Stationary mean-field games with logistic effects

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Abstract

In its standard form, a mean-field game is a system of a Hamilton-Jacobi equation coupled with a Fokker-Planck equation. In the context of population dynamics, it is natural to add to the Fokker-Planck equation features such as seeding, birth, and non-linear death rates. Here, we consider a logistic model for the birth and death of the agents. Our model applies to situations in which crowding increases the death rate. The new terms in this model require novel ideas to obtain the existence of a solution. Here, the main difficulty is the absence of monotonicity. Therefore, we construct a regularized model, establish a priori estimates for the solution, and then use a limiting argument to obtain the result.

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1 Introduction

Mean-field games have been the focus of intense research since the seminal works [24, 25, 23, 22]. These games model a large population of competing rational agents. In most models, either the agents never leave the domain, so that their number is constant and their distribution arises from a probability measure, or they are subject to exponential death rates or exit through the boundary and a replacement by a population-independent source, see, for example, [7]. Here, we take into consideration non-linear effects: the death rate of the agents increases with the density. These effects give rise to a logistic term in the Fokker-Planck equation, which breaks the monotonicity properties of the model and requires careful analysis.

More precisely, we consider the following stationary problem.

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Problem 1. *Given a Hamiltonian,* $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, $H \in C^\infty$, *and* $g : E \to \mathbb{R}$, *where* $E = \mathbb{R}^+ = (0, \infty)$ *or* $E = [0, \infty)$, $g \in C^\infty(E)$ *with* g *increasing, and positive reals,* δ *and* α, *find* $u : \mathbb{T}^d \to \mathbb{R}$ *and* $m : \mathbb{T}^d \to \mathbb{R}$ *with* $m \geq 0$ *that solve*

$$\begin{cases} H(x, Dv) = g(m) - m^\alpha v + \Delta v \\ -\operatorname{div}(m D_p H(x, Dv)) = (1 - m^\alpha)m + \delta + \Delta m. \end{cases} \tag{1}$$

The prior system models a population of agents whose state is a point in $\mathbb{T}^d$, the standard d-dimensional torus. We denote the distribution of the agents by the density $m \geq 0$. The integral of m in a set gives the (normalized) number of agents in that set. Here, we look for an equilibrium distribution arising from the following mechanisms. First, agents move both by diffusion and by controlling their drift seeking to optimize an integral functional. The integrand of this functional takes into account individual location preferences, the cost of motion, and the density of agents. Second, agents reproduce and die; this effect appears in the term $(1 - m^\alpha)m$ in the second equation of (1). Finally, there are incoming agents as determined by the seeding rate parameter $\delta > 0$. The positivity of δ implies that m never vanishes, as we show in Section 4, particularly in Subsection 4.3.

In (1), the Hamiltonian H satisfies the assumptions listed in Section 2. A model Hamiltonian for which these are valid is

$$H(x, p) = a(x)|p|^2 + V(x),$$

with $a, V \in C^\infty(\mathbb{T}^d, [0, \infty[)$, $a > 0$. The Lagrangian associated with the Hamiltonian is

$$L(x, v) = \sup_p \{-pv - H(x, p)\}.$$

In particular, for the model Hamiltonian above, $L(x, v) = \frac{|v|^2}{4a(x)} - V(x)$.

Now, we use the Lagrangian to formulate the control problem for each agent. Fix a reference agent whose state at the initial time is x and suppose that the distribution of the remaining agents is stationary and given by a probability density m. This agent controls the diffusion

$$\begin{aligned} dX_s &= u(s)ds + \sqrt{2}dw_s \\ X_0 &= x \end{aligned}$$

and aims at finding a progressively measurable control, $u : [0, \infty) \to \mathbb{R}^d$, that realizes the infimum for the value function

$$v(x) = \inf_u \mathbb{E}\left[\int_0^\infty e^{-\int_0^s m^\alpha(X_r)dr} \left(L(s, X_s, u_s) + g\left(m(X_s)\right)\right) ds\right].$$

We see from the above definition, that L determines the cost associated with movement; the second term in the integral, corresponding to the function g, encodes crowding effects. If g is increasing, staying in or passing through high-density regions is more expensive than low-density regions. For concreteness, here, we consider two main cases, $g(z) = \ln z$ and $g(z) = z^\gamma$. In the first case, for the problem to be well-posed, we need $m > 0$. As we will see, however, strictly positive lower bounds for m play an essential role in our results and will be established for both cases.

Consider the value function v. If it is smooth enough, then it solves the Hamilton-Jacobi equation

$$H(x, Dv) = g(m) - m^\alpha v + \Delta v.$$

Moreover, the optimal control is given in feedback form

$$u(s) = -D_p H(X_s, Dv(X_s)).$$

Accordingly, if all agents behaved optimally and if there were no sources, as in the usual mean-field game system, m would solve the corresponding Fokker-Planck equation

$$-\operatorname{div}(mD_pH(x, Dv)) = \Delta m.$$

However, there are three additional terms in the Fokker-Plank equation in Problem 1. The term $-m^{1+\alpha}$, which accounts for the non-linear death rate of the agents, the term m, which encodes the birth rate for the agents, and finally the constant source δ, which we see as an exogenous source of agents.

The theory of mean-field games has seen substantial progress; see, for instance, the books [2, 12, 18, 5]. In particular, stationary MFGs have been examined in the context of regular solutions [19, 21, 15, 29], explicit solutions [13, 28], numerical methods [1], weak solutions [3, 27, 4], and monotone operators [10, 9] by multiple authors. Additional problems such as mean-field games with congestion [11, 7, 6], the obstacle problem [14], and extended mean-field games [15] were also investigated in detail.

The present model was introduced in [20], where the authors addressed the one-dimensional case. The high-dimension case is substantially harder and presents many mathematical challenges. One key point is that the Hamilton-Jacobi equation changes behavior depending on whether or not m vanishes. To illustrate this point, consider the following simplified version of (1)

$$\begin{cases} H(x, Dv) = g(m) - m^\alpha v + \Delta v \\ 0 = (1 - m^\alpha)m + \Delta m. \end{cases}$$

In the simplified system, both $m = 0$ and $m = 1$ solve the second equation. For $m = 1$, there are universal bounds for v using the standard methods in Hamilton-Jacobi equations. However, when $m = 0$, v is only defined up to constants. Even though the seeding rate, δ, in (1) prevents m from vanishing identically, quantitative lower bounds for m are substantially harder. Thus, the critical estimates developed in this paper are the lower bounds for m developed in Section 4.3.

Our main result is the following theorem:

Theorem 1. *Suppose Assumptions 1, 3, 4, 6, 8, and 11 from Section 2 hold. Then, there exists a solution (v, m) for (1) in the classic sense.*

The preceding theorem extends the result from [20] in two ways. There, we obtained the a priori estimates for the unregularized 1-dimensional system. Here, we generalize to dimensions 2 and 3 (Assumption 11) and also prove the existence of solutions for the systems. Moreover, we obtain several new estimates that are valid in higher dimensions.

The basis of the proof of Theorem 1 is a regularization argument. After discussing the key assumptions on the Hamiltonian, the non-linearity, and the parameters, in Section 2, we proceed to Section 3, where we define a new problem, Problem 2 which is a regularized version of Problem 1 depending on a regularizing parameter, ϵ. We choose the regularization so that formal arguments using integration by parts for Problem 1 can be undertaken rigorously for Problem 2.

There, we prove the existence of solutions, (v^ϵ, m^ϵ), of Problem 2 by considering the system (7), a dynamic approximation to the stationary system (4), and finding positive and periodic solutions of (7) with arbitrarily small periods. We note that in (7), only the Fokker-Planck equation is explicitly time-dependent. Once we have positive solutions for Problem 2, we prove estimates that show that such solutions are classical and that provide bounds uniform in the regularization parameter. In Section 4, we develop preliminary estimates on v^ϵ, Dv^ϵ, m^ϵ, and $\frac{1}{m^\epsilon}$. Next, in Section 5, we prove the Lipschitz regularity for v^ϵ using Evans' [8] non-linear adjoint method. After obtaining the Lipschitz regularity of the solutions to the Hamilton-Jacobi equation, we use the techniques from Section 6 to improve the regularity of solutions to the system (4) through bootstrapping. Finally, in Section 7, we gather the previous results to prove Theorem 1 by passing to the limit in the regularizing parameter and obtain a solution to Problem 1.

2 Assumptions

In this section, we list the main hypotheses employed in the paper. The first assumption imposes a coercivity condition on the Hamiltonian, a necessary condition for the application of the Bernstein method.

Assumption 1. *For any bounded function, $a : \mathbb{T}^d \to \mathbb{R}$ and any bounded vector field, $b : \mathbb{T}^d \to \mathbb{R}$, the Hamiltonian satisfies the following estimate*

$$\lim_{|p|\to\infty} \frac{|H(x,p)+a(x)|^2}{2} + (D_xH(x,p)+b(x))\cdot p = +\infty,$$

uniformly in x.

Here, we work with the coupling g that satisfies certain growth assumptions. The next assumption states the minimal properties for our a priori bounds. That is also the case for Assumption 8 below. Later to simplify, in Assumption 9, we require g to have either polynomial or logarithmic growth, but other functions with similar growth are possible.

Assumption 2. *There exist positive constants c and C such that*

$$g(y) \leq \frac{1}{2} y g(y) + C$$

and

$$-C + cy \ln y \leq y g(y) \tag{2}$$

for all $y > 0$.

Note that (2) implies that $yg(y)$ is bounded by below and requires g to grow at least logarithmically.

The next two Assumptions, together with Assumption 6, impose natural, quadratic-like growth on the Hamiltonian.

Assumption 3. *There exist positive constants c and C such that*

$$D_p H(x,p) \cdot p - H(x,p) \geq cH(x,p) - C.$$

Assumption 4. *There exist positive constants c and C such that*

$$H(x,p) \geq c|p|^2 - C.$$

In the various estimates that we establish, we need several constraints on the parameter α. The next assumption gives a preliminary constraint that is refined in Assumption 7 and, later, in Assumption 11.

Assumption 5. *The restrictions on α according to the dimension d are:*

$$\begin{cases} \alpha > 0, & \text{if } d = 2 \\ 0 < \alpha < \frac{d+2}{d-2}, & \text{if } d > 2. \end{cases}$$

Assumption 6. *There exists a positive constant C such that*

$$|D_p H|^2 \leq C|p|^2 + C.$$

Later, we restrict further the range of α, and, thus, replace Assumption 5 with the following.

Assumption 7.

$$0 < \alpha < 1.$$

As we explained before, the next assumption is the second assumption on the structure of g.

Assumption 8. *The coupling g satisfies the following: for any $\beta > 0$, there exists $C > 0$ such that*

$$\left| \int_{\mathbb{T}^d} g(\theta) dx \right|^{\frac{1}{1-\alpha}} \leq C + \beta \int_{\mathbb{T}^d} \theta g(\theta) dx$$

for any probability density $\theta : \mathbb{T}^d \to \mathbb{R}^+$.

Assumption 9. *The dimension is $d = 2$ or $d = 3$ and the coupling g is of the form*

A. $g(m) = \ln m$

B. $g(m) = m^\gamma$ *where the exponent γ satisfies*

$$\begin{cases} \gamma > 0, & \text{if } d = 2 \\ 0 < \gamma < 2, & \text{if } d = 3. \end{cases}$$

Note that the above assumption implies Assumptions 2 and 8. The restriction on γ is updated, in Assumption 11, in case Assumption 9B holds and $d = 3$.

The following is a thechnical assumption which appears only in the adjoint method.

Assumption 10.

$$|D_x H(x,p)| \leq C + C|p|^\beta$$

for some $0 < \beta < 2$.

Assumption 11. *The dimension is $d = 2$ or $d = 3$,*

$$0 < \alpha < \frac{1}{d-1},$$

and $g(m) = \ln m$ or $g(m) = m^\gamma$ with

$$\begin{cases} \gamma > 0, & \text{if } d = 2 \\ 0 < \gamma < 1, & \text{if } d = 3. \end{cases}$$

Remark 2. *A class of Hamiltonians that satisfy Assumptions 1, 3, 4, 6, and 10 is*

$$H(x,p) = |p|^2 + b(x) \cdot p + V(x), \tag{3}$$

for $c > 0$ and smooth b and V. Observe that Assumptions 4 and 6 imply quadratic-like growth of the Hamiltonian, and that Assumption 10 excludes terms like $a(x)|p|^2$.

3 The regularized system

We introduce an auxiliary problem, a non-local regularization of Problem 1, Problem 2 below. Take the standard Gaussian, $\eta : \mathbb{R}^d \to \mathbb{R}$, which is given by $\eta(x) = \frac{1}{(2\pi)^{(d/2)}} e^{-|x|^2/2}$, and consider, for each $\epsilon > 0$, the function $\eta_\epsilon(x) := \epsilon^d \eta(x/\epsilon)$. This defines a family of mollifiers which we only employ in the context of Problem 2. The choice of the mollifiers is motivated by convenience, namely we rely on the positivity of $\eta_\epsilon * m$, for $m \geq 0$, to justify the bounds (13) in the proof Lemma 5.

Problem 2. *Let H, g, α and δ be as in Problem 1. For $\epsilon > 0$, let $\eta_\epsilon : \mathbb{R}^d \to \mathbb{R}^+$ be a family of Gaussian mollifiers. Define $g_\epsilon(f) := \eta_\epsilon * g(\eta_\epsilon * f)$, where $*$ denotes the convolution. Find $v^\epsilon : \mathbb{T}^d \to \mathbb{R}$ and $m^\epsilon : \mathbb{T}^d \to \mathbb{R}$ with $m^\epsilon \geq 0$ satisfying*

$$\begin{cases} H(x, Dv^\epsilon) = g_\epsilon(m^\epsilon) - (\eta_\epsilon * (m^\epsilon_\epsilon)^\alpha)\, v^\epsilon + \Delta v^\epsilon \\ -\operatorname{div}(m^\epsilon D_p H(x, Dv^\epsilon)) = (1 - \eta_\epsilon * (m^\epsilon_\epsilon)^\alpha)\, m^\epsilon + \Delta m^\epsilon + \delta, \end{cases} \tag{4}$$

where $m^\epsilon_\epsilon = \eta_\epsilon * m^\epsilon$.

