# SOME ESTIMATES FOR THE PLANNING PROBLEM WITH POTENTIAL

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Abstract. In this paper, we study a priori estimates for a first-order mean-field planning problem with a potential. In the theory of mean-field games (MFGs), a priori estimates play a crucial role to prove the existence of classical solutions. In particular, uniform bounds for the density of players' distribution and its inverse are of utmost importance. Here, we investigate a priori bounds for those quantities for a planning problem with a non-vanishing potential. The presence of a potential raises non-trivial difficulties, which we overcome by exploring a displacement-convexity property for the mean-field planning problem with a potential together with Moser's iteration method. We show that if the potential satisfies a certain smallness condition, then a displacement-convexity property holds. This property enables $L^q$ bounds for the density. In the one-dimensional case, the displacement-convexity property also gives $L^q$ bounds for the inverse of the density. Finally, using these $L^q$ estimates and Moser's iteration method, we obtain $L^\infty$ estimates for the density of the distribution of the players and its inverse. We conclude with an application of our estimates to prove existence and uniqueness of solutions for a particular first-order mean-field planning problem with a potential.

## 1. Introduction

The theory of mean-field games (MFGs) was proposed by J.-M. Lasry and P.-L. Lions (see [13, 14, 15]) and, independently, by M. Huang, R. Malhamé, and P. Caines (see [12]). These games describe the interaction between identical rational agents, where each agent minimizes the same value function. A standard MFG is determined by a system of PDEs, a Hamilton–Jacobi and a Fokker–Planck equation:

$$\begin{cases} -u_t - \varepsilon\Delta u + H(x, Du) = f(x, m) \\ m_t - \varepsilon\Delta m - \operatorname{div}(mH_p(x, Du)) = 0 \end{cases} \quad \text{in } (0,T)\times\mathbb{T}^d, \tag{1.1}$$

with the initial and terminal conditions

$$\begin{cases} m(0, x) = m_0(x) \\ u(T, x) = u_T(x) \end{cases} \quad \text{in } \mathbb{T}^d. \tag{1.2}$$

Here, $u$ represents the value function of a typical agent and $m$ the distribution of the agents. Under mild condition on the problem data, the existence of weak solutions of (1.1)–(1.2) is addressed in [1] and in [3] using monotonicity methods. Regarding classical solutions, it is proved in [5, 6, 7, 8] that (1.1)–(1.2) has a unique classical solution under suitable conditions on the problem data. A priori estimates play a crucial role in the proof of the existence of classical solutions. In particular, the uniform boundedness of the functions $m$ and $m^{-1}$,

$$\|m\|_{L^\infty} < \infty \quad \text{and} \quad \|m^{-1}\|_{L^\infty} < \infty, \tag{1.3}$$

is crucial to obtain classical solutions.

In his lectures in Collège de France [17], P.-L. Lions introduced *mean-field planning problems*. This problem amounts to solve (1.1) with initial and terminal conditions only on the density, $m$; that is,

$$\begin{cases} m(0, x) = m_0(x) \\ m(T, x) = m_T(x) \end{cases} \quad \text{in } \mathbb{T}^d. \tag{1.4}$$

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In those lectures, P.-L. Lions proved the existence and uniqueness (up to an additive constant in $u$) of classical solutions for the planning problem with a quadratic Hamiltonian, $H(x,p) = \frac{|p|^2}{2}$, within both the second-order case ($\varepsilon > 0$ in (1.1)) and the first-order case ($\varepsilon = 0$ in (1.1)), and with $f = f(m)$ an increasing function (see [17]). S. Muñoz addressed in [18] the existence and uniqueness of classical solutions of the first-order strictly elliptic planning problem (in the sense that $f(\cdot, 0) \equiv -\infty$) with quadratic growth Hamiltonians. In [20, 21], A. Porretta proved the existence and uniqueness of weak solutions for the second-order case with a more general Hamiltonian. For the first-order case with $f = f(m)$ an increasing function, D. Gomes and T. Seneci explored in [10] the displacement convexity property to obtain $L^p$ and $L^\infty$ estimates. Recently, the existence and uniqueness of weak solutions for the first-order case with a wide range of Hamiltonian has been addressed in [11, 19].

We consider the case where the coupling function $f = f(x,m)$ is separated: $f(x,m) = g(m) - V(x)$. The potential, $V$, describes the spatial preferences of each agent. In our setting, the potential can also depend on time, $V = V(t,x)$. More precisely, we investigate the following first-order mean-field planning problem with a time-dependent potential and a quadratic Hamiltonian.

**Problem 1.** *Suppose that $m_0, m_T \in C^2(\mathbb{T}^d)$ are probability densities and $V \in C^2([0,T] \times \mathbb{T}^d)$. Let $g : [0,\infty) \to [-\infty,\infty)$ be a non-decreasing and twice continuously differentiable function in $\mathbb{R}^+$. Find $(u,m) \in C^3([0,T] \times \mathbb{T}^d) \times C^2([0,T] \times \mathbb{T}^d)$ satisfying $m \geqslant 0$ and*

$$\begin{cases} -u_t + \frac{|Du|^2}{2} + V(t,x) = g(m) & \text{in } (0,T) \times \mathbb{T}^d \\ m_t - \operatorname{div}(mDu) = 0 & \text{in } (0,T) \times \mathbb{T}^d \\ m(0,x) = m_0(x), \;\; m(T,x) = m_T(x) & \text{in } \mathbb{T}^d. \end{cases} \tag{1.5}$$

In [16], the authors use a flow interchange technique to obtain $L^\infty$ estimates on the density of the solution of mean-field games without a potential. This flow interchange technique is a discrete analog of the displacement convexity. In that same reference, a key technical tool is the Moser method to iterate $L^p$ estimates and obtain $L^\infty$ estimates. This method is also used here, although in a somewhat different manner. In particular, our focus is on proving a priori bounds of the type (1.3). Such bounds were established in [10] for solutions of the first-order mean-field planning problem without a potential, Problem 1 with $V \equiv 0$. In this manuscript, we concentrate on exploring similar a priori estimates for the first-order mean-field planning problem with a potential ($V \not\equiv 0$). The presence of a potential raises technical difficulties in establishing a priori estimates, which we overcome through new techniques that combine displacement convexity with Moser's iteration method.

Next, we state our main results. We first outline our assumptions on the data of the Problem 1. The first one is a smallness condition on the potential, $V$. As we show at the end of Section 3, we cannot, in general, expect bounds of the type (1.3) to hold without this smallness condition.

**Assumption 1.** *There exists $p > 0$ such that the potential, $V : [0,T] \times \mathbb{T}^d \to \mathbb{R}$, satisfies*

$$\|\Delta V\|_{L^\infty([0,T]\times\mathbb{T}^d)} < \frac{2}{T^2}\frac{1}{p}.$$

Further, we impose a positive lower bound on the planning problem initial-terminal data, $m_0$ and $m_T$. This lower bound, together with the smoothness of $m_0$ and $m_T$, guarantees that any power of $m_0^{-1}$ and $m_T^{-1}$ is an integrable function on $\mathbb{T}^d$. As it will become clear within our proofs, the value of those integrals are key in our estimates.

**Assumption 2.** *There exists a positive constant, $k_0$, such that the initial-terminal functions, $m_0$ and $m_T$, satisfy*

$$m_0(x),\, m_T(x) \geqslant k_0 > 0, \quad x \in \mathbb{T}^d.$$

**Theorem 1.1.** Let $(u,m)$ solve Problem 1, and suppose that Assumption 1 holds for some $p > 0$. Then, there exists a positive constant, $C$, depending only on the problem data and on $p$, such that

$$\max_{t\in[0,T]} \|m\|_{L^{p+1}(\mathbb{T}^d)} \leqslant C. \tag{1.6}$$

Moreover, in the $d = 1$ case, if Assumptions 1 and 2 hold for some $p \geqslant 2$, then there exists a positive constant, $C$, depending only on the problem data and on $p$, such that

$$\max_{t\in[0,T]} \|m^{-1}\|_{L^{p-1}(\mathbb{T})} \leqslant C. \tag{1.7}$$

**Remark 1.2 (On the $p = 0$ case).** We observe that the estimate in (1.6) holds for $p = 0$ and for an arbitrary smooth potential, $V \in C^2([0, T] \times \mathbb{T}^d)$. In fact, by the mass-conservation property of the Fokker–Planck equation together with the initial condition, we have $\int_{\mathbb{T}^d} m \, \mathrm{d}x = 1$ for all $t \in [0, T]$.

In Section 2, we explore a displacement-convexity property of Problem 1. We refer the reader to [10] (and the references therein) for further insights on the concept of displacement convexity for MFGs. Relying on this property, we prove Theorem 1.1.

