Parameter and Differentiation Order Estimation for a Two Dimensional Fractional Partial Differential Equation

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Abstract

This paper deals with the estimation of coefficients and differentiation orders for two-dimensional fractional partial differential equations. Recently, a hybrid method based on modulating functions has been proposed by the authors to estimate the coefficients and a differentiation order for a one dimensional fractional advection dispersion equation in [**?**]. We propose to extend this method to the two-dimensional case. First, the coefficients are estimated using a modulating functions method, where the problem is transferred into solving a system of algebraic equations. Then, the modulating functions method combined with a Newton algorithm is proposed to estimate the coefficients and the differentiation orders simultaneously. Numerical example is presented with noisy measurements to show the effectiveness and the robustness of the method.

Keywords: Fractional partial differential equations; modulating functions; parameter estimation; Newton algorithm

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1. Introduction

Fractional calculus has gained increasing interest in many scientific fields, such as physics [**? ? ?**], biology [**?**], control [**?**], electrical and mechanical engineering [**?**], signal processing [**?**], and finance [**?**]. Thanks to their interesting non-locality and memory properties [**? ?**], fractional derivatives have been proposed as a powerful tool for modeling complex phenomena such as anomalous diffusion in porous media. However, fractional differentiation orders are usually unknown and model calibration methods are required to estimate model's parameters and differentiation orders.

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In this paper, we are interested in the estimation problem of coefficients and differentiation orders for a two-dimensional fractional partial differential equation. This is an inverse problem, often ill-posed in the sense of Hadamard. Several methods have been proposed to solve such estimation problems with their advantages and limitations.

Parameters' identification for a one-dimensional fractional partial differential equation has been the focus of many researchers. However, when modeling real physical phenomena such as groundwater flow and transport, two-dimensional model is more realistic.

Very little progress has been made on the inverse problem for two-dimensional fractional partial differential equations. Among the existing work in the literature is Xiong et al. [?] who investigated an inverse problem for a two-dimensional time-fractional diffusion equation and established the stability of determining heat flux from a measured temperature history at a fixed point using Fourier regularizing method. Moreover, Kirane et al. [?], showed the existence and the uniqueness of an inverse source problem for a two-dimensional fractional diffusion equation using the properties of the biorthogonal system of functions. Qian et al. [?], presented a numerical solution for the two-dimensional inverse heat conduct problem based on Kernel approximation in the frequency domain.

Inspired by the algorithm introduced in [?], in this paper, we propose to combine a modulating functions based method with a Newton algorithm to estimate the parameters and the differentiation orders of a two-dimensional fractional partial differential equation. The extension of the algorithm to two dimensional case is not straightforward and therefore the main contributions of this paper compared to [?], are as follows. First, three sets of modulating functions are used and integrations with respect to more than one variable are introduced to solve the problem. Second, a Jacobian matrix is exactly characterized instead of computing the derivative of the equation's coefficients with respect to one variable. Third, unlike the work done in [?] where a single differentiation order is computed, several differentiation orders are estimated in this paper which requires solving a system of nonlinear equations using Newton's method.

The remainder of the paper is organized as follows: in Section 2, some definitions are re-called. In Section 3, the considered problem is introduced. In Section 4, the modulating functions method is applied to the fractional partial differential equation, where the coefficients are estimated by solving a system of algebraic equations. Then, the modulating functions method is combined with the Newton's type algorithm to estimate all parameters simultaneously, in Section 5. In Section 6, some numerical results are given to show the efficiency and robustness of the method. Finally, a conclusion summarizes the obtained results.

2. Preliminaries

In this section, we recall the definition of the so-called modulating functions and some useful properties.

Definition 2.1. ([?] p. 62) The α^{th} order Riemann-Liouville fractional derivative of a continuous function f defined on $\mathbb{R}$, with $\alpha \in \mathbb{R}$, is defined as follows: $\forall t \in \mathbb{R}$, $n \in \mathbb{N}^*$,

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-\tau)^{n-\alpha-1} f(\tau) d\tau, \quad n-1 \leq \alpha < n. \tag{1}$$

Definition 2.2. [?] A function $\phi(x) \in C^n$, defined over the interval $[a, b]$, is called a modulating function of order k with $k \in \mathbb{N}^*$ if:

$$\phi^{(i)}(a) = \phi^{(i)}(b) = 0, \quad i = 0, 1, \ldots, k-1. \tag{2}$$

One of the remarkable properties of the modulating functions is given in the below lemma, where the convolution theorem of the Laplace transform was applied to obtain a useful general integration by parts formula [?].

Lemma 2.1. *[?] If the α^{th} order Riemann-Liouville derivative of f exists where $n-1 \leq \alpha < n$, and g is an n^{th} order modulating function defined on $[0, L]$. Then, we have:*

$$\int_0^L g(L-x) D_x^\alpha f(x) dx = \int_0^L D_x^\alpha g(x) f(L-x) dx. \tag{3}$$

Proof. By applying the convolution theorem of Laplace transform (see [?], p. 1020), we get:

$$\mathcal{L}\left\{ \int_0^L g(L-x) D_x^\alpha y(x) dx \right\}(s) = \hat{g}(s) \mathcal{L}\{D_x^\alpha y(x)\}(s).$$

Then by using the Laplace transformation for a fractional derivative (see [?], p.284), we get

$$\hat{g}(s)\mathcal{L}\{D_x^\alpha y(x)\}(s) = \hat{g}(s) s^\alpha \hat{y}(s) - \sum_{i=0}^{l-1} s^i \hat{g}(s) [D_x^{\alpha-i-1} y(x)]_{x=0}. \tag{4}$$

By using the Laplace transformation of a fractional derivative and by the properties of the modulating function, we have

$$\mathcal{L}\{D_x^\alpha g(x)\}(s) = s^\alpha \hat{g}(s), \tag{5}$$

$$\mathcal{L}\{g^{(i)}(x)\}(s) = s^i \hat{g}(s), \tag{6}$$

for $i = 0, \ldots, l-1$. Consequently, by applying (**??**), (**??**) and the inverse of the Laplace transform to (**??**), we obtain:

$$\mathcal{L}^{-1}\left\{\hat{g}(s)\mathcal{L}\{D_x^\alpha y(x)\}(s)\right\}(L) =$$

$$\mathcal{L}^{-1}\{\mathcal{L}\{D_x^\alpha g(x)\}(s).\hat{y}(s)\}(L) - \sum_{i=0}^{l-1} g^{(i)}(L)[D_x^{\alpha-i-1}y(x)]_{x=0}. \tag{7}$$

By the properties of the modulating functions, the initial conditions $[D_x^{\alpha-i-1}y(x)]_{t=0}$, for $i = 0, \ldots, l-1$, can be eliminated.

Finally, this proof can be completed by applying the convolution theorem of the Laplace transform to (**??**). □

In the next proposition, we present the derivative of the fractional derivative with respect to the fractional order α. We consider the left Riemann-Liouville derivative. However, similar results can be obtained using other definitions.

Proposition 2.1. *If the α^{th} order Riemann-Liouville derivative of f exists where $n-1 \leq \alpha < n$, then the derivative of $\dfrac{\partial^\alpha f}{\partial x^\alpha}$ with respect to α is given by*

$$\frac{\partial}{\partial \alpha}\frac{\partial^\alpha f(x)}{\partial x^\alpha} = \psi_0(n-\alpha)\frac{\partial^\alpha f(x)}{\partial x^\alpha} - \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-\tau)^{n-\alpha-1}\ln(x-\tau)\, f(\tau)\, \mathrm{d}\tau, \tag{8}$$

where $\psi_0(n-\alpha) = \frac{\Gamma'(n-\alpha)}{\Gamma(n-\alpha)}$.

