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Ultimate Force Boundary of Trusses

Caigui Jiang[*,a]

[*] King Abdullah University of Science and Technology
Thuwal, KSA
caigui.jiang@kaust.edu.sa

[a] ICSI/UC Berkeley

Abstract

In this paper, we introduce an effective way of limit analysis on the bearing capacity of given trusses. To illustrate the maximum external forces that a truss can carry on different directions, we propose the concept of ultimate force boundary (UFB) which is usually a closed polygon for a 2D truss and a closed polygonal mesh for a 3D truss. We prove that UFB is convex, and we introduce an efficient algorithm to compute the precise UFB based on its convexity. UFB can be used as direct visualization of bearing capacity of trusses and also provides insight of reducing the complexity of continuous boundary condition in truss optimization.

Keywords: trusses, optimization, ultimate force boundary, linear programming, limit analysis, cutting line, cutting plane.

1 Introduction

Trusses as fundamental structures have been studied in the field of civil engineering for decades. Recently, more attention has been paid to truss optimization in the community of computer graphics as the development of digital fabrication and geometric modeling such as research works in [1, 2, 3]. For a given truss, one of the important concern is the bearing capacity of truss when external forces are applied. To illustrate the maximum external forces that a truss can carry along different directions, we propose the concept of *ultimate force boundary*, UFB for short. We first set the joint where the external force applied as the origin. The maximum affordable external force along an arbitrary direction can be represented as a vertex on the boundary. UFB is a collection of all possible external force vectors which form a closed curve in 2D and a closed surface in 3D. Because of discreteness of trusses, more precisely, UFB is a closed polygon for a 2D truss and a closed polygonal mesh for a 3D truss. The region enclosed by the boundary represents all the affordable forces for a given truss. UFB can be used as a direct illustration of the bearing capacity of a truss, and more importantly, it provides an easy way to access the information of the maximum affordable force for any possible external force direction.

In the following chapters, we will first provide a formulation of finding the maximum external force along a given direction and provide a rough method of estimating UFB for given trusses. Then we will analysis the properties of UFB and provide an algorithm which can generate an exact ultimate force boundary.

2 Plastic Limit Analysis

2.1 A Simple Example

We first look at a simple 2D example, as shown in Fig.1(a), a truss connected by two bars i and j. Each bar has one end fixed as supporting point (red) and another end connected at the joint p. The bars can afford compression(-) and tension(+) along axial directions. Assume the limited compression and tension in bar i is $-\mathbf{F}_i$ and $\mathbf{F}_i$, and in bar j is $-\mathbf{F}_j$ and $\mathbf{F}_j$. We would like to know the bearing capacity of the truss when an external force is applied at the pin-joint p. The truss can afford an external force if the force can be decomposed into two forces along bar i and j and less than their limits. Then the solution of bearing capacity boundary is simply defined by a parallelogram with four vertices $\mathbf{F}_i + \mathbf{F}_j$, $\mathbf{F}_i - \mathbf{F}_j$, $-\mathbf{F}_i - \mathbf{F}_j$, and $-\mathbf{F}_i + \mathbf{F}_j$. Any external force within the parallelogram is affordable for the truss in Fig.1(a). Here the affordable force region bounded by the boundary is shown in yellow. In Fig.1(b), one more supporting bar k is connected at the joint p and its limited internal force range is $-\mathbf{F}_k$ and $\mathbf{F}_k$ shown in blue. The new ultimate force boundary in Fig.1(d) is the outline of region when moving the parallelogram by vector $-\mathbf{F}_k$ and $\mathbf{F}_k$ illustrated in Fig.1(c). This simple example provides a basic sense of the ultimate bearing capacity boundary and a simple way of calculation. However, when the truss gets complex, the analysis and calculation of the bearing capacity region are not straightforward.

