A PRICE MODEL WITH FINITELY MANY AGENTS

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Resumo: Neste trabalho, estudamos um modelo de formação de preços numa população com um número finito de agentes que compram e vendem uma mercadoria. A oferta desta mercadoria é exógena e os agentes são racionais uma vez que pretendem minimizar os custos de transacção. O problema em estudo é formulado como um jogo dinâmico entre N jogadores com uma condição de equilíbrio de mercado. O limite deste problema de N jogadores é um "mean field game". Posteriormente, mostramos como reformular o nosso jogo como um problema de optimização do custo total. Mostramos a existência de uma solução usando o método direto do cálculo das variações. Por fim, mostramos que o preço é o multiplicador de Lagrange para a condição de equilíbrio entre a oferta e a procura.

Abstract Here, we propose a price-formation model, with a population consisting of a finite number of agents storing and trading a commodity. The supply of this commodity is determined exogenously, and the agents are rational as they seek to minimize their trading costs. We formulate our problem as an N-player dynamic game with a market-clearing condition. The limit of this N-player problem is a mean-field game (MFG). Subsequently, we show how to recast our game as an optimization problem for the overall trading cost. We show the existence of a solution using the direct method in the calculus of variations. Finally, we show that the price is the Lagrange multiplier for the balance condition between supply and demand.

palavras-chave: Formação de preço, jogos dinâmicos, equilíbrio de mercado.

keywords: Price formation model, dynamic games, market equilibrium

1 Prologue

This document is the result of the second KAUST Summer Camp in Applied Partial Differential Equations that took place from August 25 to September 8 of 2019. The purpose of this summer camp is to give an intense hands-on research experience in cutting edge topics to BS/BSc and MS students. Participants attended mini-courses that provide them with the tools to reach the results we present here. For the research project, the participants worked in small groups. These were coordinated by Professor Diogo Gomes, together with his Ph.D. Students and Postdocs and Research Scientist Rita Ferreira.

Participants also had the opportunity to get acquainted with a variety of research topics pursued by KAUST scholars as a means of broadening their mathematical perspectives and future opportunities at KAUST. On the weekends, there were cultural activities, such as sightseeing in the UNESCO Cultural Heritage neighborhood of Al Balad, a snorkeling trip, and a Hejazi Fish Dinner.

2 Introduction

Mean-field game (MFG) theory studies the behavior of large populations of identical rational agents in competition, where the behavior of each agent is determined by their state and by statistical information of the remaining players. In [9], Gomes and Saúde studied a price formation problem using an MFG approach. In this paper, we address a similar price formation problem (**Problem 1**) in a market with N identical rational agents who trade continuously a commodity whose supply, Q, is a given exogenous variable and whose price, ϖ, is determined by the balance between supply and demand. The agents are rational, in the sense that they seek to minimize their trading cost. The collective behavior of the agents, coupled with the market clearing condition, determines the evolution of the price, ϖ. More precisely, we consider the following problem:

Problem 1. *Let $Q \in C^1([0,T])$ be the supply rate per agent. Let $L \in C^2(\mathbb{R} \times \mathbb{R})$, the Lagrangian, be a non-negative function, convex in the second component. Let $\Psi \in C^1(\mathbb{R})$ be a non-negative terminal cost. Let $N \in \mathbb{N}$ be the number of agents. At time 0, each agent i owns x_0^i units of the commodity.*

Find a price, $\varpi : [0,T] \to \mathbb{R}$*, and trajectories,* $\mathbf{x}_i : [0,T] \to \mathbb{R}$*, with initial conditions* $\mathbf{x}_i(0) = x_0^i$*, such that for each* $1 \leqslant i \leqslant N$*,* $\mathbf{x}_i$ *minimizes the functional*

$$\int_0^T \Big(L(\mathbf{x}_i(s), \dot{\mathbf{x}}_i(s)) + \varpi(s)\dot{\mathbf{x}}_i(s) \Big) ds + \Psi(\mathbf{x}_i(T)) \tag{2.1}$$

subjected to the balance condition

$$\frac{1}{N}\sum_{i=1}^{N} \dot{\mathbf{x}}_i(t) = Q(t), \quad \forall t \in [0,T]. \tag{2.2}$$