A *positive solution* of Problem 2 is a pair (v^ϵ, m^ϵ), with $v^\epsilon : \mathbb{T}^d \to \mathbb{R}$ and $m^\epsilon : \mathbb{T}^d \to \mathbb{R}$, $v^\epsilon, m^\epsilon \in C^\infty(\mathbb{T}^d)$, and $m^\epsilon > 0$. As we state next, Problem 2 has positive solutions.

Theorem 3. *Suppose that Assumption 1 holds. Then, there exists a positive solution to Problem 2.*

Before proving the preceding theorem, we lay out the notation and establish several preliminary results. For Λ_0 and Λ_1 given positive real numbers, let

$$\mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d) := \left\{ \mu \in \mathcal{M}(\mathbb{T}^d) \mid \Lambda_0 \leq \int_{\mathbb{T}^d} \mu dx \leq \Lambda_1 \right\},$$

where we denote by $\mathcal{M}(\mathbb{T}^d)$ the family of measurable functions on $\mathbb{T}^d$.

Lemma 4. *Let H and g be as in Problem 1. Suppose that Assumption 1 holds. Fix $0 < \Lambda_0 \leq \Lambda_1$ and let*

$$\mu \in \mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d). \tag{5}$$

Then, the Hamilton-Jacobi equation

$$H(x, DU) = g_\epsilon(\mu) - (\eta_\epsilon * (\eta_\epsilon * \mu)^\alpha)\, U + \Delta U \tag{6}$$

has a unique classical solution $U(x;\mu)$. Moreover, the map $\mu \mapsto U(x;\mu)$ is continuous from $\mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d)$ to $C^k(\mathbb{T}^d)$ for any $k \geq 0$.

Proof. To prove that $x \mapsto U(x;\mu)$ is smooth, it is enough to show that U is Lipschitz in the x variable. Once this is done, one can retrieve more regularity by bootstrapping. Because (5) holds, there are constants $c_0, c_1 > 0$, depending only on α, ϵ, and Λ_i such that

$$c_0 \leq \eta_\epsilon * (\eta_\epsilon * \mu)^\alpha \leq c_1.$$

With these bounds, the maximum principle ensures the existence of a unique solution to (6). Assumption (1), implies that

$$\lim_{|p|\to\infty} \frac{|H(x,p) - g_\epsilon(\mu)|^2}{2} + (D_x H(x,p) - D_x g_\epsilon(\mu)) \cdot p = +\infty,$$

uniformly in x. Hence, applying the Bernstein method, we have that the solution $U(x;\mu)$ is a Lipschitz function of x.

The continuity concerning μ as a map from $\mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d)$ to $C(\mathbb{T}^d)$ is a consequence of the stability of viscosity solutions and the uniqueness of the solution. Next, given μ we define $u(x) = U(x;\mu)$, take a sequence $\mu_n \to m$ in $\mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d)$ and let $u_n(x) = U(x;\mu_n)$. Then,

$$|H(x, Du_n) - H(x, Du)| \leq C|Du_n - Du|.$$

Using the Gagliardo-Nirenberg inequality, we obtain

$$\|Du - Du_n\|_{L^2} \leq C\left(\|u\|_{L^\infty} + \|u_n\|_{L^\infty}\right)^{1/2} \|D^2u - D^2u_n\|_{L^2}^{1/2}.$$

Therefore, using the equation once more, we see that

$$\|D^2u - D^2u_n\|_{L^2} \to 0,$$

from which we bootstrap stability with respect to higher derivatives. □

We introduce the dynamic approximation to (4)

$$\begin{cases} H(x, Dv) = g_\epsilon(m) - (\eta_\epsilon * (\eta_\epsilon * m)^\alpha)\, v + \Delta v \\ m_t - \operatorname{div}(mD_pH(x, Dv)) = (1 - \eta_\epsilon * (\eta_\epsilon * m)^\alpha)\, m + \Delta m + \delta. \end{cases} \tag{7}$$

Lemma 5. *Let $b : \mathbb{T}^d \times [0, T] \to \mathbb{R}^d$, with $b \in C([0, T], C^k(\mathbb{T}^d, \mathbb{R}^d))$ for any $k > 0$, and let $\alpha > 0$. Consider the non-linear Fokker-Planck equation*

$$m_t - \operatorname{div}(mb) = (1 - \eta_\epsilon * (\eta_\epsilon * m)^\alpha)\, m + \Delta m + \delta. \tag{8}$$

Then there exist constants $0 < \Lambda_0 < \Lambda_1$ for which $\mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d)$ is invariant with respect to the flow induced by (8), that is, for any $m(\cdot, 0) = m_0 \in \mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d)$ there exists a corresponding solution, m, of (8) with $m(\cdot, t) \in \mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d)$ for all $t \in [0, T]$.

Proof. It suffices to prove the existence of a solution for $m_0 \geq 0$ and $m_0 \in C^\infty(\mathbb{T}^d)$ since the general case follows by approximation.

If T is small enough, the existence of a solution for (8) follows from the semigroup method by writing (8) as

$$m_t - \Delta m = \operatorname{div}(mb) + (1 - \eta_\epsilon * (\eta_\epsilon * m)^\alpha)\, m + \delta,$$

and using the properties of the heat semigroup to establish local existence of a solution $m \in C([0, T], \mathcal{M}(\mathbb{T}^d))$.

Next, we set

$$M(t) = \int_{\mathbb{T}^d} m(x, t)dx \tag{9}$$

and use the notation $\dot{M}(t) = \frac{dM}{dt}(t)$. By integrating (8), we obtain

$$\dot{M}(t) = \delta + M(t) - \int_{\mathbb{T}^d} (\eta_\epsilon * m)^{\alpha+1} dx.$$

Because the Gaussian mollifier η_ϵ is non-vanishing, there exist positive constants, c_0 and c_1 (that may depend on ϵ), such that

$$c_0 M \leq \eta_\epsilon * m \leq c_1 M.$$

Accordingly,

$$\delta + M - \tilde{c_1} M^{\alpha+1} \leq \dot{M} \leq \delta + M - \tilde{c_0} M^{\alpha+1}. \tag{10}$$

From the right-hand side of (10), we obtain that M is bounded up to time T. Consequently, m solves the following linear equation up to time T

$$m_t - \operatorname{div}(mb) = \mu(x,t)m + \Delta m + \delta, \tag{11}$$

with the smooth coefficient

$$\mu(x,t) = 1 - \eta_\epsilon * (\eta_\epsilon * m)^\alpha.$$

Thus, we can continue the solution for arbitrary $T > 0$.

The left-hand side of (10) has a positive root, Λ_0, whereas the right-hand side has a positive root Λ_1. Hence, we obtain the invariance of $\mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d)$.

Then, a simple bootstrapping argument shows that $m \in C([0,T], C^k)$ for any $k > 0$. $\square$

Remark 6. *The solution of* (8) *is continuous in b; that is, if $b_n \to b$ in $C([0,T], C^k(\mathbb{T}^d))$, we have $m_n \to m$ in $C([0,T], \mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d))$.*

Lemma 7. *Let H and g be as in Problem 1. Suppose that Assumption 1 holds. Then, for any $m_0 \in \mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d)$, there is a solution (v,m) to* (7) *with the initial condition $m(x,0) = m_0$ such that $m \in C^\infty(\mathbb{T}^d \times (0,+\infty))$.*

Proof. The existence of solutions to (7) follows from a standard fixed-point argument. First, we define $\Psi : C([0,T], \mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d)) \to C([0,T], \mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d))$ as follows. Given $\mu \in C([0,T], \mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d))$, we let U be such that

$$H(x, DU) = g_\epsilon(\mu) - (\eta_\epsilon * (\eta_\epsilon * \mu)^\alpha)\, U + \Delta U. \tag{12}$$

According to Lemma 5, there exists a solution, m, of

$$\begin{cases} m_t - \operatorname{div}(m D_p H(x, DU)) = (1 - \eta_\epsilon * (\eta_\epsilon * m)^\alpha)\, m + \Delta m + \delta \\ m(\cdot, 0) = m_0. \end{cases} \tag{13}$$

The map $m \mapsto U$ is continuous and compact as a map from $C([0,T], \mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d))$ to $C([0,T], C^k(\mathbb{T}^d))$. The map $U \mapsto m$ is continuous and compact as a map from $C([0,T], C^k(\mathbb{T}^d))$ to $C([0,T], \mathcal{M}_{\Lambda_0,\Lambda_1}(\mathbb{T}^d))$. Thus, according to Schauder's fixed-point theorem, Ψ has a fixed point. The smoothness of the solution follows from standard regularity theory for (12) and (13). $\square$

Lemma 8. *Let H and g be as in Problem 1. Suppose that Assumption 1 holds. Then, for any initial condition m_0, there exists C_T, that depends on T, such that the solution of (8) satisfies*

$$\int_{\mathbb{T}^d} |Dm(x,T)|^2 dx \leq C_T.$$

Hence, the map $m_0 \in L^2(\mathbb{T}^d) \mapsto m(\cdot,T) \in H^1(\mathbb{T}^d)$ is compact.

Proof. We provide some preliminary results first. Interpolation inequality between L^1 and L^{2^*} (if $d=2$, we set as usual 2^* to be a sufficiently large real number) gives, for θ such that $1/2 = \theta + (1-\theta)/2^*$,

$$\|m\|_2 \leq \|m\|_1^{\theta} \|m\|_{2^*}^{1-\theta};$$

that is,

$$\int m^2 \leq \left(\int m\right)^{\frac{2^*-2}{2^*-1}} \left(\int m^{2^*}\right)^{\frac{1}{2^*-1}} \leq C\|m\|_{2^*}^{\frac{2^*}{2^*-1}}.$$

Since $m \in \mathcal{M}_{\Lambda_0,\Lambda_1}$, we have, using the Sobolev's inequality that

$$\|m\|_{L^{2^*}}^{\frac{2^*}{2^*-1}} \leq C\, \|m\|_{W^{1,2}}^{\frac{2^*}{2^*-1}} \leq C\left(\int m^2 + \int |Dm|^2\right)^{\frac{2^*}{2(2^*-1)}},$$

from which we obtain

$$\int m^2 \leq C + C\left(\int |Dm|^2\right)^{\frac{2^*}{2(2^*-1)}}.$$

Again, since $m \in \mathcal{M}_{\Lambda_0,\Lambda_1}$, but this time using the Gagliardo-Nirenberg interpolation inequality with $1/2 = 1/d + (1/2 - 2/d)(1-\theta) + \theta$, that is, $\theta = \frac{2}{4+d} = \frac{2^*-2}{2^*3-4}$, we have

$$\left(\int |Dm|^2\right)^{\frac{1}{2}} \leq C\left(\int m\right)^{\theta} \left(\int |\Delta m|^2\right)^{\frac{1-\theta}{2}}$$

so that

$$\int |\Delta m|^2 \geq \frac{1}{C}\left(\int |Dm|^2\right)^{\frac{4-2^*3}{2-2^*2}} = \frac{1}{C}\left(\int |Dm|^2\right)^{1+\frac{2}{2+d}},$$

where we note that $\frac{2}{2+d} > 0$.

We observe that, if m satisfies (11) with $b \in C^\infty(\mathbb{T}^d \times [0,T])$, then the function $t \mapsto \int_{\mathbb{T}^d} |Dm(x,t)|^2 dx$ is a subsolution of

$$Z' = C - \frac{1}{C} Z^{1+\theta}, \tag{14}$$

for some $\theta > 0$. This implies that

$$t \mapsto \int |Dm(x,t)|^2 dx$$

has a non-linear, faster-than-exponential, decay. In fact,

$$\begin{aligned}
\frac{d}{dt}\int \frac{|Dm|^2}{2} &= \int DmDm_t = -\int \Delta m m_t \\
&= -\int \Delta m \left[\mu m - \operatorname{div}(mb) + \Delta m + \delta\right] \\
&= -\int \Delta m \left[m(\mu - \operatorname{div} b) - Dm\cdot b + \Delta m\right] \\
&\leq \|\mu - \operatorname{div} b\|_{L^\infty}\left(C_{\kappa_1}\int m^2 + \kappa_1 \int |\Delta m|^2\right) \\
&+ \|b\|_{L^\infty}\left(C_{\kappa_2}\int |Dm|^2 + \kappa_2\int |\Delta m|^2\right) - \int |\Delta m|^2 \\
&\leq C\left(\int m^2 + \int |Dm|^2\right) - (1-\kappa_3)\int |\Delta m|^2 \\
&\leq C + C\left(\int |Dm|^2\right)^{\frac{2^*}{2(2^*-1)}} - (1-\kappa_3)\frac{1}{C}\left(\int |Dm|^2\right)^{\frac{4-2^*3}{2-2^*2}} \\
&\leq C - \frac{1}{C}\left(\int |Dm|^2\right)^{1+\theta},
\end{aligned}$$

for κ_1 and κ_2 sufficiently small such that $\kappa_3 := \|\mu - \operatorname{div} b\|_{L^\infty}\kappa_1 + \|b\|_{L^\infty}\kappa_2 < 1$ and $\theta > 0$. Let $\kappa_4 > 0$ be the constant solution of (14). Let Z solve (14) with initial condition $Z(0) = \int |Dm_0|^2 > \kappa_4$ and consider $\kappa_5 = Z(T)$, then

$$\int |Dm(x,T)|^2 dx \leq \max\{\kappa_4, \kappa_5\}.$$

□

Lemma 9. *Let H and g be as in Problem 1. Suppose that Assumption 1 holds. Then, for any $T > 0$, (7) admits a smooth T-periodic solution.*

Proof. It suffices to see that the map $m(x,0) \mapsto m(x,T)$, as defined in Lemma 8, has a fixed point. The bounds in Lemma 8 also give that

$$\left\{m_0 \in L^2(\mathbb{T}^d) \mid m_0 = \lambda m(\cdot, T) \text{ for some } 0 \leq \lambda \leq 1\right\}$$

is bounded. So, Schaefer's fixed-point theorem applies. We get $H^k(\mathbb{T}^d \times [0,T])$ regularity, for any k, by bootstrapping. □

Lemma 10. *Let H and g be as in Problem 1. Suppose that Assumption 1 holds. Then there exists C such that any T-periodic solutions of the second equation in 7 satisfies*

$$\int_{\mathbb{T}^d} |Dm(x,T)|^2 dx \leq C$$

for any initial condition $m_0 \in H^2(\mathbb{T}^d)$.

