

MEASURE-VALUED SOLUTIONS FOR THE EQUATIONS OF POLYCONVEX ADIABATIC THERMOELASTICITY

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ABSTRACT. For the system of polyconvex adiabatic thermoelasticity, we define a notion of dissipative measure-valued solution, which can be considered as the limit of a viscosity approximation. We embed the system into a symmetrizable hyperbolic one in order to derive the relative entropy. However, we base our analysis in the original variables, instead of the symmetric ones (in which the entropy is convex) and we prove measure-valued weak versus strong uniqueness using the averaged relative entropy inequality.

1. INTRODUCTION

For systems of hyperbolic conservation laws, the class of measure-valued solutions [18] provides a notion of solvability vast enough to support a global existence theory. These solutions usually arise as limits of converging sequences satisfying an approximating parabolic problem [13]. As these solutions are considered to be very weak, it is crucial to examine their stability properties with respect to classical solutions and to attempt that in their natural energy framework. The relative entropy method of Dafermos [12, 11] and DiPerna [17] provides an analytical framework upon which one can examine such questions, and has been tested in a variety of contexts (e.g. [5, 16, 22, 9, 20, 6]).

In this article, we derive (in the Appendix) a framework of dissipative measure-valued solutions for the system of adiabatic polyconvex thermoelasticity, motivated by approximating that system by the system of thermoviscoelasticity on the natural energy framework. The relative entropy method is used to show weak-strong uniqueness for polyconvex thermoelasticity in the class of measure-valued solutions. The main novelty of this work is the derivation of the averaged relative entropy inequality with respect to a dissipative measure-valued solution. This solution is defined by means of generalized Young measures, describing both oscillatory and concentration effects. The analysis is based on the embedding of polyconvex thermoelasticity into an augmented, symmetrizable, hyperbolic system, [8]. However, the embedding cannot be used in a direct manner, and notably, instead of working with the extended variables, we base our analysis on the parent system in the original variables using the weak stability properties of some transport-stretching identities, which allow us to carry out the calculations by placing minimal regularity assumptions in the energy framework.

Consider the system of adiabatic thermoelasticity,

$$\begin{aligned}\partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t v_i &= \partial_\alpha \Sigma_{i\alpha} \\ \partial_t \left(\frac{1}{2} |v|^2 + e \right) &= \partial_\alpha (\Sigma_{i\alpha} v_i) + r\end{aligned}\tag{1.1}$$

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describing the evolution of a thermomechanical process $(y(x, t), \theta(x, t)) \in \mathbb{R}^3 \times \mathbb{R}^+$ with $(x, t) \in \mathbb{R}^3 \times \mathbb{R}^+$. Here, $F \in \mathbb{M}^{3 \times 3}$ stands for the deformation gradient, $F = \nabla y$, while $v = \partial_t y$ is the velocity of the motion y and θ is the temperature. The condition

$$\partial_\alpha F_{i\beta} = \partial_\beta F_{i\alpha}, \quad i, \alpha, \beta = 1, 2, 3, \quad (1.2)$$

imposes that F is a gradient and comes from equation (1.1)₁ as an involution inherited from the initial data. The stress is denoted as $\Sigma_{i\alpha}$, the internal energy as e and the radiative heat supply as r . The requirement of consistency with the Clausius-Duhem inequality imposes that the elastic stresses Σ , the entropy η and the internal energy e are related to the free-energy function ψ via the constitutive theory

$$\psi = \psi(F, \theta), \quad \Sigma = \frac{\partial \psi}{\partial F}, \quad \eta = -\frac{\partial \psi}{\partial \theta}, \quad e = \psi + \theta \eta. \quad (1.3)$$

In addition, the frame indifference principle for constitutive theories [25, 10] postulates invariance of the free energy under rotations and rules out convexity of $\psi(F, \theta)$ in F . To account for frame indifference, we impose a variant of the assumption of polyconvexity, familiar from the context of isothermal elasticity [3, 15]. This states that the free energy $\psi(F, \theta)$ factorizes

$$\psi(F, \theta) = \hat{\psi}(\Phi(F), \theta), \quad (1.4)$$

where $\Phi(F) = (F, \operatorname{cof} F, \det F) \in \mathbb{M}^{3 \times 3} \times \mathbb{M}^{3 \times 3} \times \mathbb{R}$ is the vector of null-Lagrangians. Conforming with the usual definition of (isothermal) polyconvexity [3], $\hat{\psi}(\xi, \theta)$ is assumed strictly convex in ξ , while following natural assumptions on thermodynamic potentials we require concavity of $\hat{\psi}(\xi, \theta)$ in θ :

$$\hat{\psi}_{\xi\xi}(\xi, \theta) > 0, \quad \hat{\psi}_{\theta\theta}(\xi, \theta) < 0. \quad (1.5)$$

Condition (1.5)₂ is tantamount to requiring the entropy $\eta(\cdot, \theta)$ to be an increasing function of the temperature θ , and is equivalent to the natural assumption of convexity of the internal energy as a function of the entropy (see Appendix B). We call the set of assumptions (1.4)-(1.5) as polyconvexity in the non-isothermal context.

The main result of this article is the weak-strong uniqueness of polyconvex adiabatic thermoelasticity (1.1)-(1.4) in the class of dissipative measure-valued solutions. The advantage of the dissipative framework is that the averaged energy equation holds in its integrated form. Even though this notion of solutions is generally considered to be very weak, not possessing detailed information, this result contributes to a long list of similar works [1, 19, 15, 16, 9] on hyperbolic systems of conservation laws, pointing out the importance of this framework in the analysis of such physical problems. Unlike the case of scalar conservation laws [18, 24], where the theory of Young measures suffices to deal with nonlinearities and overcome oscillatory behaviors, when it comes to hyperbolic systems, one must take into account the formation of both oscillations and concentrations. In our case, the concentration effects are described through a concentration measure, which appears in the energy equation since the Fundamental Lemma of Young measures cannot represent the weak limits of $\frac{1}{2}|v|^2 + e$, due to lack of L^1 precompactness. This is illustrated in Appendix A. Thus we turn our attention to the theory of generalized Young measures [1, 19, 5, 16, 9, 22] and apply the relative entropy formulation to compare a dissipative measure-valued solution to polyconvex thermoelasticity against a strong solution.

We organize this paper as follows: In Section 2, we define the notion of dissipative measure-valued solutions for polyconvex thermoelasticity. This definition comes as a result of the limiting process we discuss in Appendix A, starting from the associated viscous problem. Section 3 is

dedicated to the study of the generated Young measure and the concentration measure, which is a well-defined, nonnegative Radon measure for a subsequence of approximate solutions coming from a uniform bound on the energy. In Section 4 we calculate the averaged relative entropy inequality (4.16) and in Section 5 we use it to prove the main theorem on uniqueness of strong solutions in the class of measure-valued solutions. The proof is heavily based on the estimates (5.2) and (5.4)-(5.7) on the relative entropy, namely Lemmas 5.1, 5.2, which are stated and proved at the level of the original variables, instead of the extended ones. As a result, we only assume quite minimal growth hypotheses on the constitutive functions, which guarantee all the necessary technical requirements for the dissipative measure-valued versus strong uniqueness to hold. Additionally, the proof is carried on with respect to a dissipative solution which satisfies an averaged and integrated version of the energy equation, where the concentration measure appears. This setting has the strong advantage that we need no artificial integrability restrictions on the energy equation. Similar results are available for the incompressible Euler equations [5], for polyconvex elastodynamics [16], and for the isothermal gas dynamics system [22]. Finally, Appendix B outlines the relation between Legendre transforms and the definitions and properties of thermodynamic potentials, which in turn determine the associated thermoelasticity theory. In particular, it justifies the compatibility between assumption (1.5)₂ (or (B.2)) and the coercivity assumption (3.1), which plays an important role on the technical aspects of this work.

2. MEASURE-VALUED SOLUTIONS FOR POLYCONVEX ADIABATIC THERMOELASTICITY

Consider the system of adiabatic thermoelasticity (1.1), (1.2) together with the entropy production identity

$$\partial_t \eta = \frac{r}{\theta}, \quad (2.1)$$

under the constitutive theory (1.3) and the polyconvexity hypothesis (1.4). To avoid unnecessary technicalities, henceforth we work in a domain $Q_T = \mathbb{T}^d \times [0, T]$, where \mathbb{T}^d is the torus, $d = 3$ and $T \in [0, \infty)$. In the polyconvex case, the Euler-Lagrange equation

$$\partial_\alpha \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}} (\nabla y) \right) = 0, \quad B = 1, \dots, 19, \quad (2.2)$$

formulated for the vector of the minors $\Phi(F) = (F, \operatorname{cof} F, \det F) \in \mathbb{M}^{3 \times 3} \times \mathbb{M}^{3 \times 3} \times \mathbb{R}$, holds for any motion $y(x, t)$ and together with the kinematic compatibility equation (1.1)₁ and (1.2), allows to express $\partial_t \Phi^B(F)$ as

$$\partial_t \Phi^B(F) = \frac{\partial \Phi^B}{\partial F_{i\alpha}} (F) \partial_\alpha v_i = \partial_\alpha \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}} (F) v_i \right) \quad (2.3)$$

for any deformation gradient F and velocity field v . Additionally, the stress tensor Σ becomes

$$\Sigma_{i\alpha} = \frac{\partial \psi}{\partial F_{i\alpha}} (F, \theta) = \frac{\partial}{\partial F_{i\alpha}} \left(\hat{\psi}(\Phi(F), \theta) \right) = \frac{\partial \hat{\psi}}{\partial \xi^B} (\Phi(F), \theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}} (F). \quad (2.4)$$

The first nine components of $\Phi(F)$ are the components of F , therefore (1.3) implies that we can express the entropy η and the internal energy e with respect to the null-Lagrangian vector $\Phi(F)$,

namely

$$\begin{aligned}\eta(F, \theta) &= -\frac{\partial \psi}{\partial \theta}(F, \theta) = -\frac{\partial \hat{\psi}}{\partial \theta}(\Phi(F), \theta) =: \hat{\eta}(\Phi(F), \theta), \\ e(F, \theta) &= \psi(F, \theta) - \theta \frac{\partial \psi}{\partial \theta}(F, \theta) \\ &= \hat{\psi}(\Phi(F), \theta) - \theta \frac{\partial \hat{\psi}}{\partial \theta}(\Phi(F), \theta) =: \hat{e}(\Phi(F), \theta),\end{aligned}\tag{2.5}$$

where we have set

$$\hat{\eta}(\xi, \theta) := -\frac{\partial \hat{\psi}}{\partial \theta}(\xi, \theta), \quad \hat{e}(\xi, \theta) := \hat{\psi}(\xi, \theta) - \theta \frac{\partial \hat{\psi}}{\partial \theta}(\xi, \theta).\tag{2.6}$$