In the preceding problem, $\mathbf{x}_i(t)$ is the amount of commodity held by the agent i at time t; hence, $\dot{\mathbf{x}}_i(t)$ denotes the rate at which the agent i trades. The functional in (2.1) represents the cost for each agent. The running cost is composed of the trading cost that comprises the instantaneous cost of the commodity $\varpi\dot{\mathbf{x}}_i$ and indirect costs such as storage or market impact encoded in the term $L(\mathbf{x}, \dot{\mathbf{x}})$. The preference of the agents at the final time, T, is encoded in the term $\Psi(\mathbf{x}_i(T))$, the terminal cost. The equation in (2.2) is the requirement that the market clears; that is, supply equals demand at all times.

Formally, MFGs model the mean-field limit of N-player games as $N \to \infty$. However, the rigorous justification of this limit is unknown in the general case, despite recent substantial progress [1]. In our price formation problem, the N-player game is relatively tractable. The main goal of this work is to study this N-player problem, which is the first step towards the rigorous justification of the mean-field limit as N goes to infinity. We expect our price to approximate the one presented [9] as the number of players increases. Also, each trajectory $\mathbf{x}_i$ should converge to the trajectory of the representative player of the continuum of agents model solved as an MFG. Notice that in this limiting process, the function Q remains the same for both the finite and the continuous player models.

In their seminal paper, Lasry and Lions [15] presented three examples of mean-field modeling in economics. They were concerned with situations involving a large number of rational players with little individual effect on the game. Inspired by [15], Markowich et al. [18] discussed the existence and uniqueness of the solution for a one-dimensional parabolic evolution equation with a free boundary that models price formation. Caffarelli et al. [2] established the global existence and asymptotic behavior of a price formation model with free boundaries. Their results rely on a transformation, which takes the equation in their problem into the heat equation. Burger et al. [1] extended this problem to a Boltzman-type price formation model. Their solutions converge to the Lasry–Lions model as the transaction rate tends to infinity. The study of the behavior of rational agents in energy markets appeared in [17, 16] in the context of load-control problems. Switching space heaters on and off controls the load, for an MFG approach see [13, 14, 12]. Previous authors addressed the price issue by assuming that the demand is a given function of the price [11] or that the price is a given function of the demand, see [4], [3], [5], [6], and [10]. An N-player version of an economic growth model was presented in [8]. In a more recent paper [9], Gomes and Saúde introduced a price-formation model where a large number of small players seek to store and trade electricity. This model was a constrained MFG where the price is a Lagrange multiplier for the supply vs. demand balance condition.

Here, we prove the following main theorem:

Theorem 2.1. *Assume that Ψ, the terminal cost, is non-negative and uniformly convex, and $L \in C^2(\mathbb{R} \times \mathbb{R})$, the Langrangian, is non-negative, uniformly convex in the second component, and satisfies the following inequality uniformly in $(z, v) \in \mathbb{R} \times \mathbb{R}$:*

$$L(z, v) \geqslant \alpha |v|^q - \beta, \quad q \in (1, \infty), \alpha > 0, \beta \geqslant 0. \tag{2.3}$$

*Then, **Problem 1** has a unique solution.*

The existence is established in **Proposition 4.1** and the uniqueness in **Proposition 4.3**.

The condition (2.3) means that high trading rates are expensive. The utility function in Economics is the negative of our value function. Convexity properties of the value function translate into concavity for the utility function. Therefore, our convexity assumptions are natural from the Economics point of view.

This work starts with the description of the single-agent control problem and derives the Euler–Lagrange equation. It then deals with the N-agent problem. For this, we first show the existence of the minimizers by applying the direct method in the calculus of variations. Then, we provide an interpretation of the price of the commodity as the Lagrange multiplier of the corresponding multi-agent problem. Subsequently, we find necessary conditions for the trajectories to be minimizers, via a slight variation on the Euler–Lagrange equation. We conclude the work by proving the existence of a unique solution for **Problem 1** under convexity assumptions on L and Ψ.

Finally, we point out that **Problem 1** can be coupled with a control problem for Q on the production side, where the producer seeks to maximize profits.