Proof. Fix T and let m be T-periodic solution, which exists from Lemma 9. Let $C = \kappa_4 + \epsilon_0$ for some $\epsilon_0 > 0$. For a sufficiently large k, we iterate the map from Lemma 8 k times to get

$$\int_{\mathbb{T}^d} |Dm(x,T)|^2 dx = \int_{\mathbb{T}^d} |Dm(x,kT)|^2 dx \leq C_{kT} < C.$$

□

Proof of Theorem 3. Consider a sequence $T_n = \frac{1}{n}$ and a corresponding sequence of T_n-periodic solutions $m_n \in H^k(\mathbb{T}^d \times [0,1]), \forall k$ given by Lemma 9. Consider the corresponding solution, (v_n, m_n), to (7). Set $b_n = D_p H(x, Dv_n)$ in Lemmas 5 and 8. Since $m_n \in C^k(\mathbb{T}^d \times [0,1]), \forall k$, Lemma 10 and Ascoli-Arzela theorem implies that $m_n \to m$ in $C^\infty(\mathbb{T}^d \times [0,1])$ and m is constant in time ($(m_n)_t \to 0$ as $n \to \infty$). □

4 Uniform estimates

In this section, we prove estimates for solutions of Problem 2. These hold uniformly in ϵ and, thus, can be used to study the limit as $\epsilon \to 0$. Because (4) does not possess the standard MFG adjoint structure, our estimates are distinct from the ones in the literature as in [17, 16, 21, 19]. For example, m^ϵ may not be a probability measure, and $\int_{\mathbb{T}^d} v^\epsilon$ cannot be bounded using an elementary argument. Our first task, in Proposition 11 below, is to obtain integral bounds for m^ϵ. Subsequently, we obtain various bounds for the value function in terms of $\int_{\mathbb{T}^d} v^\epsilon$ and the norms of m^ϵ. These are summarized in Corollary 18. Finally, in Section 4.3, we control m^ϵ by below, which then allows us to obtain closure in our bounds, see Corollary 21.

4.1 Estimates for the regularized system

Because we do not assume that m^ϵ is a probability measure, we begin by bounding m^ϵ.

Proposition 11. *Let* (v^ϵ, m^ϵ) *solve Problem 2. Then*

$$\int_{\mathbb{T}^d} m^\epsilon dx \leq C \quad \textit{and} \quad \int_{\mathbb{T}^d} (m^\epsilon_\epsilon)^{\alpha+1} dx \leq C.$$

Proof. After integrating the second equation of (4) on $\mathbb{T}^d$, by parts, we get

$$\int m dx = \int m^{1+\alpha} dx + \delta.$$

Young's inequality for the product $m \cdot 1$ states that

$$m \leq \frac{1}{1+\alpha} m^{1+\alpha} + 1.$$

With this we show that $\int m^{1+\alpha} \leq C$ and therefore $\int m \leq C$. □

Next, we apply the energy method introduced in [26] to obtain higher integrability for m^ϵ and Sobolev estimates for v^ϵ.

Proposition 12. *Let (v^ϵ, m^ϵ) solve Problem 2. Then*

$$\int_{\mathbb{T}^d} H(x, Dv^\epsilon)dx = \int_{\mathbb{T}^d} g(m_\epsilon^\epsilon) - (m_\epsilon^\epsilon)^\alpha v_\epsilon^\epsilon dx. \tag{15}$$

Moreover, if Assumptions 2 and 3 hold, we have

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{T}^d} m_\epsilon^\epsilon g(m_\epsilon^\epsilon)dx + \int_{\mathbb{T}^d} (1 + cm^\epsilon)H(x, Dv^\epsilon)dx \\ &\leq C + \int_{\mathbb{T}^d} v^\epsilon(m^\epsilon + \delta) - (m_\epsilon^\epsilon)^\alpha v_\epsilon^\epsilon dx \end{aligned} \tag{16}$$

and

$$\int_{\mathbb{T}^d} (1 + cm^\epsilon)H(x, Dv^\epsilon)dx \leq C + \int_{\mathbb{T}^d} v^\epsilon(m^\epsilon + \delta) - (m_\epsilon^\epsilon)^\alpha v_\epsilon^\epsilon dx. \tag{17}$$

Proof. We integrate the first equation in (4) to get (15). Next, we multiply the first identity in (4) by m^ϵ. Then, using integration by parts, we obtain

$$\int_{\mathbb{T}^d} H(x, Dv^\epsilon)m^\epsilon dx = \int_{\mathbb{T}^d} m^\epsilon g_\epsilon(m^\epsilon) - (\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha)\, v^\epsilon m^\epsilon + v^\epsilon \Delta m^\epsilon dx.$$

Subsequently, we multiply the second equation in (4) by v^ϵ and integrate by parts to conclude that

$$\int_{\mathbb{T}^d} D_pH \cdot Dv^\epsilon m^\epsilon dx = \int_{\mathbb{T}^d} (1 - \eta_\epsilon * (m_\epsilon^\epsilon)^\alpha)\, m^\epsilon v^\epsilon + v^\epsilon\delta + v^\epsilon \Delta m^\epsilon dx.$$

By combining the two previous identities, we obtain

$$\int_{\mathbb{T}^d} m^\epsilon g_\epsilon(m^\epsilon) + m^\epsilon\,(D_pH \cdot Dv^\epsilon - H(x, Dv^\epsilon))\, dx = \int_{\mathbb{T}^d} v^\epsilon(m^\epsilon + \delta)dx.$$

From Assumption 3, we conclude that

$$\int_{\mathbb{T}^d} m_\epsilon^\epsilon g(m_\epsilon^\epsilon) + cm^\epsilon H(x, Dv^\epsilon)dx \leq C\int_{\mathbb{T}^d} m^\epsilon dx + \int_{\mathbb{T}^d} v^\epsilon(m^\epsilon + \delta)dx. \tag{18}$$

Therefore, adding (15) to the inequality above, we have

$$\int_{\mathbb{T}^d} m_\epsilon^\epsilon g(m_\epsilon^\epsilon) + H(x, Dv^\epsilon)(1 + cm^\epsilon)dx \leq C + \int_{\mathbb{T}^d} v^\epsilon(m^\epsilon + \delta) + g(m_\epsilon^\epsilon) - (m_\epsilon^\epsilon)^\alpha v_\epsilon^\epsilon dx.$$

Finally, according to Assumption 2, there exists $C > 0$ such that $g(y) \leq \frac{1}{2}yg(y) + C$ for any $y > 0$. Thus, (16) follows. Finally, because $yg(y) \geq -C$, we obtain (17). □

4.2 Estimates for the value function

Next, we state a Poincaré-like inequality that controls the difference between the average of a function with respect to two (distinct) probability measures.

Proposition 13. *Let $p > 1$ if $d = 2$ or $p \geq \frac{2d}{d+2}$ if $d > 2$. Then, for any non-negative $\theta \in L^p(\mathbb{T}^d)$, we have*

$$\left| \int_{\mathbb{T}^d} \theta v dx - \left(\int_{\mathbb{T}^d} \theta dx \right) \left(\int_{\mathbb{T}^d} v dx \right) \right| \leq C \|\theta\|_{L^p} \|Dv\|_{L^2}. \tag{19}$$

Proof. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Note that for $d > 2$, $q = \frac{p}{p-1} \leq \frac{2d}{d-2} =: 2^*$, where 2^* is the Sobolev conjugated exponent to 2. By Hölder's inequality, we have

$$\begin{aligned} \left| \int_{\mathbb{T}^d} \theta v dx - \left(\int_{\mathbb{T}^d} \theta dx \right) \left(\int_{\mathbb{T}^d} v dx \right) \right| &\leq \int_{\mathbb{T}^d} \theta \left| v(x) - \int_{\mathbb{T}^d} v dx \right| dx \\ &\leq \|\theta\|_{L^p} \left\| v(x) - \int_{\mathbb{T}^d} v dx \right\|_{L^q}. \end{aligned}$$

Finally, (19) follows from the Gagliardo-Nirenberg-Sobolev inequality. □

Now, we use the prior Proposition to establish additional estimates on Dv^ϵ and v^ϵ. The main difficulty is that, up to now, we do not control

$$\int_{\mathbb{T}^d} v^\epsilon (m^\epsilon + \delta) - (m^\epsilon_\epsilon)^\alpha v^\epsilon_\epsilon dx.$$

If we knew that the preceding term was bounded, Proposition 12 would imply the following estimates. However, since this is not the case, we need to circumvent this difficulty.

Proposition 14. *Let (v^ϵ, m^ϵ) solve Problem 2. Assume that Assumptions 2-5 hold. Let $1 < p < 1 + \frac{1}{\alpha}$ if $d = 2$ or $\frac{2d}{d+2} \leq p < 1 + \frac{1}{\alpha}$ if $d > 2$. Then, we have*

$$\|Dv^\epsilon\|^2_{L^2} + \int_{\mathbb{T}^d} |Dv^\epsilon|^2 m^\epsilon + m^\epsilon_\epsilon g(m^\epsilon_\epsilon) dx \leq C + C \left| \int_{\mathbb{T}^d} v^\epsilon dx \right| + C \|m^\epsilon + \delta\|_{L^p} \|Dv^\epsilon\|_{L^2}. \tag{20}$$

Proof. Let p be as in the statement. After applying Proposition 13 to each of the terms $\int (m + \delta) v$ and $\int m^\alpha v$, we bound the right-hand side of estimate (16) in Proposition 12 as

$$\begin{aligned} \int_{\mathbb{T}^d} v^\epsilon (m^\epsilon + \delta) - v^\epsilon_\epsilon (m^\epsilon_\epsilon)^\alpha dx &\leq \left(\int_{\mathbb{T}^d} m^\epsilon + \delta - (m^\epsilon_\epsilon)^\alpha dx \right) \int_{\mathbb{T}^d} v^\epsilon dx \\ &\quad + C \left(\|m^\epsilon + \delta\|_{L^p} + \|(m^\epsilon_\epsilon)^\alpha\|_{L^p} \right) \|Dv^\epsilon\|_{L^2}. \end{aligned}$$

From Proposition 11, we have $\|m^\epsilon\|_{L^1} \leq C$ and $\|(m^\epsilon_\epsilon)^\alpha\|_{L^p} \leq C$ provided $\alpha p \leq \alpha + 1$; that is, $p < 1 + \frac{1}{\alpha}$.

For $d > 2$, the condition $\frac{2d}{d+2} < p < 1+\frac{1}{\alpha}$ is satisfiable if Assumption 5 holds; that is $\alpha < \frac{d+2}{d-2}$. For $d = 2$, there is no additional constraint. The result follows from a weighted Cauchy inequality on the term $C\|Dv^\epsilon\|$ and Assumption 4 to bound the left-hand side in (16). □

For $\epsilon = 0$, estimate (16) in Proposition 12 provides control for $\int mg(m)$. However, for $\epsilon > 0$, we only control $\int m_\epsilon^\epsilon g(m_\epsilon^\epsilon)$, which is not enough for our purposes. Thus, we use the ellipticity of the second equation in (4) to obtain further estimates for m^ϵ.

