t), r\psi(t)\rangle dt = \varepsilon \langle p^\varepsilon(0), r\rangle \psi(0) + \int_0^T \varepsilon \langle p^\varepsilon(t), r\rangle \psi'(t) dt. \tag{5.32}$$

If we compare (5.29) with (5.31) and (5.30) with (5.32), we observe that

$$\begin{aligned} \forall \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega), \quad \langle \boldsymbol{u}^\varepsilon(0) - \boldsymbol{u}_0, \boldsymbol{w}\rangle \psi(0) &= 0, \\ \forall r \in L^2(\Omega), \quad \langle p^\varepsilon(0) - p_0, r\rangle \psi(0) &= 0. \end{aligned}$$

Choosing $\psi(0) = 1$ shows that the conditions $\boldsymbol{u}^\varepsilon(0) = \boldsymbol{u}_0$ and $p^\varepsilon(0) = p_0$ are verified. □

The proof of the previous Theorem used the property presented in Lemma (5.2) and we prove it here.

Proof of Lemma (5.2)**.** We write the following:

$$\Big| \int_0^T \langle |\boldsymbol{u}_m^\varepsilon(t)|^\alpha \boldsymbol{u}_m^\varepsilon(t), \boldsymbol{w}_k \psi(t)\rangle dt - \int_0^T \langle |\boldsymbol{u}^\varepsilon(t)|^\alpha \boldsymbol{u}^\varepsilon(t), \boldsymbol{w}_k \psi(t)\rangle dt \Big|$$

$$
\begin{aligned}
&\leq \int_0^T \Big| |\boldsymbol{u}_m^\varepsilon(t)|^\alpha \boldsymbol{u}_m^\varepsilon(t) - |\boldsymbol{u}^\varepsilon(t)|^\alpha \boldsymbol{u}^\varepsilon(t) \Big| |\boldsymbol{w}_k| |\psi(t)|, \\
&\leq (\alpha+1) \sup_{x\in\Omega} |\boldsymbol{w}_k(x)| \sup_{t\in[0,T]} |\psi(t)| \\
&\quad \times \int_0^T \int_\Omega \Big| \boldsymbol{u}_m^\varepsilon(t) - \boldsymbol{u}^\varepsilon(t) \Big| \Big(|\boldsymbol{u}_m^\varepsilon(t)|^\alpha + |\boldsymbol{u}^\varepsilon(t)|^\alpha \Big) dxdt, \\
&\leq (\alpha+1) \sup_{x\in\Omega} |\boldsymbol{w}_k(x)| \sup_{t\in[0,T]} |\psi(t)| \\
&\quad \times \int_0^T \|\boldsymbol{u}_m^\varepsilon(t) - \boldsymbol{u}^\varepsilon(t)\| \Big(\|\boldsymbol{u}_m^\varepsilon(t)\|_{2\alpha}^\alpha + \|\boldsymbol{u}^\varepsilon(t)\|_{2\alpha}^\alpha \Big) dt, \\
&\leq C \|\boldsymbol{u}_m^\varepsilon(t) - \boldsymbol{u}^\varepsilon(t)\|_{L^2(0,T,\boldsymbol{L}^2(\Omega))} \\
&\quad \times \Big(\|\boldsymbol{u}_m^\varepsilon(t)\|_{L^{\alpha+2}(0,T,\boldsymbol{L}^{\alpha+2}(\Omega))}^\alpha + \|\boldsymbol{u}^\varepsilon(t)\|_{L^{\alpha+2}(0,T,\boldsymbol{L}^{\alpha+2}(\Omega))}^\alpha \Big), \\
&\leq C \|\boldsymbol{u}_m^\varepsilon(t) - \boldsymbol{u}^\varepsilon(t)\|_{L^2(0,T,\boldsymbol{L}^2(\Omega))}.
\end{aligned}
$$

The last term converges obviously to 0 as $m \to \infty$, this finishes the proof. □