Next, we address the particular case of Problem 1 corresponding to the coupling $g(m) = m^\alpha$ for some $\alpha > 0$. For such coupling functions, which feature many MFGs models, we improve the estimates in Theorem 1.1, as stated below.

**Theorem 1.3.** Let $(u, m)$ solve Problem 1 with $g(m) = m^\alpha$ for some $\alpha > 0$. Suppose that Assumption 1 holds for some $p > 0$. Then, there exists a positive constant, $C$, depending only on the problem data and on $p$, such that

$$\max_{t\in[0,T]} \|m\|_{L^\infty(\mathbb{T}^d)} \leqslant C.$$

Moreover, in the $d = 1$ case, if Assumptions 1 and 2 hold with $p \geqslant 2$ and $p > \alpha + 1$, then there exists a positive constant, $C$, depending only on the problem data and on $p$, such that

$$\max_{t\in[0,T]} \|m^{-1}\|_{L^\infty(\mathbb{T})} \leqslant C.$$

We prove Theorem 1.3 in Section 3 by combining the arguments in the proof of Theorem 1.1 with the Moser iteration method. We further show in Section 3 that we cannot expect Theorem 1.3 to hold for general potentials. In fact, we exhibit in Example 3.4 an instance of (1.5) with an unbounded potential, $V$, for which the solution $(u, m)$ is such that $m \in C^\infty([0, T] \times \mathbb{T})$ and $m$ attains the zero. Thus, in particular, the inverse density function, $m^{-1}$, is unbounded.

As we mentioned before, a priori estimates are crucial for the theory of classical solutions. Hence, in addition to our estimates (1.6) and (1.7), we prove, in Section 4, the existence and uniqueness of classical solutions for a particular case of Problem 1. The proof combines the methods of [18] with Theorem 1.3.

**Theorem 1.4.** Let $m_0, m_T, V \in C^4(\mathbb{T})$, and $g(m) = m^\alpha$ for some $\alpha > 0$. Suppose that Assumptions 1 and 2 hold for some $p \geqslant 2$ such that $p > \alpha + 1$. Then, there exists a unique (up to constants) classical solution $(u, m) \in C^3([0, T] \times \mathbb{T}) \times C^2([0, T] \times \mathbb{T})$ to Problem 1.

## 2. Displacement Convexity for the Planning Problem with a Potential

Here, we explore displacement-convexity properties for Problem 1, which are key to prove Theorem 1.1.

Let $(u, m)$ solve Problem 1. As shown in [10], for certain functions $U : \mathbb{R}_0^+ \to \mathbb{R}$, the map

$$t \mapsto \int_{\mathbb{T}^d} U(m(t, x)) \, \mathrm{d}x \tag{2.1}$$

is convex when $V \equiv 0$. The convexity of the map in (2.1) implies that we can control

$$\max_{0\leqslant t\leqslant T} \int_{\mathbb{T}^d} U(m(t, x)) \, \mathrm{d}x$$

in terms of its values at $t = 0$ and $t = T$. In contrast with the case without potential, that property is, in general, false for the case with a potential (see Example 3.4).

Next, using this displacement convexity, we explore conditions on $V$ and $U$ under which the maximum of the map in (2.1) on a given interval is controlled by its values at the endpoints of the interval.

First, we set

$$P(z) = zU'(z) - U(z). \tag{2.2}$$

Accordingly,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} U(m)\,\mathrm{d}x &= \int_{\mathbb{T}^d} U'(m)m_t\,\mathrm{d}x = \int_{\mathbb{T}^d} U'(m)\,\mathrm{div}(mDu)\,\mathrm{d}x\\ &= \int_{\mathbb{T}^d}\big(U'(m)m\Delta u + U'(m)Dm\cdot Du\big)\,\mathrm{d}x\\ &= \int_{\mathbb{T}^d}\big(U'(m)m\Delta u + D(U(m))\cdot Du\big)\,\mathrm{d}x\\ &= \int_{\mathbb{T}^d}\big(U'(m)m\Delta u - U(m)\Delta u\big)\,\mathrm{d}x = \int_{\mathbb{T}^d} P(m)\Delta u\,\mathrm{d}x.\end{aligned}$$

Differentiating the preceding identity, we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\mathbb{T}^d} U(m)\,\mathrm{d}x = \int_{\mathbb{T}^d}\big(P'(m)m_t\Delta u + P(m)\Delta u_t\big)\,\mathrm{d}x. \tag{2.3}$$

On the other hand, applying $\Delta$ to the first equation in (1.5), we obtain

$$\Delta u_t = |D^2u|^2 + DuD\Delta u + \Delta V - \mathrm{div}(g'(m)Dm).$$

Using this equality and taking into account the second equation in (1.5), we deduce from (2.3) that

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\mathbb{T}^d} U(m)\,\mathrm{d}x = \int_{\mathbb{T}^d}\big[&P'(m)(\Delta u)^2 m + P'(m)\Delta uDu\cdot Dm + P(m)|D^2u|^2\\ &+ P(m)D\Delta u\cdot Du + P(m)\Delta V - P(m)\,\mathrm{div}(g'(m)Dm)\big]\,\mathrm{d}x.\end{aligned}$$

To estimate the right-hand side of the preceding equality, we observe that integrating by parts yields the identity

$$\begin{aligned}&\int_{\mathbb{T}^d} P(m)D\Delta u\cdot Du\,\mathrm{d}x - \int_{\mathbb{T}^d} P(m)\,\mathrm{div}(g'(m)Dm)\,\mathrm{d}x\\ &\quad= -\int_{\mathbb{T}^d}\mathrm{div}(P(m)Du)\Delta u\,\mathrm{d}x + \int_{\mathbb{T}^d} P'(m)g'(m)|Dm|^2\,\mathrm{d}x\\ &\quad= \int_{\mathbb{T}^d}\big(-P'(m)Dm\cdot Du\Delta u - P(m)(\Delta u)^2 + P'(m)g'(m)|Dm|^2\big)\,\mathrm{d}x.\end{aligned}$$

Moreover, by the Cauchy-Schwarz inequality, we have

$$|D^2u|^2 \geqslant \sum_{i=1}^d u_{x_ix_i}^2 \geqslant \frac{1}{d}\left(\sum_{i=1}^d u_{x_ix_i}\right)^2 = \frac{1}{d}(\Delta u)^2. \tag{2.4}$$

Hence, if $U$ is such that $P(m)\geqslant 0$, we deduce that

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\mathbb{T}^d} U(m)\,\mathrm{d}x \geqslant \int_{\mathbb{T}^d}\Big[&(\Delta u)^2\left(P'(m)m - P(m) + \tfrac{1}{d}P(m)\right)\\ &+ P'(m)g'(m)|Dm|^2 + P(m)\Delta V\Big]\,\mathrm{d}x.\end{aligned} \tag{2.5}$$

When $U$ is a power function,

$$U(z) = z^s, \quad s\geqslant 1, \tag{2.6}$$

from (2.5) and (2.2), we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\mathbb{T}^d} U(m)\,\mathrm{d}x \geqslant -|s-1|\|\Delta V\|_{L^\infty(\mathbb{T}^d)}\int_{\mathbb{T}^d} U(m)\,\mathrm{d}x. \tag{2.7}$$

**Remark 2.1.** In one-dimensional case, $d = 1$, the estimate (2.4) holds with equality. In particular, we do not need the condition $P(m) \geqslant 0$ to get (2.5) (which then holds with equality). Hence, a direct computation shows that, in $d = 1$ case, (2.7) holds for $U(z) = z^s$ with $s \in \mathbb{R} \setminus (0, 1)$.

Next, we introduce a simplified notation to denote the class of functions that satisfy a condition of the type (2.7). Under such type of condition, the subsequent lemma provides a smallness constraint on $V$ under which the maximum of map in (2.1) on a given interval is controlled by its values at the endpoints of the interval, as detailed in the proof of Theorem 1.1.

**Definition 2.2.** *Given $a$, $b$, $c \geqslant 0$, we denote by $\mathcal{F}_a^b(c)$ the set of all functions $f \in C^2([0, T])$ that are non-negative and satisfy*

$$\begin{cases} f''(t) + cf(t) \geqslant 0 & \text{for all } t \in [0, T], \\ f(0) = a, \quad f(T) = b. \end{cases} \tag{2.8}$$

**Lemma 2.3.** Suppose that $0 < \varepsilon \leqslant 2$ and $a$, $b \geqslant 0$. Let $0 \leqslant c \leqslant \frac{2-\varepsilon}{T^2}$. Then, the family of functions $\mathcal{F}_a^b(c)$ introduced in Definition 2.2 is uniformly bounded; more precisely, for any $f \in \mathcal{F}_a^b(c)$, we have

$$0 \leqslant f(t) \leqslant \frac{2(a+b)}{\varepsilon} \quad \text{for all } t \in [0, T].$$

*Proof.* Set

$$M = \max_{t \in [0,T]} f(t), \tag{2.9}$$

and let

$$h(t) = f(t) + kt^2,$$

where $k = \frac{cM}{2}$. We claim that $h$ is convex. In fact, by (2.8) and (2.9), we have

$$h''(t) = f''(t) + 2k = f''(t) + cf(t) + 2k - cf(t) = f''(t) + cf(t) + c(M - f(t)) \geqslant 0,$$

which proves the claim.