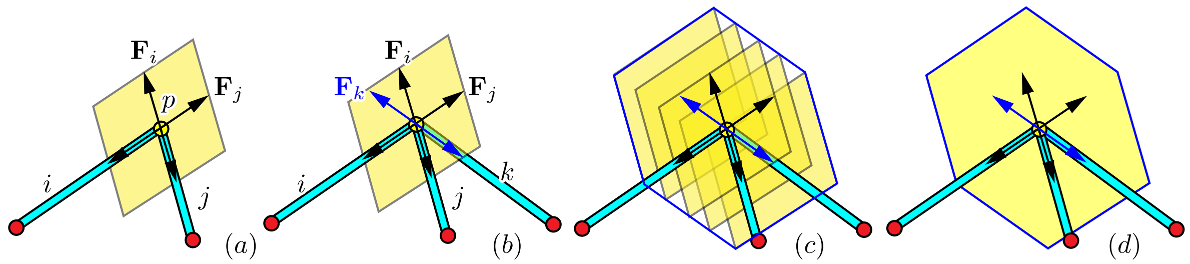

Figure 1: A simple example of ultimate force boundary.

2.2 LP Formulation

The LP formulation has been used commonly in truss optimization, such as the *ground structure method* (GSM) mentioned in [4], and methods used for truss limit analysis such in [5] and [6]. Given an external force direction and detailed information of a truss, such as its joint positions, connectivity, and cross-sections of bars, we can formulate the problem of finding the maximum force magnitude along this external force by a linear programming as shown below,

$$\underset{\lambda,\mathbf{s}}{\text{minimize}} \quad -\lambda, \tag{1}$$

$$\text{subject to} \quad \mathbf{B}^T\mathbf{s} = -\lambda\mathbf{f}, \tag{1a}$$

$$\sigma_C a_i + s_i \geq 0, \quad i = 1, \ldots, |E| \tag{1b}$$

$$\sigma_T a_i - s_i \geq 0, \quad i = 1, \ldots, |E| \tag{1c}$$

where $\mathbf{B}^T$ is the nodal equilibrium matrix, built from the directional cosines of the bars. a_i is the cross-section area of the i-th bar, and s_i is its internal force. $\mathbf{s}$ is a vector with the internal forces for all bars and $|E|$ is the number of bars. $\mathbf{f}$ is a unit vector of the external force. Here we assume there is only one external force, and λ is its magnitude. The problem formulation

used in this paper is based on the plastic analysis. σ_C and σ_T are stress limits in tension and compression, which are constant values when the material of bars is determined. The limited force the i-th bar can afford is $\sigma_C a_i$ in compression and $\sigma_T a_i$ in tension. As formulated in the constraints, the internal forces of each bar should be within its limit. Solving this LP problem, we can easily find the maximum affordable external force for any given direction.
This formulation as a building block itself doesn't provide a direct computation of the bearing capacity boundary. This means for each given external force direction, to find its maximum bearing force, we need to solve this LP problem once. It may be not efficient because the number of possible directions could be infinitely many. However, we can still use this formulation to estimated a rough bearing capacity boundary when we sample a limited number of external force directions. For instance, we sample the external force directions along a unit circle for the 2D case and along a unit sphere for the 3D case. One example is shown in Fig. 2. The upper row is for a 2D truss, and the lower one is for a 3D truss. From left to right, (a1) and (a2) show the input trusses with given joint positions, topologies and cross-sections of bars, the trusses are supported at the red joints and external forces are applied at the blue joints, (b1) and (b2) illustrate different sampled external force directions, (c1) and (c2) show the maximum magnitude of external forces that the given trusses can afford, and (d1) and (d2) show the color coding of the magnitude, here the magnitude increases from blue to red. Note that the boundary curve of 2D shape in (d1) and the surface of 3D shape in (d2) are the so-called ultimate bearing capacity curve and surface which are convex polygon and polygonal mesh respectively. However, the results from this kind of sampling method only provide a limited number of vertices on UFB which is a rough limit analysis.