Proof. The result can be obtained by differentiating (**??**) with respect to α. □

3. Problem Statement

We consider the following general fractional differential equation [? ? ? ?]: for any $0 < x < L_1$, $0 < y < L_2$ and $t > 0$,

$$\sum_{j=1}^{J} a_j \frac{\partial^{\beta_j} u(x,y,t)}{\partial t^{\beta_j}} = \sum_{i=1}^{I} b_i \frac{\partial^{\alpha_i} u(x,y,t)}{\partial x^{\alpha_i}} + \sum_{i=1}^{I} c_i \frac{\partial^{\lambda_i} u(x,y,t)}{\partial y^{\lambda_i}} + f(x,y,t), \tag{9}$$

with initial and Dirichlet boundary conditions:

$$\begin{cases} u(x,y,0) &= g_0(x,y), \\ u(0,y,t) &= u(x,0,t) = 0, \\ u(L_1,y,t) &= h_1(y,t), \\ u(x,L_2,t) &= h_2(x,t), \end{cases} \tag{10}$$

where $a_j, b_i, c_i \in \mathbb{R}$, $0 \leq \beta_j < 1$ and $1 \leq \alpha_i, \lambda_i < 2$ and $J, I \in \mathbb{N}$. Moreover, we specify that the boundary conditions g_0, h_1 and h_2 are assumed to be continuous on the intervals $[0, L_1] \times [0, L_2]$, $[0, L_2] \times \mathbb{R}$ and $[0, L_1] \times \mathbb{R}$, respectively.

We are interested in identifying the coefficients a_j, b_i, c_i and the differentiation orders $\beta_j, \alpha_i, \lambda_i$ for $i = 1, \ldots I$ and $j = 1, \ldots J$ for the fractional partial differential equation defined by (**??**) - (**??**) using the measurement of the concentration u at time $0 < t < T_1$:

$$u(x,y,t) + \xi \quad 0 < x < L_1 \quad 0 < y < L_2, \tag{11}$$

where ξ is noise contaminating the data.

4. Modulating Functions Method for estimating the coefficients

In this section, we present our first result, where the modulating functions method is used to estimate the coefficients by assuming that the differentiation orders are known.

Proposition 4.1. *Let $\{\phi_m(x)\}_{m=1}^{M}$, $\{\psi_n(y)\}_{n=1}^{N}$ and $\{\eta_\kappa(t)\}_{\kappa=1}^{K}$ be sets of 2^{nd} order modulating functions defined on the intervals $[0, L_1]$, $[0, L_2]$ and $[0, T]$ respectively, where $M + N + K \geqslant J + 2I$, $L_x \leq L_1$, $L_y \leq L_2$ and $T \leq T_1$, then the solution of the following linear system gives the estimations of the parameters $a_1, a_2, \ldots, a_J, b_1, b_2, \ldots, b_I, c_1, c_2, \ldots, c_I$:*

$$\begin{pmatrix} \mathbf{A} & | & \mathbf{B} & | & \mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \mathbf{E}, \tag{12}$$

where

$$\mathbf{a}=\begin{pmatrix}a_1\\a_2\\\vdots\\a_J\end{pmatrix},\quad \mathbf{b}=\begin{pmatrix}b_1\\b_2\\\vdots\\b_I\end{pmatrix},\quad \mathbf{c}=\begin{pmatrix}c_1\\c_2\\\vdots\\c_I\end{pmatrix}, \tag{13}$$

$$\mathbf{A}=\left(\begin{array}{cccc}A_{1,1,1,1} & A_{2,1,1,1} & \dots & A_{J,1,1,1}\\ A_{1,2,1,1} & A_{2,2,1,1} & \dots & A_{J,2,1,1}\\ \vdots & \vdots & & \vdots\\ A_{1,M,1,1} & A_{2,M,1,1} & \dots & A_{J,M,1,1}\\ \hline A_{1,1,2,1} & A_{2,1,2,1} & \dots & A_{J,1,2,1}\\ \vdots & \vdots & & \vdots\\ A_{1,2,2,1} & A_{2,2,2,1} & \dots & A_{J,2,2,1}\\ A_{1,M,2,1} & A_{2,M,2,1} & \dots & A_{J,M,2,1}\\ \hline \vdots & \vdots & & \vdots\\ \vdots & \vdots & & \vdots\\ A_{1,M,N,K} & A_{2,M,N,K} & \dots & A_{J,M,N,K}\end{array}\right),\quad \mathbf{E}=\left(\begin{array}{c}E_{1,1,1}\\ E_{2,1,1}\\ \vdots\\ E_{M,1,1}\\ \hline E_{1,2,1}\\ E_{2,2,1}\\ \vdots\\ E_{M,2,1}\\ \hline \vdots\\ \vdots\\ E_{M,N,K}\end{array}\right), \tag{14}$$

and $\mathbf{B}$, $\mathbf{C}$ *have the same structure as* $\mathbf{A}$ *with*

$$A_{j,m,n,\kappa}=\int_0^T\int_0^{L_x}\int_0^{L_y}\phi_m(L_x-x)\psi_n(L_y-y)\frac{\partial^{\beta_j}\eta^*_\kappa(t)}{\partial t^{\beta_j}}u^*(x,y,T-t)dx\,dy\,dt,$$

$$B_{i,m,n,\kappa}=-\int_0^T\int_0^{L_x}\int_0^{L_y}\frac{\partial^{\alpha_i}\phi^*_m(x)}{\partial x^{\alpha_i}}\psi_n(L_y-y)\eta_\kappa(t-T)u^*(L_x-x,y,t)dx\,dy\,dt,$$

$$C_{i,m,n,\kappa}=-\int_0^T\int_0^{L_x}\int_0^{L_y}\phi_m(L_x-x)\frac{\partial^{\lambda_i}\psi^*_n(y)}{\partial y^{\lambda_i}}\eta_\kappa(t-T)u^*(x,L_y-y,t)dx\,dy\,dt,$$

$$E_{m,n,\kappa}=\int_0^T\int_0^{L_x}\int_0^{L_y}\phi_m(L_x-x)\psi_n(L_y-y)\eta_\kappa(t-T)f(x,y,t)dx\,dy\,dt.$$

Proof. First we multiply (**??**) by the modulating functions $\phi_m(L_x-x)$,$\psi_n(L_y-y)$ $\eta_\kappa(T-t)$ for

$n = 1, \dots, N$, $m = 1, \dots, M$ and $\kappa = 1, \dots, K$, then we get:

$$\begin{aligned}\sum_{j=1}^{J} a_j \phi_m(L_x - x)\psi_n(L_y - y)\eta_\kappa(T - t)\frac{\partial^{\beta_j} u(x,y,t)}{\partial t^{\beta_j}} \\ = \sum_{i=1}^{I} b_i \phi_m(L_x - x)\psi_n(L_y - y)\eta_\kappa(T - t)\frac{\partial^{\alpha_i} u(x,y,t)}{\partial x^{\alpha_i}} \\ + \sum_{i=1}^{I} c_i \phi_m(L_x - x)\psi_n(L_y - y)\eta_\kappa(T - t)\frac{\partial^{\lambda_i} u(x,y,t)}{\partial y^{\lambda_i}} \\ + \phi_m(L_x - x)\psi_n(L_y - y)\eta_\kappa(T - t) f(x,y,t).\end{aligned} \tag{15}$$