By the convexity and non-negativity of $h$, we conclude that, for all $t \in [0, T]$, we have

$$f(t) \leqslant h(t) \leqslant h(0) + h(T) = f(0) + f(T) + kT^2 = a + b + \frac{cM}{2} T^2.$$

Taking the maximum over $t \in [0, T]$ in the preceding estimate, we get

$$M \leqslant a + b + \frac{cM}{2} T^2,$$

from which we deduce that $M \leqslant \frac{2(a+b)}{2-cT^2} \leqslant \frac{2(a+b)}{\varepsilon}$ because $c \leqslant \frac{2-\varepsilon}{T^2}$. □

**Remark 2.4.** The claim in Lemma 2.3 is false for an arbitrary positive constant $c$. For instance, let $c \geqslant \frac{\pi^2}{T^2}$ and

$$f_k(t) = k \sin \tfrac{\pi t}{T} + 1, \quad k \in \mathbb{N}.$$

Then, $f_k \in \mathcal{F}_1^1(c)$ for all $k \in \mathbb{N}$, which shows that, in this case, the claim in Lemma 2.3 fails for any fixed constant $\varepsilon > 0$ and $c \geqslant \frac{\pi^2}{T^2}$.

*Proof of Theorem 1.1.* We start by proving the estimate in (1.6). Let $s = p+1$. Then, from Assumption 1, it follows that there exists $0 < \varepsilon < 2$ such that

$$|s - 1| \|\Delta V\|_{L^\infty([0,T]\times\mathbb{T}^d)} \leqslant \frac{2-\varepsilon}{T^2}.$$

Applying Lemma 2.3 to $f(t) := \int_{\mathbb{T}^d} m^s(x, t)\, dx$ and taking into account (2.6)–(2.7), we deduce that

$$\int_{\mathbb{T}^d} m^s \, dx \leqslant \frac{2}{\varepsilon} \left( \int_{\mathbb{T}^d} m_0^s \, dx + \int_{\mathbb{T}^d} m_T^s \, dx \right), \tag{2.10}$$

which together with the smothness of $m_0$ and $m_T$ concludes the proof of (1.6).

The proof of the estimate in (1.7) is analogous, as we outline next. In this case, we take $s=-p+1$ and observe that $s\leqslant -1$ for $p\geqslant 2$. The conclusion follows by applying Lemma 2.3 using Remark 2.1. Note also that Assumption 2 guarantees that the right-hand side of (2.10) is finite whenever $s<0$. □

## 3. Further Estimates

In this section, we consider the mean-field planning problem in Problem 1 with $g(m)=m^\alpha$ for some $\alpha>0$. In this case, we establish $L^\infty$ estimates for $m$ and $m^{-1}$, as stated in Theorem 1.3, which we prove by combining the next two propositions.

**Proposition 3.1.** Let $(u,m)$ solve Problem 1 with $g(m)=m^\alpha$ for some $\alpha>0$ and $d=1$. Suppose that Assumption 2 holds and that there exists $r\geqslant 1$ with $r>\alpha$ for which we can find a positive constant, $c$, such that

$$\max_{t\in[0,T]}\int_{\mathbb{T}}\frac{1}{m^r}\,\mathrm{d}x<c. \tag{3.1}$$

Then, there exists a positive constant, $C$, depending only on the problem data and on the constant in (3.1), such that

$$\max_{t\in[0,T]}\|m^{-1}\|_{L^\infty(\mathbb{T})}\leqslant C. \tag{3.2}$$

*Proof.* To simplify the notation, throughout this proof, we denote by the same letter $C$ any positive constant that depends only on the problem data or on the constant in (3.1) or on universal constants such as the constant in the Sobolev inequality. However, we keep track of the relevant power dependencies of these constants. Moreover, we assume, without loss of generality, that any such constant $C$ satisfies $C\geqslant 1$.

For $s\geqslant 1$, set

$$M_s=\max_{t\in[0,T]}\int_{\mathbb{T}}\frac{1}{m^s}\,\mathrm{d}x. \tag{3.3}$$

We aim at proving bounds for $M_s^{\frac{1}{s}}$ that are uniform in $s$, from which (3.2) follows.

Fix

$$q>r+\alpha, \tag{3.4}$$

and let $\ell=\frac{2r}{q-\alpha}$. By (3.1), we have $M_r<\infty$ and, without loss of generality, we may assume that $M_r\geqslant 1$. As we are in one-dimensional case, $d=1$, by Remark 2.1 and (2.5) with $s=-q$, we have

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\mathbb{T}}\frac{1}{m^q}\,\mathrm{d}x&=q(q+1)\int_{\mathbb{T}}\frac{(\Delta u)^2}{m^q}\,\mathrm{d}x+\alpha q(q+1)\int_{\mathbb{T}}\frac{|Dm|^2}{m^{q+2-\alpha}}\,\mathrm{d}x-(q+1)\int_{\mathbb{T}}\frac{\Delta V}{m^q}\,\mathrm{d}x\\&\geqslant \alpha q(q+1)\int_{\mathbb{T}}\frac{|Dm|^2}{m^{q-\alpha+2}}\,\mathrm{d}x-(q+1)C\int_{\mathbb{T}}\frac{1}{m^q}\,\mathrm{d}x\\&=4\alpha\frac{q(q+1)}{(q-\alpha)^2}\int_{\mathbb{T}}\left|D\left(\frac{1}{m^{\frac{q-\alpha}{2}}}\right)\right|^2\mathrm{d}x-(q+1)C\int_{\mathbb{T}}\frac{1}{m^q}\,\mathrm{d}x.\end{aligned} \tag{3.5}$$

To simplify the notation, we denote the first integral term on the right-hand side of (3.5) by

$$f(t)=\int_{\mathbb{T}}\left|D\left(\frac{1}{m^{\frac{q-\alpha}{2}}}\right)\right|^2\mathrm{d}x.$$

Next, we prove bounds for the last integral term on the right-hand side of (3.5) that allows us to incorporated it in the term associated with $f$. Ultimately, these bounds will allow us to replace the right-hand side of (3.5) by an expression only involving $q$-dependent constants and $M_r$ (see (3.20) below).

According to the generalized Poincaré inequality (see [9, Proposition 4.10]), for any $0<a<2$, there exists a positive constant, $C_a\geqslant 1$, such that, for any function $h\in W^{1,2}(\mathbb{T})$, we have

$$\left(\int_{\mathbb{T}}|h|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\leqslant C_a\left[\left(\int_{\mathbb{T}}|Dh|^2\,\mathrm{d}x\right)^{\frac{1}{2}}+\left(\int_{\mathbb{T}}|h|^a\,\mathrm{d}x\right)^{\frac{1}{a}}\right]. \tag{3.6}$$

Taking $h = m^{\frac{-q+\alpha}{2}}$ and $a = \ell = \frac{2r}{q-\alpha}$ in (3.6), we obtain

$$\int_{\mathbb{T}} \frac{1}{m^{q-\alpha}} \,\mathrm{d}x \leqslant C_\ell^2 \left( f^{\frac{1}{2}}(t) + M_r^{\frac{q-\alpha}{2r}} \right)^2 \leqslant 2C_\ell^2 \left( f(t) + M_r^{\frac{q-\alpha}{r}} \right). \tag{3.7}$$

On the other hand, by Morrey's embedding theorem (see [2, Section 5.6, Theorem 4]), we have

$$\left\| \frac{1}{m^{\frac{q-\alpha}{2}}} \right\|_{L^\infty(\mathbb{T})} \leqslant C \left( \int_{\mathbb{T}} \frac{1}{m^{q-\alpha}} \,\mathrm{d}x + f(t) \right)^{\frac{1}{2}}. \tag{3.8}$$

Raising the preceding estimate to the power of $2/(q-\alpha)$ first, and then using (3.7), we deduce that

$$\left\| \frac{1}{m} \right\|_{L^\infty(\mathbb{T})} \leqslant C^{\frac{2}{q-\alpha}} C_\ell^{\frac{2}{q-\alpha}} \left( M_r^{\frac{1}{r}} + (f(t))^{\frac{1}{q-\alpha}} \right). \tag{3.9}$$