3 Single-agent control problem

To build an N-player model, we first analyze a single-agent control problem. Using optimal control theory and calculus of variations, we derive the Euler–Lagrange equation and the boundary conditions.

Let $\mathcal{B}_t^q = W^{1,q}(t,T)$ be the set of admissible functions with $q \in (1,\infty)$ as in (2.3). Each agent seeks to find an optimal trajectory, $\mathbf{x} \in \mathcal{B}_0^q$, minimizing the functional

$$I[\mathbf{x}] = \int_0^T \Big(L(\mathbf{x}(t), \dot{\mathbf{x}}(t)) + \varpi(t)\dot{\mathbf{x}}(t) \Big)\, dt + \Psi(\mathbf{x}(T))$$

with an initial position, $\mathbf{x}(0) = x_0$.

If $\mathbf{x}$ is a minimizer, then for any $\mathbf{y} \in C_c^\infty((0,T])$, and every $\epsilon \in \mathbb{R}$, we have

$$I[\mathbf{x}] \leqslant I[\mathbf{x} + \epsilon \mathbf{y}] \ .$$

Thus, the function $i : \mathbb{R} \to \mathbb{R}$ defined by $i(\epsilon) = I[\mathbf{x} + \epsilon\mathbf{y}]$ attains a local minimum at $\epsilon = 0$. Then, $i'(\epsilon)|_{\epsilon=0} = 0$. Accordingly, computing $i'(0)$ and using the fact that $\mathbf{y}$ is arbitrary, we obtain the Euler–Lagrange equation

$$D_x L(\mathbf{x}, \dot{\mathbf{x}}) - \frac{d}{dt}(D_v L(\mathbf{x}, \dot{\mathbf{x}}) + \varpi) = 0$$

and the *natural boundary condition*

$$D_v L(\mathbf{x}(T), \dot{\mathbf{x}}(T)) + \varpi(T) + \Psi'(\mathbf{x}(T)) = 0.$$

Example 3.1. Consider a Lagrangian of the form

$$L(x, v) = L(v).$$

Then, the Euler–Lagrange equation becomes

$$\frac{d}{dt}(D_vL(\dot{\mathbf{x}})+\varpi)=0 \Leftrightarrow D_vL(\dot{\mathbf{x}})+\varpi=K,$$

where K is some constant. Since L is uniformly convex, D_vL is strictly monotone and, thus, invertible. Therefore,

$$\dot{\mathbf{x}}=(D_vL)^{-1}(K-\varpi)\ .$$

So, if the price $\varpi(t)$ increases, the agents buy less or sell. In particular, if $L(v)=\frac{v^2}{2}$, then

$$D_vL(\dot{\mathbf{x}})=\dot{\mathbf{x}}.$$

Hence, the Euler–Lagrange equation becomes

$$\dot{\mathbf{x}}=K-\varpi. \tag{3.1}$$

Equation (3.1) shows that as the price increases, $\dot{\mathbf{x}}$ decreases.

Let $\Psi(x)=\frac{x^2}{2}$, which means that agents seek to minimize $|\mathbf{x}(T)|^2$. This choice of Ψ corresponds to the portfolio liquidation problem. The Euler–Lagrangian equation and corresponding natural boundary condition give

$$\begin{cases}\dfrac{d}{dt}\left(\dot{\mathbf{x}}(t)+\varpi(t)\right)=0\\ \dot{\mathbf{x}}(T)+\mathbf{x}(T)=-\varpi(T)\\ \mathbf{x}(0)=x_0\ .\end{cases} \tag{3.2}$$

Thus, from (3.1) and (3.2), we get

$$K=\frac{1}{1+T}\left[\int_0^T\varpi(t)dt-x_0\right].$$

Define the average price

$$\hat{\varpi}=\frac{1}{T}\int_0^T\varpi(t)dt.$$

The agent buys when

$$\dot{\mathbf{x}}(t)>0.$$

According to (3.1), the above inequality holds if

$$\varpi(t)<\frac{T\hat{\varpi}-x_0}{T+1}.$$

Thus, an agent buys when the price is below the threshold price on the right-hand side of the preceding inequality.