Proposition 15. *Let (v^ϵ, m^ϵ) solve Problem 2. Suppose that Assumptions 2–6 hold. Let $1 < p < 1+\frac{1}{\alpha}$ if $d = 2$ or $\frac{2d}{d+2} \leq p < 1+\frac{1}{\alpha}$ if $d > 2$. Then, we have*

$$\int_{\mathbb{T}^d} \left|D\left((m^\epsilon)^{\frac{1}{2}}\right)\right|^2 \leq C + C\left|\int_{\mathbb{T}^d} v^\epsilon dx\right| + C\left\|m^\epsilon+\delta\right\|_{L^p}\left\|Dv^\epsilon\right\|_{L^2}. \tag{21}$$

Proof. After multiplying the second identity in (4) by $\ln m^\epsilon$, we integrate by parts to get

$$\begin{aligned}
&\int_{\mathbb{T}^d} \frac{|Dm^\epsilon|^2}{m^\epsilon} + (m_\epsilon^\epsilon)^\alpha \eta_\epsilon * (m^\epsilon \ln m^\epsilon) dx \\
&= \int_{\mathbb{T}^d} (\ln m^\epsilon)(m^\epsilon+\delta) - D_pH(x, Dv^\epsilon)\sqrt{m^\epsilon}\cdot\frac{Dm^\epsilon}{\sqrt{m^\epsilon}} dx \\
&\leq \int_{\mathbb{T}^d} m^\epsilon \ln m^\epsilon + \delta \ln m^\epsilon + C\left|D_pH(x, Dv^\epsilon)\right|^2 m^\epsilon + \frac{|Dm^\epsilon|^2}{4m^\epsilon} dx \\
&\leq C + \int_{\mathbb{T}^d} m^\epsilon \ln m^\epsilon + C\left|Dv^\epsilon\right|^2 m^\epsilon + \frac{|Dm^\epsilon|^2}{4m^\epsilon} dx,
\end{aligned}$$

using Assumption 6 in the second inequality above. Therefore, according to Proposition 14, we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^d} \frac{|Dm^\epsilon|^2}{m^\epsilon} + (m_\epsilon^\epsilon)^\alpha \eta_\epsilon * (m^\epsilon \ln m^\epsilon) dx \\
&\leq C + C\int_{\mathbb{T}^d} m^\epsilon \ln m^\epsilon dx + C\left|\int_{\mathbb{T}^d} v^\epsilon dx\right| + C\left\|m^\epsilon\right\|_p \left\|Dv^\epsilon\right\|_2 + \int_{\mathbb{T}^d} \frac{|Dm^\epsilon|^2}{4m^\epsilon} dx.
\end{aligned}$$

Next, we take into account that $m^\epsilon \ln m^\epsilon$ is bounded by below and the estimate in Proposition 11 to conclude that

$$\int_{\mathbb{T}^d} (m_\epsilon^\epsilon)^\alpha \eta_\epsilon * (m^\epsilon \ln m^\epsilon) dx \geq -C.$$

Finally, we note that Assumption 2 gives $m^\epsilon \ln m^\epsilon \leq m^\epsilon g(m^\epsilon)$ and, therefore, part of estimate (20) in Proposition 14 yields (21). □

Now, we combine the previous results to get preliminary bounds on $\int v^\epsilon$.

Proposition 16. *Let (v^ϵ, m^ϵ) solve Problem 2. Suppose that Assumptions 2-5 hold. Then, we have*

$$c\|Dv^\epsilon\|_{L^2}^2 + \lambda^\epsilon \int_{\mathbb{T}^d} v^\epsilon dx \le C + \int_{\mathbb{T}^d} g(m_\epsilon^\epsilon)dx, \tag{22}$$

where

$$\lambda^\epsilon = \int_{\mathbb{T}^d} (m_\epsilon^\epsilon)^\alpha dx. \tag{23}$$

Furthermore, for $1 < p < 1 + \frac{1}{\alpha}$ if $d = 2$ or $\frac{2d}{d+2} \le p < 1 + \frac{1}{\alpha}$ if $d > 2$, we have

$$\left| \int_{\mathbb{T}^d} v^\epsilon dx \right| \le C + C \left\| m^\epsilon{+}\delta \right\|_{L^p} \left\| Dv^\epsilon \right\|_{L^2} + \int_{\mathbb{T}^d} v^\epsilon dx,$$

Proof. Using Assumption 4 in identity (15), we get

$$c\|Dv^\epsilon\|_{L^2}^2 \le C + \int_{\mathbb{T}^d} g(m_\epsilon^\epsilon)dx - \int_{\mathbb{T}^d} v_\epsilon^\epsilon (m_\epsilon^\epsilon)^\alpha dx.$$

Proposition 13, applied to the term $\int_{\mathbb{T}^d} v_\epsilon^\epsilon (m_\epsilon^\epsilon)^\alpha dx$, gives

$$\begin{aligned} c\|Dv^\epsilon\|_{L^2}^2 + \lambda^\epsilon \int v^\epsilon dx &\le C + \int_{\mathbb{T}^d} g(m_\epsilon^\epsilon)dx + C\|(m_\epsilon^\epsilon)^\alpha\|_{L^p} \left\| D(v_\epsilon^\epsilon) \right\|_{L^2} \\ &\le C + \int_{\mathbb{T}^d} g(m_\epsilon^\epsilon)dx + \frac{c}{2} \left\| Dv^\epsilon \right\|_{L^2}^2, \end{aligned}$$

where we used Proposition 11, the condition on p, and Assumption 5 to bound $\|(m_\epsilon^\epsilon)^\alpha\|_p$. This gives the first estimate in the statement.

Now, from (18), Proposition 11, and Assumption 4, we obtain

$$\int_{\mathbb{T}^d} v^\epsilon (m^\epsilon + \delta) dx \ge -C.$$

Application of Proposition 13 on the above estivate gives

$$\eta \int_{\mathbb{T}^d} v^\epsilon dx \ge -C - C \left\| m^\epsilon + \delta \right\|_{L^p} \left\| Dv \right\|_{L^2},$$

where $\eta = \int_{\mathbb{T}^d} m^\epsilon + \delta dx$. Because $\eta > \delta > 0$, we obtain

$$\int_{\mathbb{T}^d} v^\epsilon dx \ge -C - C \left\| m^\epsilon{+}\delta \right\|_{L^p} \left\| Dv^\epsilon \right\|_{L^2}.$$

The analysis of the cases where $\int v^\epsilon$ is negative or positive gives the result. □

Remark 17. *If Assumption 7 holds, i.e. $0 < \alpha < 1$, then*

$$\lambda^\epsilon \ge \int_{\mathbb{T}^d} (m^\epsilon)^\alpha dx. \tag{24}$$

Because η_ϵ is a probability measure, Jensen's inequality gives

$$\begin{aligned}(m_\epsilon^\epsilon)^\alpha &= \left(\int_{\mathbb{R}^d} m^\epsilon(x-y)\eta_\epsilon(y)dy\right)^\alpha \\ &\geq \int_{\mathbb{R}^d} (m^\epsilon(x-y))^\alpha \eta_\epsilon(y)dy = \eta_\epsilon * ((m^\epsilon)^\alpha)\,.\end{aligned}$$

In the next Corollary, we record a few elementary consequences of the estimates in this section.

Corollary 18. *Let (v^ϵ, m^ϵ) solve Problem 2. Suppose that Assumptions 2–7 hold. Let $1 < p < 1+\frac{1}{\alpha}$ if $d=2$ or $\frac{2d}{d+2} \leq p < 1+\frac{1}{\alpha}$ if $d>2$. Then, the following estimates hold:*

$$\int_{\mathbb{T}^d} \left|D\left((m^\epsilon)^{\frac{1}{2}}\right)\right|^2 dx \leq C + C\int_{\mathbb{T}^d} v^\epsilon dx + C\left\|m^\epsilon + \delta\right\|_{L^p}\left\|Dv^\epsilon\right\|_{L^2} \tag{25}$$

and

$$\left\|Dv^\epsilon\right\|_{L^2}^2 + \int_{\mathbb{T}^d} \left|Dv^\epsilon\right|^2 m^\epsilon + m_\epsilon^\epsilon g(m_\epsilon^\epsilon) dx \leq C + C\int_{\mathbb{T}^d} v^\epsilon dx + C\left\|m^\epsilon + \delta\right\|_{L^p}\left\|Dv^\epsilon\right\|_{L^2}. \tag{26}$$

Proof. Combining the second estimate in Proposition 16 with the estimate in Proposition 15, we obtain (25). Proceeding analogously with the estimate in Proposition 14, we obtain (26). □

4.3 Lower bounds for the density

We now establish upper bounds for $\frac{1}{m^\epsilon}$ and, hence, prove that m^ϵ is bounded by below. These bounds are useful to estimate $\int v^\epsilon dx$.

Proposition 19. *Let (v^ϵ, m^ϵ) solve Problem 2. Suppose that Assumption 6 holds. Then,*

$$\int_{\mathbb{T}^d} \frac{\delta}{m^\epsilon} dx + \frac{1}{2}\left\|D(\ln m^\epsilon)\right\|_{L^2}^2 \leq C + C\left\|Dv^\epsilon\right\|_{L^2}^2. \tag{27}$$

Proof. First, we divide the second identity in (4) by m^ϵ and then integrate by parts to obtain

$$\int_{\mathbb{T}^d} \frac{\delta}{m^\epsilon} + \left|D(\ln m^\epsilon)\right|^2 dx = \int_{\mathbb{T}^d} (m_\epsilon^\epsilon)^\alpha - 1 - D_pH(x, Dv^\epsilon)\cdot D\left(\ln m^\epsilon\right) dx.$$

Then, Cauchy's inequality yields

$$\int_{\mathbb{T}^d} \frac{\delta}{m^\epsilon} dx + \left\|D(\ln m^\epsilon)\right\|_{L^2}^2 \leq C + \int_{\mathbb{T}^d} \frac{1}{2}\left|D_pH(x, Dv^\epsilon)\right|^2 dx + \frac{1}{2}\left\|D\left(\ln m^\epsilon\right)\right\|_{L^2}^2,$$

which, combined with Assumption 6, implies the result. □

Now, we give bounds by below on λ^ϵ

Proposition 20. *Let (v^ϵ, m^ϵ) solve Problem 2. Suppose that Assumptions 6 and 7 hold. For λ^ϵ as in (23), we have*

$$\frac{1}{\lambda^\epsilon} \leq C + C\|Dv^\epsilon\|_{L^2}^{2\alpha}. \tag{28}$$

Proof. Let $\beta = \frac{\alpha}{1+\alpha}$, $p = 1 + \alpha$ and $q = \frac{\alpha+1}{\alpha}$. Then, we have

$$1 = \int_{\mathbb{T}^d} (m^\epsilon)^\beta (m^\epsilon)^{-\beta} \leq \left(\int_{\mathbb{T}^d} \frac{1}{m^\epsilon} \right)^{\frac{1}{q}} \left(\int_{\mathbb{T}^d} (m^\epsilon)^\alpha \right)^{\frac{1}{p}}.$$

Now, (24) gives

$$\frac{1}{\lambda^\epsilon} \leq \frac{1}{\int_{\mathbb{T}^d} (m^\epsilon)^\alpha} \leq \left(\int_{\mathbb{T}^d} \frac{1}{m^\epsilon} \right)^\alpha.$$

Hence, using Proposition 19, we obtain (28). □

Corollary 21. *Let (v^ϵ, m^ϵ) solve Problem 2. Suppose that Assumptions 2–8 hold and that $d = 2$ or $d = 3$. Then, we have the following estimates*

$$\int_{\mathbb{T}^d} \left| D\left((m^\epsilon)^{\frac{1}{2}} \right) \right|^2 \leq C, \tag{29}$$

$$\left| \int_{\mathbb{T}^d} v^\epsilon dx \right| \leq C, \tag{30}$$

$$\|Dv^\epsilon\|_{L^2}^2 + \int_{\mathbb{T}^d} |Dv^\epsilon|^2 m^\epsilon + m_\epsilon^\epsilon g(m_\epsilon^\epsilon) dx \leq C, \tag{31}$$

and

$$\int_{\mathbb{T}^d} \frac{\delta}{m^\epsilon} dx + \frac{1}{2} \|D(\ln m^\epsilon)\|_{L^2}^2 \leq C. \tag{32}$$

Proof. The lower bound for λ^ϵ, (24) in Remark 17, implies that $\lambda^\epsilon > 0$. Thus, using (22) in Proposition 16 divided by λ^ϵ and using the bound given by (28) in Proposition 20, we obtain

$$\int_{\mathbb{T}^d} v^\epsilon dx \leq \left(C + \int_{\mathbb{T}^d} g(m_\epsilon^\epsilon) dx \right) \left(C + C\|Dv^\epsilon\|_{L^2}^{2\alpha} \right).$$

Thus, using Young's inequality with weights, for any $r, s > 1$ such that $\frac{1}{r} + \frac{1}{s} = 1$ and any $\mu > 0$, there exists a positive constant C, that depends on μ, such that

$$\int_{\mathbb{T}^d} v^\epsilon dx \leq C + C \left(\int_{\mathbb{T}^d} g(m_\epsilon^\epsilon) dx \right)^r + \mu\|Dv^\epsilon\|_{L^2}^{2\alpha s}. \tag{33}$$

Fix $1 < p < 1 + \frac{1}{\alpha}$ if $d = 2$ or $\frac{2d}{d+2} \leq p < 1 + \frac{1}{\alpha}$ if $d > 2$. According to estimate (26) in Corollary 18, we conclude that for any $\mu > 0$ there exists $C > 0$ such that

$$\begin{aligned}
&\|Dv^\epsilon\|_{L^2}^2 + \int_{\mathbb{T}^d} |Dv^\epsilon|^2 m^\epsilon + m_\epsilon^\epsilon g(m_\epsilon^\epsilon) dx \\
&\leq C + C\left(\int_{\mathbb{T}^d} g(m_\epsilon^\epsilon) dx\right)^r + \mu \|Dv^\epsilon\|_{L^2}^{2\alpha s} + C\,\|m^\epsilon + \delta\|_{L^p}^2 .
\end{aligned}$$

Now, we observe that $\|m^\epsilon + \delta\|_{L^p}^2 \leq C + C\,\|m^\epsilon\|_{L^p}^2$. Next, we select $2\alpha s = 2$, thus $r = \frac{1}{1-\alpha}$, and we also select μ small enough to get

$$\begin{aligned}
&c\,\|Dv^\epsilon\|_{L^2}^2 + \int_{\mathbb{T}^d} |Dv^\epsilon|^2 m^\epsilon dx + \int_{\mathbb{T}^d} m_\epsilon^\epsilon g(m_\epsilon^\epsilon) dx \\
&\leq C + C\left(\int_{\mathbb{T}^d} g(m_\epsilon^\epsilon) dx\right)^{\frac{1}{1-\alpha}} + C\,\|m^\epsilon\|_{L^p}^2 .
\end{aligned}$$