5.2. **The solutions of perturbed problem converge to solution of BF problem.** The aim of this subsection is to consider the limit as $\epsilon \to 0$. It is useful to establish some *a priori* estimates for $\boldsymbol{u}^\epsilon$ and p^ϵ, independent of ϵ. Since we are interested in $\epsilon \to 0$, we can always suppose that $\epsilon \leq 1$. It follows from ((5.9), (5.10), (5.18)-(5.20)) and the lower semi-continuity of the norm that for any $\epsilon \in (0,1]$, it holds

$$\|\boldsymbol{u}^\epsilon\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \leq \liminf_{m\to\infty} \|\boldsymbol{u}_m^\epsilon\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \leq \sqrt{d_1}, \tag{5.33}$$

$$\|\boldsymbol{u}^\epsilon\|_{L^2(0,T;\boldsymbol{H}_0^1(\Omega))} \leq \liminf_{m\to\infty} \|\boldsymbol{u}_m^\epsilon\|_{L^2(0,T;\boldsymbol{H}_0^1(\Omega))} \leq \sqrt{\frac{d_1}{2\gamma}}, \tag{5.34}$$

$$\sqrt{\varepsilon}\|p^\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq \sqrt{\varepsilon} \liminf_{m\to\infty} \|p_m^\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq \sqrt{d_1}. \tag{5.35}$$

Eventually, from (5.17) it holds for $0 < \theta < 1/4$ and for any $\epsilon \in (0,\, 1]$

$$\int_{-\infty}^{+\infty} |\tau|^{2\theta} \|\widehat{\boldsymbol{u}}^\varepsilon(\tau)\|^2 d\tau \leq \liminf_{m\to\infty} \int_{-\infty}^{+\infty} |\tau|^{2\theta} \|\widehat{\boldsymbol{u}}_m^\varepsilon(\tau)\|^2 d\tau \leq \text{ Const}, \tag{5.36}$$

where the constant is independent of ϵ.

Theorem 5.3. *Let* $(\boldsymbol{u}^\epsilon,\, p^\epsilon)$ *be the solution of problem* (5.1)*. Then, there exists a subsequence still denoted by* $(\boldsymbol{u}^\epsilon,\, p^\epsilon)$ *so that*

$$\boldsymbol{u}^\varepsilon \longrightarrow \boldsymbol{u} \quad \textit{in } L^2(0,T;\boldsymbol{H}_0^1(\Omega)) \quad \textit{weakly}, \tag{5.37}$$

$$\boldsymbol{u}^\varepsilon \longrightarrow \boldsymbol{u} \quad \textit{in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \quad \textit{weak-star}, \tag{5.38}$$

$$\boldsymbol{u}^{\varepsilon} \longrightarrow \boldsymbol{u} \quad \textit{in } L^2(0,T;\boldsymbol{L}^2(\Omega)) \quad \textit{strongly}, \tag{5.39}$$

$$p^{\varepsilon} \longrightarrow p \quad \textit{in } L^\infty(0,T;L^2(\Omega)) \quad \textit{weak-star}, \tag{5.40}$$

where $(\boldsymbol{u}, p)$ *is the solution of the (BF) equations* (1.1).

Proof. By virtue of ((5.33)-(5.36)), there exists a subsequence still denoted by $(\boldsymbol{u}^\epsilon, p^\epsilon)$ such that

$$\boldsymbol{u}^{\varepsilon} \longrightarrow \boldsymbol{u}^* \quad \text{in } L^2(0,T;\boldsymbol{H}_0^1(\Omega)) \quad \text{weakly}, \tag{5.41}$$

$$\boldsymbol{u}^{\varepsilon} \longrightarrow \boldsymbol{u}^* \quad \text{in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \quad \text{weak-star}, \tag{5.42}$$

$$\boldsymbol{u}^{\varepsilon} \longrightarrow \boldsymbol{u}^* \quad \text{in } L^2(0,T;\boldsymbol{L}^2(\Omega)) \quad \text{strongly}, \tag{5.43}$$

$$\sqrt{\epsilon}\, p^{\varepsilon} \longrightarrow \chi \quad \text{in } L^\infty(0,T;L^2(\Omega)) \quad \text{weak-star}. \tag{5.44}$$