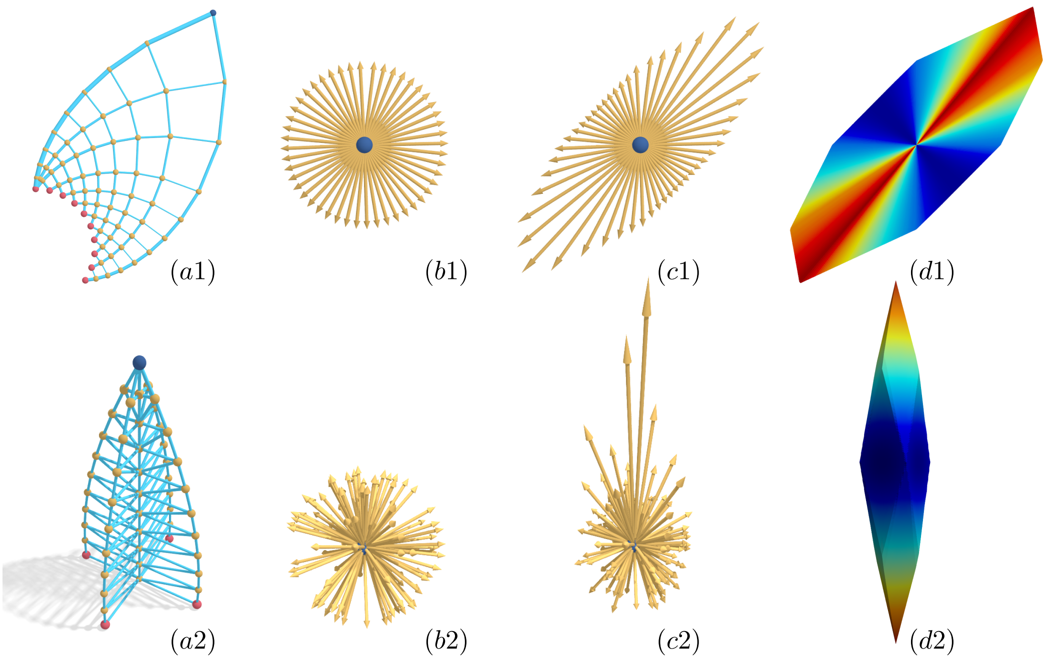

Figure 2: A rough limit analysis of given trusses.

To estimate the exact bearing capacity boundary, in the next chapters, we will first analyze the properties of the boundary and provide an efficient algorithm of computing the bearing capacity boundary using the properties and the basic formulation in Eq. 1.

3 Properties of UFB

In the following, we first provide some properties of the ultimate force boundary and prove these properties briefly. These properties can be used to find the exact force bearing capacity boundary for 2D and 3D trusses.

Property I *Ultimate force boundary is convex.*

This property can also be described as if a truss can bear the external forces $\mathbf{F}_1$ and $\mathbf{F}_2$, it can always bear an external force $\mathbf{F}_3 = \delta\mathbf{F}_1 + (1-\delta)\mathbf{F}_2$, where $0 \leq \delta \leq 1$. Assume $\mathbf{F}_1 = \lambda_1\mathbf{f}_1$ and $\mathbf{F}_2 = \lambda_2\mathbf{f}_2$, where $\mathbf{f}_1$, $\mathbf{f}_2$ are unit vectors and λ_1, λ_2 are force magnitudes as shown in Fig. 3 left. To prove the convexity of ultimate force boundary, we take a look at the constraint part of Eq. 1. As $\mathbf{F}_1$ and $\mathbf{F}_2$ are affordable external forces, we can always obtain the internal forces $\mathbf{s}_1$ and $\mathbf{s}_2$, such that the following constraints are satisfied.

$$\begin{aligned} &B^T\mathbf{s}_1 = -\lambda_1\mathbf{f}_1 = -\mathbf{F}_1, \\ &-\sigma_C\mathbf{a} \leq \mathbf{s}_1 \leq \sigma_T\mathbf{a}. \end{aligned} \qquad \textit{and} \qquad \begin{aligned} &B^T\mathbf{s}_2 = -\lambda_2\mathbf{f}_2 = -\mathbf{F}_2, \\ &-\sigma_C\mathbf{a} \leq \mathbf{s}_2 \leq \sigma_T\mathbf{a}. \end{aligned}$$