Then, integrate over the intervals $[0, L_x]$, $[0, L_y]$ and $[0, T]$, we will get:

$$\begin{aligned}\sum_{j=1}^{J} a_j \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x)\psi_n(L_y - y)\eta_\kappa(T - t)\frac{\partial^{\beta_j} u(x,y,t)}{\partial t^{\beta_j}} dx\, dy\, dt \\ = \sum_{i=1}^{I} b_i \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x)\psi_n(L_y - y)\eta_\kappa(T - t)\frac{\partial^{\alpha_i} u(x,y,t)}{\partial x^{\alpha_i}} dx\, dy\, dt \\ + \sum_{i=1}^{I} c_i \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x)\psi_n(L_y - y)\eta_\kappa(T - t)\frac{\partial^{\lambda_i} u(x,y,t)}{\partial y^{\lambda_i}} dx\, dy\, dt \\ + \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x)\psi_n(L_y - y)\eta_\kappa(T - t) f(x,y,t) dx\, dy\, dt.\end{aligned} \tag{16}$$

Now, apply integration by parts and Lemma **??** to equation (**??**), we obtain:

$$\begin{aligned}\sum_{j=1}^{J} a_j \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x)\psi_n(L_y - y)\frac{\partial^{\beta_j} \eta_\kappa^*(t)}{\partial t^{\beta_j}} u^*(x,y,T-t) dx\, dy\, dt \\ = \sum_{i=1}^{I} b_i \int_0^T \int_0^{L_x} \int_0^{L_y} \frac{\partial^{\alpha_i} \phi_m^*(x)}{\partial x^{\alpha_i}} \psi_n(L_y - y)\eta_\kappa(t - T) u^*(L_x - x, y, t) dx\, dy\, dt \\ + \sum_{i=1}^{I} c_i \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x)\frac{\partial^{\lambda_i} \psi_n^*(y)}{\partial y^{\lambda_i}} \eta_\kappa(t - T) u^*(x, L_y - y, t) dx\, dy\, dt \\ + \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x)\psi_n(L_y - y)\eta_\kappa(t - T) f(x,y,t) dx\, dy\, dt,\end{aligned} \tag{17}$$

with

$$\phi_m^*(x) = \begin{cases} -\phi_m(L - x) & : \quad \alpha_i = 1 \\ \phi_m(x) & : \quad o.w \end{cases}, \tag{18}$$

$$\psi_n^*(y) = \begin{cases} -\psi_n(L_y - y) & : \quad \lambda_i = 1 \\ \psi_n(y) & : \quad o.w \end{cases}, \tag{19}$$

$$\eta_\kappa^*(t) = \begin{cases} -\eta_\kappa(T - t) & : \quad \beta_j = 1 \\ \eta_\kappa(t) & : \quad o.w \end{cases}, \tag{20}$$

$$u^*(L_x - x, y, T) = \begin{cases} u(x, y, t) & : \quad \alpha_i = 1 \\ u(L_x - x, y, t) & : \quad o.w \end{cases}, \tag{21}$$

$$u^*(x, L_y - y, T) = \begin{cases} u(x, y, t) & : \quad \lambda_i = 1 \\ u(x, L_y - y, t) & : \quad o.w \end{cases}, \tag{22}$$

$$u^*(x, y, t - T) = \begin{cases} u(x, y, t) & : \quad \eta_j = 1 \\ u(x, y, T - t) & : \quad o.w \end{cases}. \tag{23}$$

Finally, the unknown coefficients can be estimated by solving the linear system given in (**??**). □

5. Parameter and differentiation order estimation

In this section, we combine the modulating functions method with a Newton's type algorithm to simultaneously estimate the coefficients and the differentiation orders for the fractional differential equation given in (**??**).

An important problem for fractional models is the estimation of the parameters that best fit the model. However, the problem becomes more difficult for 2D models as the number of unknown parameters increases and using the standard optimization techniques may lead to an ill-conditioning problem. This difficulty and complexity can be reduced by applying the algorithm presented in [**?**]. However, to extend the combined algorithm to the 2D case we need to (i) use more sets of modulating functions, (ii) represent the unknown coefficients in terms of two variables, (iii) compute the Jacobian matrix and (iv) solve a system of nonlinear equations.

Now, we introduce the two-stage algorithm to estimate coefficients and the differentiation orders and provide an exact characterization of the Jacobian matrix which is the main challenge for most optimization methods.

Stage 1: In this stage, we apply Proposition **??** to re-write the coefficients as functions of the unknown differentiation orders $\beta_i, \alpha_j, \lambda_i$ and provides an exact characterization of the Jacobian matrix which is the main challenge in most existing methods.

Then, we consider the following equation:

$$\sum_{j=1}^{J} a_j(\mathbf{z}) \frac{\partial^{\beta_j} u(x,y,t)}{\partial t^{\beta_j}} = \sum_{i=1}^{I} \left[b_i(\mathbf{z}) \frac{\partial^{\alpha_i} u(x,y,t)}{\partial x^{\alpha_i}} + c_i(\mathbf{z}) \frac{\partial^{\lambda_i} u(x,y,t)}{\partial y^{\lambda_i}} \right] + f(x,y,t), \tag{24}$$

where $\mathbf{z} = (\beta_1, \beta_2, \ldots, \beta_J, \alpha_1, \alpha_2, \ldots, \alpha_I, \lambda_1, \lambda_2, \ldots, \lambda_I)$.

If $\{\phi_m(x)\}_{m=1}^{M+1}$, $\{\psi_n(y)\}_{n=1}^{N+1}$ and $\eta_{K+1}(t)$ are 2^{nd} order modulating functions, then using a similar way of obtaining (**??**), we get:

$$\begin{aligned}
&\sum_{j=1}^{J} a_j(\mathbf{z}) \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x)\psi_n(L_y - y) \frac{\partial^{\beta_j} \eta^*_{K+1}(t)}{\partial t^{\beta_j}} u^*(x,y,T-t) dx\, dy\, dt \\
&= \sum_{i=1}^{I} b_i(\mathbf{z}) \int_0^T \int_0^{L_x} \int_0^{L_y} \frac{\partial^{\alpha_i} \phi^*_m(x)}{\partial x^{\alpha_i}} \psi_n(L_y - y)\eta_{K+1}(t-T) u^*(L_x - x, y, t) dx\, dy\, dt \\
&\quad + \sum_{i=1}^{I} c_i(\mathbf{z}) \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x) \frac{\partial^{\lambda_i} \psi^*_n(y)}{\partial y^{\lambda_i}} \eta_{K+1}(t-T) u^*(x, L_y - y, t) dx\, dy\, dt \\
&\quad + \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x)\psi_n(L_y - y)\eta_{K+1}(t-T) f(x,y,t) dx\, dy\, dt.
\end{aligned} \tag{25}$$

Since $\mathbf{z}$ is the unknown in equation (**??**), we can write it as follows:

$$K_{m,n,K+1}(\mathbf{z}) = U_{m,n,K+1}, \tag{26}$$

where

$$\begin{aligned}
K_{m,n,K+1}(\mathbf{z}) &:= \sum_{j=1}^{J} a_j(\mathbf{z}) \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x)\psi_n(L_y - y) \frac{\partial^{\beta_j} \eta^*_{K+1}(t)}{\partial t^{\beta_j}} u^*(x,y,T-t) dx\, dy\, dt \\
&- \sum_{i=1}^{I} b_i(\mathbf{z}) \int_0^T \int_0^{L_x} \int_0^{L_y} \frac{\partial^{\alpha_i} \phi^*_m(x)}{\partial x^{\alpha_i}} \psi_n(L_y - y)\eta_{K+1}(t-T) u^*(L_x - x, y, t) dx\, dy\, dt \\
&\quad - \sum_{i=1}^{I} c_i(\mathbf{z}) \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x) \frac{\partial^{\lambda_i} \psi^*_n(y)}{\partial y^{\lambda_i}} \eta_{K+1}(t-T) u^*(x, L_y - y, t) dx\, dy\, dt,
\end{aligned} \tag{27}$$

and

$$U_{m,n,K+1} := \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x)\psi_n(L_y - y)\eta_{K+1}(t-T) f(x,y,t) dx\, dy\, dt. \tag{28}$$