Passing to the limit $\epsilon \to 0$ for the subsequence in (5.3), we obtain in the sense of distributions $\sqrt{\epsilon}\langle \partial_t p^\epsilon(t), q\rangle \longrightarrow \langle \partial_t \chi, q\rangle$, and then $\epsilon\langle \partial_t p^\epsilon(t), q\rangle \longrightarrow 0$. This limit with (5.3) gives

$$\langle \operatorname{div}\boldsymbol{u}^*(t), q\rangle = 0, \quad \forall q \in L^2(\Omega),$$

which in turn implies that $\boldsymbol{u}^*$ is divergence free, *i.e.,* $\operatorname{div}\boldsymbol{u}^* = 0$. Therefore, $\boldsymbol{u}^* \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H})$. In order to show that $\boldsymbol{u}^*$ verifies the variational formulation (2.1), we take $\boldsymbol{v} \in \boldsymbol{V}$ in (5.2) and we multiply both sides of this last equation by $\psi \in \mathcal{D}(0,T)$ and we integrate over $(0,T)$, we get

$$- \int_0^T \langle \boldsymbol{u}^\varepsilon(t), \boldsymbol{v}\rangle \psi'(t)dt + \gamma \int_0^T \langle \nabla \boldsymbol{u}^\varepsilon(t), \psi(t)\nabla \boldsymbol{v}\rangle dt + a \int_0^T \langle \boldsymbol{u}^\varepsilon(t), \boldsymbol{v}\psi(t)\rangle dt$$
$$+ b \int_0^T \langle |\boldsymbol{u}^\varepsilon(t)|^\alpha \boldsymbol{u}^\varepsilon(t), \boldsymbol{v}\psi(t)\rangle dt = \int_0^T \langle \boldsymbol{f}(t), \boldsymbol{v}\psi(t)\rangle dt. \tag{5.45}$$

Using the convergence properties ((5.41)-(5.42)), we can easily pass to the limit $\varepsilon \to 0$ in the linear terms in (5.45). Next, we use the same arguments of the proof of (5.25) to check that

$$\int_0^T \langle |\boldsymbol{u}^\varepsilon(t)|^\alpha \boldsymbol{u}^\varepsilon(t), \boldsymbol{v}\psi(t)\rangle dt \longrightarrow \int_0^T \langle |\boldsymbol{u}^*(t)|^\alpha \boldsymbol{u}^*(t), \boldsymbol{v}\psi(t)\rangle dt. \tag{5.46}$$

We obtain in the limit

$$- \int_0^T \langle \boldsymbol{u}^*(t), \boldsymbol{v}\rangle \psi'(t)dt + \gamma \int_0^T \langle \nabla \boldsymbol{u}^*(t), \psi(t)\nabla \boldsymbol{v}\rangle dt + a \int_0^T \langle \boldsymbol{u}^*(t), \boldsymbol{v}\psi(t)\rangle dt$$
$$+ b \int_0^T \langle |\boldsymbol{u}^*(t)|^\alpha \boldsymbol{u}^*(t), \boldsymbol{v}\psi(t)\rangle dt = \int_0^T \langle \boldsymbol{f}(t), \boldsymbol{v}\psi(t)\rangle dt, \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{5.47}$$

The solution of problem (5.47) satisfies then (2.1) in the sense of distributions. It follows that in order to show that $\boldsymbol{u}^* = \boldsymbol{u}$ and $p^\varepsilon = p$, it remains only to prove $\boldsymbol{u}^*(0) = \boldsymbol{u}_0$ and $p^*(0) = p_0$. For this one proceeds exactly as in the proof of the initial conditions of problem (5.1). □

6. NUMERICAL EXPERIMENTS

In this section, we carry out numerical experiments for the perturbed Brinkman-Forchheimer equations (5.1) but with non-homogeneous boundary conditions

$$\begin{array}{l} \partial_t \boldsymbol{u}_\epsilon + \gamma \Delta \boldsymbol{u}_\epsilon + a\boldsymbol{u}_\epsilon + b\,|\boldsymbol{u}_\epsilon|^\alpha\, \boldsymbol{u}_\epsilon + \nabla p_\epsilon = \boldsymbol{f}, \quad \text{ in } \ \Omega_T, \\ \nabla \cdot \boldsymbol{u}_\epsilon + \epsilon \partial_t p_\epsilon = 0, \quad \text{ in } \ \Omega_T, \\ \boldsymbol{u}_\epsilon = \boldsymbol{u}_B, \quad \text{ on } \ \Sigma_T, \\ \boldsymbol{u}_\epsilon(0) = \boldsymbol{u}_0, \quad \text{ in } \ \Omega, \\ p_\epsilon(0) = p_0, \quad \text{ in } \ \Omega, \end{array} \tag{6.1}$$

where $\boldsymbol{u}_B$ is an imposed velocity on the boundary and is assumed to be time-independent below for convenience.