On the other hand, by (3.4), we have

$$\theta = \frac{r}{q} \in (0,1).$$

Hence, Hölder's interpolation inequality yields

$$\left\| \frac{1}{m} \right\|_{L^q(\mathbb{T})} \leqslant \left\| \frac{1}{m} \right\|_{L^r(\mathbb{T})}^{\theta} \left\| \frac{1}{m} \right\|_{L^\infty(\mathbb{T})}^{1-\theta} \leqslant M_r^{\frac{\theta}{r}} \left\| \frac{1}{m} \right\|_{L^\infty(\mathbb{T})}^{1-\theta}. \tag{3.10}$$

Then, setting

$$\gamma = \frac{q(1-\theta)}{q-\alpha} = \frac{q-r}{q-\alpha}, \tag{3.11}$$

we conclude from (3.9) and (3.10) that

$$\int_{\mathbb{T}} \frac{1}{m^q} \,\mathrm{d}x \leqslant C^{2\gamma+1} C_\ell^{2\gamma} M_r^{\frac{q\theta}{r}} \left( M_r^{\frac{q(1-\theta)}{r}} + (f(t))^\gamma \right). \tag{3.12}$$

Observing that $\frac{q\theta}{r} = 1$ and $\frac{q(1-\theta)}{r} = \frac{q-r}{r}$, estimates (3.12) and (3.5) yield

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_{\mathbb{T}} \frac{1}{m^q} \,\mathrm{d}x \geqslant 4\alpha \frac{q(q+1)}{(q-\alpha)^2} f(t) - (q+1) C^{2\gamma+2} C_\ell^{2\gamma} M_r \left( M_r^{\frac{q-r}{r}} + (f(t))^\gamma \right). \tag{3.13}$$

Next, we estimate the right-hand side of (3.13) using Young's inequality with $\varepsilon$; namely, the estimate

$$ab \leqslant \varepsilon a^\sigma + C(\varepsilon) b^{\frac{\sigma}{\sigma-1}} \tag{3.14}$$

that is valid for all $a$, $b \geqslant 0$, $\varepsilon > 0$, and $1 < \sigma < \infty$, with $C(\varepsilon) = \frac{\sigma-1}{\sigma}(\varepsilon\sigma)^{-\frac{1}{\sigma-1}}$. Note that $\gamma \in (0,1)$ because $r > \alpha$ and $q > \min\{r, \alpha\}$ by (3.4). Then, taking $\sigma = \gamma^{-1}$, $a = (f(t))^\gamma$, $b = 1$, and

$$\varepsilon = 4\alpha \frac{q}{(q-\alpha)^2} \frac{1}{C^{2\gamma+2} C_\ell^{2\gamma} M_r}. \tag{3.15}$$

in (3.14), we conclude that

$$(f(t))^\gamma \leqslant \varepsilon f(t) + C(\varepsilon), \tag{3.16}$$

where

$$C(\varepsilon) = (1-\gamma) \left( \frac{\gamma}{\varepsilon} \right)^{\frac{\gamma}{1-\gamma}} = (1-\gamma) \gamma^{\frac{\gamma}{1-\gamma}} \left( \frac{(q-\alpha)^2 C^{2\gamma+2} C_\ell^{2\gamma} M_r}{4\alpha q} \right)^{\frac{\gamma}{1-\gamma}}. \tag{3.17}$$

Hence, using (3.16) in (3.13) yields

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_{\mathbb{T}} \frac{1}{m^q} \,\mathrm{d}x \geqslant -(q+1) C^{2\gamma+2} C_\ell^{2\gamma} \left( M_r^{\frac{q}{r}} + M_r C(\varepsilon) \right). \tag{3.18}$$

From the conditions $0 < \gamma < 1$ and $q > \alpha > 0$, we get the estimates

$$(1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} < 1 \quad \text{ and } \quad \frac{(q-\alpha)^2}{4\alpha q} < \frac{q+1}{\alpha}. \tag{3.19}$$

Because $q+1 \leqslant 2q$, it follows from (3.19) and (3.17) that

$$C(\varepsilon) \leqslant \left( \frac{2q}{\alpha} C^{2\gamma+2} C_\ell^{2\gamma} M_r \right)^{\frac{\gamma}{1-\gamma}} \leqslant \left( q C^{2\gamma+3} C_\ell^{2\gamma} M_r \right)^{\frac{\gamma}{1-\gamma}},$$

which combined with (3.18) yields

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\mathbb{T}^d}\frac{1}{m^q}\,\mathrm{d}x \geqslant -qC^{2\gamma+3}C_\ell^{2\gamma}M_r^{\frac{q}{r}} - \left(qC^{2\gamma+3}C_\ell^{2\gamma}M_r\right)^{\frac{1}{1-\gamma}}.$$

Further, taking into account that $\frac{1}{1-\gamma}=\frac{q-\alpha}{r-\alpha}>\frac{q}{r}>1$ and $q$, $C$, $C_\ell$, $M_r\geqslant 1$, we deduce that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\mathbb{T}}\frac{1}{m^q}\,\mathrm{d}x \geqslant -2\left(qC^{2\gamma+3}C_\ell^{2\gamma}\right)^{\frac{1}{1-\gamma}} M_r^{\frac{1}{1-\gamma}}. \tag{3.20}$$

Using a convexity argument and the preceding estimate, we prove next an estimate for $M_q$. Defining

$$h(t)=\int_{\mathbb{T}}\frac{1}{m^q}\,\mathrm{d}x + \left(qC^{2\gamma+3}C_\ell^{2\gamma}\right)^{\frac{1}{1-\gamma}} M_r^{\frac{1}{1-\gamma}}t^2,$$

the preceding estimate gives that $h$ is a (non-negative) convex function. Thus, $h(t)\leqslant h(0)+h(T)$, which, together with Assumption 2, implies that

$$\begin{aligned}\int_{\mathbb{T}}\frac{1}{m^q}\,\mathrm{d}x &\leqslant \int_{\mathbb{T}}\frac{1}{m_0^q}\,\mathrm{d}x + \int_{\mathbb{T}}\frac{1}{m_T^q}\,\mathrm{d}x + \left(qC^{2\gamma+3}C_\ell^{2\gamma}\right)^{\frac{1}{1-\gamma}} M_r^{\frac{1}{1-\gamma}}T^2\\ &\leqslant 2\left(qC^{2\gamma+4}C_\ell^{2\gamma}\right)^{\frac{1}{1-\gamma}} M_r^{\frac{1}{1-\gamma}}.\end{aligned}$$

Consequently, recalling the notation introduced in (3.3), we have

$$M_q \leqslant \left(qC^{2\gamma+5}C_\ell^{2\gamma}\right)^{\frac{1}{1-\gamma}} M_r^{\frac{1}{1-\gamma}}. \tag{3.21}$$

We observe that from (3.21), the arguments above show that for any $q$ satisfying (3.4), there exists a positive constant, $C_{q,r}$, depending only on $r$, $M_r$, $q$, and $C$, such that

$$M_q \leqslant C_{q,r}. \tag{3.22}$$

Next, to prove bounds for $M_q^{\frac{1}{q}}$ that are uniform in $q\geqslant 1$, we apply Moser's iteration method to the estimate in (3.21).

Set

$$\beta=\frac{3}{2}, \tag{3.23}$$

and, for each $n\in\mathbb{N}$, define $q_n=\beta^n$. Note that $q_n\to\infty$ as $n\to\infty$ because $\beta>1$. Thus, there exists $n_0\in\mathbb{N}$ such that for all $n\geqslant n_0$, we have

$$q_{n+1}>q_n+\alpha, \quad q_n>\alpha+r. \tag{3.24}$$

In view of (3.24), we argue as before to conclude that (3.21) holds with $q=q_{n+1}$ and $r=q_n$ for all $n\geqslant n_0$. Thus,

$$M_{q_{n+1}} \leqslant \left(q_{n+1}C^{2\gamma_n+5}C_{\ell_n}^{2\gamma_n}\right)^{\frac{1}{1-\gamma_n}} M_{q_n}^{\frac{1}{1-\gamma_n}}, \tag{3.25}$$

where

$$\ell_n=\frac{2q_n}{q_{n+1}-\alpha} \quad\text{and}\quad \gamma_n=\frac{q_{n+1}-q_n}{q_{n+1}-\alpha}. \tag{3.26}$$

Recalling that $q_n=\beta^n$ and (3.23), we get

$$\lim_{n\to\infty}\ell_n=\frac{4}{3} \quad\text{and}\quad \lim_{n\to\infty}(1-\gamma_n)=\frac{2}{3}. \tag{3.27}$$