Stage 2: In this stage, the solution to the following nonlinear system of equations will give an estimate for the unknown $\mathbf{z}$:

$$F_{m,n,K+1}(\mathbf{z}) = K_{m,n,K+1}(\mathbf{z}) - U_{m,n,K+1} = 0, \tag{29}$$

where $m = 1, 2, \cdots M$ and $n = 1, 2, \cdots N$. Which can be written in the following matrix form:

$$\mathbf{F} = \mathbf{K} - \mathbf{U} = 0, \tag{30}$$

where

$$\mathbf{F} = \left(\begin{array}{c} K_{1,1,K+1} - U_{1,1,K+1} \\ K_{1,2,K+1} - U_{1,2,K+1} \\ \vdots \\ K_{1,N+1,K+1} - U_{1,N+1,K+1} \\ \hline K_{2,1,K+1} - U_{2,1,K+1} \\ K_{2,2,K+1} - U_{2,2,K+1} \\ \vdots \\ K_{2,N+1,K+1} - U_{2,N+1,K+1} \\ \hline \vdots \\ \vdots \\ \hline K_{M+1,1,K+1} - U_{M+1,1,K+1} \\ K_{M+1,2,K+1} - U_{M+1,2,K+1} \\ \vdots \\ K_{M+1,N+1,K+1} - U_{M+1,N+1,K+1} \end{array} \right), \tag{31}$$

with $K_{m,n,K+1}$ and $U_{m,n,K+1}$ defined as in (**??**) and (**??**), respectively.

Newton's approach:

The nonlinear system given in (**??**) will be solved using a first-order Newton's type method, where at each iteration, $\mathbf{z}$ will be updated using:

$$\mathbf{z}_{k+1} = \mathbf{z}_k + \Delta\mathbf{z}_k, \tag{32}$$

where $\Delta \mathbf{z}_k$ is the solution of the following equation

$$F(\mathbf{z}_k) = -\Delta \mathbf{z}_k J[F(\mathbf{z}_k)], \tag{33}$$

and the Jacobian matrix $J[F(\mathbf{z})]$ is exactly characterized using the following propositions.

Proposition 5.1. *Let $\{\phi_m(x)\}_{m=1}^{M}$, $\{\psi_n(y)\}_{n=1}^{N}$ and $\{\eta_\kappa(t)\}_{\kappa=1}^{K}$ be sets of 2^{nd} order modulating functions defined on the intervals $[0, L_1]$, $[0, L_2]$ and $[0, T]$ respectively, where $M + N + K \geq J + 2I$, $L_1 \leq L_x$, $L_2 \leq L_y$ and $T \leq T_1$, then the solution of the following linear system gives the estimations of the derivatives of the parameters a_j with respect to the differentiation order β_{j^*}:*

$$\begin{pmatrix} \mathbf{A} & | & \mathbf{B} & | & \mathbf{C} \end{pmatrix} \frac{\partial}{\partial \beta_{j^*}} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = -a_{j^*}(\mathbf{z})\mathbf{A}', \tag{34}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are defined as in proposition **??** and

$$\mathbf{A}' = \frac{\partial}{\partial \beta_{j^*}} \left(\begin{array}{c} A_{j^*,1,1,1} \\ A_{j^*,2,1,1} \\ \vdots \\ A_{j^*,M,1,1} \\ \hline A_{j^*,1,2,1} \\ A_{j^*,2,2,1} \\ \vdots \\ A_{j^*,M,2,1} \\ \hline \vdots \\ A_{j^*,M,N,K} \end{array} \right) \tag{35}$$

and

$$\frac{\partial}{\partial \beta_{j^*}} A_{j^*,m,n,\kappa} = \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x)\psi_n(L_y - y) \frac{\partial}{\partial \beta_{j^*}} \frac{\partial^{\beta_{j^*}} \eta_\kappa^*(t)}{\partial t^{\beta_{j^*}}} u^*(x, y, T - t)\, dx\, dy\, dt. \tag{36}$$

Proof. This proof can be obtained by differentiating (**??**) with respect to β_{j^*}. □

Remark 5.1. In the previous proposition, we determined the derivatives of the coefficients a_j with respect to the differentiation order β_{j^*}, where $0 \leq j^* \leq J$. However, the derivatives of the coefficients b_i and c_i with respect to the differentiation orders α_i^* and λ_i^* can be obtained similarly where $0 \leq i^* \leq I$.

Proposition 5.2. *Using the coefficients $a_j(\mathbf{z}), b_i(\mathbf{z})$ and $c_i(\mathbf{z})$ which are the estimations given in proposition (??) and $\mathbf{F}$ is given in (??), the Jacobian $J[\mathbf{F}(\mathbf{z})]$ exists and is given as follows:*

$$J[\mathbf{F}(\mathbf{z})] = \begin{pmatrix} \frac{\partial \mathbf{F}}{\partial \beta_1} \\ \frac{\partial \mathbf{F}}{\partial \beta_2} \\ \vdots \\ \frac{\partial \mathbf{F}}{\partial \beta_J} \\ \frac{\partial \mathbf{F}}{\partial \alpha_1} \\ \frac{\partial \mathbf{F}}{\partial \alpha_2} \\ \vdots \\ \frac{\partial \mathbf{F}}{\partial \alpha_J} \\ \frac{\partial \mathbf{F}}{\partial \lambda_1} \\ \frac{\partial \mathbf{F}}{\partial \lambda_2} \\ \vdots \\ \frac{\partial \mathbf{F}}{\partial \lambda_I} \end{pmatrix}^T, \quad \text{where} \quad \frac{\partial \mathbf{F}}{\partial \cdot} = \left(\begin{array}{c} \frac{\partial K_{1,1,K+1}}{\partial \cdot} \\ \frac{\partial K_{1,2,K+1}}{\partial \cdot} \\ \vdots \\ \frac{\partial K_{1,N+1,K+1}}{\partial \cdot} \\ \hline \frac{\partial K_{2,1,K+1}}{\partial \cdot} \\ \frac{\partial K_{2,2,K+1}}{\partial \cdot} \\ \vdots \\ \frac{\partial K_{2,N+1,K+1}}{\partial \cdot} \\ \vdots \\ \frac{\partial K_{M+1,1,K+1}}{\partial \cdot} \\ \frac{\partial K_{M+1,2,K+1}}{\partial \cdot} \\ \vdots \\ \frac{\partial K_{M+1,N+1,K+1}}{\partial \cdot} \end{array}\right), \tag{37}$$

and

$$\begin{aligned} \frac{\partial K_{m,n,K+1}}{\partial \beta_{j^*}} &= \sum_{j=1}^{J} \frac{\partial a_j(\mathbf{z})}{\partial \beta_{j^*}} \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x)\psi_n(L_y - y) \frac{\partial^{\beta_j} \eta_{K+1}^*(t)}{\partial t^{\beta_j}} u^*(x, y, T - t) dx\, dy\, dt \\ &+ a_{j^*}(\mathbf{z}) \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x)\psi_n(L_y - y) \frac{\partial}{\partial \beta_{j^*}} \frac{\partial^{\beta_j} \eta_{K+1}^*(t)}{\partial t^{\beta_j}} u^*(x, y, T - t) dx\, dy\, dt \\ &- \sum_{i=1}^{I} \frac{\partial b_i(\mathbf{z})}{\partial \beta_{j^*}} \int_0^T \int_0^{L_x} \int_0^{L_y} \frac{\partial^{\alpha_i} \phi_m^*(x)}{\partial x^{\alpha_i}} \psi_n(L_y - y) \eta_{K+1}(t - T) u^*(L_x - x, y, t) dx\, dy\, dt \\ &- \sum_{i=1}^{I} \frac{\partial c_i(\mathbf{z})}{\partial \beta_{j^*}} \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x) \frac{\partial^{\lambda_i} \psi_n^*(y)}{\partial y^{\lambda_i}} \eta_{K+1}(t - T) u^*(x, L_y - y, t) dx\, dy\, dt, \end{aligned} \tag{38}$$