Again, we first convert the perturbed Brinkman-Forchheimer equations (6.1) into a variational formulation: Find $\boldsymbol{u}_\epsilon \in E_{\Omega_T}(\boldsymbol{u}_B) + L^2(0,T,\boldsymbol{H}_0^1(\Omega))$ and $p_\epsilon \in L^2(0,T,Q)$ such that

$$\begin{aligned} \frac{d}{dt}\left\langle \boldsymbol{u}_\epsilon(t), \boldsymbol{v}\right\rangle + \gamma \left\langle \nabla \boldsymbol{u}_\epsilon(t), \nabla \boldsymbol{v}\right\rangle + a\left\langle \boldsymbol{u}_\epsilon(t), \boldsymbol{v}\right\rangle & \\ +b\left\langle |\boldsymbol{u}_\epsilon(t)|^\alpha\, \boldsymbol{u}_\epsilon(t), \boldsymbol{v}\right\rangle - \left\langle p_\epsilon(t), \nabla\cdot \boldsymbol{v}\right\rangle &= \left\langle \boldsymbol{f}(t), \boldsymbol{v}\right\rangle, \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \end{aligned}$$

$$\begin{aligned} \left\langle \nabla \cdot \boldsymbol{u}_\epsilon(t), q\right\rangle + \epsilon \left\langle \partial_t p_\epsilon(t), q\right\rangle = 0, \quad \forall q \in Q, \\ \left\langle \boldsymbol{u}_\epsilon(0), \boldsymbol{v}\right\rangle = \left\langle \boldsymbol{u}_0, \boldsymbol{v}\right\rangle, \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \\ \left\langle p_\epsilon(0), q\right\rangle = \left\langle p_0, q\right\rangle, \quad \forall q \in Q, \end{aligned}$$

where $Q = L^2(\Omega)$ and $E_{\Omega_T}(\boldsymbol{u}_B)$ is the extension of $\boldsymbol{u}_B$ from Σ_T to Ω_T.

To obtain numerical (approximated) solution of the above equations, we first apply semi-implicit Euler time step to integrate with time: Find $\boldsymbol{u}_\epsilon^{k+1} \in E_\Omega(\boldsymbol{u}_B) + \boldsymbol{H}_0^1(\Omega)$ and $p_\epsilon^{k+1} \in Q$, $k = 0, 1, 2, \cdots$, such that

$$\begin{aligned} \frac{\langle \boldsymbol{u}_\epsilon^{k+1}, \boldsymbol{v}\rangle - \langle \boldsymbol{u}_\epsilon^{k}, \boldsymbol{v}\rangle}{t_{k+1} - t_k} + \gamma \langle \nabla \boldsymbol{u}_\epsilon^{k+1}, \nabla \boldsymbol{v}\rangle + a\langle \boldsymbol{u}_\epsilon^{k+1}, \boldsymbol{v}\rangle & \\ +b\langle |\boldsymbol{u}_\epsilon^{k}|^\alpha \boldsymbol{u}_\epsilon^{k+1}, \boldsymbol{v}\rangle - \langle p_\epsilon^{k+1}, \nabla\cdot \boldsymbol{v}\rangle = \langle \boldsymbol{f}(t_{k+1}), \boldsymbol{v}\rangle, \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), & \end{aligned}$$

$$\langle \nabla \cdot \boldsymbol{u}_\epsilon^{k+1}, q\rangle + \epsilon \frac{\left\langle p_\epsilon^{k+1}, q\right\rangle - \left\langle p_\epsilon^{k}, q\right\rangle}{t_{k+1} - t_k} = 0, \quad \forall q \in Q,$$

$$\left\langle \boldsymbol{u}_\epsilon^0, \boldsymbol{v}\right\rangle = \left\langle \boldsymbol{u}_0, \boldsymbol{v}\right\rangle, \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \quad \left\langle p_\epsilon^0, q\right\rangle = \left\langle p_0, q\right\rangle, \quad \forall q \in Q,$$

where $E_\Omega(\boldsymbol{u}_B)$ is the extension of $\boldsymbol{u}_B$ from Σ to Ω.