Then, we have

$$\begin{aligned} B^T(\delta\mathbf{s}_1 + (1-\delta)\mathbf{s}_2) = -(\delta\mathbf{F}_1 + (1-\delta)\mathbf{F}_2)) = -\mathbf{F}_3, \\ -\sigma_C\mathbf{a} \leq (\delta\mathbf{s}_1 + (1-\delta)\mathbf{s}_2) \leq \sigma_T\mathbf{a}. \end{aligned}$$

The above formulation means we can always find the internal forces $\mathbf{s}_3 = \delta\mathbf{s}_1 + (1-\delta)\mathbf{s}_2$ which satisfy the constrain in Eq. 1 when the external force is $\mathbf{F}_3 = \delta\mathbf{F}_1 + (1-\delta)\mathbf{F}_2$. Assume $\mathbf{f}_3$ and λ_3 are the unit direction and magnitude of $\mathbf{F}_3$. When the external force direction $\mathbf{f} = \mathbf{f}_3$, the above formulation also means $(\lambda_3, \mathbf{s}_3)$ is in the feasible region of the LP formulation with $\mathbf{f}_3$ as the external force. Then solving the LP problem can always lead to an optima $\lambda_3^* \geq \lambda_3$. This means the maximum external force along $\mathbf{f}_3$ is

$$\mathbf{F}_3^* = \lambda_3^*\mathbf{f}_3 \geq \lambda_3\mathbf{f}_3 = \mathbf{F}_3 = \delta\mathbf{F}_1 + (1-\delta)\mathbf{F}_2 \tag{2}$$

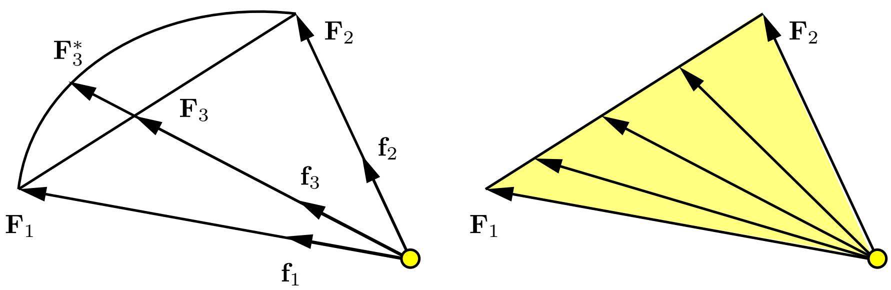

Figure 3: Convexity of UFB.

Similarly in 3D, we can prove if a truss can afford external forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$, the truss can always afford an external forces which is the linear interpolation of these three vertices.

According to the definition of ultimate force boundary and observations in Fig. 1 and 2. UFB is a polygon in 2D and a polygonal mesh in 3D which is enclosed by a limited number of lines or planes. We call such lines and planes as *cutting lines* and *cutting planes.* The following property can be used as a tool to find the cutting lines and planes.

Property II *For a 2D truss, if three vertices* $\mathbf{F}_1$, $\mathbf{F}_2$ *and* $\mathbf{F}_3$ *are on the ultimate force boundary and lie on a common line, then this line is a cutting line of the ultimate force boundary.*
This property also means the affordable force region is bounded by the line connected by $\mathbf{F}_1$ and $\mathbf{F}_2$, or the line segment $\mathbf{F}_1\mathbf{F}_2$ is a part of the ultimate force boundary. As show in Fig. 4, the maximum affordable forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ are collinear. For an arbitrary direction $\mathbf{f}_4$ between force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$, its maximum affordable force $\mathbf{F}_4$ can only be the case in Fig.4 (c). For the other cases, we can simply find the concavity of $\mathbf{F}_1\mathbf{F}_3\mathbf{F}_4$ in Fig. 4(a) and $\mathbf{F}_3\mathbf{F}_4\mathbf{F}_2$ in Fig. 4(b) is conflict with Property I.