4 A constrained minimization problem for N agents

We use the single-agent control problem to formulate an N-agent minimization problem that includes the balance condition. We prove existence, uniqueness, and then we provide a characterization of such minimizer by showing that the price is the Lagrange multiplier of an equivalent minimization problem.

4.1 A variational problem

Notice that, for each agent, (2.1) is a functional that is independent of the dynamics of the other agents. Hence, **Problem 1** is equivalent to the following minimization problem

$$x \min_{\mathbf{x},\mathbf{x}(0)=x_0} \frac{1}{N}\sum_{i=1}^{N}\left(\int_0^T \Big(L(\mathbf{x}_i(s),\dot{\mathbf{x}}_i(s)) + \varpi(s)\dot{\mathbf{x}}_i(s)\Big)ds + \Psi(\mathbf{x}_i(T))\right) \tag{4.1}$$

$$\text{subject to} \quad \frac{1}{N}\sum_{i=1}^{N}\dot{\mathbf{x}}_i(t) = Q(t) \quad \forall t \in [0,T]\ . \tag{4.2}$$

Substituting (4.2) into (4.1) we get

$$\begin{aligned}&\frac{1}{N}\sum_{i=1}^{N}\left(\int_0^T \Big(L(\mathbf{x}_i(s),\dot{\mathbf{x}}_i(s)) + \varpi(s)\dot{\mathbf{x}}_i(s)\Big)ds + \Psi(\mathbf{x}_i(T))\right)\\ &= \frac{1}{N}\sum_{i=1}^{N}\left(\int_0^T L(\mathbf{x}_i(s),\dot{\mathbf{x}}_i(s))ds + \Psi(\mathbf{x}_i(T))\right) + \int_0^T \varpi(s)Q(s)ds\ ,\end{aligned}$$

and since $\varpi(s)Q(s)$ is independent of $\mathbf{x}$ at every s, the minimization problem is equivalent to

$$\min_{\mathbf{x},\mathbf{x}(0)=x_0} \frac{1}{N}\sum_{i=1}^{N}\int_0^T L(\mathbf{x}_i(s),\dot{\mathbf{x}}_i(s))ds + \Psi(\mathbf{x}_i(T)) \tag{4.3}$$

$$\text{subject to} \quad \frac{1}{N}\sum_{i=1}^{N}\dot{\mathbf{x}}_i(t) = Q(t) \quad \forall t \in [0,T]\ . \tag{4.4}$$

We now prove the existence of optimal trajectories.

4.2 Existence of a solution

We use the direct method in the calculus of variations to obtain the existence of a minimizer of (4.3) and (4.4). For that, let

$$\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{N}\sum_{i=1}^{N}\left(L(\mathbf{x}_i, \dot{\mathbf{x}}_i) + \frac{1}{T}\Psi(\mathbf{x}_i(T))\right),$$

and

$$I_N[\mathbf{x}] = \int_0^T \mathcal{L}(\mathbf{x}, \dot{\mathbf{x}})ds.$$

Then (4.3) and (4.4) becomes

$$\begin{aligned} &\min_{\mathbf{x},\mathbf{x}(0)=\mathbf{x}_0} I_N[\mathbf{x}] \\ &\text{s.t.} \quad \langle \mathbf{x}(t)\rangle = Q(t). \end{aligned} \tag{4.5}$$

Proposition 4.1. *Let L satisfy* (2.3)*. Then* ***Problem 1*** *has a solution.*

Proof. We show that $\mathcal{L}$ is coercive and lower semicountinuous in $W^{1,q}$. It is enough to show that there exist $\bar{\alpha} > 0$, $\bar{\beta} \geqslant 0$, and $q > 1$ such that

$$\mathcal{L}(\mathbf{x}, \mathbf{p}) \geqslant \alpha|\mathbf{p}|^q - \beta$$

to obtain coercivity. The condition on the Lagrangian for each agent implies the coercivity on $\mathcal{L}$, since, by the non-negativity of Ψ, we have:

$$\begin{aligned} \mathcal{L}(\mathbf{x}, \mathbf{p}) = \frac{1}{N}\sum_{i=1}^{N}\left(L(\mathbf{x}_i, \mathbf{p}_i) + \frac{1}{T}\Psi(\mathbf{x}_i(T))\right) &\geqslant \frac{1}{N}\sum_{i=1}^{N}(\alpha|\mathbf{p}_i|^q - \beta) \\ &\geqslant \alpha\sum_{i=1}^{N}|\mathbf{p}_i|^q - \beta = \alpha||\mathbf{p}||_{L^q}^q - \beta \\ &\geqslant \frac{\alpha C}{N}|\mathbf{p}|^q - \beta. \end{aligned}$$

The last inequality follows from the fact that in $\mathbb{R}^N$ all the p-norms are equivalent. The above establishes the coercivity of $\mathcal{L}$.