This allows to supplement the equations of polyconvex thermoelasticity (1.1), (1.2) with (2.3) and write

$$\begin{aligned}\partial_t \Phi^B(F) &= \partial_\alpha \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) v_i \right) \\ \partial_t v_i &= \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F), \theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) \right) \\ \partial_t \left(\frac{1}{2} |v|^2 + \hat{e}(\Phi(F), \theta) \right) &= \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F), \theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) v_i \right) + r \\ \partial_\alpha F_{i\beta} &= \partial_\beta F_{i\alpha}\end{aligned}\tag{2.7}$$

while the entropy production identity (2.1) becomes

$$\partial_t \hat{\eta}(\Phi(F), \theta) = \frac{r}{\theta}.\tag{2.8}$$

One then observes that $\xi = (\Phi(F), v, \theta)$ satisfies the augmented system

$$\begin{aligned}\partial_t \xi^B &= \partial_\alpha \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) v_i \right) \\ \partial_t v_i &= \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\xi, \theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) \right) \\ \partial_t \left(\frac{1}{2} |v|^2 + \hat{e}(\xi, \theta) \right) &= \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\xi, \theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) v_i \right) + r \\ \partial_\alpha F_{i\beta} &= \partial_\beta F_{i\alpha}\end{aligned}\tag{2.9}$$

that consists of conservation laws in \mathbb{R}^{23} subject to the involution (2.9)₄. The augmented system satisfies the entropy production identity

$$\partial_t \hat{\eta}(\xi, \theta) = \frac{r}{\theta}.\tag{2.10}$$

System (2.9) belongs to a class of conservation laws for $U : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$, studied in [9], of the form

$$\partial_t A(U) + \partial_\alpha f_\alpha(U) = 0,\tag{2.11}$$

that are endowed with an additional conservation law

$$\partial_t H(U) + \partial_\alpha Q_\alpha(U) = 0,\tag{2.12}$$

resulting by multiplying (2.11) by the multiplier $G(U)$, where $G(U) \cdot \nabla A(U) = \nabla H(U)$. It turns out that the hypothesis

$$\nabla^2 H(U) - G(U)^T \nabla^2 A(U) > 0\tag{2.13}$$

renders (2.11) symmetrizable [9] and when viewed in terms of the conserved variable $V = A(U)$ the resulting system is endowed with a convex entropy. This theory applies to (2.9), (2.10) and shows that by virtue of (1.5), augmented system (2.9) symmetrizable and hyperbolic [8].

Several implications of the embedding of (2.7) to an augmented hyperbolic system arise. One can apply to (2.9), expressed in terms of the conserved variables, the general existence theory in [13, Thm 5.5.3] to obtain local existence of classical solutions. One also expects the embedding to lead to an existence theory of dissipative measure-valued solutions for (2.7), but this is at present an open problem (such a result follows by the analogous embedding for the system of isothermal polyconvex elasticity [15]).

Another direction is that of weak-strong uniqueness theorems for weak or for measure-valued solutions. A theorem establishing recovery of classical solutions from dissipative measure-valued solutions for hyperbolic systems endowed with a convex entropy, was developed in [9]. We note that since in the variables (F, v, θ) system (2.7) is not equipped with a convex entropy, we cannot treat this problem as a direct application of the general setting of [9]. In [8], system (1.1)–(1.5) was studied by augmenting it to (2.9) using the relative entropy method in order to prove convergence from thermoviscoelasticity to (1.1)–(1.5). The objective in the present paper is to prove a weak-strong uniqueness theorem in the context of measure-valued solutions. This requires to work at the level of the original rather than the augmented system, which presents various technical challenges.

Following the theory on generalized Young measures [1, 9, 19], we define a dissipative measure-valued solution to polyconvex thermoelasticity, which involves a parametrized Young measure $\nu = \nu_{(x,t)}$ describing the oscillatory behavior of the solution and a Radon measure $\gamma \in \mathcal{M}^+(Q_T)$ describing concentration effects. According to the analysis in Appendix A, we can treat dissipative measure-valued solutions as limits of an approximating solution for the associated viscous problem, that satisfy an averaged and integrated energy equation. The reason behind the formation of concentrations, lies with the fact that the energy function $(x, t) \mapsto |v|^2 + e(F, \theta)$ is not weakly precompact in L^1 and thus, the Young measure representation fails. Since the only uniform bound at one's disposal is on the energy, the way we construct these solutions corresponds to a minimal framework obtained from this natural bound, for viscosity approximations of the adiabatic thermoelasticity system. The analysis in Appendix A leads to the following definition:

Definition 2.1. *A dissipative measure valued solution to polyconvex thermoelasticity (2.7), (2.8) consists of a thermomechanical process*

$$(y(t, x), \theta(t, x)) : [0, T] \times \mathbb{T}^3 \rightarrow \mathbb{R}^3 \times \mathbb{R}^+,$$

$$y \in W^{1,\infty}(L^2(\mathbb{T}^3)) \cap L^\infty(W^{1,p}(\mathbb{T}^3)), \quad \theta \in L^\infty(L^\ell(\mathbb{T}^3)), \quad (2.14)$$

a parametrized family of probability measures $\nu = \nu_{(x,t) \in \bar{Q}_T}$, with averages

$$F = \langle \nu, \lambda_F \rangle, \quad v = \langle \nu, \lambda_v \rangle, \quad \theta = \langle \nu, \lambda_\theta \rangle,$$

and a nonnegative Radon measure $\gamma \in \mathcal{M}^+(Q_T)$, where

$$F = \nabla y \in L^\infty(L^p), \quad v = \partial_t y \in L^\infty(L^2), \quad (2.15)$$

$$\Phi(F) = (F, \text{cof} F, \det F) \in L^\infty(L^p) \times L^\infty(L^q) \times L^\infty(L^\rho),$$

$p \geq 4$, $q \geq 2$, $\rho > 1$, $\ell > 1$, which satisfy the averaged equations

$$\begin{aligned} \partial_t \Phi^B(F) &= \partial_\alpha \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) v_i \right) \\ \partial_t \langle \nu, \lambda_{v_i} \rangle &= \partial_\alpha \left\langle \nu, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) \right\rangle \end{aligned} \quad (2.16)$$

$$\partial_t \langle \nu, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta) \rangle \geq \left\langle \nu, \frac{r}{\lambda_\theta} \right\rangle$$

in the sense of distributions, together with the integrated form of the averaged energy equation,

$$\begin{aligned} & \int \varphi(0) \left(\left\langle \nu, \frac{1}{2} |\lambda_v|^2 + \hat{e}(\Phi(\lambda_F), \lambda_\theta) \right\rangle(x, 0) dx + \gamma_0(dx) \right) \\ & + \int_0^T \int \varphi'(t) \left(\left\langle \nu, \frac{1}{2} |\lambda_v|^2 + \hat{e}(\Phi(\lambda_F), \lambda_\theta) \right\rangle(x, t) dx dt + \gamma(dx dt) \right) \\ & = - \int_0^T \int \langle \nu, r \rangle \varphi(t) dx dt, \end{aligned} \quad (2.17)$$

for all $\varphi \in C_c^1[0, T]$.

In this definition, the first equation (2.16)₁ holds in a classical weak sense under the regularity conditions (2.15), (2.14) placed on the motion and its derivatives for $p \geq 4$, $q \geq 2$, $\rho, \ell > 1$, as a consequence of the weak continuity of the null-Lagrangian vector $(F, \text{cof} F, \det F)$ and the weak continuity of the transport-stretching identities

$$\begin{aligned} \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t \det F &= \partial_\alpha ((\text{cof} F)_{i\alpha} v_i) \\ \partial_t (\text{cof} F)_{k\gamma} &= \partial_\alpha (\epsilon_{ijk} \epsilon_{\alpha\beta\gamma} F_{j\beta} v_i). \end{aligned} \quad (2.18)$$

We summarize the corresponding results, taken out of [3] and [15], in the following lemma. As the weak continuity property is important for the forthcoming analysis, we present the proof here for the reader's convenience.

Lemma 2.1. [3, Lemma 6.1], [15, Lemmas 4,5] *The following hold true:*

(i) For $y \in W^{1,\infty}(L^2(\mathbb{T}^3)) \cap L^\infty(W^{1,p}(\mathbb{T}^3))$ with $p \geq 4$, $F = \nabla y$ and $v = \partial_t y$, formulas (2.18) hold in the sense of distributions.

(ii) Suppose the family $\{y^\varepsilon\}_{\varepsilon>0}$, where $y^\varepsilon : [0, \infty) \times \mathbb{T}^3 \rightarrow \mathbb{R}^3$ satisfies

$$y^\varepsilon \text{ is uniformly bounded in } W^{1,\infty}(L^2(\mathbb{T}^3)) \cap L^\infty(W^{1,p}(\mathbb{T}^3)), \quad (2.19)$$

and let $v^\varepsilon = \partial_t y^\varepsilon$, $F^\varepsilon = \nabla y^\varepsilon$. Then, along a subsequence,

$$(F^\varepsilon, \text{cof} F^\varepsilon, \det F^\varepsilon) \rightharpoonup (F, \text{cof} F, \det F),$$

weakly in $L^\infty(L^p) \times L^\infty(L^q) \times L^\infty(L^r)$ with $p \geq 2$, $q \geq \frac{p}{p-1}$, $r \geq \frac{4}{3}$, $\rho > 1$. Moreover, if $p \geq 4$ the identities (2.18) are weakly stable in the regularity class (2.19).