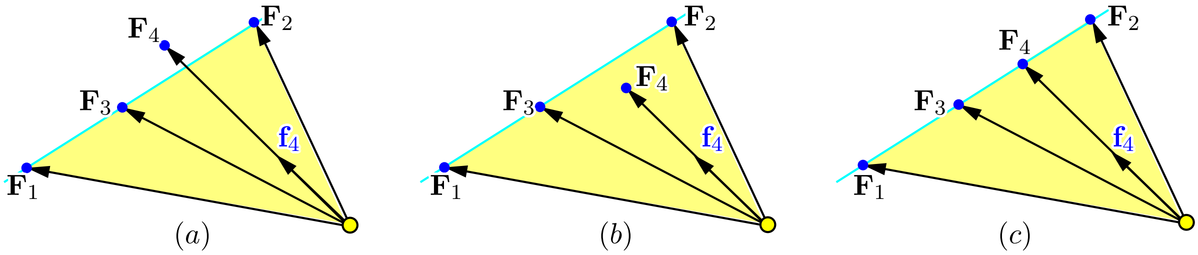

Figure 4: Cutting line for finding UFB of 2D trusses.

Similarly for 3D trusses, if a linear interpolation of three vertices $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ on the ultimate force boundary is also on the boundary, the plane defined by $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ is a cutting plane of ultimate force boundary as shown in Fig. 5.

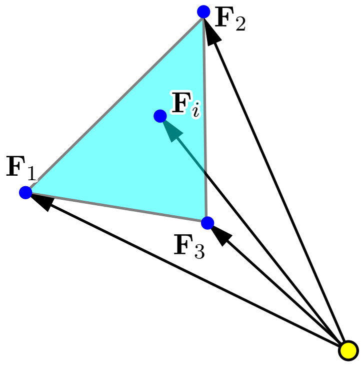

Figure 5: Cutting plane for finding UFB of 3D trusses.

4 Our Algorithm

Based on the properties mentioned in the last chapter. We propose an efficient algorithm of computing the bearing capacity boundary for given trusses.

4.1 Calculating of Cutting Line and Plane

Given an external force direction $\mathbf{f}_i$, the formulation in Section 2.2 tells us how to compute its maximum affordable force $\mathbf{F}_i$. After that, we would like to know how to calculate the boundary line or plane (3D) passing through $\mathbf{F}_i$, here we also call them cutting line or plane. To calculate the cutting line or plane, we estimate the maximum external forces of its several neighboring directions such as the ones with an angle of θ between $\mathbf{f}_i$ for the 2D case shown in Fig. 6 (a). Then we check whether the neighboring maximum external forces are collinear. If the case is at least three neighboring maximum external forces lie on a common line, as shown in Fig. 6 (a), then this common line (cyan) is the cutting line. Otherwise, in the case shown in Fig. 6 (b), we need to narrow down the neighboring searching directions further and check whether the cutting line or plane can be found. Usually, a cutting line or plane can be achieved within a few iterations as shown in Fig. 6 (c) and (d). Similar idea of cutting plane has been investigated in the solutions of mixed integer programming such as in [7].

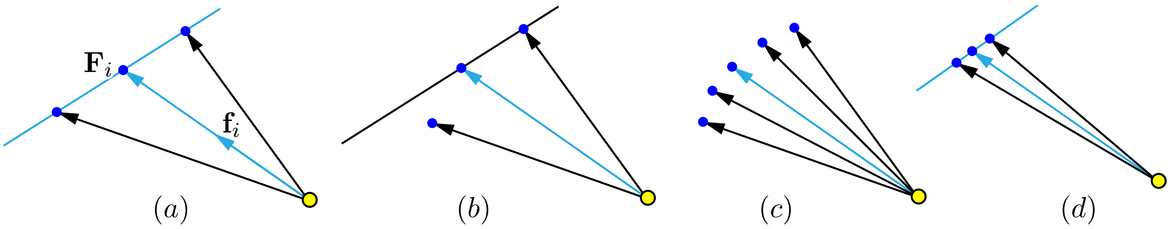

Figure 6: Procedure of finding cutting line.