To show lower semicontinuity, we need to ensure the convexity of $\mathcal{L}$ on the second variable, and that $\mathcal{L}$ is bounded from below. Convexity follows from the convexity of L in $\dot{\mathbf{x}}$. Boundedness from below follows from the coercivity condition.

We use the direct method in the calculus of variations to determine the existence of a minimizer for our problem. Define the admissible set

$$\mathcal{A}_t = \left\{ \mathbf{x} \in W^{1,q}(t,T) \mid \frac{\sum_{i=1}^{N} \dot{\mathbf{x}}_i(s)}{N} = Q(s), \mathbf{x}_i(0) = x_0^i, 1 \leqslant i \leqslant N,\ t \leqslant s \leqslant T \right\},$$

and set $\mathcal{A} = \mathcal{A}_0$. We notice that $\mathcal{A}$ is nonempty by taking $\dot{\mathbf{x}}_i = Q(t), \mathbf{x}_i(0) = x_0^i$. Since $\mathcal{L}$ is bounded from below, there exists a minimizing sequence, $(\mathbf{x}^n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that

$$\lim_{n\to+\infty} I_N[\mathbf{x}^n] = \inf_{\mathbf{x}} I_N[\mathbf{x}].$$

By the coercivity of $\mathcal{L}$, we have

$$I_N[\mathbf{x}^n] \geqslant \alpha \|\dot{\mathbf{x}}^n\|_{L^q}^q - \beta T.$$

Thus, by Poincaré's inequality, $(\mathbf{x}^n)_{n\in\mathbb{N}}$ is bounded in $W^{1,q}(0,T)$. Then, there exists $\mathbf{x}^* \in W^{1,q}(0,T)$ such that, up to a subsequence, $\mathbf{x}^n$ converges weakly to $\mathbf{x}^*$. We notice that $\mathcal{A}$ is convex. Since $q > 1$, Morrey's theorem (see [7]) gives that $\mathcal{A}$ is closed. Thus, by Mazur's theorem, see ([7] Appendix D.4), $\mathcal{A}$ is weakly closed in $W^{1,q}(0,T)$, which implies that $\mathbf{x}^* \in \mathcal{A}$. Then, since $\mathcal{L}$ is bounded from below and convex in p, I is sequentially weakly lower semicontinuous in $W^{1,q}(0,T)$. Thus, $\mathbf{x}^*$ is the minimizer of I since

$$\inf_{\mathbf{x}} I_N[\mathbf{x}] = \lim_{n\to+\infty} I_N[\mathbf{x}^n] \geqslant I_N[\mathbf{x}^*] \geqslant \inf_{\mathbf{x}} I_N[\mathbf{x}]. \qquad \square$$

4.3 Uniqueness of solutions

Assume that

Assumption 4.2.

1. the map $(x,v) \mapsto L(x,v)$ is convex and for each $x \in \mathbb{R}$, the map $v \mapsto L(x,v)$ is uniformly convex; that is, there exists $\theta > 0$ such that for all $x, y, v, w \in \mathbb{R}$, we have

   $$L(\lambda x+(1-\lambda)y, \lambda v+(1-\lambda)w) \leqslant \lambda L(x,v)+(1-\lambda)L(y,w)-\theta\lambda(1-\lambda)|v-w|^2.$$

2. Ψ is uniformly convex.

We notice that the term $\varpi \dot{x}$ is linear in the velocity, thus convex.

Proposition 4.3. *Let $x \in \mathbb{R}^N$. Under Assumptions 1. and 2., the solution of the problem*

$$\min_{\mathbf{x} \in \mathcal{A}_t, \mathbf{x}(t)=x} I_N[\mathbf{x}]$$

is unique.