Proof. We note the formulas, for smooth maps,

$$\begin{aligned} (\text{cof} F)_{i\alpha} &= \frac{1}{2} \epsilon_{ijk} \epsilon_{\alpha\beta\gamma} F_{j\beta} F_{k\gamma}, \\ \det F &= \frac{1}{6} \epsilon_{ijk} \epsilon_{\alpha\beta\gamma} F_{i\alpha} F_{j\beta} F_{k\gamma} = \frac{1}{3} (\text{cof} F)_{i\alpha} F_{i\alpha} \end{aligned} \quad (2.20)$$

and

$$\partial_t \det F = \partial_\alpha ((\text{cof} F)_{i\alpha} v_i)$$

$$\partial_t(\operatorname{cof} F)_{k\gamma} = \partial_\alpha(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{j\beta}v_i).$$

Step 1. For $y \in W^{1,\infty}(L^2(\mathbb{T}^3)) \cap L^\infty(W^{1,p}(\mathbb{T}^3))$, we extend y to a function defined for all times, by putting $y(t, x) = y(0, x)$, for $t \leq 0$. The extended y belongs to the same regularity class. Define the convolution (in space and time) $y_\epsilon := y \star f_\epsilon$, where $f_\epsilon = \varrho_\epsilon(t) \prod_{i=1}^3 \varrho_\epsilon(x_i)$, $\varrho_\epsilon = \frac{1}{\epsilon} \varrho(\frac{\cdot}{\epsilon})$, for $\varrho \in C_0^\infty(\mathbb{R})$ positive, $\int \varrho(s) ds = 1$. Then $y_\epsilon \in C^\infty(\mathbb{R} \times \mathbb{T}^3)$ and such that for all $s < \infty$, $T > 0$ there holds

$$\|\partial_t y_\epsilon - \partial_t y\|_{L^s([-T, T]; L^2)} + \|y_\epsilon - y\|_{L^\infty([-T, T]; W^{1,p})} \rightarrow 0.$$

Let $F_\epsilon = \nabla y_\epsilon$ and $v_\epsilon = \partial_t y_\epsilon$. Since the cofactor matrix is bilinear in the components of F , and the determinant is trilinear, it follows by repeated use of Hölder inequalities that for some numerical constant C ,

$$\begin{aligned} \|\operatorname{cof} F_\epsilon - \operatorname{cof} F\|_{L^s(L^{p/2})} &\leq C \|F_\epsilon - F\|_{L^{2s}(L^p)} \left(\|F_\epsilon\| + \|F\| \right)_{L^{2s}(L^p)} \\ \|\det F_\epsilon - \det F\|_{L^s(L^{p/3})} &\leq C \|F_\epsilon - F\|_{L^{3s}(L^p)} \left(\|F_\epsilon\| + \|F\| \right)_{L^{3s}(L^p)}^2. \end{aligned}$$

We thus conclude:

$$\operatorname{cof} F_\epsilon \rightarrow \operatorname{cof} F \quad \text{in } L^s(L^{p/2}), \quad \det F_\epsilon \rightarrow \det F \quad \text{in } L^s(L^{p/3}).$$

Passing to the limit $\epsilon \rightarrow 0$, for $p \geq 4$, in the formulas

$$\begin{aligned} \partial_t(\operatorname{cof} F_\epsilon)_{k\gamma} &= \partial_\alpha(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}(F_\epsilon)_{j\beta}(v_\epsilon)_i) \\ \partial_t(\det F_\epsilon) &= \partial_\alpha((\operatorname{cof} F_\epsilon)_{i\alpha}(v_\epsilon)_i) \end{aligned}$$

we obtain (2.18) in the sense of distributions and complete the proof of (i).

Step 2. Let $\{y^\epsilon\}_{\epsilon>0}$ be a family satisfying the uniform bound (2.19) and let $F^\epsilon = \nabla y^\epsilon$ and $v^\epsilon = \partial_t y^\epsilon$. We adapt the proof of [3, Lemma 6.1] suggesting to write the cofactor and the determinant in divergence form:

$$\begin{aligned} (\operatorname{cof} F^\epsilon)_{i\alpha} &= \frac{1}{2} \epsilon_{ijk}\epsilon_{\alpha\beta\gamma} F_{j\beta}^\epsilon F_{k\gamma}^\epsilon = \frac{1}{2} \partial_\beta(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma} y_j^\epsilon F_{k\gamma}^\epsilon), \\ \det F^\epsilon &= \frac{1}{3} (\operatorname{cof} F^\epsilon)_{i\alpha} F_{i\alpha}^\epsilon = \frac{1}{3} \partial_\alpha(y_i^\epsilon (\operatorname{cof} F^\epsilon)_{i\alpha}). \end{aligned}$$

With $p \geq 2$, $q \geq \frac{p}{p-1}$, hypothesis (2.19) implies $y^\epsilon \rightharpoonup y$ weakly in $W_{loc}^{1,2}([0, \infty) \times \mathbb{T}^3)$ along subsequences and Rellich's theorem (for dimension 3+1) implies $y^\epsilon \rightarrow y$ strongly in $L_{loc}^z([0, \infty) \times \mathbb{T}^3)$ for $z < 4$. Additionally, $F^\epsilon \rightharpoonup F$ weak-* in $L^\infty(L^p)$, for $p \geq 2 > \frac{4}{3}$ the dual exponent to 4. Therefore, we can pass to the limit in the sense of distributions:

$$\frac{1}{2} \partial_\beta(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma} y_j^\epsilon F_{k\gamma}^\epsilon) \rightharpoonup \frac{1}{2} \partial_\beta(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma} y_j F_{k\gamma}) = (\operatorname{cof} F)_{i\alpha} \quad (2.21)$$

and similarly for the determinant

$$\frac{1}{3} \partial_\alpha(y_i^\epsilon (\operatorname{cof} F^\epsilon)_{i\alpha}) \rightharpoonup \frac{1}{3} \partial_\alpha(y_i (\operatorname{cof} F)_{i\alpha}) = \det F, \quad (2.22)$$

since $\operatorname{cof} F^\epsilon \rightharpoonup \operatorname{cof} F$ weak-* in $L^\infty(L^q)$, for $q > \frac{4}{3}$ the dual exponent to 4. The distributional limits in (2.21) and (2.22) coincide with the limits in the weak-* topology. Altogether we have

$$\begin{aligned} (\operatorname{cof} F^\epsilon)_{i\alpha} &\rightharpoonup (\operatorname{cof} F)_{i\alpha}, \quad \text{weak-* in } L^\infty(L^q), \quad \text{for } q \geq \frac{p}{p-1}, \quad q > \frac{4}{3} \\ \det F^\epsilon &\rightharpoonup \det F, \quad \text{weak-* in } L^\infty(L^p). \end{aligned}$$

Next, note that y^ε , F^ε , v^ε satisfy

$$\begin{aligned}\partial_t(\operatorname{cof} F^\varepsilon)_{k\gamma} &= \partial_t \partial_\alpha \left(\frac{1}{2} \varepsilon_{ijk} \varepsilon_{\alpha\beta\gamma} y_i^\varepsilon F_{j\beta}^\varepsilon \right) = \partial_\alpha (\varepsilon_{ijk} \varepsilon_{\alpha\beta\gamma} F_{j\beta}^\varepsilon v_i^\varepsilon), \\ \partial_t \det F^\varepsilon &= \partial_t \partial_\alpha \left(\frac{1}{3} y_i^\varepsilon (\operatorname{cof} F^\varepsilon)_{i\alpha} \right) = \partial_\alpha (\operatorname{cof} F^\varepsilon_{i\alpha} v_i^\varepsilon).\end{aligned}$$

Using the weak continuity properties of $\operatorname{cof} F$ and $\det F$ and that for $p \geq 4$ equations (2.18) hold for functions y of class (2.19), we conclude that equations (2.18) are weakly stable. \square

Remark 2.1. On the definition of the dissipative measure-valued solution:

1. Combining the requirements of Lemma 2.1, with those of Lemmas 5.1 and 5.2, we must assume the exponents $p \geq 4$, $q \geq 2$, $\rho, \ell > 1$.
2. Henceforth, we assume the measure $\gamma_0 = 0$, meaning that we consider initial data with no concentrations at time $t = 0$.
3. Next, we highlight why we choose to work with the system in the physical variables $(\Phi(F), v, \theta)$ instead of the extended ones (ξ, v, θ) : This allows to avoid imposing restrictive growth conditions on the constitutive functions with respect to the cofactor and the determinant derivatives. From previous works in isothermal polyconvex elastodynamics (e.g. [16]) or even in [8], it becomes evident that when considering the extended system, one has to impose growth condition on terms

$$\frac{\partial \hat{\psi}}{\partial F}(\xi, \theta), \quad \frac{\partial \hat{\psi}}{\partial \zeta}(\xi, \theta) \frac{\partial(\operatorname{cof} F)}{\partial F}, \quad \frac{\partial \hat{\psi}}{\partial w}(\xi, \theta) \frac{\partial(\det F)}{\partial F}$$

where $\xi = (F, \zeta, w)$, in order to achieve representation of the associated weak limits via Young measures. The resulting regularity class of functions is far too restrictive and in particular functions with general power-like behavior do not satisfy such assumptions and their weak-limits cannot be represented. By contrast, if one works with the original variables, the growth hypotheses (3.1)–(3.2) placed on $\psi(F, \theta)$ and $e(F, \theta)$, which are compatible with the constitutive theory, are also sufficient to allow representation of the corresponding weak limits.

4. The reasoning behind studying the integrated form of the averaged energy equation lies in the technical advantage that, one does not need to place any integrability condition on the right hand-side of the energy equation (2.7)₃, namely on the term

$$\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F), \theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) v_i,$$

since it appears as a divergence and its contribution integrates to zero.