$$
\begin{aligned}
\frac{\partial K_{m,n,K+1}}{\partial\, \alpha_{i^*}} &= \sum_{j=1}^{J} \frac{\partial a_j(\mathbf{z})}{\partial\, \alpha_{i^*}} \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x-x)\psi_n(L_y-y)\frac{\partial^{\beta_j}\eta^*_{K+1}(t)}{\partial t^{\beta_j}} u^*(x,y,T-t)dx\,dy\,dt \\
&\quad - \sum_{i=1}^{I} \frac{\partial b_i(\mathbf{z})}{\partial\, \alpha_{i^*}} \int_0^T \int_0^{L_x} \int_0^{L_y} \frac{\partial^{\alpha_i}\phi^*_m(x)}{\partial x^{\alpha_i}}\psi_n(L_y-y)\eta_{K+1}(t-T)u^*(L_x-x,y,t)dx\,dy\,dt \\
&\quad - b_{i^*}(\mathbf{z}) \int_0^T \int_0^{L_x} \int_0^{L_y} \frac{\partial}{\partial \alpha_{i^*}} \frac{\partial^{\alpha_i}\phi^*_m(x)}{\partial x^{\alpha_i}}\psi_n(L_y-y)\eta_{K+1}(t-T)u^*(L_x-x,y,t)dx\,dy\,dt \\
&\quad - \sum_{i=1}^{I} \frac{\partial c_i(\mathbf{z})}{\partial\, \alpha_{i^*}} \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x-x)\frac{\partial^{\lambda_i}\psi^*_n(y)}{\partial y^{\lambda_i}}\eta_{K+1}(t-T)u^*(x,L_y-y,t)dx\,dy\,dt,
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
\frac{\partial K_{m,n,K+1}}{\partial\, \lambda_{i^*}} &= \sum_{j=1}^{J} \frac{\partial a_j(\mathbf{z})}{\partial\, \lambda_{i^*}} \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x-x)\psi_n(L_y-y)\frac{\partial^{\beta_j}\eta^*_{K+1}(t)}{\partial t^{\beta_j}} u^*(x,y,T-t)dx\,dy\,dt \\
&\quad - \sum_{i=1}^{I} \frac{\partial b_i(\mathbf{z})}{\partial\, \lambda_{i^*}} \int_0^T \int_0^{L_x} \int_0^{L_y} \frac{\partial^{\alpha_i}\phi^*_m(x)}{\partial x^{\alpha_i}}\psi_n(L_y-y)\eta_{K+1}(t-T)u^*(L_x-x,y,t)dx\,dy\,dt \\
&\quad - \sum_{i=1}^{I} \frac{\partial c_i(\mathbf{z})}{\partial\, \lambda_{i^*}} \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x-x)\frac{\partial^{\lambda_i}\psi^*_n(y)}{\partial y^{\lambda_i}}\eta_{K+1}(t-T)u^*(x,L_y-y,t)dx\,dy\,dt \\
&\quad - c_{i^*}(\mathbf{z}) \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x-x)\frac{\partial}{\partial \alpha_{i^*}} \frac{\partial^{\lambda_i}\psi^*_n(y)}{\partial y^{\lambda_i}}\eta_{K+1}(t-T)u^*(x,L_y-y,t)dx\,dy\,dt,
\end{aligned}
\tag{40}
$$

and

$$
\begin{aligned}
\frac{\partial K_{m,n,K+1}}{\partial\, \beta_{j^*}} &= \sum_{j=1}^{J} \frac{\partial a_j(\mathbf{z})}{\partial\, \beta_{j^*}} A_{j,m,n,K+1} + a_{j^*}(\mathbf{z}) \frac{\partial}{\partial \beta_{j^*}} A_{j^*,m,n,K+1} - \sum_{i=1}^{I} \frac{\partial b_i(\mathbf{z})}{\partial\, \beta_{j^*}} B_{i,m,n,K+1} \\
&\quad - \sum_{i=1}^{I} \frac{\partial c_i(\mathbf{z})}{\partial\, \beta_{j^*}} C_{i,m,n,K+1},
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
\frac{\partial K_{m,n,K+1}}{\partial\, \alpha_{i^*}} &= \sum_{j=1}^{J} \frac{\partial a_j(\mathbf{z})}{\partial\, \alpha_{i^*}} A_{j,m,n,K+1} - \sum_{i=1}^{I} \frac{\partial b_i(\mathbf{z})}{\partial\, \alpha_{i^*}} B_{i,m,n,K+1} - b_{i^*}(\mathbf{z}) \frac{\partial}{\partial \alpha_{i^*}} B_{i^*,m,n,K+1} \\
&\quad - \sum_{i=1}^{I} \frac{\partial c_i(\mathbf{z})}{\partial\, \alpha_{i^*}} C_{i,m,n,K+1},
\end{aligned}
\tag{42}
$$

$$\frac{\partial K_{m,n,K+1}}{\partial\,\lambda_{i^*}} = \sum_{j=1}^{J}\frac{\partial a_j(\mathbf{z})}{\partial\,\lambda_{i^*}}A_{j,m,n,K+1} - \sum_{i=1}^{I}\frac{\partial b_i(\mathbf{z})}{\partial\,\lambda_{i^*}}B_{i,m,n,K+1} - \sum_{i=1}^{I}\frac{\partial c_i(\mathbf{z})}{\partial\,\lambda_{i^*}}C_{i,m,n,K+1} - c_{i^*}(\mathbf{z})\frac{\partial}{\partial\alpha_{i^*}}C_{i^*,m,n,K+1}, \tag{43}$$

Proof. The elements of $J[\mathbf{F}(\mathbf{z})]$ can be obtained directly by differentiating (**??**) with respect to the fractional orders β_j, α_i and λ_i. The existence of the Jacobian $J[\mathbf{F}(\mathbf{z})]$ depends on the existence of of the derivative of (**??**) with respect to the fractional orders β_j, α_i and λ_i, which always exists as shown in Proposition **??** the gamma function is differentiable. □

Algorithm 1: Two stage algorithm to estimate the parameters and the fractional order.

Step1: Start with an initial guess $\mathbf{z_o} = (\beta_1^o, \beta_2^o, \ldots, \beta_J^o, \alpha_1^o, \alpha_2^o, \ldots, \alpha_I^o, \lambda_1^o, \lambda_2^o, \ldots, \lambda_I^o)$.

Step 2: Compute the corresponding $a_j(\mathbf{z_k}), b_i(\mathbf{z_k})$ and $c_i(\mathbf{z_k})$.