Proof. We prove the statement via contradiction. Assume that there exist two different minimizers, $\mathbf{x}$, $\mathbf{y} \in \mathcal{A}_t$ with $\mathbf{x}(t) = x$. Then, taking the middle point, $\dfrac{\mathbf{x}+\mathbf{y}}{2}$, we obtain

$$\begin{aligned}
I_N\left[\frac{\mathbf{x}+\mathbf{y}}{2}\right] &= \frac{1}{N}\sum_{i=1}^{N}\left[\int_0^T L\left(\frac{\mathbf{x}_i+\mathbf{y}_i}{2}, \frac{\dot{\mathbf{x}}_\mathbf{i}+\dot{\mathbf{y}}_\mathbf{i}}{2}\right)dt + \Psi\left(\frac{\mathbf{x}_\mathbf{i}(T)+\mathbf{y}_\mathbf{i}(T)}{2}\right)\right] \\
&\leqslant \frac{1}{2N}\sum_{i=1}^{N}\Bigg[\int_0^T L(\mathbf{x}_\mathbf{i}, \dot{\mathbf{x}}_\mathbf{i})dt + \int_0^T L(\mathbf{y}_\mathbf{i}, \dot{\mathbf{y}}_\mathbf{i})dt - \frac{\theta}{2}\int_0^T |\dot{\mathbf{x}}_\mathbf{i} - \dot{\mathbf{y}}_\mathbf{i}|^2 dt \\
&\qquad + \Psi(\mathbf{x}_\mathbf{i}(T)) + \Psi(\mathbf{y}_\mathbf{i}(T))\Bigg] \\
&= \frac{1}{2}I_N[\mathbf{x}] + \frac{1}{2}I_N[\mathbf{y}] - \sum_{i=1}^{N}\frac{\theta}{4N}||\dot{\mathbf{x}}_\mathbf{i} - \dot{\mathbf{y}}_\mathbf{i}||^2_{L^2(0,T)} \\
&= \min_{\mathbf{z} \in \mathcal{A}_t, \mathbf{z}(t)=x} I_N[\mathbf{z}] - \sum_{i=1}^{N}\frac{\theta}{4N}||\dot{\mathbf{x}}_\mathbf{i} - \dot{\mathbf{y}}_\mathbf{i}||^2_{L^2(0,T)}.
\end{aligned}$$

Because $\theta > 0$, the preceding inequality can hold only if $||\dot{\mathbf{x}}_\mathbf{i} - \dot{\mathbf{y}}_\mathbf{i}||^2_{L^2(0,T)} = 0$ for all $i = 1, \ldots, N$. Consequently, there exists a constant, $c \in \mathbb{R}^N$, such that $\mathbf{x}(s) - \mathbf{y}(s) = c$ for all $s \in [t, T]$. Using the initial condition $\mathbf{x}(t) = \mathbf{y}(t) = x$, we get $c = 0$, which contradicts the fact that $\mathbf{x}$ and $\mathbf{y}$ are distinct. Thus, $\mathbf{x} = \mathbf{y}$ and this concludes the proof. □

4.4 Price as a Lagrange Multiplier

For $F = (f_1, \ldots, f_N) \in \mathbb{R}^N$, we denote its entry-wise average by

$$\langle F \rangle := \frac{1}{N}\sum_{k=1}^{N} f_k.$$

Before deriving the necessary optimality conditions, we introduce the following auxiliary result.

Lemma 4.4. *Let $F = (f_1, \dots, f_N) \in C((0,T);\mathbb{R}^N)$ be such that for all $P \in C_c^\infty([0,T];\mathbb{R}^N)$ with $\langle P(s)\rangle = 0$, for all $s \in [0,T]$, F satisfies*

$$\int_0^T F(s) \cdot P(s)\, ds = 0.$$

Then, there exists $c \in C(0,T)$ such that, for all $k = 1, \dots, N$, we have

$$f_k(t) = c(t).$$

Proof. Fix $R \in C_c^\infty([0,T];\mathbb{R}^N)$. Set P by

$$P = R - \langle R\rangle \mathbb{1}.$$

Because $\langle P(s)\rangle = 0$, we have

$$0 = \int_0^T F \cdot P\, ds = \int_0^T (F \cdot R - N\langle R\rangle\langle F\rangle)\, ds = \int_0^T (F - \langle F\rangle \mathbb{1}) \cdot R\, ds.$$