3. YOUNG MEASURES AND CONCENTRATION MEASURES

For $\hat{\psi} \in C^3(\mathbb{R}^{19} \times [0, \infty))$, we assume the following growth conditions

$$c(|F|^p + \theta^\ell) - c \leq e(F, \theta) \leq c(|F|^p + \theta^\ell) + c, \quad (3.1)$$

$$|\psi(F, \theta)| \leq c(|F|^p + \theta^\ell) + c, \quad (3.2)$$

$$\lim_{|F|^p + \theta^\ell \rightarrow \infty} \frac{|\partial_\theta \psi(F, \theta)|}{|F|^p + \theta^\ell} = \lim_{|F|^p + \theta^\ell \rightarrow \infty} \frac{|\eta(F, \theta)|}{|F|^p + \theta^\ell} = 0, \quad (3.3)$$

and

$$\lim_{|F|^p + \theta^\ell \rightarrow \infty} \frac{|\partial_F \psi(F, \theta)|}{|F|^p + \theta^\ell} = \lim_{|F|^p + \theta^\ell \rightarrow \infty} \frac{|\Sigma(F, \theta)|}{|F|^p + \theta^\ell} = 0 \quad (3.4)$$

-which are consistent with the constitutive theory (1.3)- for some constant $c > 0$ and $p \geq 4, \ell > 1$. The main hypotheses are (3.1), (3.3) and (3.4). Hypothesis (3.2) essentially follows from the others. Note, that the key hypothesis (3.1) of coercivity of $e(F, \theta)$ is consistent with hypothesis (1.5). This is clear regarding the growth in F ; regarding the growth in θ the consistency is seen from the equivalence between (B.2) and (B.5) (see Appendix B). As presented in Appendix A, we consider measure-valued solutions as limits of approximations that satisfy the uniform bound

$$\int_{\mathbb{T}^d} \hat{e}(\Phi(F^\varepsilon), \theta^\varepsilon) + \frac{1}{2}|v^\varepsilon|^2 dx \stackrel{(2.5)}{=} \int_{\mathbb{T}^d} e(F^\varepsilon, \theta^\varepsilon) + \frac{1}{2}|v^\varepsilon|^2 dx < C, \quad (3.5)$$

coming from the energy conservation equation (2.7)₃, given that the radiative heat supply r is a bounded function in $L^1(Q_T)$. Growth condition (3.1) in combination with (3.5) suggests that the functions $F^\varepsilon \in L^p$, $v^\varepsilon \in L^2$, $\theta^\varepsilon \in L^\ell$ are all (uniformly) bounded in the respective spaces. The approximating sequence $U^\varepsilon = (F^\varepsilon, v^\varepsilon, \theta^\varepsilon)$, represents weak limits of the form

$$\text{wk-} \lim_{\varepsilon \rightarrow 0} f(F^\varepsilon, v^\varepsilon, \theta^\varepsilon) = \langle \nu, f(F, v, \theta) \rangle \quad (3.6)$$

for all continuous functions $f = f(\lambda_F, \lambda_v, \lambda_\theta)$ such that

$$\lim_{|\lambda_F|^p + |\lambda_v|^2 + \lambda_\theta^\ell \rightarrow \infty} \frac{|f(\lambda_F, \lambda_v, \lambda_\theta)|}{|\lambda_F|^p + |\lambda_v|^2 + \lambda_\theta^\ell} = 0,$$

where $\lambda_F \in \mathbb{M}^{3 \times 3}$, $\lambda_v \in \mathbb{R}^3$, $\lambda_\theta \in \mathbb{R}^+$. The generated Young measure $\nu_{(x,t)}$ is associated with the motion $y : Q_T \rightarrow \mathbb{R}^3$ through F and v , by imposing that a.e.

$$F = \langle \nu, \lambda_F \rangle, \quad v = \langle \nu, \lambda_v \rangle, \quad \theta = \langle \nu, \lambda_\theta \rangle$$

and its action is well-defined for all functions f that grow slower than the energy. To take into account the formation of concentration effects, we introduce the concentration measure γ , depending on the total energy. This is a well-defined nonnegative Radon measure for a subsequence of $e(F^\varepsilon, \theta^\varepsilon) + \frac{1}{2}|v^\varepsilon|^2$. To prove this claim, let us define the sets

$$\begin{aligned} \mathcal{F}_0 &= \left\{ h \in C^b(\mathbb{R}^d) : h^\infty(z) = \lim_{\substack{s \rightarrow \infty \\ z' \rightarrow z}} \frac{h(sz')}{s} \text{ exists and is continuous on } S^{d-1} \right\} \\ \mathcal{F}_1 &= \left\{ g \in C(\mathbb{R}^d) : g(z) = h(z)(1 + |z|), h \in \mathcal{F}_0 \right\}. \end{aligned}$$

where $C^b(\mathbb{R}^d)$ denotes the set of all bounded and continuous functions on \mathbb{R}^d and S^{d-1} is the unit sphere on \mathbb{R}^d . Let X be a locally compact Hausdorff space, where we define the set of all Radon measures $\mathcal{M}(X)$ and all positive Radon measures $\mathcal{M}^+(X)$, while $\text{Prob}(X)$ denotes all probability measures on X . Let Ω be any open subset of \mathbb{R}^d and fix a Radon measure λ on Ω . We denote by $\mathcal{P}(\lambda; X) = L_w^\infty(d\lambda; \text{Prob}(X))$ the parametrized families of probability measures $(\nu_z)_{z \in \Omega}$ acting on X which are weakly-* measurable with respect to $z \in \Omega$. When λ is the Lebesgue measure, we use the notation $\mathcal{P}(\lambda; X) = \mathcal{P}(\Omega; X)$.

The following theorem as it appears in [1, 19] uses the theory of generalized Young measures to describe weak limits of the form

$$\lim_{n \rightarrow \infty} \int_{\Omega} \phi(x) g(u_n(x)) dx,$$

for $\phi \in C^0(\Omega)$, any bounded sequence u_n in L^1 , and test functions g such that

$$g(z) = \bar{g}(z)(1 + |z|), \quad \bar{g} \in C^b(\mathbb{R}^d).$$

For $g \in \mathcal{F}_1$, the L^1 -recession function

$$g^\infty(z) = \lim_{\substack{s \rightarrow \infty \\ z' \rightarrow z}} \frac{g(sz')}{s}, \quad z \in S^{d-1}, \quad (3.7)$$

coincides with $h^\infty(z)$, where $h \in \mathcal{F}_0$ and $g(z) = h(z)(1 + |z|)$.

Theorem 3.1. *Let $\{u_n\}$ be bounded in $L^1(\Omega; \mathbb{R}^d)$. There exists a subsequence $\{u_{n_k}\}$, a nonnegative Radon measure $\mu \in \mathcal{M}^+(\bar{\Omega})$ and parametrized families of probability measures*

$$\nu \in \mathcal{P}(\Omega; \mathbb{R}^d), \quad \nu^\infty \in \mathcal{P}(\lambda; S^{d-1})$$

such that

$$g(u_{n_k}) \rightharpoonup \langle \nu, g \rangle + \langle \nu^\infty, g^\infty \rangle \mu \quad \text{weak-* in } \mathcal{M}^+(\bar{\Omega}),$$

for any $g \in \mathcal{F}_1$.

Given that the only available bound for the approximate sequence U^ε is of the form

$$\int_{\mathbb{T}^d} f(F^\varepsilon, v^\varepsilon, \theta^\varepsilon) dx < C,$$

we want to represent the weak limits $\text{wk-*} \lim_{\varepsilon \rightarrow 0} f(F^\varepsilon, v^\varepsilon, \theta^\varepsilon)$, for a continuous test function f satisfying the growth condition

$$|f(F^\varepsilon, v^\varepsilon, \theta^\varepsilon)| \leq C(1 + |F|^p + |v|^2 + \theta^\ell).$$

In order to apply Theorem 3.1, we perform the change of variables

$$(A, b, c) = (|F|^{p-1}F, |v|v, \theta^\ell)$$

and define

$$f(F, v, \theta) := g(|F|^{p-1}F, |v|v, \theta^\ell),$$

imposing that the function g grows like

$$|g(A, b, c)| \leq C(1 + |A| + |b| + |c|).$$

Then Theorem 3.1 applies to represent the weak-* limits of g :

$$g^\infty(A, b, c) = \lim_{s \rightarrow \infty} \frac{g(sA, sb, sc)}{1 + s(A, b, c)},$$

for all $(A, b, c) \in S^{d^2+d} \cap \{c > 0\}$. Consequently, there exist a positive Radon measure $M \in \mathcal{M}^+(\bar{Q}_T)$ and probability measures $N \in \mathcal{P}(\bar{Q}_T; \mathbb{R}^{d^2+d+1})$, $N^\infty \in \mathcal{P}(\bar{Q}_T; S^{d^2+d})$ which, up to a subsequence, satisfy

$$g(A_n, b_n, c_n) \rightharpoonup \langle N, g(\lambda_A, \lambda_b, \lambda_c) \rangle + \langle N^\infty, g^\infty(\lambda_A, \lambda_b, \lambda_c) \rangle M.$$

Then, property (3.7) implies that

$$f(F_n, v_n, \theta_n) \rightharpoonup \langle \nu, f(\lambda_F, \lambda_v, \lambda_\theta) \rangle + \langle \nu^\infty, f^\infty(\lambda_F, \lambda_v, \lambda_\theta) \rangle M$$

where

$$\langle \nu, f(\lambda_F, \lambda_v, \lambda_\theta) \rangle = \langle N, g(|\lambda_F|^{p-1}\lambda_F, |\lambda_v|\lambda_v, \lambda_\theta^\ell) \rangle$$

and

$$\langle \nu^\infty, f(\lambda_F, \lambda_v, \lambda_\theta) \rangle = \langle N, g^\infty(|\lambda_F|^{p-1}\lambda_F, |\lambda_v|\lambda_v, \lambda_\theta^\ell) \rangle.$$

Therefore, given the bound (3.5) and assuming that the recession function

$$\left(e(F, \theta) + \frac{1}{2}|v|^2 \right)^\infty = \lim_{s \rightarrow \infty} \frac{e(s^{1/p}F, s^{1/\ell}\theta) + \frac{s}{2}|v|^2}{1 + s(|F|^p, |v|^2, \theta^\ell)},$$

exists and is continuous for all $(|F|^{p-1}F, |v|v, \theta^\ell) \in S^{d^2+d} \cap \{c > 0\}$, we have that (along a subsequence)

$$\begin{aligned} \text{wk-}* \lim_{\varepsilon \rightarrow 0} \left(\hat{e}(\Phi(F^\varepsilon), \theta^\varepsilon) + \frac{1}{2}|v^\varepsilon|^2 \right) &= \left\langle \nu, e(\lambda_F, \lambda_\theta) + \frac{1}{2}|\lambda_v|^2 \right\rangle \\ &\quad + \left\langle \nu^\infty, \left(e(\lambda_F, \lambda_\theta) + \frac{1}{2}|\lambda_v|^2 \right)^\infty \right\rangle_M, \end{aligned}$$

recalling (2.5)₂. Then (3.1) implies that $(e(\lambda_F, \lambda_\theta) + \frac{1}{2}|\lambda_v|^2)^\infty > 0$, therefore

$$\begin{aligned} \gamma &:= \left\langle \nu^\infty, \left(\frac{1}{2}|\lambda_v|^2 + e(\lambda_F, \lambda_\theta) \right)^\infty \right\rangle_M \\ &= \left\langle \nu^\infty, \left(\frac{1}{2}|\lambda_v|^2 + \hat{e}(\Phi(\lambda_F), \lambda_\theta) \right)^\infty \right\rangle_M \in \mathcal{M}^+(\bar{Q}_T). \end{aligned} \quad (3.8)$$