Therefore, because Assumption 8 holds, we have, for any $\beta > 0$, that there exists $C > 0$ such that

$$\left(\int_{\mathbb{T}^d} g(m_\epsilon^\epsilon) dx\right)^{\frac{1}{1-\alpha}} \leq C + \beta \int_{\mathbb{T}^d} m_\epsilon^\epsilon g(m_\epsilon^\epsilon) dx$$

and, consequently,

$$\begin{aligned}
&c\,\|Dv^\epsilon\|_{L^2}^2 + \int_{\mathbb{T}^d} |Dv^\epsilon|^2 m^\epsilon dx + c\int_{\mathbb{T}^d} m_\epsilon^\epsilon g(m_\epsilon^\epsilon) dx \\
&\leq C + C\,\|m^\epsilon\|_{L^p}^2 .
\end{aligned} \tag{34}$$

Next, we use the previous estimate in (33), with the same choice of s and r, to conclude that

$$\int_{\mathbb{T}^d} v^\epsilon dx \leq C + C\,\|m^\epsilon\|_{L^p}^2 . \tag{35}$$

The two preceding estimates combined with (21) give

$$\int_{\mathbb{T}^d} \left| D\left((m^\epsilon)^{\frac{1}{2}}\right)\right|^2 \leq C + C\,\|m^\epsilon\|_{L^p}^2 .$$

Then, using an interpolation inequality with κ as the interpolating parameter, $0 \leq \kappa \leq 1$ and

$$\frac{1}{p} = \kappa + \frac{1-\kappa}{2^*/2}, \tag{36}$$

we have

$$\int_{\mathbb{T}^d} \left| D\left((m^\epsilon)^{\frac{1}{2}}\right)\right|^2 \leq C + C\|m^\epsilon\|_{L^1}^{2\kappa} \|m^\epsilon\|_{L^{\frac{2^*}{2}}}^{2(1-\kappa)}.$$

Using the Gagliardo-Nirenberg-Sobolev inequality (with the convention that, for $d = 2$, 2^* stands for an arbitrarily large number), we obtain

$$\int_{\mathbb{T}^d} \left| D\left((m^\epsilon)^{\frac{1}{2}}\right)\right|^2 \leq C + C\|m^\epsilon\|^{2(1-\kappa)}_{L^{\frac{2^*}{2}}} \leq C + C\left(\int_{\mathbb{T}^d} \left| D\left((m^\epsilon)^{\frac{1}{2}}\right)\right|^2\right)^{2(1-\kappa)}$$

For $d = 2$ or $d = 3$ and $0 < \alpha < \min\left\{1, \frac{d+2}{d-2}\right\}$, elementary computations show that there exists $\frac{2d}{d+2} < p < 1 + \frac{1}{\alpha}$ and $0 \leq \kappa \leq 1$ satisfying (36) such that $2(1-\kappa) < 1$, and (29) also follows. Therefore, if $d = 2$ or $d = 3$, $\|m^\epsilon\|_p$ is bounded.

Thus, (30) follows from (35) and (31) follows from (34). Finally, (32) results from (27) combined with (31). □

The methods used above are, unfortunately, not enough to guarantee (29) for $d > 3$.

5 Regularity by the adjoint method

Here, we use the non-linear adjoint method from [8] to establish additional regularity for the solutions. This technique is similar to the one used in [17, 16, 21]. First, we introduce $\rho^\epsilon : \mathbb{T}^d \times [0,1] \to \mathbb{R}^+$ as the adjoint variable. For that, we fix a solution (v^ϵ, m^ϵ) of Problem 2 and consider the time-dependent PDE

$$\begin{cases} \rho^\epsilon_t - \operatorname{div}\left(D_pH(x, Dv^\epsilon)\rho^\epsilon\right) = -\eta_\epsilon * (m^\epsilon_\epsilon)^\alpha \rho^\epsilon + \Delta\rho^\epsilon, & \text{in } \mathbb{T}^d \times [0,1] \\ \rho^\epsilon(\cdot, 0) = \delta_{x_0}, & \text{on } \mathbb{T}^d \end{cases} \tag{37}$$

for some $x_0 \in \mathbb{T}^d$. The first equation in (37) is the adjoint of the linearization in v^ϵ of

$$-v^\epsilon_t + H(x, Dv^\epsilon) = g_\epsilon(m^\epsilon) - \eta_\epsilon * (m^\epsilon_\epsilon)^\alpha v^\epsilon + \Delta v^\epsilon. \tag{38}$$

Note that if (v^ϵ, m^ϵ) solves Problem 2 then v^ϵ solves (38) since $v^\epsilon_t = 0$. Next, we record some elementary properties of the solution of (37).

Proposition 22. *Consider the setting of Problem 2 and let ρ^ϵ solve* (37)*. Then, $\rho^\epsilon_\epsilon \geq 0$ and $\int_{\mathbb{T}^d} \rho^\epsilon(x,t) \leq 1$. Hence, $\|\rho^\epsilon(\cdot,t)\|_{L^1} \leq 1$ for all $t \in [0,1]$.*

Proof. By the maximum principle, $\rho^\epsilon \geq 0$ and, as a consequence, $\rho^\epsilon_\epsilon \geq 0$. We integrate the first equation in (37) and get

$$\frac{d}{dt}\int_{\mathbb{T}^d} \rho^\epsilon dx = -\int_{\mathbb{T}^d} (m^\epsilon_\epsilon)^\alpha \rho^\epsilon_\epsilon dx \leq 0.$$

Therefore, $\int_{\mathbb{T}^d} \rho^\epsilon(x,t)dx \leq \int_{\mathbb{T}^d} \rho^\epsilon(x,0)dx = 1$ for all $t \in [0,1]$. □

Proposition 23. *Let (v^ϵ, m^ϵ) solve Problem 2 and let ρ^ϵ solve* (37)*. Then*

$$v^\epsilon(x_0) = \int_{\mathbb{T}^d} v^\epsilon(x)\rho^\epsilon(x,0)dx =$$
$$\int_{\mathbb{T}^d} v^\epsilon(x)\rho^\epsilon(x,1)dx + \int_0^1\int_{\mathbb{T}^d} \left(D_pH \cdot Dv^\epsilon - H + g_\epsilon(m^\epsilon)\right)\rho^\epsilon dxdt.$$

Proof. We integrate (38) multiplied by ρ^ϵ. Then, we multiply the first equation in (37) by $-v^\epsilon$, add the resulting expressions, and, finally, use integration by parts combined with the initial condition for ρ^ϵ to obtain the result. □

We define

$$\|\rho\|_{L^m(L^q)} = \|\rho\|_{L^m([0,1],L^q(\mathbb{T}^d))} = \left(\int_0^1 \|\rho(\cdot,t)\|_{L^q}^m \, dt\right)^{\frac{1}{m}}$$

and set $\operatorname{osc} f = \sup_x f - \inf_x f$ for any bounded function, $f : \mathbb{T}^d \to \mathbb{R}$.

Proposition 24. *Let (v^ϵ, m^ϵ) solve Problem 2 and let ρ^ϵ solve (37). Suppose that Assumptions 3 and 4 hold. Then, for any $q, r \geq 1$ with $\frac{1}{q} + \frac{1}{r} = 1$, we have*

$$\int_0^1 \int_{\mathbb{T}^d} \rho^\epsilon |Dv^\epsilon|^2 \, dxdt \leq C + C \operatorname{Lip}(v^\epsilon) + C \|g(m^\epsilon)\|_{L^r} \|\rho^\epsilon\|_{L^1(L^q)} .$$

Proof. We use Assumptions 3 and 4 on the identity in Proposition 23 to conclude that

$$\int_0^1 \int_{\mathbb{T}^d} \left(c\,|Dv^\epsilon|^2 - C + g_\epsilon(m^\epsilon)\right) \rho^\epsilon dxdt \quad \leq v^\epsilon(x_0) + \int_{\mathbb{T}^d} -v^\epsilon(x)\rho^\epsilon(x,1)dx.$$

Now,

$$\begin{aligned}\int_0^1 \int_{\mathbb{T}^d} \left(c\,|Dv^\epsilon|^2 - C + g_\epsilon(m^\epsilon)\right) \rho^\epsilon dxdt &\leq \sup_x v^\epsilon - \inf_x v^\epsilon \int_{\mathbb{T}^d} \rho^\epsilon(x,1)dx \\ &\leq \sup_x v^\epsilon \left(1 - \int_{\mathbb{T}^d} \rho^\epsilon(x,1)dx\right) + \operatorname{osc} v^\epsilon \int_{\mathbb{T}^d} \rho^\epsilon(x,1)dx \\ &\leq C \int_{\mathbb{T}^d} v^\epsilon(x)dx + C \operatorname{Lip}(v^\epsilon) \leq C + C \operatorname{Lip}(v^\epsilon),\end{aligned}$$

where we use Proposition 22, $\sup v \leq \operatorname{osc} v + \int v$, $\operatorname{osc} v \leq C \operatorname{Lip}(v)$, and estimate (30) from Corollary 21.

Finally, Hölder's inequality gives

$$\int_0^1 \int_{\mathbb{T}^d} |g_\epsilon(m^\epsilon)| \, \rho^\epsilon dxdt \leq \|g_\epsilon(m^\epsilon)\|_{L^r} \|\rho^\epsilon\|_{L^1(L^q)} ,$$

which ends the proof. □

Proposition 25. *Let (v^ϵ, m^ϵ) solve Problem 2 and let ρ^ϵ solve (37). Suppose that Assumptions 2–8 hold and that $d = 2$ or $d = 3$. Then, for $0 < \nu < 1$, $q \geq 1$, and $\hat{\epsilon} > 0$, we have*

$$\begin{aligned}\int_0^1 \int_{\mathbb{T}^d} \left|D\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)\right|^2 dxdt \leq & C + \hat{\epsilon} \int_0^1 \int_{\mathbb{T}^d} \rho^\epsilon |Dv^\epsilon|^2 \, dxdt \\ & + C \|\rho^\epsilon\|_{L^1(L^q)}^\nu \|(m_\epsilon^\epsilon)^\alpha\|_{L^{\frac{q}{q-\nu}}} .\end{aligned} \tag{39}$$

Proof. By multiplying the differential equation in (37) by $(\rho^\epsilon)^{\nu-1}$, integrating by parts, and using $\int_{\mathbb{T}^d}(\rho^\epsilon)^\nu(x,t)dx \le C$, we obtain, for any $\epsilon_1 > 0$,

$$\begin{aligned} 4\frac{(1-\nu)}{\nu^2}\int_0^1\int_{\mathbb{T}^d}\left|D\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)\right|^2 dxdt &= \frac{1}{\nu}\int_{\mathbb{T}^d}(\rho^\epsilon(x,1))^\nu - \rho^\nu(x,0)dx+ \\ &+\int_0^1\int_{\mathbb{T}^d}\eta_\epsilon*(m_\epsilon^\epsilon)^\alpha(\rho^\epsilon)^\nu + (\nu-1)(\rho^\epsilon)^{\nu-1}D_pHD\rho^\epsilon dxdt \\ &\le C + \int_0^1\int_{\mathbb{T}^d}\eta_\epsilon*(m_\epsilon^\epsilon)^\alpha(\rho^\epsilon)^\nu dxdt + \epsilon_1\int_0^1\int_{\mathbb{T}^d}\left|D\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)\right|^2 dxdt \\ &\quad + C\int_0^1\int_{\mathbb{T}^d}|D_pH|^2(\rho^\epsilon)^\nu dxdt. \end{aligned} \tag{40}$$

Now, given $\epsilon_2 > 0$ there exists $C_{\epsilon_2} > 0$ such that $(\rho^\epsilon)^\nu \le C_{\epsilon_2} + \epsilon_2\rho^\epsilon$. This estimate implies, after selecting ϵ_1 sufficiently small, that

$$\int_0^1\int_{\mathbb{T}^d}|D_pH|^2(\rho^\epsilon)^\nu dxdt \le C_{\epsilon_2} + \epsilon_2\int_0^1\int_{\mathbb{T}^d}\rho^\epsilon|Dv^\epsilon|^2 dxdt,$$

where we used Assumption 6 and estimate (31) from Corollary 21. Moreover, using Hölder and Jensen inequalities, we have

$$\begin{aligned} \int_0^1\int_{\mathbb{T}^d}\eta_\epsilon*(m_\epsilon^\epsilon)^\alpha(\rho^\epsilon)^\nu dxdt &\le \int_0^1 \|\eta_\epsilon*(m_\epsilon^\epsilon)^\alpha\|_{L^{\frac{q}{q-\nu}}}\|(\rho^\epsilon)^\nu(\cdot,t)\|_{L^{\frac{q}{\nu}}}dt \\ &\le C\int_0^1\|\rho^\epsilon(\cdot,t)\|_{L^q}^\nu dt\,\|\eta_\epsilon*(m_\epsilon^\epsilon)^\alpha\|_{L^{\frac{q}{q-\nu}}} \\ &\le C\|\rho^\epsilon\|_{L^1(L^q)}^\nu\|\eta_\epsilon*(m_\epsilon^\epsilon)^\alpha\|_{L^{\frac{q}{q-\nu}}}. \end{aligned}$$

Combining (40) with the two preceding estimates yields (39). □

Corollary 26. *Let (v^ϵ, m^ϵ) solve Problem 2 and let ρ_ϵ solve* (37)*. Suppose that Assumptions 2-8 hold and that $d = 2$ or $d = 3$. Then, for $0 < \nu < 1$, and $q, r \ge 1$ with $\frac{1}{q} + \frac{1}{r} = 1$, and $\hat\epsilon > 0$ we have*

$$\begin{aligned} \int_0^1\int_{\mathbb{T}^d}\left|D\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)\right|^2 dxdt \le & C + \hat\epsilon\,\mathrm{Lip}(v^\epsilon) \\ & + C\|\rho^\epsilon\|_{L^1(L^q)}(\|g(m^\epsilon)\|_{L^r} + 1) + C\|(m_\epsilon^\epsilon)^\alpha\|_{L^{\frac{q}{q-\nu}}}^{\frac{1}{1-\nu}}. \end{aligned}$$