Step 3: Compute $\|\mathbf{F}(\mathbf{z_k})\|_2^2$,

if $\|\mathbf{F}(\mathbf{z_k})\|_2^2 < \epsilon$ **then**

  output: $a_j(\mathbf{z_k}), b_i(\mathbf{z_k})$ and $c_i(\mathbf{z_k})$

else

  update $\mathbf{z_{k+1}} = \mathbf{z_k} + \Delta\mathbf{z_k}$,

  where $\Delta\mathbf{z_k}$ can be computed by solving the following system $J[\mathbf{F}(\mathbf{z_k})]\Delta\mathbf{z_k} = -\mathbf{F}(\mathbf{z_k})$

  and go back to step 2.

end

For the convenience, we present a description of the proposed algorithm in figure (**??**).

Remark 5.2. We would like to point out that the accuracy of the presented algorithm can be affected by the number of modulating functions. Increasing this number improves the estimation and especially in the presence of noise. However, increasing the noise may lead to numerical instability as the number of equations of the algebraic system increases. Therefore, a regularization technique might be required. However, in the extensive numerical investigations performed, the algorithm was stable and regularization was not needed.

6. Numerical results

In this section, we present some numerical results to show the efficiency and the robustness of the presented method. We consider the following two dimensional space fractional advection dispersion equation

$$\frac{\partial u(x,y,t)}{\partial t} = -v_1 \frac{\partial u(x,y,t)}{\partial x} - v_2 \frac{\partial u(x,y,t)}{\partial y} + d_1 \frac{\partial^\alpha u(x,y,t)}{\partial x^\alpha} + d_2 \frac{\partial^\beta u(x,y,t)}{\partial y^\beta} + f(x,y,t). \quad (44)$$

In this case $b_1 = -\nu_1$, $c_1 = -\nu_2$, $b_2 = d_1$, $c_2 = d_2$, $\alpha_1 = \lambda_1 = 1$, $\alpha_2 = \alpha$, $\lambda_2 = \beta$ and $a_i = b_j = c_j = 0$ for $i = 1, \dots I$ and $j = 3, \dots J$.

First, we estimate the coefficients ν_1, ν_2, d_1 and d_2 using proposition **??**, where the system (**??**) will be reduced to:

$$\begin{pmatrix} \mathbf{B} & | & \mathbf{C} \end{pmatrix}_{(M\times N\times K)\times 4} \begin{pmatrix} -\nu_1 \\ d_1 \\ -\nu_2 \\ d_2 \end{pmatrix}_{4\times 1} = \mathbf{E}, \quad (45)$$

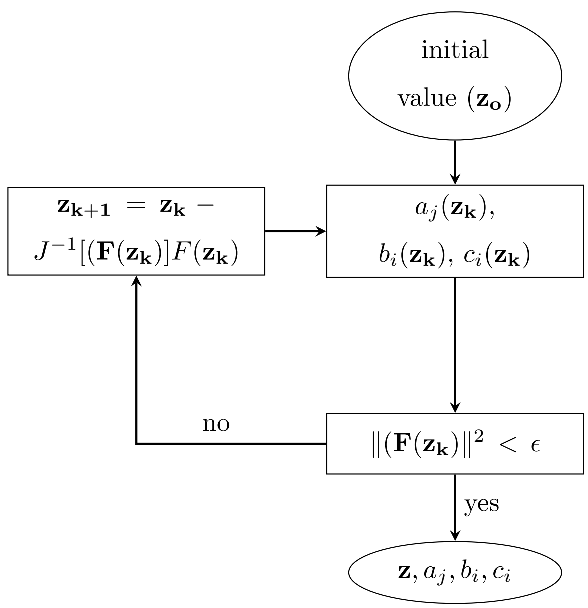

Figure 1: Two stage algorithm

with

$$B_{1,m,n,\kappa} = -\int_0^T \int_0^{L_y} \int_0^{L_x} \frac{\partial \phi_m(L_x - x)}{\partial x} \psi_n(L_y - y)\eta_\kappa(T-t)u(x,y,t)\,dx\,dy\,dt,$$

$$B_{2,m,n,\kappa} = \int_0^T \int_0^{L_y} \int_0^{L_x} \frac{\partial^\alpha \phi_m(x)}{\partial x^\alpha} \psi_n(L_y - y)\eta_\kappa(T-t)u(L_x - x,y,t)\,dx\,dy\,dt,$$

$$C_{1,m,n,\kappa} = -\int_0^T \int_0^{L_y} \int_0^{L_x} \phi_m(L_x - x)\frac{\partial \psi_n(L_y - y)}{\partial y} \eta_\kappa(T-t)u(x,y,t)\,dx\,dy\,dt,$$

$$C_{2,m,n,\kappa} = \int_0^T \int_0^{L_y} \int_0^{L_x} \phi_m(L_x - x)\frac{\partial^\beta \psi_n(y)}{\partial y^\beta} \eta_\kappa(T-t)u(x,L_y - y,t)\,dx\,dy\,dt,$$

$$E_{m,n,\kappa} = \int_0^T \int_0^{L_y} \int_0^{L_x} [\phi_m(L_x - x)\psi_n(L_y - y)] \\ [\frac{\partial \eta_\kappa(T-t)}{\partial t} u(x,y,t) - \eta_\kappa(T-t)f(x,y,t)]\,dx\,dy\,dt.$$

Then, we use the algorithm given in Section 5 to estimate the parameters ν_1, ν_2, d_1 and d_2 and $\mathbf{z} = (\alpha, \beta)$, where the derivatives of the parameters ν_1, ν_2, d_1 and d_2 with respect to the differentiation order α can be estimated by solving the following system:

$$\begin{pmatrix}\mathbf{B} & | & \mathbf{C}\end{pmatrix}_{(M\times N\times K)\times 4} \frac{\partial}{\partial \alpha} \begin{pmatrix} -\nu_1 \\ d_1 \\ -\nu_2 \\ d_2 \end{pmatrix}_{4\times 1} = -d_1(\mathbf{z})\mathbf{B}'_{(M\times N\times K)\times 1}, \tag{46}$$

with

$$\frac{\partial}{\partial \alpha} B_{2,m,n,\kappa} = \int_0^T \int_0^{L_x} \int_0^{L_y} \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha \phi_m(x)}{\partial x^\alpha} \psi_n(L_y - y)\eta_\kappa(T-t)u(L_x - x,y,t)dx\,dy\,dt, \tag{47}$$

where

$$\mathbf{B}' = \frac{\partial}{\partial \alpha} \left(\begin{array}{c} B_{2,1,1,1} \\ B_{2,2,1,1} \\ \vdots \\ B_{2,M,1,1} \\ \hline B_{2,1,2,1} \\ B_{2,2,2,1} \\ \vdots \\ B_{2,M,2,1} \\ \hline \vdots \\ B_{2,M,N,K} \end{array} \right). \tag{48}$$