4. THE AVERAGED RELATIVE ENTROPY INEQUALITY

As mentioned in Section 1, augmented system (2.9) belongs to a general class of hyperbolic systems of the form

$$\partial_t A(U) + \partial_\alpha f_\alpha(U) = 0$$

where $U = U(x, t) \in \mathbb{R}^n$, is the unknown with $x \in \mathbb{R}^d$, $t \in \mathbb{R}^+$ and $A, f_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are given smooth functions of U . It is symmetrizable in the sense of Friedrichs and Lax [21], under appropriate hypotheses: The map $A(U)$ is globally invertible and there exists an entropy-entropy flux pair (H, q) , *i.e.* there exists a smooth multiplier $G(U) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\begin{aligned} \nabla H &= G \cdot \nabla A \\ \nabla q_\alpha &= G \cdot \nabla f_\alpha, \quad \alpha = 1, \dots, d. \end{aligned}$$

In our case

$$\begin{aligned} U &= \begin{pmatrix} \Phi(F) \\ v \\ \theta \end{pmatrix}, \quad A(U) = \begin{pmatrix} \Phi(F) \\ v \\ \frac{1}{2}|v|^2 + \hat{e}(\Phi(F), \theta) \end{pmatrix}, \\ f_\alpha(U) &= \begin{pmatrix} \frac{\partial \Phi^B}{\partial F_{i\alpha}}(F)v_i \\ \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F), \theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) \\ \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F), \theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(F)v_i \end{pmatrix} \end{aligned}$$

while the (mathematical) entropy is given by $H(U) = -\hat{\eta}(\Phi(F), \theta)$, the entropy flux $q_\alpha = 0$ and the associated multiplier is

$$G(U) = \frac{1}{\theta} \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F), \theta), v, -1 \right)^T, \quad B = 1, \dots, 19$$

see [9, 8].

Consider a strong solution $(\Phi(\bar{F}), \bar{v}, \bar{\theta})^T \in W^{1,\infty}(Q_T)$ to (2.7) that satisfies the entropy identity (2.8) and a dissipative measure valued solution to (2.7), (2.8) according to Definition 2.1. We write the difference of the weak form of equations (2.7), (2.8) and (2.16), (2.17) to obtain the following

three integral identities

$$\begin{aligned} & \int (\Phi^B(F) - \Phi^B(\bar{F}))(x, 0) \phi_1(x, 0) dx \\ & + \int_0^T \int (\Phi^B(F) - \Phi^B(\bar{F})) \partial_t \phi_1(x, t) dx dt \\ & = \int_0^T \int \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) v_i - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \bar{v}_i \right) \partial_\alpha \phi_1(x, t) dx dt, \end{aligned} \quad (4.1)$$

$$\begin{aligned} & \int (\langle \nu, \lambda_{v_i} \rangle - \bar{v}_i)(x, 0) \phi_2(x, 0) dx \\ & + \int_0^T \int (\langle \nu, \lambda_{v_i} \rangle - \bar{v}_i) \partial_t \phi_2(x, t) dx dt \\ & = \int_0^T \int \left(\left\langle \nu, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) \right\rangle \right. \\ & \quad \left. - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \partial_\alpha \phi_2(x, t) dx dt, \end{aligned} \quad (4.2)$$

and

$$\begin{aligned} & \int \left(\left\langle \nu, \frac{1}{2} |\lambda_v|^2 + \hat{e}(\Phi(\lambda_F), \lambda_\theta) \right\rangle - \frac{1}{2} |\bar{v}|^2 - \hat{e}(\Phi(\bar{F}), \bar{\theta}) \right) (x, 0) \phi_3(x, 0) dx \\ & + \int_0^T \int \left\{ \left(\left\langle \nu, \frac{1}{2} |\lambda_v|^2 + \hat{e}(\Phi(\lambda_F), \lambda_\theta) \right\rangle - \frac{1}{2} |\bar{v}|^2 - \hat{e}(\Phi(\bar{F}), \bar{\theta}) \right) \right. \\ & \quad \left. + \gamma \right\} \partial_t \phi_3(x, t) dx dt = - \int_0^T \int (\langle \nu, r \rangle - \bar{r}) \phi_3(x, t) dx dt, \end{aligned} \quad (4.3)$$

for any $\phi_i \in C_c^1(Q_T)$, $i = 1, 2$ and $\phi_3 \in C_c^1[0, T]$. Similarly, testing the difference of (2.8) and (2.16)₃ against $\phi_4 \in C_c^1(Q_T)$, with $\phi_4 \geq 0$, we have

$$\begin{aligned} & - \int (\langle \nu, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta) \rangle - \hat{\eta}(\Phi(\bar{F}), \bar{\theta}))(x, 0) \phi_4(x, 0) dx \\ & - \int_0^T \int (\langle \nu, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta) \rangle - \hat{\eta}(\Phi(\bar{F}), \bar{\theta})) \partial_t \phi_4(x, t) dx dt \\ & \geq \int_0^T \int \left(\left\langle \nu, \frac{r}{\lambda_\theta} \right\rangle - \frac{\bar{r}}{\bar{\theta}} \right) \phi_4(x, t) dx dt. \end{aligned} \quad (4.4)$$

We then choose $(\phi_1, \phi_2, \phi_3) = -\bar{\theta} G(\bar{U}) \varphi(t) = (-\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}), -\bar{v}, 1)^T \varphi(t)$, for some $\varphi \in C_c^1[0, T]$, thus (4.1), (4.2) and (4.3) become

$$\begin{aligned} & \int \left(-\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta})(\Phi^B(F) - \Phi^B(\bar{F})) \right) (x, 0) \varphi(0) dx \\ & + \int_0^T \int \left(-\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta})(\Phi^B(F) - \Phi^B(\bar{F})) \right) \varphi'(t) dx dt \\ & = \int_0^T \int \left[\partial_t \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta})(\Phi^B(F) - \Phi^B(\bar{F})) \right) \right. \\ & \quad \left. - \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) v_i - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \bar{v}_i \right) \right) \right] \varphi(t) dx dt, \end{aligned} \quad (4.5)$$

$$\begin{aligned}
& \int (-\bar{v}_i(\langle \boldsymbol{\nu}, \lambda_{v_i} \rangle - \bar{v}_i))(x, 0) \varphi(0) dx \\
& + \int_0^T \int -\bar{v}_i(\langle \boldsymbol{\nu}, \lambda_{v_i} \rangle - \bar{v}_i) \varphi'(t) dx dt \\
& = - \int_0^T \int \left[-\partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) (\langle \boldsymbol{\nu}, \lambda_{v_i} \rangle - \bar{v}_i) \right. \\
& \quad \left. + \partial_\alpha \bar{v}_i \left(\left\langle \boldsymbol{\nu}, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) \right\rangle \right. \right. \\
& \quad \left. \left. - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \right] \varphi(t) dx dt, \tag{4.6}
\end{aligned}$$

and

$$\begin{aligned}
& \int \left(\left\langle \boldsymbol{\nu}, \frac{1}{2} |\lambda_v|^2 + \hat{e}(\Phi(\lambda_F), \lambda_\theta) \right\rangle - \frac{1}{2} |\bar{v}|^2 - \hat{e}(\Phi(\bar{F}), \bar{\theta}) \right) (x, 0) \varphi(0) dx \\
& + \int_0^T \int \left\{ \left(\left\langle \boldsymbol{\nu}, \frac{1}{2} |\lambda_v|^2 + \hat{e}(\Phi(\lambda_F), \lambda_\theta) \right\rangle - \frac{1}{2} |\bar{v}|^2 - \hat{e}(\Phi(\bar{F}), \bar{\theta}) \right) \right. \\
& \quad \left. + \gamma \right\} \varphi'(t) dx dt = - \int_0^T \int (\langle \boldsymbol{\nu}, r \rangle - \bar{r}) \varphi(t) dx dt. \tag{4.7}
\end{aligned}$$

For inequality (4.4), we choose accordingly $\phi_4 := \bar{\theta} \varphi(t) \geq 0$, $\varphi \geq 0$ so that

$$\begin{aligned}
& - \int \bar{\theta} (\langle \boldsymbol{\nu}, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta) \rangle - \hat{\eta}(\Phi(\bar{F}), \bar{\theta})) (x, 0) \varphi(0) dx \\
& - \int_0^T \int \bar{\theta} (\langle \boldsymbol{\nu}, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta) \rangle - \hat{\eta}(\Phi(\bar{F}), \bar{\theta})) \varphi'(t) dx dt \\
& \geq \int_0^T \int \left[\partial_t \bar{\theta} (\langle \boldsymbol{\nu}, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta) \rangle - \hat{\eta}(\Phi(\bar{F}), \bar{\theta})) \right. \\
& \quad \left. + \bar{\theta} \left(\left\langle \boldsymbol{\nu}, \frac{r}{\lambda_\theta} \right\rangle - \frac{\bar{r}}{\bar{\theta}} \right) \right] \varphi(t) dx dt. \tag{4.8}
\end{aligned}$$

Adding together (4.5), (4.6), (4.7) and (4.8), we obtain the integral inequality

$$\begin{aligned}
& \int \varphi(0) \left[- \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) (\Phi^B(F) - \Phi^B(\bar{F})) (x, 0) - \langle \boldsymbol{\nu}, \bar{v}_i (\lambda_{v_i} - \bar{v}_i) \rangle (x, 0) \right. \\
& \quad \left. + \left\langle \boldsymbol{\nu}, \frac{1}{2} |\lambda_v|^2 + \hat{e}(\Phi(\lambda_F), \lambda_\theta) - \frac{1}{2} |\bar{v}|^2 - \hat{e}(\Phi(\bar{F}), \bar{\theta}) \right\rangle (x, 0) \right. \\
& \quad \left. - \bar{\theta} \langle \boldsymbol{\nu}, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta) - \hat{\eta}(\Phi(\bar{F}), \bar{\theta}) \rangle (x, 0) \right] dx \\
& + \int_0^T \int \varphi'(t) \left[- \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) (\Phi^B(F) - \Phi^B(\bar{F})) - \langle \boldsymbol{\nu}, \bar{v}_i (\lambda_{v_i} - \bar{v}_i) \rangle \right. \\
& \quad \left. + \left\langle \boldsymbol{\nu}, \frac{1}{2} |\lambda_v|^2 + \hat{e}(\Phi(\lambda_F), \lambda_\theta) - \frac{1}{2} |\bar{v}|^2 - \hat{e}(\Phi(\bar{F}), \bar{\theta}) \right\rangle \right. \\
& \quad \left. - \bar{\theta} \langle \boldsymbol{\nu}, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta) - \hat{\eta}(\Phi(\bar{F}), \bar{\theta}) \rangle + \gamma \right] dx dt \\
& \geq - \int_0^T \int \varphi(t) \left[- \partial_t \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) (\Phi^B(F) - \Phi^B(\bar{F})) \right. \\
& \quad \left. + \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) v_i - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \bar{v}_i \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \partial_\alpha \bar{v}_i \left\langle \boldsymbol{\nu}, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right\rangle \\
& - \partial_t \bar{\theta} \langle \boldsymbol{\nu}, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta) - \hat{\eta}(\Phi(\bar{F}), \bar{\theta}) \rangle \\
& - \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \langle \boldsymbol{\nu}, (\lambda_{v_i} - \bar{v}_i) \rangle \\
& - \bar{\theta} \left\langle \boldsymbol{\nu}, \frac{r}{\lambda_\theta} - \frac{\bar{r}}{\bar{\theta}} \right\rangle + \langle \boldsymbol{\nu}, r - \bar{r} \rangle \Big] dx dt \\
= & - \int_0^T \int \varphi(t) K(x, t) dx dt . \tag{4.9}
\end{aligned}$$