Proof. The result follows by combining Propositions 24, 25, and using Young's inequality for the term $C\|\rho^\epsilon\|_{L^1(L^q)}^\nu\|(m_\epsilon^\epsilon)^\alpha\|_{L^{\frac{q}{q-\nu}}}$ with exponents $\frac{1}{\nu}$ and $\frac{1}{1-\nu}$. □

Proposition 27. *Let (v^ϵ, m^ϵ) solve Problem 2 and let ρ_ϵ solve* (37)*.*
Suppose that q and ν satisfy

$$1 \le q < \frac{2^*}{2} \tag{41}$$

and
$$1-\frac{1}{q}+\frac{2}{2^*}<\nu<1. \tag{42}$$

Let
$$\mu=\frac{q-1}{q}\frac{2^*}{2^*\nu-2}. \tag{43}$$
Then, $0\leq\mu<1$*. Moreover, for any* m *such that*
$$m\geq 1 \quad \text{and} \qquad m\mu<1, \tag{44}$$
we have
$$\|\rho^\epsilon\|_{L^m(L^q)}\leq C+C\left(\int_0^1\int_{\mathbb{T}^d}\left|D\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)\right|^2dxdt\right)^{m\mu}. \tag{45}$$

Proof. Set $p_0=1$, $p_\theta=q$, and $p_1=\frac{\nu}{2}2^*$, note that $p_0<p_\theta<p_1$ and for
$$\theta=\frac{q-1}{q}\frac{2^*\nu}{2^*\nu-2},$$
we have
$$\|\rho^\epsilon\|_{L^q}\leq\|\rho^\epsilon\|_{L^1}^{1-\theta}\|\rho^\epsilon\|_{L^{\frac{\nu}{2}2^*}}^{\theta}.$$

Proposition 22 implies $\|\rho^\epsilon\|_{L^1}^{1-\theta}\leq C$. Thus, by Sobolev's inequality, we obtain
$$\|\rho^\epsilon\|_{L^q}\leq C\left\|(\rho^\epsilon)^{\frac{\nu}{2}}\right\|_{L^{2^*}}^{\frac{2}{\nu}\theta}\leq C+C\left\|D\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)\right\|_{L^2}^{2\frac{\theta}{\nu}}=C+C\left(\int_{\mathbb{T}^d}\left|D\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)\right|^2dx\right)^{\frac{\theta}{\nu}}.$$
Elementary computations show that the restrictions on q and ν imply $0\leq\mu<1$. Now, we write
$$\|\rho^\epsilon\|_{L^q}^m\leq C+C\left(\int_{\mathbb{T}^d}\left|D\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)\right|^2dx\right)^{m\mu},$$
integrate it in t from 0 to 1, and use Jensen's inequality to get the result. □

Remark 28. *Because work in dimension 2 or 3, the restriction* (41) *becomes*
$$\begin{cases} q\geq 1, & \text{if } d=2\\ 1\leq q<3, & \text{if } d=3,\end{cases} \tag{46}$$
and, then, (42) *can be written as*
$$2-\frac{2}{d}-\frac{1}{q}<\nu<1. \tag{47}$$

Corollary 29. *Let* (v^ϵ,m^ϵ) *solve Problem 2 and let* ρ_ϵ *solve* (37)*. Suppose that Assumptions 2-8 hold and that* $d=2$ *or* $d=3$*. Suppose that* q *and* ν *satisfy* (46) *and* (47)*. Let* μ *be given by* (43)*. Then, for every* $\hat\epsilon>0$*, we have*
$$\int_0^1\int_{\mathbb{T}^d}\left|D\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)\right|^2dxdt\leq C+\hat\epsilon\operatorname{Lip}(v^\epsilon)+C\left\|g(m^\epsilon)\right\|_{L^{\frac{q}{q-1}}}^{\frac{1}{1-\mu}}+C\left\|(m_\epsilon^\epsilon)^\alpha\right\|_{L^{\frac{q}{q-\nu}}}^{\frac{1}{1-\nu}}.$$

Proof. Taking into account Remark 28, q satisfies (41) and ν satisfies (42). Therefore, $0 \le \mu < 1$. Using Proposition 27 with $m = 1$, we conclude that

$$\|\rho^\epsilon\|_{L^1(L^q)} \le C + C\left(\int_0^1 \int_{\mathbb{T}^d} \left|D\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)\right|^2 dxdt\right)^\mu .$$

After using this inequality in the estimate of Corollary 26, we only need to observe that, for any $\epsilon_2 > 0$ and $\frac{1}{r} + \frac{1}{q} = 1$, we have

$$\left(\int_0^1 \int_{\mathbb{T}^d} \left|D\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)\right|^2 dxdt\right)^\mu \|g(m)\|_{L^r}$$
$$\le \epsilon_2 \int_0^1 \int_{\mathbb{T}^d} \left|D\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)\right|^2 dxdt + C\|g(m)\|_{L^r}^{\frac{1}{1-\mu}},$$

which follows by a weighted Young's inequality with exponents $\frac{1}{\mu}$ and $\frac{1}{1-\mu}$. $\square$

Proposition 30. *Let (v^ϵ, m^ϵ) solve Problem 2. Suppose that Assumptions 2–8 hold. Let $0 < \nu < 1$ and $q \ge 1$. If*

$$\begin{cases} \alpha > 0 & \text{if } d = 2 \\ 0 < \alpha \le 3 - \frac{3\nu}{q} & \text{if } d = 3, \end{cases} \tag{48}$$

then

$$\|(m^\epsilon)^\alpha\|_{L^{\frac{q}{q-\nu}}} \le C.$$

Proof. Corollary 21 states that $\left\|D\left((m^\epsilon)^{\frac{1}{2}}\right)\right\|_{L^2} \le C$, from which we get that $\left\|(m^\epsilon)^{\frac{1}{2}}\right\|_{L^{2^*}} \le C$ and consequently $\|m^\epsilon\|_{L^{\frac{2^*}{2}}} \le C$. So, if $\alpha\frac{q}{q-\nu} \le \frac{2^*}{2}$, we have the result. $\square$

Proposition 31. *Let (v^ϵ, m^ϵ) solve Problem 2. Suppose that Assumptions 2–8 hold. Let $r \ge 1$. Then, if Assumption 9A holds or if Assumption 9B holds with*

$$\begin{cases} \gamma > 0 & \text{if } d = 2 \\ 0 < \gamma \le \frac{3}{r} & \text{if } d = 3, \end{cases} \tag{49}$$

we have

$$\|g(m^\epsilon)\|_{L^r(\mathbb{T}^d)} \le C.$$

Proof. By estimate (32) from Corollary 21, $\frac{1}{m^\epsilon}$ is integrable. Hence, the first case follows immediately from the existence of a C such that $|\ln m^\epsilon|^r \le \frac{C}{m^\epsilon} + Cm^\epsilon$. The second case follows from the fact that $\|m^\epsilon\|_{L^{\frac{2^*}{2}}} \le C$, again by Corollary 21. $\square$

Corollary 32. *Let (v^ϵ, m^ϵ) solve Problem 2 and let ρ_ϵ solve* (37)*. Suppose that Assumptions 2-9 hold. Suppose that q, ν, and α satisfy* (46)*,* (47)*, and* (48)*,* *respectively. Let μ be defined by* (43)*. If Assumption 9B holds, suppose further that γ satisfies* (49) *with $r = \frac{q}{q-1}$. Then, for any $\hat{\epsilon} > 0$, we have*

$$\int_0^1 \int_{\mathbb{T}^d} \left| D\left((\rho^\epsilon)^{\frac{\nu}{2}} \right) \right|^2 dxdt \leq C_{\hat{\epsilon}} + \hat{\epsilon}\,\mathrm{Lip}(v^\epsilon). \tag{50}$$

Additionally, if $m \geq 1$ is such that (44) *holds, we have*

$$\|\rho^\epsilon\|_{L^m(L^q)} \leq C_{\hat{\epsilon}} + \hat{\epsilon}\,\mathrm{Lip}(v^\epsilon)^{m\mu} \tag{51}$$

for any $\hat{\epsilon} > 0$.

Finally, we have

$$\int_0^1 \int_{\mathbb{T}^d} \rho^\epsilon \left| Dv^\epsilon \right|^2 dxdt \leq C + C\,\mathrm{Lip}(v^\epsilon). \tag{52}$$

Proof. To establish (50), we combine Corollary 29 with Propositions 30 and 31. Next, to establish (51), we use (50) in the estimate (45) in Proposition 27. Finally, we combine the estimate in Proposition 24 and the estimates in Proposition 31 with (51) for $m = 1$ in the statement to obtain (52). □

Remark 33. *We observe that, if Assumptions 2-9 hold, it is possible to select q, ν, m, μ, r that satisfy all the inequalities required by the above corollary, as can be checked directly.*

Proposition 34. *Let (v^ϵ, m^ϵ) solve Problem 2. Suppose that Assumptions 2-11 hold. Then*

$$\mathrm{Lip}(v^\epsilon) \leq C.$$

Proof. Let ρ_ϵ solve (37). Let $\zeta = D_{x_i} v^\epsilon$. Then, differentiating the first equation in (4) with respect to x_i, we have

$$\begin{aligned} &D_{x_i} H(x, Dv^\epsilon) + D_p H(x, Dv^\epsilon) \cdot D\zeta \\ &\qquad = D_{x_i}\left(g_\epsilon(m^\epsilon)\right) - D_{x_i}\left(\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\right) v^\epsilon - \eta_\epsilon * (m_\epsilon^\epsilon)^\alpha \zeta + \Delta\zeta. \end{aligned} \tag{53}$$

Next, we select a smooth function, $\phi(t)$, such that $\phi(0) = 1$ and $\phi(1) = 0$ and define $w(x,t) = \zeta(x)\phi(t)$. Let ρ_ϵ solve (37). We multiply (53) by $\phi(t)\rho^\epsilon(x,t)$, after integration, we obtain

$$\begin{aligned} &\int_0^1 \int_{\mathbb{T}^d} D_{x_i} H(x, Dv^\epsilon)\phi\rho^\epsilon - D_{x_i}(g_\epsilon(m^\epsilon))\phi\rho^\epsilon + D_{x_i}\left(\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\right) v^\epsilon \phi\rho^\epsilon dxdt \\ &\qquad = \int_0^1 \int_{\mathbb{T}^d} -D_p H(x, Dv^\epsilon) \cdot Dw\rho^\epsilon - \eta_\epsilon * (m_\epsilon^\epsilon)^\alpha w\rho^\epsilon + \Delta w\rho^\epsilon dxdt. \end{aligned}$$

Multiplying the diffusion equation in (37) by w, we obtain the identity

$$\int_0^1 \int_{\mathbb{T}^d} w\rho_t^\epsilon dxdt = \int_0^1 \int_{\mathbb{T}^d} -D_p H \cdot Dw\rho^\epsilon - \left(\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\right) w\rho^\epsilon + \rho^\epsilon \Delta w dxdt.$$

Taking into account that $w(x,1)=0,$ and $w(x,0)=\zeta(x)$, we have

$$\int_0^1 w\rho_t^\epsilon dt = -\zeta(x)\delta_{x_0} - \int_0^1 \phi'\zeta\rho^\epsilon dt.$$

Thus,

$$-\zeta(x_0) = \int_0^1 \int_{\mathbb{T}^d} \underbrace{\rho^\epsilon \phi D_{x_i} H(x, Dv^\epsilon)}_{\mathbb{A}} + \underbrace{\rho^\epsilon \phi D_{x_i} \left(\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\right) v^\epsilon}_{\mathbb{B}} + \underbrace{\phi'\zeta\rho^\epsilon}_{\mathbb{C}} - \underbrace{\rho^\epsilon \phi D_{x_i}(g_\epsilon(m^\epsilon))}_{\mathbb{D}} dxdt.$$

From the preceding identity, we deduce the estimate

$$|\zeta(x_0)| \leq \left| \int_0^1 \int_{\mathbb{T}^d} \mathbb{A} + \mathbb{B} + \mathbb{C} - \mathbb{D} dxdt \right|. \tag{54}$$

We use Assumption 10, Proposition 22, and estimate (52) in Corollary 32 to obtain that, for any $\epsilon_1 > 0$,

$$\begin{aligned} \int_0^1 \int_{\mathbb{T}^d} |\mathbb{A}|\, dxdt &\leq \int_0^1 \int_{\mathbb{T}^d} C\rho^\epsilon + C\,|Dv^\epsilon|^\beta \rho^\epsilon dxdt \\ &\leq C_{\epsilon_1} + \epsilon_1 \int_0^1 \int_{\mathbb{T}^d} \rho^\epsilon\, |Dv^\epsilon|^2\, dxdt \\ &\leq C_{\epsilon_1} + \epsilon_1 \operatorname{Lip}(v^\epsilon). \end{aligned} \tag{55}$$

Because $|\mathbb{C}| \leq C\,|Dv^\epsilon|\,\rho^\epsilon$, we use a weighted Cauchy inequality on the product $\left((\rho^\epsilon)^{\frac{1}{2}}\,|Dv^\epsilon|\right)(\rho^\epsilon)^{\frac{1}{2}}$ and get

$$\int_0^1 \int_{\mathbb{T}^d} |\mathbb{C}|\, dxdt \leq C_{\epsilon_1} + \epsilon_1 \operatorname{Lip}(v^\epsilon) \tag{56}$$

for any $\epsilon_1 > 0$.