The derivative of the parameters with respect to β can be estimated similarly and the Jacobian $J[\mathbf{F}]$ is given as follows:

$$J[\mathbf{F}(\mathbf{z})] = \begin{pmatrix} \frac{\partial \mathbf{F}}{\partial \alpha} \\ \frac{\partial \mathbf{F}}{\partial \beta} \end{pmatrix}^T, \quad \text{where} \quad \frac{\partial \mathbf{F}}{\partial \cdot} = \left(\begin{array}{c} \frac{\partial K_{1,1,K+1}}{\partial \cdot} \\ \frac{\partial K_{1,2,K+1}}{\partial \cdot} \\ \vdots \\ \frac{\partial K_{1,N+1,K+1}}{\partial \cdot} \\ \hline \frac{\partial K_{2,1,K+1}}{\partial \cdot} \\ \frac{\partial K_{2,2,K+1}}{\partial \cdot} \\ \vdots \\ \frac{\partial K_{2,N+1,K+1}}{\partial \cdot} \\ \vdots \\ \frac{\partial K_{M+1,1,K+1}}{\partial \cdot} \\ \frac{\partial K_{M+1,2,K+1}}{\partial \cdot} \\ \vdots \\ \frac{\partial K_{M+1,N+1,K+1}}{\partial \cdot} \end{array} \right)_{(M+1 \times N+1) \times 1}, \tag{49}$$

with

$$
\begin{aligned}
&\frac{\partial K_{m,n,K+1}}{\partial \alpha} = \\
&\quad - \frac{\partial \nu_1(\mathbf{z})}{\partial \alpha} \int_0^T \int_0^{L_y} \int_0^{L_x} \frac{\partial \phi_m(L_x - x)}{\partial x} \psi_n(L_y - y) \eta_\kappa(T - t) c(x, y, t)\, dx\, dy\, dt \\
&\quad - \frac{\partial \nu_2(\mathbf{z})}{\partial \alpha} \int_0^T \int_0^{L_y} \int_0^{L_x} \phi_m(L_x - x) \frac{\partial \psi_n(L_y - y)}{\partial y} \eta_\kappa(T - t) c(x, y, t)\, dx\, dy\, dt \\
&\quad + d_1(\mathbf{z}) \int_0^T \int_0^{L_y} \int_0^{L_x} \frac{\partial}{\partial \alpha} \frac{\partial^\alpha \phi_m(x)}{\partial x^\alpha} \psi_n(L_y - y) \eta_\kappa(T - t) c(L_x - x, y, t)\, dx\, dy\, dt \\
&\quad + \frac{\partial d_1(\mathbf{z})}{\partial \alpha} \int_0^T \int_0^{L_y} \int_0^{L_x} \frac{\partial^\alpha \phi_m(x)}{\partial x^\alpha} \psi_n(L_y - y) \eta_\kappa(T - t) c(L_x - x, y, t)\, dx\, dy\, dt \\
&\quad + \frac{\partial d_2(\mathbf{z})}{\partial \alpha} \int_0^T \int_0^{L_y} \int_0^{L_x} \phi_m(L_x - x) \frac{\partial^\beta \psi_n(y)}{\partial y^\beta} \eta_\kappa(T - t) c(x, L_y - y, t)\, dx\, dy\, dt,
\end{aligned}
\tag{50}
$$

and

$$
\begin{aligned}
&\frac{\partial K_{m,n,K+1}}{\partial \beta} = \\
&\quad - \frac{\partial \nu_1(\mathbf{z})}{\partial \beta} \int_0^T \int_0^{L_x} \int_0^{L_y} \frac{\partial \phi_m(L_x - x)}{\partial x} \psi_n(L_y - y) \eta_\kappa(T - t) c(x, y, t)\, dx\, dy\, dt \\
&\quad - \frac{\partial \nu_2(\mathbf{z})}{\partial \beta} \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x) \frac{\partial \psi_n(L_y - y)}{\partial y} \eta_\kappa(T - t) c(x, y, t)\, dx\, dy\, dt \\
&\quad + \frac{\partial d_1(\mathbf{z})}{\partial \beta} \int_0^T \int_0^{L_x} \int_0^{L_y} \frac{\partial^\alpha \phi_m(x)}{\partial x^\alpha} \psi_n(L_y - y) \eta_\kappa(T - t) c(L_x - x, y, t)\, dx\, dy\, dt \\
&\quad + \frac{\partial d_2(\mathbf{z})}{\partial \beta} \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x) \frac{\partial^\beta \psi_n(y)}{\partial y^\beta} \eta_\kappa(T - t) c(x, L_y - y, t)\, dx\, dy\, dt \\
&\quad + d_2(\mathbf{z}) \int_0^T \int_0^{L_x} \int_0^{L_y} \phi_m(L_x - x) \frac{\partial}{\partial \beta} \frac{\partial^\beta \psi_n(y)}{\partial y^\beta} \eta_\kappa(T - t) c(x, L_y - y, t)\, dx\, dy\, dt.
\end{aligned}
\tag{51}
$$

We consider the following polynomial modulating functions whose fractional derivatives are simple to calculate. Other modulating functions can be used, but this will require the numerical computation of the fractional derivatives of the modulating functions if it can not be easily determined analytically: $\phi_m(x) = x^{M+b+1-m}(L_x - x)^{b+m}, \psi_n(y) = y^{N+b+1-n}(L_y - y)^{b+n}$ and , $\eta_\kappa(t) = t^{K+b+1-\kappa}(T - t)^{b+\kappa}$ where $L_x \leq L_1$, $L_y \leq L_2$, $T \leq T_1$, $b = 3$, and $m = 1, 2, \cdots M$, $n = 1, 2, \ldots, N$ and $\kappa = 1, 2, \cdots K$, where M, N and K are the number of modulating functions. The fractional derivatives and the derivative with respect to the fractional derivatives of the modulating functions can be computed as given in (**??**) and (**??**), respectively. Moreover, we apply the trapezoidal rule to numerically approximate the integrals.

Example 6.1. *Let us consider the following space fractional advection-dispersion equation:*

$$\frac{\partial u(x,y,t)}{\partial t} = -v_1\frac{\partial u(x,y,t)}{\partial x} - v_2\frac{\partial u(x,y,t)}{\partial y} + d_1\frac{\partial^\alpha u(x,y,t)}{\partial x^\alpha} + d_2\frac{\partial^\beta u(x,y,t)}{\partial y^\beta} + f(x,y,t), \quad (52)$$

with the following initial and Dirichlet boundary conditions:

$$\begin{cases} u(x,y,0) = x^{4.8}y^3, \\ u(0,y,t) = 0, \\ u(x,0,t) = 0, \\ u(10,y,t) = -e^{-t}10^{4.8}y^3, \\ u(x,10,t) = -e^{-t}x^{4.8}10^3, \end{cases} \quad (53)$$

where

$$f(x,y,t) = exp(-t)[-x^{4.8}y^3 + 4.8v_1x^{3.8}y^3 + 3v_2x^{4.8}y^2 - d_1\frac{\Gamma(5.8)}{\Gamma(5.8-\alpha)}x^{4.8-\alpha}y^3$$

$$-d_2\frac{\Gamma(4)}{\Gamma(4-\beta)}x^{4.8}y^{3-\beta}]$$

The exact solution of the forward problem is $u(x,y,t) = e^{-t}x^{4.8}y^3$ *and the flux is* $\frac{\partial u(x,y,t)}{\partial t} = -e^{-t}x^{4.8}y^3$.

Estimating ν, d when α is known

In this part, we assume that the differentiation orders α and β are known and we estimate ν_1, ν_2, d_1 and d_2. We set the exact values of the average velocities as $\nu_1 = 0.2$ and $\nu_2 = 0.5$, the dispersion coefficients $d_1 = 1$ and $d_2 = 0.8$, the differentiation orders $\alpha = 1.5$ and $\beta = 1.6$. Figure **??**, represents the estimated values of ν_1, ν_2, d_1, and d_2 when adding a white Gaussian noise with $\sigma = 2\%$ to the measurements.

Estimating ν, d and α

In this part, we will use the combined Newton's and modulating functions method to estimate all parameters simultaneously. We set the exact values of the average velocities $\nu_1 = 0.5, \nu_2 = 0.3$, and the dispersion coefficients $d_1 = 1$ and $d_2 = 0.8$, the differentiation orders $\alpha = 1.5$ $\beta = 1.6$, the final time $T = 1$, the initial guess $z_o = (\alpha_o, \beta_o) = (1.2, 1.2)$. As we can see in Table **??**, the results are satisfactory even with different number of modulating functions.