Using the entropy identity (2.8) and the null-Lagrangian property (2.2), the quantity $K(x, t)$ in the integrand on the right hand-side of (4.9) becomes

$$\begin{aligned}
K = & - \partial_t \bar{\theta} \langle \boldsymbol{\nu}, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) \rangle + \partial_t \Phi^B(\bar{F}) \left\langle \boldsymbol{\nu}, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) \right\rangle \\
& - \partial_t \bar{\theta} \frac{\partial \hat{\eta}}{\partial \theta}(\Phi(\bar{F}), \bar{\theta}) \langle \boldsymbol{\nu}, \lambda_\theta - \bar{\theta} \rangle \\
& - \partial_t \Phi^B(\bar{F}) \left(\left\langle \boldsymbol{\nu}, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right\rangle \right. \\
& \quad \left. - \frac{\partial^2 \hat{\psi}}{\partial \xi^B \partial \theta}(\Phi(\bar{F}), \bar{\theta}) \langle \boldsymbol{\nu}, \lambda_\theta - \bar{\theta} \rangle \right) \\
& + \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) v_i - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \bar{v}_i \right) \\
& - \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \langle \boldsymbol{\nu}, (\lambda_{v_i} - \bar{v}_i) \rangle \\
& + \partial_\alpha \bar{v}_i \left\langle \boldsymbol{\nu}, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right\rangle \\
& - \bar{\theta} \left\langle \boldsymbol{\nu}, \frac{r}{\lambda_\theta} - \frac{\bar{r}}{\bar{\theta}} \right\rangle + \langle \boldsymbol{\nu}, r - \bar{r} \rangle \\
= & - \partial_t \bar{\theta} \langle \boldsymbol{\nu}, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) \rangle + \partial_t \Phi^B(\bar{F}) \left\langle \boldsymbol{\nu}, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) \right\rangle \\
& - \frac{\bar{r}}{\bar{\theta}} \langle \boldsymbol{\nu}, \lambda_\theta - \bar{\theta} \rangle - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \partial_\alpha \bar{v}_i \left\langle \boldsymbol{\nu}, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right\rangle \\
& + \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) v_i - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \bar{v}_i \right) \\
& - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \langle \boldsymbol{\nu}, (\lambda_{v_i} - \bar{v}_i) \rangle \\
& + \partial_\alpha \bar{v}_i \left\langle \boldsymbol{\nu}, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right\rangle \\
& - \bar{\theta} \left\langle \boldsymbol{\nu}, \frac{r}{\lambda_\theta} - \frac{\bar{r}}{\bar{\theta}} \right\rangle + \langle \boldsymbol{\nu}, r - \bar{r} \rangle \tag{4.10}
\end{aligned}$$

employing (2.7)₁ and (2.2). Here, we use the quantities

$$\begin{aligned}
\langle \boldsymbol{\nu}, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) \rangle & := \left\langle \boldsymbol{\nu}, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta) - \hat{\eta}(\Phi(\bar{F}), \bar{\theta}) \right. \\
& \quad \left. - \frac{\partial \hat{\eta}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) (\Phi^B(\lambda_F) - \Phi^B(\bar{F})) - \frac{\partial \hat{\eta}}{\partial \theta}(\Phi(\bar{F}), \bar{\theta}) (\lambda_\theta - \bar{\theta}) \right\rangle , \tag{4.11}
\end{aligned}$$

and

$$\begin{aligned} \left\langle \boldsymbol{\nu}, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) \right\rangle &:= \left\langle \boldsymbol{\nu}, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right. \\ &\quad \left. - \frac{\partial^2 \hat{\psi}}{\partial \xi^B \partial \xi^A}(\Phi(\bar{F}), \bar{\theta})(\Phi^B(\lambda_F) - \Phi^B(\bar{F})) - \frac{\partial^2 \hat{\psi}}{\partial \xi^B \partial \theta}(\Phi(\bar{F}), \bar{\theta})(\lambda_\theta - \bar{\theta}) \right\rangle. \end{aligned} \quad (4.12)$$

Next, we rewrite the terms

$$\begin{aligned} & - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \partial_\alpha \bar{v}_i \left\langle \boldsymbol{\nu}, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right\rangle \\ & + \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) v_i - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \bar{v}_i \right) \\ & - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \langle \boldsymbol{\nu}, (\lambda_{v_i} - \bar{v}_i) \rangle \\ & + \partial_\alpha \bar{v}_i \left\langle \boldsymbol{\nu}, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right\rangle \\ & = \partial_\alpha \bar{v}_i \left\langle \boldsymbol{\nu}, \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \right\rangle \\ & \quad + \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \langle \boldsymbol{\nu}, (\lambda_{v_i} - \bar{v}_i) \rangle \\ & \quad + \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \bar{v}_i, \end{aligned} \quad (4.13)$$

since there holds

$$\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) = \left\langle \boldsymbol{\nu}, \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) \right\rangle$$

and because of the null-Lagrangian property (2.2). Also, we observe that

$$-\frac{\bar{r}}{\bar{\theta}} \langle \boldsymbol{\nu}, \lambda_\theta - \bar{\theta} \rangle - \bar{\theta} \left\langle \boldsymbol{\nu}, \frac{r}{\lambda_\theta} - \frac{\bar{r}}{\bar{\theta}} \right\rangle + \langle \boldsymbol{\nu}, r - \bar{r} \rangle = \left\langle \boldsymbol{\nu}, \left(\frac{r}{\lambda_\theta} - \frac{\bar{r}}{\bar{\theta}} \right) (\lambda_\theta - \bar{\theta}) \right\rangle \quad (4.14)$$

Finally, if we define the averaged quantity

$$\begin{aligned} I(\lambda_U | \bar{U}) &= I(\Phi(\lambda_F), \lambda_v, \lambda_\theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \\ &:= \hat{\psi}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) \\ &\quad + (\hat{\eta}(\Phi(\lambda_F), \lambda_\theta) - \hat{\eta}(\Phi(\bar{F}), \bar{\theta}))(\lambda_\theta - \bar{\theta}) + \frac{1}{2} |\lambda_v - \bar{v}|^2, \end{aligned} \quad (4.15)$$

for

$$\begin{aligned} \hat{\psi}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) &:= \hat{\psi}(\Phi(\lambda_F), \lambda_\theta) - \hat{\psi}(\Phi(\bar{F}), \bar{\theta}) \\ &\quad - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta})(\Phi^B(\lambda_F) - \Phi^B(\bar{F})) - \frac{\partial \hat{\psi}}{\partial \theta}(\Phi(\bar{F}), \bar{\theta})(\lambda_\theta - \bar{\theta}), \end{aligned}$$

and then combine (4.9),(4.10),(4.13) and (4.14), we arrive at the relative entropy inequality

$$\begin{aligned}
& \int \varphi(0) [\langle \nu, I(\lambda_{U_0} | \bar{U}_0) \rangle dx] + \int_0^T \int \varphi'(t) [\langle \nu, I(\lambda_U | \bar{U}) \rangle dx dt + \gamma(dx dt)] \\
& \geq - \int_0^T \int \varphi(t) \left[- \partial_t \bar{\theta} \langle \nu, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) \rangle \right. \\
& \quad + \partial_t \Phi^B(\bar{F}) \left\langle \nu, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) \right\rangle \\
& \quad + \partial_\alpha \bar{v}_i \left\langle \nu, \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \right\rangle \\
& \quad + \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \langle \nu, (\lambda_{v_i} - \bar{v}_i) \rangle \\
& \quad \left. + \left\langle \nu, \left(\frac{r}{\lambda_\theta} - \frac{\bar{r}}{\bar{\theta}} \right) (\lambda_\theta - \bar{\theta}) \right\rangle \right] dx dt. \tag{4.16}
\end{aligned}$$

We note that the last term in (4.13) vanishes when we substitute into the integral relation (4.9).

5. UNIQUENESS OF SMOOTH SOLUTIONS IN THE CLASS OF DISSIPATIVE MEASURE VALUED SOLUTIONS

In this section, we state and prove the main theorem on dissipative measure-valued versus strong uniqueness. Before we proceed with the proof, we show some useful estimates on the terms appearing in (4.16), which yield the relative entropy as a “metric” measuring the distance between the two solutions. These bounds are obtained by using the convexity of the free energy function in the compact domain and the growth conditions (3.1)-(3.4) placed on the constitutive functions in the original variables, in the unbounded domain.

Lemma 5.1. *Assume that $(\bar{F}, \bar{v}, \bar{\theta})$ are defined in the compact set*

$$\Gamma_{M,\delta} := \{(\bar{F}, \bar{v}, \bar{\theta}) : |\bar{F}| \leq M, |\bar{v}| \leq M, 0 < \delta \leq \bar{\theta} \leq M\} \tag{5.1}$$

for some positive constants M and δ and let $\hat{\psi} = \hat{e} - \theta \hat{\eta} \in C^3(\mathbb{R}^{19} \times [0, \infty))$. Assuming the growth conditions (3.1)-(3.4) and $p > 3$, $\ell > 1$, then there exist $R = R(M, \delta)$ and constants $K_1 = K_1(M, \delta, c) > 0$, $K_2 = K_2(M, \delta, c) > 0$ such that

$$I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \geq \begin{cases} \frac{K_1}{2} (|F|^p + \theta^\ell + |v|^2), & \text{if } |F|^p + \theta^\ell + |v|^2 > R \\ K_2 (|\Phi(F) - \Phi(\bar{F})|^2 & \text{if } |F|^p + \theta^\ell + |v|^2 \leq R \\ \quad + |\theta - \bar{\theta}|^2 + |v - \bar{v}|^2), & \end{cases} \tag{5.2}$$

for all $(\bar{F}, \bar{v}, \bar{\theta}) \in \Gamma_{M,\delta}$.