We now carry out spatial discretization. We assume a rectangular mesh and we adopt the lowest-order Raviart-Thomas elements for our finite element spaces $\mathbf{V}_h \subset H(\Omega;\ \mathrm{div})$ and $Q_h \subset Q$. That is, we let the approximating subspaces $\mathbf{V}_h \times Q_h$ of $H(\Omega;\ \mathrm{div}) \times Q$ be the zeroth order Raviart-Thomas space (RT_0) of the rectangular partition. For example, for two-dimensional domain Ω , it is defined as

$$\begin{aligned} \mathbf{V}_h &= \{\boldsymbol{v} \in H(\Omega;\ \mathrm{div}) : \boldsymbol{v}|_R \in Q_{1,0}(R) \times Q_{0,1}(R),\ R \in \mathcal{T}_h\}, \\ Q_h &= \left\{w \in L^2(\Omega) :\ w|_R \in Q_{0,0}(R),\ R \in \mathcal{T}_h\right\}, \end{aligned}$$

where, we denote by $Q_{i,j}(R)$ the space of polynomials of degree less than or equal to i (or j) in the first (or second) variable restricted to R.

If taking $\boldsymbol{u}_\epsilon^{k+1} \in \mathbf{V}_h$ and $\boldsymbol{v} \in \mathbf{V}_h$, all terms in the above equations make sense except the term $\gamma \left\langle \nabla \boldsymbol{u}_\epsilon^{k+1}, \nabla \boldsymbol{v} \right\rangle$ because $\mathbf{V}_h$ is not a subspace of $\boldsymbol{H}_0^1(\Omega)$. To overcome this difficulty, we apply an interpolation $\mathcal{I}_{C,h}$ from $\mathbf{V}_h$ to the continuous piecewise linear space $\mathbf{V}_h^C$. Now, the fully discrete scheme reads: Find $\boldsymbol{u}_{\epsilon,h}^{k+1} \in E_\Omega(\boldsymbol{u}_B) + \mathbf{V}_h$ and $p_{\epsilon,h}^{k+1} \in Q_h$, $k = 0, 1, 2, \cdots$, such that

$$\begin{aligned} &\frac{\langle \boldsymbol{u}_{\epsilon,h}^{k+1}, \boldsymbol{v}\rangle - \langle \boldsymbol{u}_{\epsilon,h}^{k}, \boldsymbol{v}\rangle}{t_{k+1} - t_k} + \gamma \langle \nabla \mathcal{I}_{C,h} \boldsymbol{u}_{\epsilon,h}^{k+1}, \nabla \mathcal{I}_{C,h} \boldsymbol{v}\rangle + a \langle \boldsymbol{u}_{\epsilon,h}^{k+1}, \boldsymbol{v}\rangle \\ &+ b\langle |\boldsymbol{u}_{\epsilon,h}^{k}|^\alpha \boldsymbol{u}_{\epsilon,h}^{k+1}, \boldsymbol{v}\rangle - \langle p_{\epsilon,h}^{k+1}, \nabla \cdot \boldsymbol{v}\rangle = \langle \boldsymbol{f}(t_{k+1}), \boldsymbol{v}\rangle, \quad \forall \boldsymbol{v} \in \mathbf{V}_h, \end{aligned}$$

$$\begin{aligned} \langle \nabla \cdot \boldsymbol{u}_{\epsilon,h}^{k+1}, q\rangle + \epsilon \frac{\langle p_{\epsilon,h}^{k+1}, q\rangle - \langle p_{\epsilon,h}^{k}, q\rangle}{t_{k+1} - t_k} &= 0, \quad \forall q \in Q_h, \\ \langle \boldsymbol{u}_{\epsilon,h}^{0}, \boldsymbol{v}\rangle &= \langle \boldsymbol{u}_0, \boldsymbol{v}\rangle, \quad \forall \mathbf{v} \in \mathbf{V}_h, \\ \langle p_{\epsilon,h}^{0}, q\rangle &= \langle p_0, q\rangle, \quad \forall q \in Q_h. \end{aligned}$$

We implement the above finite element scheme in MATLAB. In each time step, a linear algebraic system is solved. We run a large number of time steps to ensure that we reach the steady state solution.