Since R is arbitrary, by the fundamental theorem of the calculus of variations, for all $k = 1, \dots, N$ and $t \in (0,T)$, we have

$$f_k(s) - \langle F(s)\rangle = 0.$$

Hence, we obtain $c(\cdot) = \langle F(\cdot)\rangle \in C(0,T)$. □

In the next proposition, we derive the necessary conditions (Euler–Lagrange equations) for solutions of (4.5). Let X be

$$X := \left\{ \mathbf{x} \in C^2([0,T],\mathbb{R}^N) \mid \langle \dot{\mathbf{x}}(s)\rangle = Q(s) \text{ for all } s \in [0,T] \right\}.$$

Proposition 4.5. *Assume that $L \in C^2(\mathbb{R}^2)$. Then, there exist $c \in C(0,T)$ and $\tilde{c} \in \mathbb{R}$ such that, if $\bar{\mathbf{x}} \in X \cap C^2([0,T];\mathbb{R}^N)$ is a minimizer of (4.5), then it solves*

$$\frac{\partial}{\partial x} L(\bar{\mathbf{x}}_k(t), \dot{\bar{\mathbf{x}}}_k(t)) - \frac{d}{dt}\left(\frac{\partial}{\partial v} L(\bar{\mathbf{x}}_k(t), \dot{\bar{\mathbf{x}}}_k(t))\right) = c(t)$$

and

$$\frac{\partial}{\partial v} L(\bar{\mathbf{x}}_k(T), \dot{\bar{\mathbf{x}}}_k(T)) + \Psi'(\bar{\mathbf{x}}_k(T)) = \tilde{c}$$

for all $t \in (0,T)$ and for all $k = 1, \dots, N$.

Proof. Let $\mathbf{y} \in C^\infty([0,T],\mathbb{R}^N)$ be such that $\mathbf{y}(0) = 0$ and $\langle \mathbf{y}(s)\rangle = 0$ for every $s \in [0,T]$. For $\epsilon \in \mathbb{R}$, we define $i : \mathbb{R} \to \mathbb{R}$ as

$$i(\epsilon) = \frac{1}{N}\sum_{k=1}^{N}\int_0^T \Big(L(\bar{\mathbf{x}}_k+\epsilon\mathbf{y}_k, \dot{\bar{\mathbf{x}}}_k+\epsilon\dot{\mathbf{y}}_k)+\varpi\cdot(\dot{\bar{\mathbf{x}}}_k+\epsilon\dot{\mathbf{y}}_k)\Big)\,ds+\Psi(\bar{\mathbf{x}}_k(T)+\epsilon\mathbf{y}_k(T)).$$

Since $\langle \mathbf{y}\rangle = 0$ and $\bar{\mathbf{x}} \in X$, we have

$$\langle \dot{\bar{\mathbf{x}}} + \epsilon\dot{\mathbf{y}}\rangle = Q(s).$$

Thus, we obtain

$$i(\epsilon) = \frac{1}{N}\sum_{k=1}^{N}\left[\int_0^T L(\bar{\mathbf{x}}_k + \epsilon\mathbf{y}_k, \dot{\bar{\mathbf{x}}}_k + \epsilon\dot{\mathbf{y}}_k)\,ds + \Psi(\bar{\mathbf{x}}_k(T) + \epsilon\mathbf{y}_k(T))\right] + \int_0^T \varpi\cdot Q\,ds.$$

We have that $i \in C^1(\mathbb{R})$ because $L \in C^2(\mathbb{R}^2)$. Thus, because $\bar{x}_k$ is a minimizer for all $k = 1, \ldots, N$, we have $i'(0) = 0$; that is

$$\frac{1}{N}\sum_{k=1}^{N}\left[\int_0^T \left(\frac{\partial}{\partial x}L(\bar{\mathbf{x}}_k, \dot{\bar{\mathbf{x}}}_k)\mathbf{y}_k + \frac{\partial}{\partial v}L(\bar{\mathbf{x}}_k, \dot{\bar{\mathbf{x}}}_k)\dot{\mathbf{y}}_k\right)\,ds + \Psi'(\bar{\mathbf{x}}_k(T))\mathbf{y}_k(T)\right] = 0.$$