Proof. Let $(\bar{F}, \bar{v}, \bar{\theta}) \in \Gamma_{M,\delta}$ and choose $r = r(M) := M^p + M^\ell + M^2$ for which $\Gamma_{M,\delta} \subset B_r = \{(F, v, \theta) : |F|^p + \theta^\ell + |v|^2 \leq r\}$. Taking under consideration (2.5), we can write $I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta})$ in the form

$$\begin{aligned}
& I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \\
& = \hat{e}(\Phi(F), \theta) - \hat{\psi}(\Phi(\bar{F}), \bar{\theta}) - \frac{\partial \hat{\psi}}{\partial \xi}(\Phi(\bar{F}), \bar{\theta}) \cdot (\Phi(F) - \Phi(\bar{F})) \\
& \quad - \bar{\theta} \hat{\eta}(\Phi(F), \theta) + \frac{1}{2} |v - \bar{v}|^2 \\
& = e(F, \theta) - \psi(\bar{F}, \bar{\theta}) - \frac{\partial \psi}{\partial \xi}(\Phi(\bar{F}), \bar{\theta}) \cdot (\Phi(F) - \Phi(\bar{F}))
\end{aligned}$$

$$-\bar{\theta}\eta(F, \theta) + \frac{1}{2}|v - \bar{v}|^2.$$

Using (3.1) and (3.3) we have

$$\begin{aligned} I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) &\geq c(|F|^p + \theta^\ell) - c_1 - c_2|F|^3 - c_3|\eta(F, \theta)| \\ &\quad + \frac{1}{2}|v|^2 - c_4|v|. \end{aligned}$$

Selecting now R sufficiently large such that $R > r(M) + 1$ and for $|F|^p + \theta^\ell + |v|^2 > R$ we have

$$\begin{aligned} I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) &\geq \min\left\{c, \frac{1}{2}\right\} (|F|^p + \theta^\ell + |v|^2) - c_3|\eta(F, \theta)| - c_5 \\ &\geq \frac{1}{2} \min\left\{c, \frac{1}{2}\right\} (|F|^p + \theta^\ell + |v|^2) \end{aligned}$$

and (5.2) is established within the region $|F|^p + \theta^\ell + |v|^2 > R$.

In the complementary region $|F|^p + \theta^\ell + |v|^2 \leq R$, observe that $(F, \text{cof} F, \det F, v, \theta)$ takes values in the set

$$\begin{aligned} D := \left\{ (\Phi(F), v, \theta) : |F| \leq R^{1/p}, |\text{cof} F| \leq CR^{2/p}, |\det F| \leq CR^{3/p}, \right. \\ \left. |v| \leq R^{1/2}, 0 < \theta \leq R^{1/\ell} \right\}, \end{aligned}$$

for some constant C . We use the convexity of the entropy $\tilde{H}(V)$ in the symmetric variables $V := A(U) = \left(\xi, v, \frac{|v|^2}{2} + \hat{e}(\xi, \theta)\right)^T$:

$$\begin{aligned} \frac{1}{\bar{\theta}} I(\xi, v, \theta | \bar{\xi}, \bar{v}, \bar{\theta}) &= \tilde{H}(A(U) | A(\bar{U})) \\ &= \tilde{H}(A(U)) - \tilde{H}(A(\bar{U})) - \tilde{H}_V(A(\bar{U}))(A(U) - A(\bar{U})) \\ &\geq \min_{V^* \in \mathcal{D}^*} \{ \tilde{H}_{VV}(V^*) \} |A(U) - A(\bar{U})|^2, \end{aligned}$$

since $\tilde{H}(V)$ is convex in V and D^* is the compact domain determined by the map $V = A(U)$ and the set D defined above. Moreover, using the invertibility of the map $U \mapsto A(U)$

$$\begin{aligned} |U - \bar{U}| &= \left| \int_0^1 \frac{d}{d\tau} [A^{-1}(\tau A(U) + (1-\tau)A(\bar{U}))] d\tau \right| \\ &\leq \left| \int_0^1 \nabla_V(A^{-1})(\tau A(U) + (1-\tau)A(\bar{U})) d\tau \right| |A(U) - A(\bar{U})| \\ &\leq C' |A(U) - A(\bar{U})|, \end{aligned}$$

where

$$C' = \sup_{\substack{U \in B_R \\ \bar{U} \in \Gamma_{M, \delta}}} \left| \int_0^1 \nabla_V(A^{-1})(\tau A(U) + (1-\tau)A(\bar{U})) d\tau \right| |A(U) - A(\bar{U})| < \infty.$$

Therefore

$$I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \geq \frac{K_2}{C'} |U - \bar{U}|^2,$$

for $K_2 := \delta \min\{\tilde{H}_{VV}(V^*)\} > 0$ and the proof is complete. \square

Lemma 5.2. *Under the assumptions of Lemma 5.1 and the additional growth hypothesis*

$$\left| \frac{\partial \hat{\psi}}{\partial \xi}(\xi, \theta) \right| \leq c |\hat{\psi}(\xi, \theta)|, \quad \forall \xi, \theta \quad (5.3)$$

for some positive constant c , the following bounds hold true:

(i) There exist constants $C_1, C_2, C_3, C_4 > 0$ such that

$$\left| \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F), \theta) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \right| \leq \tag{5.4}$$

$$\leq C_1 I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}),$$

$$\left| \frac{\partial \hat{\psi}}{\partial \xi}(\Phi(F), \theta | \Phi(\bar{F}), \bar{\theta}) \right| \leq C_2 I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}), \tag{5.5}$$

$$|\hat{\eta}(\Phi(F), \theta | \Phi(\bar{F}), \bar{\theta})| \leq C_3 I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}), \tag{5.6}$$

and

$$\left| \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) (v_i - \bar{v}_i) \right| \leq C_4 I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \tag{5.7}$$

for all $(\bar{F}, \bar{v}, \bar{\theta}) \in \Gamma_{M, \delta}$.

(ii) There exist constants K'_1, K'_2 and $R > 0$ sufficiently large such that

$$I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \geq \begin{cases} \frac{K'_1}{4} (|F - \bar{F}|^p & \text{if } |F|^p + \theta^\ell + |v|^2 > R \\ + |\theta - \bar{\theta}|^\ell + |v - \bar{v}|^2), & \\ K'_2 (|\Phi(F) - \Phi(\bar{F})|^2 & \text{if } |F|^p + \theta^\ell + |v|^2 \leq R \\ + |\theta - \bar{\theta}|^2 + |v - \bar{v}|^2), & \end{cases} \tag{5.8}$$

for all $(\bar{F}, \bar{v}, \bar{\theta}) \in \Gamma_{M, \delta}$.

Proof. We divide the proof into 5 steps.

Step 1. To prove (5.4), we use (2.4) to obtain

$$\begin{aligned} & \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F), \theta) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \\ &= \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F), \theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) \\ & \quad - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F), \theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) + \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \\ &= \Sigma(F, \theta) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F), \theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) + \Sigma(\bar{F}, \bar{\theta}). \end{aligned}$$

For $|F|^p + \theta^\ell + |v|^2 > R$ and $(\bar{F}, \bar{v}, \bar{\theta}) \in \Gamma_{M, \delta}$, using (3.4), (5.3), (3.2) and Young's inequality we have

$$\begin{aligned} & \left| \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F), \theta) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \right| \\ & \leq |\Sigma(F, \theta)| + c_1 \left| \frac{\partial \Phi^B}{\partial F}(F) \right| + c_2 \left| \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F), \theta) \right| + c_3 \\ & \leq c_4 |\psi(F, \theta)| + c_5 (|F|^2 + |F| + 1) + c_6 |\hat{\psi}(\Phi(F), \theta)| + c_3 \\ & \leq c_7 (|F|^p + \theta^\ell) + c_8. \end{aligned}$$

Selecting now R large enough, so that $c_8 < c(|F|^p + \theta^\ell + |v|^2)$ for $p > 3$, $\ell > 1$, we conclude that

$$\begin{aligned} & \left| \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F), \theta) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \right| \\ & \leq C'_1 I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \end{aligned}$$

by Lemma 5.1. In the region $|F|^p + |v|^2 + \theta^\ell \leq R$, we have that $(\Phi(F), v, \theta), (\Phi(\bar{F}), \bar{v}, \bar{\theta}) \in D$ so that

$$\begin{aligned} & \left| \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F), \theta) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \right| \\ & \leq \max_D \left| \nabla_\xi^2 \hat{\psi}(\xi, \theta) \right| |\Phi(F) - \Phi(\bar{F})| \left| \frac{\partial \Phi}{\partial F}(F) - \frac{\partial \Phi}{\partial F}(\bar{F}) \right| \\ & \leq c_1 |\Phi(F) - \Phi(\bar{F})|^2 \\ & \leq C_1'' I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \end{aligned}$$

using again Lemma 5.1. Choosing now $C_1 = \max\{C_1', C_1''\}$, estimate (5.4) follows.

Step 2. Using Young's inequality and (5.3), it follows

$$\begin{aligned} \left| \frac{\partial \hat{\psi}}{\partial \xi}(\Phi(F), \theta | \Phi(\bar{F}), \bar{\theta}) \right| & \leq \left| \frac{\partial \hat{\psi}}{\partial \xi}(\Phi(F), \theta) \right| + c_1 + c_2 |\Phi(F)| + c_3 |\theta| \\ & \leq c_4 |\hat{\psi}(\Phi(F), \theta)| + c_2 (|F|^3 + |F|^2 + |F|) + c_3 |\theta| + c_1 \\ & \leq c_5 (|F|^p + \theta^\ell) + c_6. \end{aligned}$$

Choosing again R large enough, such that $R > r(M) + 1$ there holds

$$\begin{aligned} \left| \frac{\partial \hat{\psi}}{\partial \xi}(\Phi(F), \theta | \Phi(\bar{F}), \bar{\theta}) \right| & \leq c_7 (|F|^p + \theta^\ell + |v|^2) \\ & \leq C_2' I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \end{aligned}$$

by (5.2) and for $|F|^p + \theta^\ell + |v|^2 > R$ and $(\bar{F}, \bar{v}, \bar{\theta}) \in \Gamma_{M, \delta}$. In the complementary region $|F|^p + \theta^\ell + |v|^2 \leq R$, there holds $(\Phi(F), v, \theta), (\Phi(\bar{F}), \bar{v}, \bar{\theta}) \in D$, therefore

$$\begin{aligned} \left| \frac{\partial \hat{\psi}}{\partial \xi}(\Phi(F), \theta | \Phi(\bar{F}), \bar{\theta}) \right| & \leq \max_D \left| \nabla_{(\xi, \theta)}^3 \hat{\psi}(\xi, \theta) \right| (|\Phi(F) - \Phi(\bar{F})|^2 + |\theta - \bar{\theta}|^2) \\ & \leq C_2'' I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \end{aligned}$$

again by (5.2). Choosing $C_2 = \max\{C_2', C_2''\}$, the proof of (5.5) is complete.