From (3.27), (3.26), and (3.24), we can find $N_0\geqslant n_0+1$, such that

$$\frac{4}{3}<\ell_n<\frac{5}{3} \quad\text{and}\quad \frac{1}{1-\gamma_n}<2 \tag{3.28}$$

for all $n\geqslant N_0$. Relying on (3.28), we may reduce (3.25) to

$$M_{q_{n+1}} \leqslant Cq_{n+1}^2 M_{q_n}^{\frac{1}{1-\gamma_n}}, \quad n\geqslant N_0. \tag{3.29}$$

To complete the proof, it is enough to prove the boundedness of the sequence $\{M_{q_n}^{\frac{1}{q_n}}\}_{n\geqslant N_0}$. Taking $q = q_{N_0}$ in (3.22), we get

$$M_{q_{N_0}}^{\frac{1}{q_{N_0}}} \leqslant C_{q_{N_0},r}. \tag{3.30}$$

For $n$, $k \in \mathbb{N}$, we define

$$\Phi_k^n = \prod_{j=k}^{n} \frac{1}{1-\gamma_j} \quad \text{and} \quad \Psi_n = \sum_{k=N_0+1}^{n} \Phi_k^n, \tag{3.31}$$

with the standard convention that an empty product equals 1 and an empty sum equals 0. Note that

$$\Phi_n^n = \frac{1}{1-\gamma_n}, \quad \frac{1}{1-\gamma_{n+1}}\Phi_k^n = \Phi_k^{n+1}, \quad \frac{1}{1-\gamma_{n+1}}(1+\Psi_n) = \Phi_{n+1}^{n+1} + \sum_{k=N_0+1}^{n} \Phi_k^{n+1} = \Psi_{n+1}.$$

Using these identities, the recurrence relation in (3.29), and a mathematical induction argument, we obtain

$$M_{q_{n+1}}^{\frac{1}{q_{n+1}}} \leqslant C^{\frac{1}{q_{n+1}}(1+\Psi_n)} \left(M_{q_{N_0}}^{\Phi_{N_0}^n}\right)^{\frac{1}{q_{n+1}}} \left(q_{n+1}^2 \prod_{k=N_0+1}^{n} q_k^{2\Phi_k^n}\right)^{\frac{1}{q_{n+1}}} \tag{3.32}$$

for all $n \geqslant N_0$.

Next, we estimate the three multiplicative factors on the right-hand side of (3.32) separately. From (3.24), we know that $q_j - \alpha > q_{j-1}$ for all $j \geqslant N_0$. Consequently, recalling (3.26) and the definition $q_n = \beta^n$, we have

$$\frac{1}{\beta(1-\gamma_j)} = 1 + \frac{\beta\alpha - \alpha}{\beta(\beta^j - \alpha)} < 1 + \frac{\alpha}{\beta^j - \alpha} < 1 + \frac{\alpha}{\beta^{j-1}}.$$

Thus, for all $k \geqslant N_0$, we conclude that

$$\begin{aligned} \prod_{j=k}^{n} \frac{1}{\beta(1-\gamma_j)} &< \prod_{j=k}^{n} \Big(1 + \frac{\alpha}{\beta^{j-1}}\Big) = \prod_{j=0}^{n-k} \Big(1 + \frac{\alpha}{\beta^{k-1}} \frac{1}{\beta^j}\Big) \\ &< \prod_{j=0}^{\infty} \Big(1 + \frac{\alpha}{\beta^{N_0}} \Big(\frac{1}{\beta}\Big)^j\Big) = \Big(-\frac{\alpha}{\beta^{N_0}}; \beta^{-1}\Big), \end{aligned}$$

where $(a;\mathfrak{q})$ denotes the $\mathfrak{q}$-Pochhammer symbol (see, for example, [4]), which is a finite number for all $a \in \mathbb{R}$ and $\mathfrak{q} \in (0,1)$. So, $\big(-\alpha\beta^{-N_0}; \beta^{-1}\big) = \rho < \infty$ because $\beta^{-1} < 1$. Hence, for all $k \geqslant N_0$, we have the following estimate for $\Phi_k^n$:

$$\Phi_k^n = \beta^{n-k+1} \prod_{j=k}^{n} \frac{1}{\beta(1-\gamma_j)} < \rho\beta^{n-k+1}. \tag{3.33}$$

From (3.33) and (3.31), we deduce that

$$\Psi_n = \sum_{k=N_0+1}^{n} \Phi_k^n < \rho \sum_{k=N_0+1}^{n} \beta^{n-k+1} = \rho\, \frac{\beta(\beta^{n-N_0} - 1)}{\beta - 1}. \tag{3.34}$$

Consequently, recalling (3.23), for all $n \geqslant N_0$, we have

$$\frac{\Phi_{N_0}^n}{q_{n+1}} = \frac{\Phi_{N_0}^n}{\beta^{n+1}} < \rho\beta^{-N_0} < \rho \quad \text{and} \quad \frac{\Psi_n}{q_{n+1}} = \frac{\Psi_n}{\beta^{n+1}} < \frac{\rho}{\beta - 1}\beta^{-N_0}\big(1 - \beta^{N_0 - n}\big) < 2\rho,$$

from which we obtain the following estimates for the first two multiplicative factors on the right-hand side of (3.32):

$$C^{\frac{1}{q_{n+1}}(1+\Psi_n)} < C^{\frac{1}{q_{n+1}} + 2\rho} \quad \text{and} \quad \left(M_{q_{N_0}}^{\Phi_{N_0}^n}\right)^{\frac{1}{q_{n+1}}} < M_{q_{N_0}}^{\rho}. \tag{3.35}$$

To estimate the third multiplicative factor on the right-hand side of (3.32), we observe that (3.33), together with the condition $\beta > 1$, implies that

$$q_{n+1}^2 \prod_{k=N_0+1}^{n} q_k^{2\Phi_k^n} = \beta^{2(n+1)} \beta^{\sum_{k=N_0+1}^n 2k\Phi_k^n} < \beta^{2(n+1)(\rho+1)} \beta^{\sum_{k=N_0+1}^n 2k(\rho+1)\beta^{n-k+1}}$$
$$= \beta^{2(\rho+1)\sum_{j=0}^{n-N_0}(n+1-j)\beta^j}.$$

Then, because

$$\sum_{j=0}^{n-N_0} (n+1-j)\beta^j = \frac{1}{(\beta-1)^2}\left(\beta(N_0+1)\beta^{n-N_0+1} - N_0\beta^{n-N_0+1} - (n+1)(\beta-1) - \beta\right)$$
$$< \frac{\beta(N_0+1)}{(\beta-1)^2}\beta^{n+1},$$

we conclude that

$$\left(q_{n+1}^2 \prod_{k=N_0+1}^{n} q_k^{2\Phi_k^n(\beta)}\right)^{\frac{1}{\beta^{n+1}}} < \beta^{2(\rho+1)\frac{\beta(N_0+1)}{(\beta-1)^2}}. \tag{3.36}$$

Finally, from (3.30), (3.32), (3.35), and (3.36), we conclude that $\{M_{q_n}^{\frac{1}{q_n}}\}_{n\geqslant N_0}$ is a bounded sequence. □

The next theorem gives the uniform boundedness of the density function, $m$.

**Proposition 3.2.** Let $(u, m)$ solve Problem 1 with $g(m) = m^\alpha$ for some $\alpha > 0$. Then, there exists a positive constant, $C_{max}$, depending only on the problem data, such that

$$\max_{t\in[0,T]} \|m\|_{L^\infty(\mathbb{T}^d)} \leqslant C_{max}.$$

*Proof.* Proposition 3.2 can be proved with similar arguments to those in the previous proof; thus, we only highlight the main differences.

First, we observe that, by the mass-conservation property (see Remark 1.2), we have

$$\max_{t\in[0,T]} \int_{\mathbb{T}^d} m \, \mathrm{d}x = 1,$$

which gives the analogue to (3.1) with $r = 1$ for any $d \in \mathbb{N}$.

In one-dimensional case, $d = 1$, the condition $r > \alpha$ was used only to guarantee that the value of $\gamma$ in (3.11) satisfies $\gamma < 1$. Arguing as in the previous proof adapted to the present case, we have

$$\gamma = (1-\theta)\frac{q}{q+\alpha},$$

which satisfies the condition $\gamma < 1$ for all $q > 1$.

The proof of the case $d > 2$ is also similar to the proof of Proposition 3.1. However, the proof is slightly different. Thus, although we omit the details, we outline next what needs to be changed.