Integrating by parts, we obtain

$$\frac{1}{N}\sum_{k=1}^{N}\Big[\int_0^T \Big(\frac{\partial}{\partial x}L(\bar{\mathbf{x}}_k, \dot{\bar{\mathbf{x}}}_k) - \frac{d}{dt}(\frac{\partial}{\partial v}L(\bar{\mathbf{x}}_k, \dot{\bar{\mathbf{x}}}_k))\Big)\mathbf{y}_k\,ds \\ + \Big(\frac{\partial}{\partial v}L(\bar{\mathbf{x}}_k(T), \dot{\bar{\mathbf{x}}}_k(T)) + \Psi'(\bar{\mathbf{x}}_k(T))\Big)\mathbf{y}_k(T)\Big] = 0.$$

If we select $\mathbf{y}$ such that $\mathbf{y}(T) = 0$, by Lemma 4.4, we conclude that there exists $c \in C(0,T)$ such that, for all $k = 1, \ldots, N$, we have

$$\frac{\partial}{\partial x}L(\bar{\mathbf{x}}_k, \dot{\bar{\mathbf{x}}}_k) + \frac{d}{dt}\Big(\frac{\partial}{\partial v}L(\bar{\mathbf{x}}_k, \dot{\bar{\mathbf{x}}}_k)\Big) = c(t). \tag{4.6}$$

Define $\tilde{f}_k(T)$ by

$$\tilde{f}_k(T) = \frac{\partial}{\partial v}L(\bar{\mathbf{x}}_k(T), \dot{\bar{\mathbf{x}}}_k(T)) + \Psi'(\bar{\mathbf{x}}_k(T)).$$

For all $t \in [0,T]$, set $\tilde{F}(t) := (\tilde{f}_1(T), ..., \tilde{f}_n(T))$. Since $\tilde{f}_k$ is constant, applying Lemma 4.4 for $\tilde{F}$, there exists $\tilde{c} \in \mathbb{R}$ such that we have $\tilde{f}_k(t) = \tilde{f}_k(T) = \tilde{c}$, from which we conclude that

$$\frac{\partial}{\partial v}L(\bar{\mathbf{x}}_k(T), \dot{\bar{\mathbf{x}}}_k(T)) + \Psi'(\bar{\mathbf{x}}_k(T)) = \tilde{c}. \qquad \square$$

Let c and $\tilde{c}$ be as in the statement of the preceding proposition, and let $\varpi \in C^1(0,T)$ solve

$$\dot{\varpi}(t) = -c(t), \quad \varpi(T) = -\tilde{c}.$$

Then, the necessary optimality conditions for $\mathbf{x}_k$ become

$$\frac{\partial}{\partial x} L(\bar{\mathbf{x}}_k, \dot{\bar{\mathbf{x}}}_k) + \frac{d}{dt}\Big(\frac{\partial}{\partial v} L(\bar{\mathbf{x}}_k, \dot{\bar{\mathbf{x}}}_k) + \varpi\Big) = 0$$

and

$$\frac{\partial}{\partial v} L(\bar{\mathbf{x}}_k(T), \dot{\bar{\mathbf{x}}}_k(T)) + \varpi(T) + \Psi'(\bar{\mathbf{x}}_k(T)) = 0$$

for all $k = 1, ..., N$.

The preceding equations are the optimality conditions for the functional

$$\frac{1}{N}\sum_{i=1}^{N}\int_0^T L(\mathbf{x}_i(s), \dot{\mathbf{x}}_i(s))ds + \Psi(\mathbf{x}_i(T)) + \int_0^T \varpi(s)\left(\frac{1}{N}\sum_{i=1}^{N}\dot{\mathbf{x}}_i(s) - Q(s)\right)ds,$$

and the solution constructed in **Proposition 4.5** satisfy the constraint (4.2). Thus, we can regard ϖ as a Lagrange multiplier for (4.2).

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