4.2 Calculating of UFB

The calculating of cutting line or cutting plane in the previous sub-section is the basic building block. Here we illustrate the workflow of our algorithm in the 2D case shown in Fig.7, the calculating of boundary for 3D is similar. As shown in Fig.7 (a), we initially calculate the maximum external forces along with axis directions and estimate their cutting lines using the method described before. Then we obtain intersections (red) of neighboring cutting lines shown in Fig.7 (b). For each intersection vertex, we can use the formulation in Section 2.2 to check whether this vertex is located on the ultimate force boundary. If it's true, we mark this vertex as blue and continue the checkup of the next intersection vertex. If it's false, we calculate the maximum affordable force along the direction pointed to this vertex from the origin, and also its cutting line as shown in Fig.7 (c). New intersection vertices will be added to the checking list as the new cutting line intersects with the region enclosed by the previous cutting lines. The procedure continues until all the intersection vertices are marked as blue. The resulting polygon is the ultimate force boundary. For 3D truss, the basic idea is the same. We first sample some direction and estimate the maximum affordable force and their cutting planes. Further, the region is refined by finding possible cutting planes related to the directions pointed to the intersection vertices.

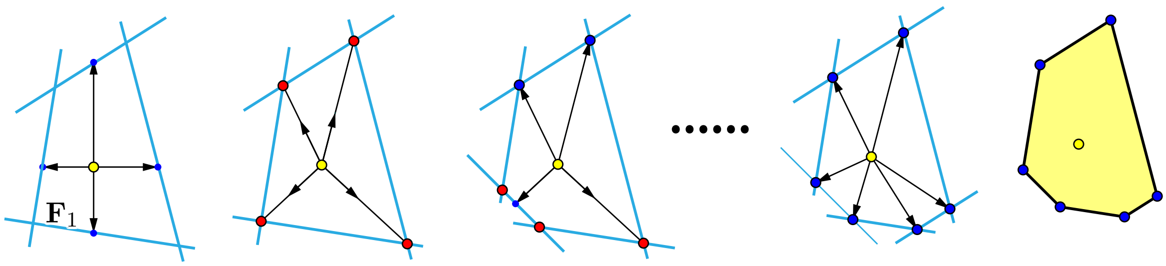

Figure 7: Basic workflow of the algorithm.

5 Applications

Our method provides an efficient way of calculating the exact ultimate force boundary which can be used as direct visualization of the bearing capacity of given trusses. For example, for the supporting truss for a dome shown in Fig.8 (a), we calculate its exact UFB when external forces applied at the top vertex of the truss using the algorithm introduced in the last chapter. The UFB is shown in Fig. 8 (b). Here, we assume $\sigma_C = \sigma_T$ and all the bars have the same cross-sections.