Figures (1)(a–f) are numerical results of the lid-driven cavity flow (a widely-used benchmark case for testing Navier-Stokes flow) applied to our Brinkman-Forchheimer equation system. In all example, we set $\epsilon = 0.01$, $a = 1$ and $\alpha = 1$, and we change the parameters γ and b to study the influence of the parameters. It is clear from the plots that the viscosity γ has a strong influence on the velocity profiles; in particular, when viscosity is increased, the dragging movement on the top of the domain (i.e. on the lid) has a stronger affect to the velocity in deeper part of the domain. The Forchheimer parameter b does not have that significant impact as the viscosity does though in this example.

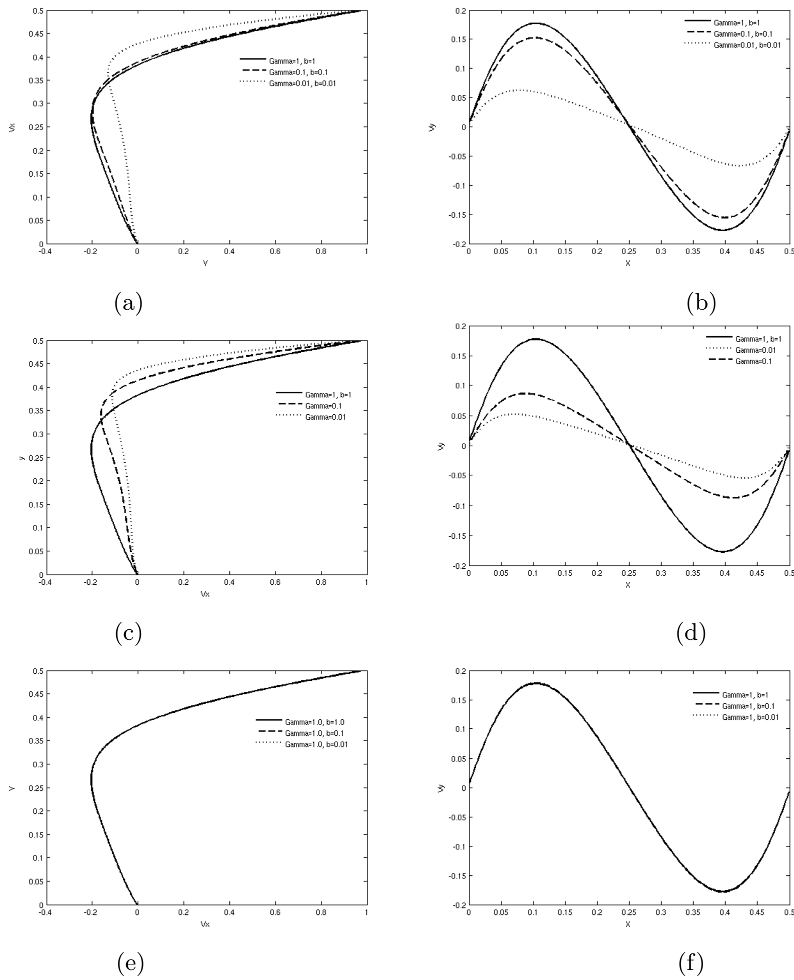

Figure 1. Computed velocity profiles through the cavity center (a) u component along the vertical line through the cavity center for various parameters gamma and b; (b) v component along the horizontal line through the cavity center for various parameters gamma and b; (c) u component along the vertical line through the cavity center for various parameters gamma; (d) v component along the horizontal line through the cavity center for various parameters gamma; (e) u component along the vertical line through the cavity center for various parameters b; (f) v component along the horizontal line through the cavity center for various parameters b.

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