Next, we estimate $\left|\iint \mathbb{D}\right|$. Let $0 < \nu < 1$. We begin by observing that

$$-\int_{\mathbb{T}^d} \mathbb{D} dx = \int_{\mathbb{T}^d} g_\epsilon(m^\epsilon)\phi D_{x_i}\rho^\epsilon dx = \frac{2}{\nu}\int_{\mathbb{T}^d} \phi g_\epsilon(m^\epsilon)(\rho^\epsilon)^{1-\frac{\nu}{2}} D_{x_i}\left((\rho^\epsilon)^{\frac{\nu}{2}}\right) dx.$$

Therefore,

$$\int_0^1 \left| \int_{\mathbb{T}^d} \mathbb{D} dx \right| dt \leq C \int_0^1 \int_{\mathbb{T}^d} g_\epsilon(m^\epsilon)^2 (\rho^\epsilon)^{2-\nu} + \left| D\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)\right|^2 dxdt.$$

Let $s > 2$. We estimate the first term in the preceding inequality as follows

$$\begin{aligned} \int_{\mathbb{T}^d} g_\epsilon(m^\epsilon)^2 (\rho^\epsilon)^{2-\nu} dx &\leq \left\| g_\epsilon(m^\epsilon)^2 \right\|_{L^{\frac{s}{2}}} \left\| (\rho^\epsilon)^{2-\nu} \right\|_{L^{\frac{s}{s-2}}} \\ &= \| g_\epsilon(m^\epsilon)\|_{L^s}^2 \, \|\rho^\epsilon\|_{L^{\frac{s(2-\nu)}{s-2}}}^{2-\nu}. \end{aligned}$$

If Assumption 9A holds, or if Assumption 9B holds with

$$s\gamma < \frac{2^*}{2}, \tag{57}$$

we have

$$\|g_\epsilon(m^\epsilon)\|_{L^s}^2 \le C.$$

In case Assumption 9A holds, elementary computations yield that, for any $0 < \alpha < 1$, there exists $s > 2$, q, r, ν, m, and μ, with μ defined by (43), $q = \frac{s(2-\nu)}{s-2}$, $m = 2-\nu$, $r = \frac{q}{q-1}$ that satisfy (44), (46), (47), and (48). In case Assumption 9B holds, we require further that (49) and (57) hold, and we obtain the following restriction on γ.

$$\begin{cases} \gamma > 0, & \text{if } d = 2 \\ 0 < \gamma < 1, & \text{if } d = 3. \end{cases}$$

Applying (50) and (51) in Corollary 32, for any $\epsilon_1 > 0$ we have

$$\int_0^1 \int_{\mathbb{T}^d} g_\epsilon(m^\epsilon)^2 (\rho^\epsilon)^{2-\nu} dxdt \le C + \epsilon_1 \operatorname{Lip}(v^\epsilon).$$

Therefore, for any $\epsilon_1 > 0$, we obtain

$$\int_0^1 \left| \int_{\mathbb{T}^d} \mathbb{D} dx \right| dt \le C + \epsilon_1 \operatorname{Lip}(v^\epsilon). \tag{58}$$

Next, we integrate by parts and obtain, for $0 < \nu < 1$, the following estimate

$$\begin{aligned} \left| \int_{\mathbb{T}^d} \mathbb{B} dx \right| &\le \int_{\mathbb{T}^d} \left| D_{x_i} \rho^\epsilon v^\epsilon \left(\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\right) + \rho^\epsilon D_{x_i} v^\epsilon \left(\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\right) \right| dx \\ &= \int_{\mathbb{T}^d} \left| \frac{2}{\nu} \underbrace{D_{x_i}\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)(\rho^\epsilon)^{1-\frac{\nu}{2}} v^\epsilon \left(\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\right)}_{\mathbb{E}} + \underbrace{\rho^\epsilon \zeta \left(\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\right)}_{\mathbb{F}} \right| dx. \end{aligned}$$

We first address the term $\mathbb{E}$ and then the term $\mathbb{F}$. Observe that

$$\begin{aligned} \int_0^1 \left| \int_{\mathbb{T}^d} \mathbb{E} dx \right| dt &\le \int_0^1 \int_{\mathbb{T}^d} \left| D\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)\right|^2 + (\rho^\epsilon)^{2-\nu} (v^\epsilon)^2 \left(\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\right)^2 dxdt \\ &\le \int_0^1 \int_{\mathbb{T}^d} \left| D\left((\rho^\epsilon)^{\frac{\nu}{2}}\right)\right|^2 dxdt + \|\rho^\epsilon\|_{L^{2-\nu} L^{a(2-\nu)}}^{2-\nu} \|v^\epsilon\|_{L^{2b}}^2 \|\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\|_{L^{2c}}^2, \end{aligned} \tag{59}$$

where

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1, \qquad a, b, c \ge 1. \tag{60}$$

If

$$2b < 2^* \text{ and } 2\alpha c < \frac{2^*}{2}, \tag{61}$$

we have, as a consequence of (31) of Corollary 21, Sobolev's inequality, and Proposition 30 that

$$\|v^\epsilon\|_{L^{2b}}^2 \|\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\|_{L^{2c}}^2 \leq C.$$

In case Assumption 9A holds, elementary computations yield that, for any $0 < \alpha < \frac{1}{d-1}$, as in Assumption 11, there exist q, r, ν, m, μ, a, b, and c, with μ defined by (43), $q = a(2-\nu)$, $m = 2-\nu$, $r = \frac{q}{q-1}$ that satisfy (44), (46), (47), (48), (60), and (61).

In case Assumption 9B holds, we require further that (49) holds.

We apply the estimates (50) and (51) from Corollary 32 in (59). We conclude that, for any $\epsilon_1 > 0$, we have

$$\int_0^1 \left| \int_{\mathbb{T}^d} \mathbb{E} dx \right| dt \leq C + \epsilon_1 \operatorname{Lip}(v^\epsilon). \tag{62}$$

The last term to estimate is $\mathbb{F}$, which we address as follows. We begin by observing that, for any $\hat{\epsilon} > 0$, we have

$$\begin{aligned}\int_0^1 \int_{\mathbb{T}^d} \mathbb{F} dx dt &= \int_0^1 \int_{\mathbb{T}^d} \rho^\epsilon \zeta \eta_\epsilon * (m_\epsilon^\epsilon)^\alpha dx dt \\ &\leq \hat{\epsilon} \int_0^1 \int_{\mathbb{T}^d} \rho^\epsilon |Dv^\epsilon|^2 dx dt + C \int_0^1 \int_{\mathbb{T}^d} \rho_\epsilon \left(\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\right)^2 dx dt \\ &\leq \hat{\epsilon} C (1 + \operatorname{Lip} v^\epsilon) + C \|\rho^\epsilon\|_{L^1 L^q} \left\| \left(\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\right)^2 \right\|_{L^r},\end{aligned}$$

where

$$\frac{1}{q} + \frac{1}{r} = 1. \tag{63}$$

If

$$2\alpha r < \frac{2^*}{2}, \tag{64}$$

we have

$$\left\| \left(\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha\right)^2 \right\|_{L^r} \leq C.$$

For any $0 < \alpha < 1$ there exist q, r, ν, m and μ, with μ defined by (43), $m = 1$ and $r = \frac{q}{q-1}$ that satisfy (44), (46), (47), (48), (63), and (64).

Therefore, by applying the estimate (51) of Proposition 32, with $m = 1$, we conclude that, for any $\hat{\epsilon} > 0$, we have

$$\int_0^1 \int_{\mathbb{T}^d} |\mathbb{F}| \leq C + \hat{\epsilon} \operatorname{Lip}(v^\epsilon). \tag{65}$$

Finally, we choose a direction $\nu \in S^{n-1} \subset \mathbb{R}^n$ and x_0 such that

$$\operatorname{Lip}(v^\epsilon) = |D_\nu v(x_0)| = |\zeta(x_0)|.$$

Then, according to (54), combined with (55), (56), (58), (62), and (65), we obtain

$$\operatorname{Lip} v^\epsilon \leq C + \epsilon_2 \operatorname{Lip}(v^\epsilon),$$

for any $\epsilon_2 > 0$. Choosing any $\epsilon_2 < 1$ finishes the proof. □

Corollary 35. *Let (v^ϵ, m^ϵ) solve Problem 2. Suppose that Assumptions 2-11 hold. Then*

$$\|v^\epsilon\|_{W^{1,\infty}} < C.$$

Proof. From Proposition 34, we have that $\|Dv^\epsilon\|_{L^\infty} \leq C$ and that

$$v^\epsilon(x) - v^\epsilon(y) \leq |v^\epsilon(x) - v^\epsilon(y)| \leq C|x-y|$$

for any $x, y \in \mathbb{T}^d$. Integrating the above inequality with respect to y, we get

$$v^\epsilon(x) \leq \int_{\mathbb{T}^d} v^\epsilon(y)dy + C\int_{\mathbb{T}^d} |x-y|dy.$$

Because we work on a compact state space, this inequality and estimate (30) from Corollary 21 imply that $\|v^\epsilon\|_{L^\infty} \leq C$. □

6 Sobolev and Hölder regularity of solutions

Now, we establish the regularity of both v^ϵ and m^ϵ using a bootstrapping argument. We begin by examining the integrability of m^ϵ.

Proposition 36. *Let (v^ϵ, m^ϵ) solve Problem 2. Suppose that Assumptions 2-11 hold.*

Then

$$\left\|D\left((m^\epsilon)^{\frac{\beta}{2}}\right)\right\|_{L^2} \leq C \tag{66}$$

for any $\beta \geq 1$. And, consequently,

$$\|m^\epsilon\|_{L^\beta} \leq C \tag{67}$$

for any $\beta \geq 1$.

Proof. For $\beta = 1$, (66) and (67) follow, respectively, from (29) in Corollary 21 and Proposition 11. Let $\beta > 1$. We begin by integrating the second identity in (4) multiplied by $\beta(m^\epsilon)^{\beta-1}$.

$$\begin{aligned}(\beta-1)\beta\int_{\mathbb{T}^d}(m^\epsilon)^{\beta-2}|Dm^\epsilon|^2dx &= \int_{\mathbb{T}^d} -(\beta-1)\beta(m^\epsilon)^{\beta-1}D_pH\cdot Dm^\epsilon dx\\ &+\int_{\mathbb{T}^d}\beta\left((m^\epsilon)^\beta(1-\eta_\epsilon*(m^\epsilon_\epsilon)^\alpha)+\delta(m^\epsilon)^{\beta-1}\right)dx.\end{aligned}$$

Using Cauchy and Young's inequalities, we obtain the estimate

$$\begin{aligned}\frac{2(\beta-1)}{\beta}&\int_{\mathbb{T}^d}\left|D\left((m^\epsilon)^{\frac{\beta}{2}}\right)\right|^2+\beta(m^\epsilon_\epsilon)^\alpha\left(\eta_\epsilon*(m^\epsilon)^\beta\right)dx\\ &\leq\int_{\mathbb{T}^d}\frac{(\beta-1)\beta}{2}(m^\epsilon)^\beta|D_pH|^2+\beta(m^\epsilon)^\beta+\beta\delta(m^\epsilon)^{\beta-1}dx\\ &\leq C+C\int_{\mathbb{T}^d}(m^\epsilon)^\beta dx,\end{aligned} \tag{68}$$

where we use Assumption 6 and the Lipschitz estimate in Corollary 35.

As before, if $d = 2$, 2^* denotes a sufficiently large real number and if $d = 3$, $2^* = 6$. Let $0 < \lambda < 1$ be such that $\frac{1}{\beta} = \lambda + (1-\lambda)\frac{2}{2^*\beta}$. We observe, using the interpolation inequality and Proposition 11, that

$$\|m^\epsilon\|_{L^\beta}^\beta \le \|m^\epsilon\|_{L^1}^{\lambda\beta} \|m^\epsilon\|_{L^{\frac{2^*\beta}{2}}}^{(1-\lambda)\beta} \le C \left\| (m^\epsilon)^{\frac{\beta}{2}} \right\|_{L^{2^*}}^{(1-\lambda)2}. \tag{69}$$

We also observe that Sobolev's inequality implies

$$\left\| (m^\epsilon)^{\frac{\beta}{2}} \right\|_{L^{2^*}}^{(1-\lambda)2} \le C \|m^\epsilon\|_{L^\beta}^{(1-\lambda)\beta} + C \left\| D\left((m^\epsilon)^{\frac{\beta}{2}} \right) \right\|_{L^2}^{(1-\lambda)2}. \tag{70}$$

Therefore, combining (69) and (70), we get the estimate

$$\|m^\epsilon\|_{L^\beta}^\beta \le C + C \left\| D\left((m^\epsilon)^{\frac{\beta}{2}} \right) \right\|_{L^2}^{(1-\lambda)2}. \tag{71}$$

Consequently, (68) and (71) imply

$$\left\| D\left((m^\epsilon)^{\frac{\beta}{2}} \right) \right\|_{L^2}^2 + \int_{\mathbb{T}^d} (m_\epsilon^\epsilon)^\alpha \left(\eta_\epsilon * (m^\epsilon)^\beta \right) dx \le C + C \left\| D\left((m^\epsilon)^{\frac{\beta}{2}} \right) \right\|_{L^2}^{(1-\lambda)2},$$

which gives (66). The second estimate follows by combining (66) and (70). ☐

Proposition 37. *Let (v^ϵ, m^ϵ) solve Problem 2. Suppose that Assumptions 2-11 hold. Then*

$$\|v^\epsilon\|_{W^{2,p}(\mathbb{T}^d)} \le C$$

for any $p \ge 1$.

Furthermore, v is continuously differentiable and, hence, bounded.