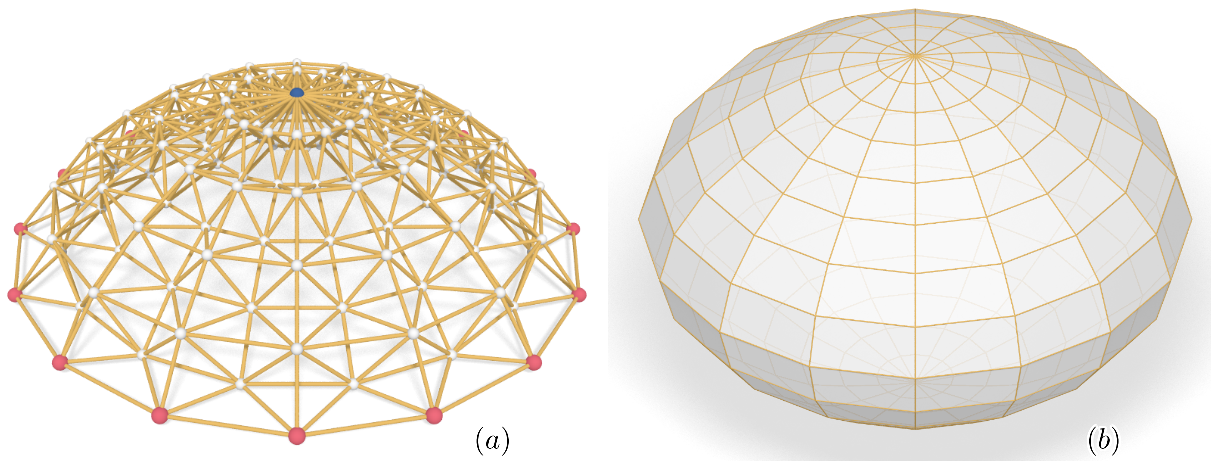

Figure 8: Left is a supporting truss. Right is the ultimate force boundary when external force applied at the top vertex (blue) of the truss.

Further, the idea used in our paper could also be used in other scenarios, such as truss optimization for dynamic external forces. For example, as shown in Fig. 9, the external forces from a hanging pendulum at the top joint change dynamically. It's challenging to incorporate infinitely many external forces in the truss optimization such as minimizing the total material consumption of truss using the traditional method such as GSM. Instead, we can consider the maximum affordable forces of several directions because of the convexity of the ultimate force boundary. For example, if the truss can afford the three external force shown in Fig.9 (c), it can always afford any external force from the pendulum shown in Fig.9 (b) because the possible external force region (gray) is covered by the affordable force region formed by three external forces. In this example, the boundary condition of external forces could be simplified dramatically. The truss optimization can be formulated as a linear programming problem as

$$
\begin{aligned}
\underset{a_i, s_i^k}{\text{minimize}} \quad & \sum_{i=1}^{|E|} l_i a_i, & \\
\text{subject to} \quad & \mathbf{B}^T \mathbf{s}^1 = -\mathbf{F}_1, & \\
& \sigma_C a_i + s_i^1 \geq 0, & i = 1, \dots, |E| \\
& \sigma_T a_i - s_i^1 \geq 0, & i = 1, \dots, |E| \\
& \mathbf{B}^T \mathbf{s}^2 = -\mathbf{F}_2, & \\
& \sigma_C a_i + s_i^2 \geq 0, & i = 1, \dots, |E| \\
& \sigma_T a_i - s_i^2 \geq 0, & i = 1, \dots, |E| \\
& \mathbf{B}^T \mathbf{s}^3 = -\mathbf{F}_3, & \\
& \sigma_C a_i + s_i^3 \geq 0, & i = 1, \dots, |E| \\
& \sigma_T a_i - s_i^3 \geq 0, & i = 1, \dots, |E|
\end{aligned}
$$

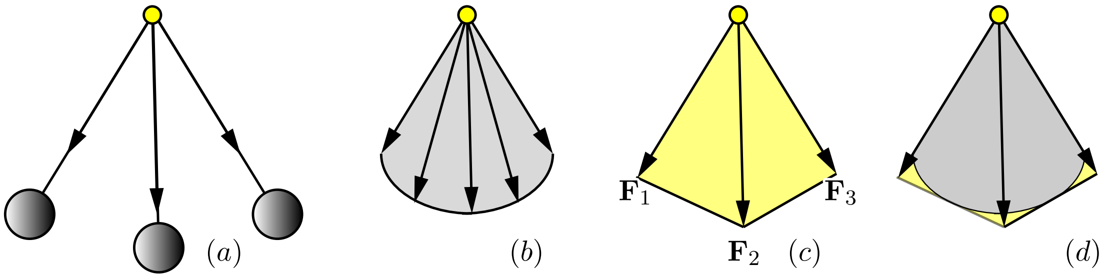

Figure 9: Dynamic external forces from a pendulum.

6 Conclusions and Future Work

We present an efficient approach to calculate the ultimate force boundary for 2D and 3D trusses. We prove that UFB is convex, and this convexity is useful for truss optimization when the external forces are continuous and dynamic. In the future, we would like to incorporate the other aspects of truss optimization such as geometry and topology, which may provide more degree of freedom for lightweight truss design. We also interested in the cases of multiple external forces applied simultaneously.

7 Acknowledgements

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