Regarding the $d > 2$ case, we first observe that the $d$-dimensional version of (3.5) holds with $m^q$ in place of $m^{-q}$ and the $d$-dimensional version of (3.7) holds with $m^{q+\alpha}$ in place of $m^{-q+\alpha}$. In contrast with the $d = 1$ case, we use Sobolev's inequality instead of Morrey's embedding theorem. So, by the Sobolev inequality (see [2, Theorem 6 in Section 5.6]), in place of (3.8), we have

$$\left\| m^{\frac{q+\alpha}{2}} \right\|_{L^{2^*}(\mathbb{T}^d)} \leqslant C \left( \int_{\mathbb{T}^d} m^{q+\alpha} \, \mathrm{d}x + \int_{\mathbb{T}^d} \left| D\left(m^{\frac{q+\alpha}{2}}\right)\right|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

Set

$$\bar\theta = \frac{2q + \alpha d}{q(d(q+\alpha) - d + 2)}.$$

It can be checked that $\bar\theta \in (0,1)$ and

$$\frac{1}{q} = \bar\theta + \frac{(1-\bar\theta)(d-2)}{d(q+\alpha)} = \bar\theta + \frac{1-\bar\theta}{\frac{2^*}{2}(q+\alpha)}.$$

Thus, by Hölder's inequality, instead of (3.10), we get

$$\|m\|_{L^q(\mathbb{T}^d)} \leqslant \|m\|_{L^1(\mathbb{T}^d)}^{\bar\theta} \|m\|_{L^{\frac{2^*}{2}(q+\alpha)}(\mathbb{T}^d)}^{1-\bar\theta} = \|m\|_{L^{\frac{2^*}{2}(q+\alpha)}(\mathbb{T}^d)}^{1-\bar\theta}.$$

Finally, here, we use

$$\bar\gamma = (1-\bar\theta)\frac{q}{q+\alpha}$$

in place of $\gamma$ in (3.11). Note that $\bar 0 < \gamma < 1$ for all $q > 1$. The remaining of the proof mimics that of Proposition 3.1.

The $d = 2$ case is similar to the $d > 2$ case, using the fact that for any $1 < a < 2$, the Sobolev inequality (see [2, Section 5.6, Theorem 6]) yields

$$\left\|m^{\frac{q+\alpha}{2}}\right\|_{L^{a^*}(\mathbb{T}^2)} \leqslant C\left(\int_{\mathbb{T}^2} m^{\frac{a(q+\alpha)}{2}}\,\mathrm{d}x + \int_{\mathbb{T}^2}\left|D\left(m^{\frac{q+\alpha}{2}}\right)\right|^a \mathrm{d}x\right)^{\frac{1}{a}}.$$

Then, by Young's inequality, we obtain

$$\left\|m^{\frac{q+\alpha}{2}}\right\|_{L^{a^*}(\mathbb{T}^2)} \leqslant C\left(\int_{\mathbb{T}^2} m^{q+\alpha}\,\mathrm{d}x + \int_{\mathbb{T}^2}\left|D\left(m^{\frac{q+\alpha}{2}}\right)\right|^2 \mathrm{d}x\right)^{\frac{1}{a}}. \qquad \square$$

**Remark 3.3.** The statements of Propositions 3.1 and 3.2 hold with the same constants $C$ and $C_{max}$, respectively, when the coupling function, $g$, is of the form

$$g(z) = z^\alpha + w(z), \quad z \in \mathbb{R}_0^+, \tag{3.37}$$

where $w : [0,\infty) \to [-\infty,\infty)$ is a non-decreasing and twice continuously differentiable function in $\mathbb{R}^+$.

To see this, we first notice that in the Proposition 3.1 case, it is enough to prove that the inequality in (3.5) also holds for the coupling function, $g$, defined by (3.37).

Because $w$ is non-decreasing, we have

$$g'(z) = \alpha z^{\alpha-1} + w'(z) \geqslant \alpha z^{\alpha-1}.$$

Relying on the preceding inequality and taking into account that $d = 1$, from Remark 2.1 and (2.5) for $s = -q$, we deduce that

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\mathbb{T}} \frac{1}{m^q}\,\mathrm{d}x &\geqslant q(q+1)\int_{\mathbb{T}} \frac{g'(m)}{m^{q+1}}|Dm|^2\,\mathrm{d}x - (q+1)\int_{\mathbb{T}}\frac{\Delta V}{m^q}\,\mathrm{d}x \\
&\geqslant \alpha q(q+1)\int_{\mathbb{T}} \frac{|Dm|^2}{m^{q-\alpha+2}}\,\mathrm{d}x - (q+1)C\int_{\mathbb{T}}\frac{1}{m^q}\,\mathrm{d}x \\
&= 4\alpha\frac{q(q+1)}{(q-\alpha)^2}\int_{\mathbb{T}}\left|D\left(\frac{1}{m^{\frac{q-\alpha}{2}}}\right)\right|^2\mathrm{d}x - (q+1)C\int_{\mathbb{T}}\frac{1}{m^q}\,\mathrm{d}x.
\end{aligned}$$

Hence, (3.5) also holds with $g$ as in (3.37). The Proposition 3.2 case can be addressed similarly, for which reason we omit here the details.

*Proof of Theorem 1.3.* The proof follows from Propositions 3.1 and 3.2 and Theorem 1.1. $\square$

As we mentioned in the Introduction, we cannot expect, in general, bounds for $m^{-1}$ without a smallness condition on $V$, as in Assumption 1. Next, we give an instance of (1.5) with an unbounded potential that does not satisfy the conditions of Proposition 3.1. In this particular case, we show that the density function, $m$, has zero values. Hence, the estimate (3.2) does not hold without further conditions on $V$.

**Example 3.4.** Let

$$m(t,x) = 1 + \sin(2\pi x)\sin(2\pi t). \tag{3.38}$$

Notice that the function $m$ defined by (3.38) is a probability density function and has two zeros, $(\frac{1}{4},\frac{3}{4})$ and $(\frac{3}{4},\frac{1}{4})$. Plugging (3.38) into the second equation in (1.5) with $\alpha=1$ and solving it for $u$, we get

$$u(t,x)=-\frac{1}{2\pi}\cot(2\pi t)\log(1+\sin(2\pi t)\sin(2\pi x)). \tag{3.39}$$

Set

$$V(t,x)=m(x,t)+u_t(x,t)-\frac{u_x^2(x,t)}{2}. \tag{3.40}$$

The functions, $m$ and $u$, given by (3.38) and (3.39), respectively, solve the mean-field planning problem

$$\begin{cases}-u_t+\frac{u_x^2}{2}+V(t,x)=m\\ m_t-(u_xm)_x=0\\ m(0,x)=m(T,x)=1,\end{cases}$$

with $V$ given by (3.40). Direct computations show that $\Delta V$ is unbounded, which means the function $V$ does not satisfy the conditions of Proposition 3.1.

## 4. Classical Solutions Of The Planning Problem With Potential

In this section, we prove the existence and uniqueness (up to constants) of classical solutions for a first-order one-dimensional planning problem with a potential. The proof relies on the a priori estimates in Theorem 1.1 and on the main result of the recent work by S. Muñoz [18].

Here, we consider the following instance of Problem 1.

**Problem 2.** *Suppose that $V\in C^4(\mathbb{T})$ and $\alpha>0$. Let $m_0$, $m_T\in C^4(\mathbb{T})$ be probability densities. Find $(u,m)\in C^3([0,T]\times\mathbb{T})\times C^2([0,T]\times\mathbb{T})$ satisfying $m\geqslant 0$ and*

$$\begin{cases}-u_t+\frac{|u_x|^2}{2}+V(x)=m^\alpha & \text{in } (0,T)\times\mathbb{T}\\ m_t-\operatorname{div}(mu_x)=0 & \text{in } (0,T)\times\mathbb{T}\\ m(0,x)=m_0(x),\ \ m(T,x)=m_T(x) & \text{in } \mathbb{T}.\end{cases} \tag{4.1}$$

To study the planning problem system (4.1), we consider an auxiliary system, which is strictly elliptic (in the sense of [18]; that is, $\lim_{z\to 0} g(z)=-\infty$ in (1.5)). Fix $\delta>0$, and let $0<\delta_1<\min\{\delta,1\}$; next, for $z\in[0,\infty)$, we define

$$w(z)=\begin{cases}0 & \text{if } z\geqslant\delta,\\ \varepsilon(z) & \text{if } z\in(\delta_1,\delta)\\ \ln z & \text{if } z\in[0,\delta_1],\end{cases} \tag{4.2}$$

where $\varepsilon(\cdot)$ is a strictly increasing function such that $w\in C^4(\mathbb{R}^+)$. Finally, we set

$$G(z)=z^\alpha+w(z),\quad z\in[0,\infty), \tag{4.3}$$

and, instead of (4.1), we consider

$$\begin{cases}-u_t+\frac{|u_x|^2}{2}+V(x)=G(m) & \text{in } (0,T)\times\mathbb{T}\\ m_t-\operatorname{div}(mu_x)=0 & \text{in } (0,T)\times\mathbb{T}\\ m(0,x)=m_0(x),\ \ m(T,x)=m_T(x) & \text{in } \mathbb{T}.\end{cases} \tag{4.4}$$

By (4.2)–(4.3), we have $G\in C^4(\mathbb{R}^+)$ and $G(0)=-\infty$, which means that the MFG system (4.4) is strictly elliptic. Next, we prove the existence of classical solutions of (4.4). The strategy of the proof follows the arguments presented in [18]. First, we get a priori estimates for (4.4), and then we transform (4.4) into a second-order quasilinear elliptic equation with non-linear oblique boundary conditions. Lastly, relying on the a priori estimates and by using the nonlinear method of continuity, we prove the existence and uniqueness of classical solutions for the second-order quasilinear elliptic equation with the nonlinear oblique boundary conditions, which implies the existence and uniqueness (up to constants) of classical solutions of (4.4).