Proof. From the first identity in (4), we get

$$\int_{\mathbb{T}^d} |\Delta v^\epsilon|^p \, dx \le C \int_{\mathbb{T}^d} |H|^p + |\eta_\epsilon * (m_\epsilon^\epsilon)^\alpha v^\epsilon|^p + |g_\epsilon(m^\epsilon)|^p \, dx. \tag{72}$$

The first term in the right-hand side is bounded because $\|Dv^\epsilon\|_{L^\infty(\mathbb{T}^d)} \le C$, by Corollary 35. The second term is bounded, for any $p \ge 1$, because $\|v^\epsilon\|_{L^\infty} \le C$, by Corollary 35, and $\|m^\epsilon\|_{L^q} \le C$ for any $q \ge 1$, by Proposition 36. The third term is bounded for any $p \ge 1$, by Proposition 31, if Assumption 9A holds or, by Proposition 36, if Assumption 9B holds.

After obtaining boundedness of (72), we get $\|v^\epsilon\|_{W^{2,p}(\mathbb{T}^d)} \le C$ for any $p \ge 1$, from standard elliptic regularity. We get the last result using Morrey's inequality and the compactness of the domain. ☐

Proposition 38. *Let (v^ϵ, m^ϵ) solve Problem 2. Suppose that Assumptions 2-11 hold. Then*

$$\|m^\epsilon\|_{W^{2,p}} \le C$$

for any $p \ge 1$.

Furthermore, m^ϵ is continuously differentiable and, hence, bounded.

Proof. We use the second identity in (4) to get

$$|\Delta m^\epsilon|^p \le C + C\,|\mathrm{div}\,(D_pHm^\epsilon)|^p + C\,|(1-\eta_\epsilon * (m^\epsilon_\epsilon)^\alpha)m^\epsilon|^p\,.$$

The last term in the right-hand side is integrable for any $p \ge 1$, by Proposition 36. Next, we notice that

$$|\mathrm{div}\,(D_pHm^\epsilon)|^p \le C\,|m^\epsilon\,\mathrm{div}\,D_pH|^p + C\,|D_pH\cdot Dm^\epsilon|^p\,.$$

By Proposition 37, we have $\|v^\epsilon\|_{W^{2,p}} \le C$ and therefore

$$\int |m^\epsilon\,\mathrm{div}\,D_pH|^p \le C.$$

Because D_pH is uniformly bounded, we take into account the L^β estimates for m^ϵ from Corollary 36 and use the Gagliardo-Nirenberg inequality to bound

$$\int |D_pH\cdot Dm^\epsilon|^p \le C\left(\int |\Delta m^\epsilon|^p\right)^\theta,$$

for any $p \ge 1$, and for $\theta = \frac{p+d(p-1)}{2p+d(p-1)} < 1$. Then, we obtain immediately that $\|m^\epsilon\|_{W^{2,p}} \le C$ for any $p \ge 1$, with estimates that are uniform in ϵ. Then, the last result follows directly from Morrey's inequality. □

Proposition 39. *Let (v^ϵ, m^ϵ) solve Problem 2. Suppose that Assumptions 2-11 hold.*

Then

$$\int_{\mathbb{T}^d} \frac{1}{(m^\epsilon)^\beta} \le C$$

for any $\beta > 0$.

Proof. Let $\beta > 0$. We proceed as in the proof of Proposition 36 to obtain

$$\begin{aligned}\beta\int_{\mathbb{T}^d}(m^\epsilon)^{-\beta}+\delta(m^\epsilon)^{-\beta-1}dx+\frac{4\,(\beta+1)}{\beta}\int_{\mathbb{T}^d}\left|D(m^\epsilon)^{\frac{-\beta}{2}}\right|^2dx\\ =\beta\int_{\mathbb{T}^d}(m^\epsilon)^{-\beta}\eta_\epsilon*(m^\epsilon_\epsilon)^\alpha dx-(\beta+1)\,\beta\int_{\mathbb{T}^d}(m^\epsilon)^{-\beta-1}D_pH\cdot Dm^\epsilon dx.\end{aligned}$$

Let $p = \frac{\beta+1}{\beta}$ and $q = 1+\beta$. We use Proposition 36 on the term $\|(m^\epsilon)^\alpha\|_{L^q}$ to observe that

$$\begin{aligned}\int_{\mathbb{T}^d}(m^\epsilon)^{-\beta}\eta_\epsilon*(m^\epsilon_\epsilon)^\alpha dx &\le \left\|(m^\epsilon)^{-\beta}\right\|_{L^p}\|(m^\epsilon)^\alpha\|_{L^q}\\ &\le C\left(\int_{\mathbb{T}^d}(m^\epsilon)^{-\beta-1}dx\right)^{\frac{\beta}{\beta+1}}\\ &\le C_{\epsilon_1}+\epsilon_1\int_{\mathbb{T}^d}(m^\epsilon)^{-\beta-1}dx\end{aligned}$$

for any $\epsilon_1 > 0$.

We use Assumption 6 and Corollary 35 to observe that, for any $\epsilon_2 > 0$, we have

$$\begin{aligned}\int_{\mathbb{T}^d} \left|(m^\epsilon)^{-\beta-1} D_pH \cdot Dm^\epsilon\right| dx &\leq C_{\epsilon_2} \int_{\mathbb{T}^d} (m^\epsilon)^{-\beta} \left|D_pH\right|^2 + \epsilon_2 \int_{\mathbb{T}^d} \left|D(m^\epsilon)^{\frac{-\beta}{2}}\right|^2 \\ &\leq C_{\epsilon_2} \int_{\mathbb{T}^d} (m^\epsilon)^{-\beta} dx + \epsilon_2 \int_{\mathbb{T}^d} \left|D(m^\epsilon)^{\frac{-\beta}{2}}\right|^2.\end{aligned}$$

By choosing $\epsilon_1 < \beta\delta$ and $\epsilon_2 < \frac{4(\beta+1)}{\beta}$ we get

$$\beta \int_{\mathbb{T}^d} m^{-\beta} dx + (\delta\beta - \epsilon_1) \int_{\mathbb{T}^d} m^{-\beta-1} dx \leq C + C \int_{\mathbb{T}^d} m^{-\beta} dx.$$

Now, using the Hölder inequality, we get

$$C \int_{\mathbb{T}^d} m^{-\beta} dx \leq C \left(\int_{\mathbb{T}^d} m^{-\beta-1} dx \right)^{\frac{\beta}{\beta+1}} \leq C_{\epsilon_3} + \epsilon_3 \int_{\mathbb{T}^d} m^{-\beta-1} dx.$$

We finally choose $\epsilon_3 < \delta\beta - \epsilon_1$ to obtain the result $\int_{\mathbb{T}^d} m^{-\beta} dx \leq C$. □

Proposition 40. *Let (v^ϵ, m^ϵ) solve Problem 2. (4). Suppose Assumptions 3-11 hold.*

Then there exists $\bar{m} > 0$ such that $m^\epsilon \geq \bar{m}$.

Proof. Let $x_0 \in \operatorname{argmin} m^\epsilon$ and set $m_0 = m^\epsilon(x_0)$. From Proposition 38, we obtain $\operatorname{Lip} m^\epsilon = k < \infty$. Also,

$$m^\epsilon(x) \leq m_0 + k|x - x_0|.$$

From Proposition 39 we obtain, for any $p \geq 1$, the first inequality below

$$\begin{aligned}C &\geq \int_{\mathbb{T}^d} \frac{1}{(m^\epsilon)^p} dx \geq \int_{B_r(x_0)} \frac{1}{(m^\epsilon)^p} dx \\ &\geq \int_{B_r(x_0)} \frac{1}{(m_0 + k|x - x_0|)^p} dx =: \int_{B_r(x_0)} f_{m_0}^{r,k,p}(x) dx.\end{aligned}$$

We fix r, k, and p and use Fatou's Lemma on $\{f_\nu^{r,k,p}\}_\nu$ to conclude that

$$\liminf_{\nu \to 0} \int_{B_r(x_0)} f_\nu^{r,k,p}(x) dx \geq \int_{B_r(x_0)} \frac{1}{(k|x - x_0|)^p} dx = +\infty.$$

This inequality implies the existence of $\bar{m} > 0$ such that $m_0 \geq \bar{m}$, otherwise a contradiction arises. □

7 Proof of the main theorem

Proof of Theorem 1. Let (v^ϵ, m^ϵ) solve Problem 2. We have that $m^\epsilon > 0$ from Theorem 3.

In Section 6, we proved in Propositions 37, 38, and 40 that $\|v^\epsilon\|_{W^{2,p}(\mathbb{T}^d)} \leq C$, $\|m^\epsilon\|_{W^{2,p}} \leq C$ for any $p \geq 1$, and that $m^\epsilon \geq \bar{m} > 0$, with constants C and $\bar{m}$ that do not depend on ϵ.

Hence, up to a subsequence, $(v^\epsilon, m^\epsilon) \to (v, m)$, where (v, m) solves Problem 1. Since (v, m) inherits the regularity of the sequence, it is in the same Sobolev spaces and is of class $C^\infty(\mathbb{T}^d)$. □

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References

[1] N. Almulla, R. Ferreira, and D. Gomes. Two numerical approaches to stationary mean-field games. *Dyn. Games Appl.*, 7(4):657–682, 2017.

[2] A. Bensoussan, J. Frehse, and P. Yam. *Mean field games and mean field type control theory.* Springer Briefs in Mathematics. Springer, New York, 2013.

[3] L. Boccardo, L. Orsina, and A. Porretta. Strongly coupled elliptic equations related to mean-field games systems. *J. Differential Equations*, 261(3):1796–1834, 2016.

[4] L. M. Briceño Arias, D. Kalise, and F. J. Silva. Proximal methods for stationary mean field games with local couplings. *SIAM J. Control Optim.*, 56(2):801–836, 2018.

[5] René Carmona and François Delarue. *Probabilistic Theory of Mean Field Games with Applications I-II.* Springer, 2018.

[6] D. Evangelista, R. Ferreira, D. Gomes, L. Nurbekyan, and V. Voskanyan. First-order, stationary mean-field games with congestion. *Nonlinear Analysis*, 173:37–74, 2018.

[7] D. Evangelista and D. Gomes. On the existence of solutions for stationary mean-field games with congestion. *Journal of Dynamics and Differential Equations*, pages 1–24, 2016.

[8] L. C. Evans. Adjoint and compensated compactness methods for Hamilton-Jacobi PDE. *Arch. Ration. Mech. Anal.*, 197(3):1053–1088, 2010.

[9] R. Ferreira and D. Gomes. Existence of weak solutions to stationary mean-field games through variational inequalities. *SIAM J. Math. Anal.*, 50(6):5969–6006, 2018.

[10] R. Ferreira, D. Gomes, and T. Tada. Existence of weak solutions to first-order stationary mean-field games with Dirichlet conditions. *To appear in Proc. Amer. Math. Society*, April 2018.

[11] D. Gomes and H. Mitake. Existence for stationary mean-field games with congestion and quadratic Hamiltonians. *NoDEA Nonlinear Differential Equations Appl.*, 22(6):1897–1910, 2015.

[12] D. Gomes, L. Nurbekyan, and E. Pimentel. *Economic models and mean-field games theory.* Publicações Matemáticas do IMPA. [IMPA Mathematical Publications]. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2015. 30o Colóquio Brasileiro de Matemática. [30th Brazilian Mathematics Colloquium].

[13] D. Gomes, L. Nurbekyan, and M. Prazeres. One-dimensional stationary mean-field games with local coupling. *Dyn. Games and Applications*, 2017.

[14] D. Gomes and S. Patrizi. Obstacle mean-field game problem. *Interfaces Free Bound.*, 17(1):55–68, 2015.

[15] D. Gomes, S. Patrizi, and V. Voskanyan. On the existence of classical solutions for stationary extended mean field games. *Nonlinear Anal.*, 99:49–79, 2014.

[16] D. Gomes, E. Pimentel, and H. Sánchez-Morgado. Time-dependent mean-field games in the subquadratic case. *Comm. Partial Differential Equations*, 40(1):40–76, 2015.

[17] D. Gomes, E. Pimentel, and H. Sánchez-Morgado. Time-dependent mean-field games in the superquadratic case. *ESAIM Control Optim. Calc. Var.*, 22(2):562–580, 2016.

[18] D. Gomes, E. Pimentel, and V. Voskanyan. *Regularity theory for mean-field game systems.* SpringerBriefs in Mathematics. Springer, [Cham], 2016.

[19] D. Gomes, G. E. Pires, and H. Sánchez-Morgado. A-priori estimates for stationary mean-field games. *Netw. Heterog. Media*, 7(2):303–314, 2012.

[20] D. Gomes and R. Ribeiro. Mean field games with logistic population dynamics. *52nd IEEE Conference on Decision and Control (Florence, December 2013)*, 2013.

[21] D. Gomes and H. Sánchez Morgado. A stochastic Evans-Aronsson problem. *Trans. Amer. Math. Soc.*, 366(2):903–929, 2014.

[22] M. Huang, P. E. Caines, and R. P. Malhamé. Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ϵ-Nash equilibria. *IEEE Trans. Automat. Control*, 52(9):1560–1571, 2007.

[23] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.*, 6(3):221–251, 2006.

[24] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I. Le cas stationnaire. *C. R. Math. Acad. Sci. Paris*, 343(9):619–625, 2006.

[25] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris*, 343(10):679–684, 2006.

[26] P. L. Lions. Collège de France course on mean-field games. 2007-2011.

[27] A.R. Mészáros and F.J. Silva. On the Variational Formulation of Some Stationary Second-Order Mean Field Games Systems. *SIAM J. Math. Anal.*, 50(1):1255–1277, 2018.

[28] L. Nurbekyan. One-dimensional, non-local, first-order stationary mean-field games with congestion: a Fourier approach. *Discrete Contin. Dyn. Syst. Ser. S*, 11(5):963–990, 2018.

[29] E. Pimentel and V. Voskanyan. Regularity for second-order stationary mean-field games. *Indiana Univ. Math. J.*, 66(1):1–22, 2017.