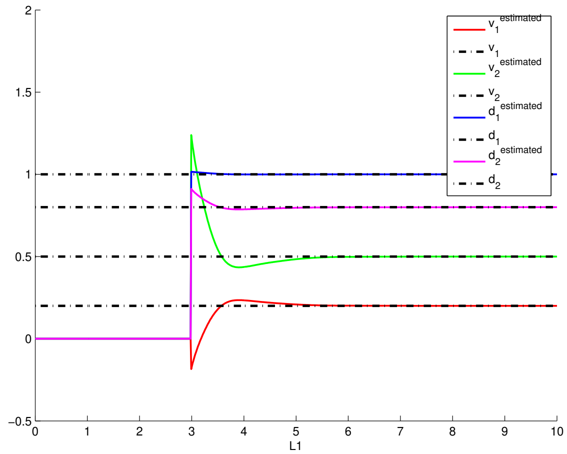

Figure 2: The estimated parameters with 2% stationary noise case with $\Delta x = \frac{1}{100}$.

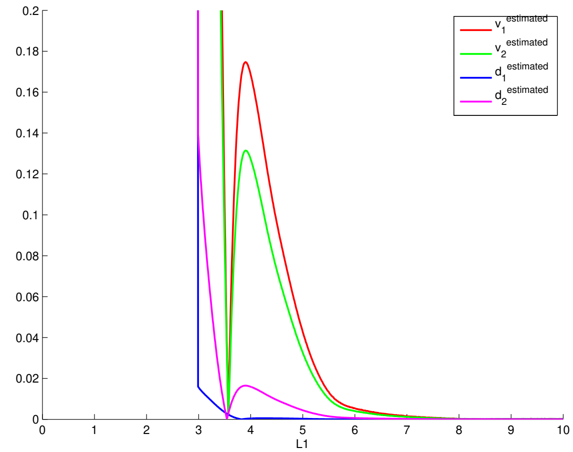

Figure 3: Relative errors for different integration interval.

Table 1: $\nu_1 = 0.5$, $\nu_2 = 0.3$ $d_1 = 1$,$d_2 = 0.8$, $\alpha = 1.5$, $\beta = 1.6$, and $\Delta x = \frac{1}{50}$, $\sigma = 0.05$.

number of modulating functions	Estimated Value $\hat{a} = (\hat{\nu_1}, \hat{\nu_2}, \hat{d_1}, \hat{d_2}, \hat{\alpha}, \hat{\beta})$	Relative Error $\frac{\|(\nu_1,\nu_2,d_1,d_2,\alpha,\beta)-(\hat{\nu_1},\hat{\nu_2},\hat{d_1},\hat{d_2},\hat{\alpha},\hat{\beta}\|_2}{\|(\nu_1,\nu_2,d_1,d_2,\alpha,\beta)\|_2}$	ϵ
5	(0.4873, 0.3102, 0.9885, 0.8087, 1.5056, 1.5930)	0.0090	1.2734e-005
6	(0.4838 , 0.3129 , 0.9853 , 0.8110 ,1.5072 , 1.5911)	0.0115	1.5879e-005
7	(0.4796 , 0.3162 , 0.9814 , 0.8138 , 1.5092 ,1.5889)	0.0145	1.9451e-005
8	(0.4745 , 0.3201 , 0.9767 , 0.8172 , 1.5116 , 1.5864)	0.0181	2.3469e-005
9	(0.4685 , 0.3246 , 0.9712 , 0.8211 , 1.5144 , 1.5835)	0.0223	2.8023e-005
10	(0.4616, 0.3297 , 0.9649 , 0.8255 , 1.5176 , 1.5802)	0.0271	3.3235e-005

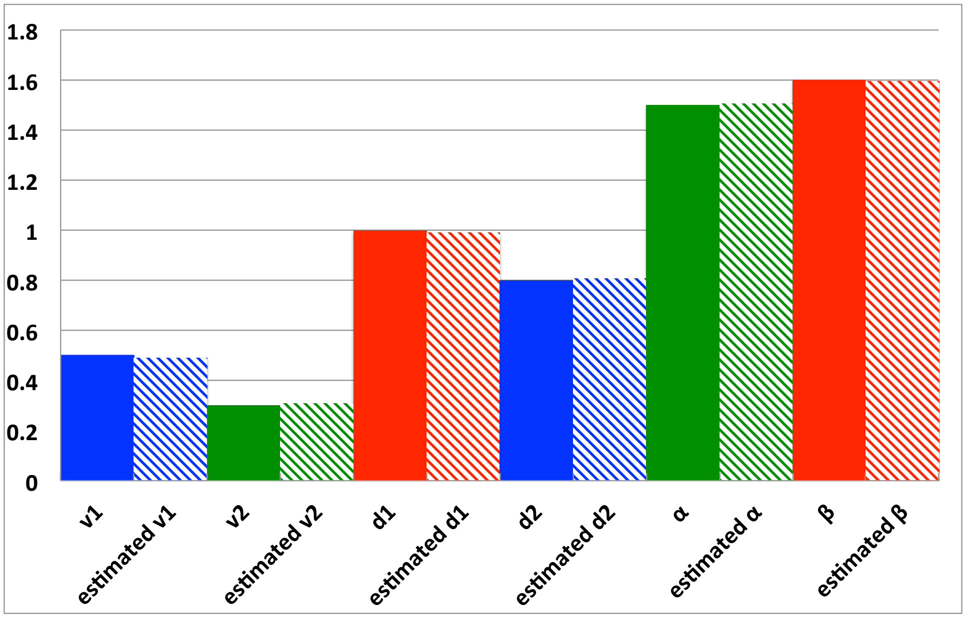

Figure 4: The estimated parameters with 4 modulating functions: $\sigma = 2\%$ and $\Delta x = \frac{1}{50}$.

Table 2: $\nu_1 = 0.5$, $\nu_2 = 0.3$ $d_1 = 1$,$d_2 = 0.8$, $\alpha = 1.5$, $\beta = 1.6$, and $\Delta x = \frac{1}{50}$, $\sigma = 0.05$.

Number of modulating functions	Relative errors $\left(\frac{\|\nu_1-\hat{\nu_1}\|_2}{\|\nu_1\|}, \frac{\|\nu_2-\hat{\nu_2}\|_2}{\|\nu_2\|}, \frac{\|d_1-\hat{d_1}\|_2}{\|d_1\|}, \frac{\|d_2-\hat{d_2}\|_2}{\|d_2\|}, \frac{\|\alpha-\hat{\alpha}\|_2}{\|\alpha\|}, \frac{\|\beta-\hat{\beta}\|_2}{\|\beta\|}\right)$
5	(0.025, 0.034, 0.012, 0.011, 0.004, 0.004)
6	(0.032, 0.043, 0.015, 0.014, 0.005, 0.006)
7	(0.041, 0.054, 0.019, 0.017, 0.006, 0.007)
8	(0.051, 0.067, 0.023, 0.022, 0.008, 0.009)
9	(0.063, 0.082, 0.023, 0.026, 0.009, 0.010)
10	(0.077, 0.099, 0.035, 0.032, 0.011, 0.012)

7. Conclusion

In this paper, we have extended the combined modulating functions method to estimate the coefficients and the differentiation orders for two-dimensional fractional differential equations. First, we have estimated the coefficients by applying the modulating functions method which transformed the coefficients identification problem into solving a linear system of algebraic equations. These estimations of the unknown coefficients are robust against high frequency noises. Second, the modulating functions method has been combined with a Newton's iterative algorithm to estimate the coefficients and the differentiation orders simultaneously, where the first order derivatives with respect to fractional differentiation orders of the coefficients have also been estimated using a modulating function approach, which simplifies the calculation of the Jacobian. Finally, numerical simulations have illustrated the effectiveness of the proposed algorithm.

References