To establish a priori estimates for (4.4), we start by observing that Remark 3.3 implies the following bounds for $m$.

**Proposition 4.1.** Let $(u,m)\in C^3([0,T]\times\mathbb{T})\times C^2([0,T]\times\mathbb{T})$ solve (4.4) with $m_0$, $m_T$, $V\in C^4(\mathbb{T})$. Suppose that Assumptions 1 and 2 hold for some $p\geqslant 2$ such that $p>\alpha+1$. Then, there exist positive constants, $C_{min}$ and $C_{max}$, depending only on the Problem 2 data, such that

$$C_{min}\leqslant m(t,x)\leqslant C_{max},\quad (t,x)\in[0,T]\times\mathbb{T}. \tag{4.5}$$

To get an $L^\infty$ bound for $u$, we need the two following auxiliary lemmas. We first note that if $(u,m)$ solves (4.4), then so does $(u+C,m)$ for any $C\in\mathbb{R}$. Thus, without loss of generality, we assume that $u(0,0)=0$.

**Lemma 4.2.** *Consider the setting of Proposition 4.1. Then, there exists a positive constant, $C$, which depends only on the problem data, such that*

$$\int_0^T\int_{\mathbb{T}}|u_x|^2\,\mathrm{d}x\mathrm{d}t\leqslant C.$$

*Proof.* We first observe that

$$\tilde{m}(t,x)=\left(1-\tfrac{t}{T}\right)m_0(x)+\tfrac{t}{T}m_T(x),\quad (t,x)\in[0,T]\times\mathbb{T},$$

satisfies the boundary conditions in (4.4), with $C_{min}\leqslant\tilde{m}\leqslant C_{max}$ and $\tilde{m}_t=\frac{1}{T}(m_T-m_0)$. Then, integrating over $[0,T]\times\mathbb{T}$ the first equation in (4.4) multiplied by $(m-\tilde{m})$, an integration by parts argument yields

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}}\Big(um_t-\tfrac{1}{T}u(m_T-m_0)+\tfrac{|u_x|^2}{2}(m-\tilde{m})+V(x)(m-\tilde{m})\Big)\,\mathrm{d}x\mathrm{d}t\\&\quad=\int_0^T\int_{\mathbb{T}}g(m)(m-\tilde{m})\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.6}$$

Similarly, integrating over $[0,T]\times\mathbb{T}$ the second equation in (4.4) multiplied by $u$ and integrating by parts, we deduce that

$$\int_0^T\int_{\mathbb{T}}um_t\,\mathrm{d}x\mathrm{d}t=-\int_0^T\int_{\mathbb{T}}|u_x|^2m\,\mathrm{d}x\mathrm{d}t. \tag{4.7}$$

Next, we note that because $s\mapsto s\ln(s)$ is uniformly bounded from below in $\mathbb{R}^+$, the bounds in (4.5) and the definition of $G$ yield $mG(m)\geqslant -C$ on $[0,T]\times\mathbb{T}$ for some positive constant $C$ independent of $m$. Using this estimate and (4.5) first, and then (4.7) together with (4.6), we get

$$\begin{aligned}C_{min}\int_0^T\int_{\mathbb{T}}|u_x|^2\,\mathrm{d}x\mathrm{d}t-C&\leqslant\int_0^T\int_{\mathbb{T}}\Big(\tfrac{|u_x|^2}{2}(m+\tilde{m})+G(m)m\Big)\,\mathrm{d}x\mathrm{d}t\\&=\int_0^T\int_{\mathbb{T}}\Big(V(x)(m-\tilde{m})+G(m)\tilde{m}-\tfrac{1}{T}u(m_T-m_0)\Big)\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.8}$$

To estimate the right-hand side of (4.8), we use the fact $G(m)\leqslant m^\alpha$ and Proposition 4.1 to conclude that

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}}\Big(V(x)(m-\tilde{m})+G(m)\tilde{m}\Big)\,\mathrm{d}x\mathrm{d}t\\&\quad\leqslant\big(\|V\|_{L^\infty(\mathbb{T})}+1\big)\int_0^T\int_{\mathbb{T}}(m+\tilde{m}+m^\alpha\tilde{m})\,\mathrm{d}x\mathrm{d}t\leqslant C.\end{aligned} \tag{4.9}$$

To estimate the remaining term on the right-hand side of (4.8), we set $\bar{u}=\int_{\mathbb{T}}u\,\mathrm{d}x$ and note that $\int_{\mathbb{T}}\bar{u}(m_T-m_0)\,\mathrm{d}x=0$ because $m_0$ and $m_T$ are probability densities on $\mathbb{T}$. Using, in addition, Schwarz's, Poincaré–Wirtinger's, and Young's inequalities, it follows that

$$\int_{\mathbb{T}^d} -\tfrac{1}{T}u(m_T - m_0)\,\mathrm{d}x = \int_{\mathbb{T}} -\tfrac{1}{T}(u-\bar{u})(m_T - m_0)\,\mathrm{d}x \leqslant \tfrac{1}{T}\|u-\bar{u}\|_{L^2(\mathbb{T})}\|m_T - m_0\|_{L^2(\mathbb{T})}$$
$$\leqslant C\|u-\bar{u}\|_{L^2(\mathbb{T})} \leqslant C\Big(\int_{\mathbb{T}}|u_x|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} \leqslant C + \tfrac{C_{min}}{2}\int_{\mathbb{T}}|u_x|^2\,\mathrm{d}x. \tag{4.10}$$

Finally, combining (4.8) with (4.9)–(4.10), we complete the proof. □

The next lemma establishes energy conservation for the problem (4.4).

**Lemma 4.3.** *Let $(u, m)$ be a solution of (4.4). Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}\Big(\tfrac{|u_x|^2}{2}m + V(x)m - Q(m)\Big)\,\mathrm{d}x = 0,$$

*where $Q$ is such that $Q' = G$.*

*Proof.* Note that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}\Big(\tfrac{|u_x|^2}{2}m + V(x)m - Q(m)\Big)\,\mathrm{d}x\\ &\quad= \int_{\mathbb{T}}\Big(\tfrac{|u_x|^2}{2} + V(x) - Q'(m)\Big)m_t\,\mathrm{d}x + \int_{\mathbb{T}}u_x m(u_t)_x\,\mathrm{d}x.\end{aligned}\tag{4.11}$$

Moreover, from the second equation in (4.4) together with the divergence theorem, it follows that

$$\int_{\mathbb{T}}u_x m(u_t)_x\,\mathrm{d}x = -\int_{\mathbb{T}}u_t(mu_x)_x\,\mathrm{d}x = -\int_{\mathbb{T}}u_t m_t\,\mathrm{d}x.$$

The conclusion follows by using the preceding identity in the last term of (4.11), and then using the first equation in (4.4). □

Next, using the preceding two lemmas, we prove an $L^\infty$ bound for $u$.

**Proposition 4.4.** Consider the setting of Proposition 4.1. Then, there exists a positive constant, $C$, which depends only on the problem data, such that

$$\|u\|_{C^0([0,T]\times\mathbb{T})} \leqslant C.$$

*Proof.* Relying on (4.5), from Lemmas 4.2 and 4.3, it follows that there exists a positive constant, $C$, such that

$$\max_{t\in[0,T]}\int_{\mathbb{T}}|u_x|^2\,\mathrm{d}x \leqslant C. \tag{4.12}$$

The preceding estimate and the condition $u(0,0) = 0$ yield

$$|u(0,x)| = \Big|\int_0^x u_x\,\mathrm{d}\tau\Big| \leqslant \int_0^x |u_x|\,\mathrm{d}\tau \leqslant C. \tag{4.13}$$

Next, we note that from the first equation in (4.4) and the estimates in (4.5) and (4.12), we get

$$\max_{t\in[0,T]}\int_{\mathbb{T}}|u_t|\,\mathrm{d}x \leqslant C. \tag{4.14}$$

By (4.13) and (4.14), we have, for all $t \in [0,T]$,

$$\Big|\int_{\mathbb{T}}u(t,x)\,\mathrm{d}x\Big| = \Big|\int_{\mathbb{T}}u(0,x)\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}}u_t\,\mathrm{d}x\mathrm{d}t\tau\Big| \leqslant \int_{\mathbb{T}}|u(0,x)|\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}}|u_t|\,\mathrm{d}x\mathrm{d}\tau \leqslant C. \tag{4.15}$$

Using the Poincaré–Wirtinger inequality, we deduce from (4.12) and (4.15) that

$$\begin{aligned}\max_{t\in[0,T]}\|u\|_{L^2(\mathbb{T})} &\leqslant \max_{t\in[0,T]}\Bigg(\Big\|u - \int_{\mathbb{T}}u(t,x)\,\mathrm{d}x\Big\|_{L^2(\mathbb{T})} + \Big|\int_{\mathbb{T}}u(t,x)\,\mathrm{d}x\Big|\Bigg)\\ &\leqslant \max_{t\in[0,T]}\|u_x\|_{L^2(\mathbb{T})} + C \leqslant C.\end{aligned}\tag{4.16}$$

Finally, using (4.12) and the preceding inequality, we get from Morrey's embedding theorem (see [2, Theorem 4 in Section 5.6]) that

$$\max_{t\in[0,T]} \|u\|_{C^0(\mathbb{T})} \leqslant C \max_{t\in[0,T]} \|u\|_{W^{1,2}(\mathbb{T})} \leqslant C. \qquad \square$$

To get uniform bounds for $u_x$ and to prove the existence of classical solutions of (4.4), we transform the problem into a second-order quasilinear elliptic equation with non-linear oblique boundary conditions. Because $G$ is a strictly monotone function, we first determine the value of $m$ from the first equation in (4.4) which we then insert into the second equation in (4.4). This leads to the system

$$\begin{cases} -u_{tt} + 2u_x u_{tx} - (\chi(-u_t + \frac{|u_x|^2}{2} + V) + |u_x|^2)u_{xx} - V'u_x = 0 & \text{in } (0,T)\times\mathbb{T} \\ B(t,x,u,(u_t,u_x)) = 0 & \text{in } \{0,T\}\times\mathbb{T}, \end{cases} \tag{4.17}$$

where $\chi(w) = G^{-1}(w)G'(G^{-1}(w))$ and

$$\begin{cases} B(0,x,z,(s,p)) = -s + \frac{|p|^2}{2} + V - G(m_0) & \text{in } \mathbb{T} \\ B(T,x,z,(s,p)) = s - \frac{|p|^2}{2} - V + G(m_T) & \text{in } \mathbb{T}. \end{cases} \tag{4.18}$$

Notice that (4.18) implies that

$$D_qB(0,x,z,q)\nu(0,x) = -B_s(0,x,z,q) = 1 > 0$$
$$D_qB(T,x,z,q)\nu(T,x) = B_s(T,x,z,q) = 1 > 0,$$

where $\nu$ is the outward pointing normal vector of $\partial([0,T]\times\mathbb{T})$ and $q = (s,p)$. This means that (4.18) is indeed a second-order quasilinear elliptic equation with non-linear oblique boundary conditions.

In the sequel, relying on the proprieties of a second-order quasilinear elliptic equation, we prove an $L^\infty$ bound for $u_x$ (for more details, see Lemma 3.18 in [18]).

**Proposition 4.5.** Consider the setting of Proposition 4.1. Then, there exists a positive constant, $C$, which depends only on the problem data, such that

$$\|u_x\|_{C^0([0,T]\times\mathbb{T})} \leqslant C. \tag{4.19}$$

*Proof.* The proof is similar to the one of Lemma 3.18 in [18]. Thus, here, we only discuss the main differences. As in [18], we set

$$\tilde{u} = u + \|u\|_{C^0([0,T]\times\mathbb{T})} + 1 - \frac{2(\|u\|_{C^0([0,T]\times\mathbb{T})} + 1)}{T}(T-t).$$

Proposition 4.4 implies that

$$|\tilde{u}| \leqslant C, \quad \tilde{u}(0,x) \leqslant -1, \quad \tilde{u}(T,x) \geqslant 1. \tag{4.20}$$

Further, we define

$$T_u v = -v_t + u_x v_x, \quad v(t,x) = \frac{|u_x|^2}{2} + V(x) + \frac{c_1}{2}\tilde{u}^2,$$

where $0 < c_1 \leqslant 1$ is a constant that will be chosen later. As a continuous function on $[0,T]\times\mathbb{T}$, $v$ achieves its maximum at a certain point $(t_0,x_0) \in [0,T]\times\mathbb{T}$. Next, we observe that we get (4.19) from the definition of $v$ and (4.20) provided we prove that

$$u_x^2(t_0,x_0) \leqslant C, \tag{4.21}$$

where C depends only on the problem data. To prove (4.21), we consider three different cases depending on the position of $t_0$ in $[0,T]$; namely, $0 < t_0 < T$, $t_0 = 0$, and $t_0 = T$. The first two cases are addressed in [18] (see Case 2 and Case 3 in the proof of Lemma 3.18 in [18]). The constraint in $c_1$ is used in the analysis of these two cases.

We are left to discuss the case where $t_0 = T$. Because $(T, x_0)$ is a point of maximum, we have $v_x = 0$ and $v_t \geqslant 0$. These two conditions and the first equation in (4.4) give

$$\begin{aligned}0 \geqslant T_u v(T,x_0) &= T_u\Big(\tfrac{|u_x^2|}{2} + V\Big)(T,x_0) + c_1\tilde{u}(T,x_0)(-u_t(T,x_0) + |u_x(T,x_0)|^2 - C)\\ &= u_x(-u_{tx} + u_x u_{xx} + V'(x_0)) + c_1\tilde{u}\Big(G(m_T(x_0)) - \tfrac{|u_x|^2}{2} + |u_x|^2 - C\Big)\\ &\geqslant u_x(T,x_0)(G(m_T))_x(x_0) + \frac{c_1}{2}\tilde{u}(T,x_0)|u_x(T,x_0)|^2 - C.\end{aligned}$$

Taking into account (4.20), Assumption 2, and using Young's inequality in the preceding inequality, we deduce that

$$|u_x|^2 \leqslant C,$$

which completes the proof. □

Next, we prove that (4.17) has a unique classical solution.

**Proposition 4.6.** Consider the setting of Proposition 4.1. Then, problem (4.17) has a unique classical solution, $(u, m) \in C^3([0,T] \times \mathbb{T}) \times C^2([0,T] \times \mathbb{T})$.

*Proof.* The proof is hinged on Propositions 4.1, 4.4, and 4.5, and uses the nonlinear method of continuity presented in the proof of Theorem 1.1 in [18]. Thus, we omit the proof. □

We are now in position to prove Theorem 1.4, which establishes the existence of classical solutions of Problem 2.

*Proof of Theorem 1.4.* First, we notice that Theorem 1.1 implies that there exists a positive constant, $C_{min}$, such that any classical solution $(u, m)$ of Problem 2 satisfies

$$m(t,x) \geqslant C_{min} > 0 \quad \text{for all } (t,x) \in [0,T] \times \mathbb{T}. \tag{4.22}$$

By Proposition 4.6, we deduce that (4.4) has a unique (up to constants) classical solution, $(\bar{u}, \bar{m}) \in C^3([0,T] \times \mathbb{T}) \times C^2([0,T] \times \mathbb{T})$. Furthermore, by using Proposition 4.1, we have, for the same $C_{min}$ in (4.22), that

$$\bar{m}(t,x) \geqslant C_{min} \quad \text{for all } (t,x) \in [0,T] \times \mathbb{T}. \tag{4.23}$$

Therefore, taking $\delta = C_{min}$ in (4.2), we deduce that $(\bar{u}, \bar{m})$ also solves Problem 2. Thus, to complete the proof, it remains to prove the uniqueness (up to constants) of solutions. The estimate (4.22) and (4.3) with $\delta = C_{min}$ imply that any classical solution of (4.1) is also a classical solution of (4.4). Consequently, because (4.4) has a unique (up to constants) classical solution, also the solution of Problem 2 is unique (up to constants). □

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