Step 3. We proceed in a similar manner as in Step 2. to prove (5.6). First we study the region $|F|^p + |v|^2 + \theta^\ell > R$ and we use growth assumption (3.3) and relation (2.5) to get

$$\begin{aligned} \lim_{|F|^p + \theta^\ell \rightarrow \infty} \frac{|\hat{\eta}(\Phi(F), \theta | \Phi(\bar{F}), \bar{\theta})|}{|F|^p + \theta^\ell} & = \lim_{|F|^p + \theta^\ell \rightarrow \infty} \frac{|\hat{\eta}(\Phi(F), \theta)|}{|F|^p + \theta^\ell} \\ & = \lim_{|F|^p + \theta^\ell \rightarrow \infty} \frac{|\eta(F, \theta)|}{|F|^p + \theta^\ell} = 0. \end{aligned}$$

So immediately we deduce

$$|\hat{\eta}(\Phi(F), \theta | \Phi(\bar{F}), \bar{\theta})| \leq C_3' I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}),$$

for R large enough. On the complementary region $|F|^p + |v|^2 + \theta^\ell \leq R$,

$$\begin{aligned} |\hat{\eta}(\Phi(F), \theta | \Phi(\bar{F}), \bar{\theta})| & \leq \max_D \left| \nabla_{(\xi, \theta)}^2 \hat{\eta}(\Phi(F), \theta) \right| (|\Phi(F) - \Phi(\bar{F})|^2 + |\theta - \bar{\theta}|^2) \\ & \leq C_3'' I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}), \end{aligned}$$

by (5.2). Choosing $C_3 = \max\{C_3', C_3''\}$, the proof of (5.6) is complete.

Step 4. Similarly, for (5.7), when $|F|^p + \theta^\ell + |v|^2 > R$ and $(\bar{F}, \bar{v}, \bar{\theta}) \in \Gamma_{M,\delta}$, we have

$$\left| \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) (v_i - \bar{v}_i) \right| \leq c_1 |\Phi(F)|^2 + c_2 |v|^2 + c_3$$

and choosing appropriately the radius R , proceeding as before, it follows

$$\begin{aligned} \left| \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) (v_i - \bar{v}_i) \right| &\leq c_4 (|F|^p + \theta^\ell + |v|^2) \\ &\leq C'_4 I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}), \end{aligned}$$

where we use again (5.2) and $p > 3$. Then, for $|F|^p + \theta^\ell + |v|^2 \leq R$ and for all $(\Phi(F), v, \theta), (\Phi(\bar{F}), \bar{v}, \bar{\theta}) \in D$, we also get

$$\begin{aligned} \left| \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) (v_i - \bar{v}_i) \right| &\leq \frac{1}{2} \left| \frac{\partial \Phi^B}{\partial F}(F) - \frac{\partial \Phi^B}{\partial F}(\bar{F}) \right|^2 + \frac{1}{2} |v - \bar{v}|^2 \\ &\leq c_1 (|\Phi(F) - \Phi(\bar{F})|^2 + |v - \bar{v}|^2) \\ &\leq C''_4 I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \end{aligned}$$

by (5.2). Choosing $C_4 = \max\{C'_4, C''_4\}$, estimate (5.7) follows.

Step 5. Since $(\bar{F}, \bar{v}, \bar{\theta}) \in \Gamma_{M,\delta} \subset B_r$ -for sufficiently large R - there holds

$$|F - \bar{F}|^p + |\theta - \bar{\theta}|^\ell + |v - \bar{v}|^2 \leq (|F| + M)^p + (\theta + M)^\ell + (|v| + M)^2$$

and

$$\lim_{|F|^p + \theta^\ell + |v|^2 \rightarrow \infty} \frac{(|F| + M)^p + (\theta + M)^\ell + (|v| + M)^2}{|F|^p + \theta^\ell + |v|^2} = 1.$$

Thus, we may select R such that

$$\begin{aligned} |F - \bar{F}|^p + |\theta - \bar{\theta}|^\ell + |v - \bar{v}|^2 &\leq 2(|F|^p + \theta^\ell + |v|^2 + 1) \\ &\leq C(|F|^p + \theta^\ell + |v|^2) \end{aligned}$$

when $|F|^p + \theta^\ell + |v|^2 \geq R$. Thus (5.8) follows from (5.2). This concludes the proof. \square

We now consider a dissipative measure-valued solution for polyconvex thermoelasticity as defined in Definition 2.1. Using the averaged relative entropy inequality (4.16), we prove that in the presence of a classical solution, given that the associated Young measure is initially a Dirac mass, the *dissipative measure-valued* solution must coincide with the *classical* one.

Theorem 5.1. *Let \bar{U} be a Lipschitz bounded solution of (2.7),(2.8) with initial data \bar{U}^0 and (ν, γ, U) be a dissipative measure-valued solution satisfying (2.16),(2.17), with initial data U^0 , both under the constitutive assumptions (1.3) and such that $r(x, t) = \bar{r}(x, t) = 0$. Suppose that $\nabla_{\xi}^2 \hat{\psi}(\Phi(F), \theta) > 0$ and $\hat{\eta}_\theta(\Phi(F), \theta) > 0$ and the growth conditions (3.1), (3.2), (3.3), (3.4), (5.3) hold for $p \geq 4$, and $\ell > 1$. If $\bar{U} \in \Gamma_{M,\delta}$, for some positive constants M, δ and $\bar{U} \in W^{1,\infty}(Q_T)$, whenever $\nu_{(0,x)} = \delta_{\bar{U}^0}(x)$ and $\gamma_0 = 0$ we have that $\nu = \delta_{\bar{U}}$ and $U = \bar{U}$ a.e. on Q_T .*

Proof. Let $\{\varphi_n\}$ be a sequence of monotone decreasing functions such that $\varphi_n \geq 0$, for all $n \in \mathbb{N}$, converging as $n \rightarrow \infty$ to the Lipschitz function

$$\varphi(\tau) = \begin{cases} 1 & 0 \leq \tau \leq t \\ \frac{t - \tau}{\varepsilon} + 1 & t \leq \tau \leq t + \varepsilon \\ 0 & \tau \geq t + \varepsilon \end{cases}$$

for some $\varepsilon > 0$. Writing the relative entropy inequality (4.16) for $r(x, t) = \bar{r}(x, t) = 0$, tested against the functions φ_n we have

$$\begin{aligned}
 & \int \varphi_n(0) \langle \nu, I(\lambda_{U_0} | \bar{U}_0) \rangle dx + \int_0^t \int \varphi_n'(\tau) [\langle \nu, I(\lambda_U | \bar{U}) \rangle dx d\tau + \gamma(dx d\tau)] \\
 & \geq - \int_0^t \int \varphi_n(\tau) \left[-\partial_t \bar{\theta} \langle \nu, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) \rangle \right. \\
 & \quad \left. + \partial_t \Phi^B(\bar{F}) \left\langle \nu, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) \right\rangle \right. \\
 & \quad \left. + \partial_\alpha \bar{v}_i \left\langle \nu, \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \right\rangle \right. \\
 & \quad \left. + \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \langle \nu, (\lambda_{v_i} - \bar{v}_i) \rangle \right] dx d\tau.
 \end{aligned} \tag{5.9}$$

Passing to the limit as $n \rightarrow \infty$ we get

$$\begin{aligned}
 & \int \langle \nu, I(\Phi(\lambda_F), \lambda_v, \lambda_\theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \rangle (x, 0) dx \\
 & - \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \int [\langle \nu, I(\Phi(\lambda_F), \lambda_v, \lambda_\theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \rangle dx d\tau + \gamma(dx d\tau)] \\
 & \geq - \int_0^{t+\varepsilon} \int \left[-\partial_t \bar{\theta} \langle \nu, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) \rangle \right. \\
 & \quad \left. + \partial_t \Phi^B(\bar{F}) \left\langle \nu, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) \right\rangle \right. \\
 & \quad \left. + \partial_\alpha \bar{v}_i \left\langle \nu, \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \right\rangle \right. \\
 & \quad \left. + \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \langle \nu, (\lambda_{v_i} - \bar{v}_i) \rangle \right] dx d\tau.
 \end{aligned}$$

Passing now to the limit as $\varepsilon \rightarrow 0^+$ and using the fact that $\gamma \geq 0$ in combination with the estimates (5.4), (5.5), (5.6) and (5.7), we arrive at

$$\begin{aligned}
 & \int \langle \nu, I(\Phi(\lambda_F), \lambda_v, \lambda_\theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \rangle dx dt \leq \\
 & \leq C \int_0^t \int \langle \nu, I(\Phi(\lambda_F), \lambda_v, \lambda_\theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \rangle dx d\tau \\
 & \quad + \int \langle \nu, I(\Phi(\lambda_F), \lambda_v, \lambda_\theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \rangle (x, 0) dx
 \end{aligned}$$

for $t \in (0, T)$. Note that the constant C depends only on the smooth bounded solution \bar{U} . Then Gronwall's inequality implies

$$\begin{aligned}
 & \int \langle \nu, I(\Phi(\lambda_F), \lambda_v, \lambda_\theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \rangle dx dt \leq \\
 & \leq C_1 e^{C_2 t} \int \langle \nu, I(\Phi(\lambda_F), \lambda_v, \lambda_\theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \rangle (x, 0) dx
 \end{aligned}$$

and the proof is complete by (5.8). \square

An extension of Theorem 5.1 holds in case we assume $r(x, t) = \bar{r}(x, t) \neq 0$. For this purpose, we need the additional assumption

$$\text{supp } \nu \subset \mathbb{R}^{19} \times \mathbb{R}^3 \times [\underline{\delta}, \infty). \tag{5.10}$$

to control the terms that arise from the radiative heat supply in (4.16). We first prove the following lemma:

Lemma 5.3. *Suppose that $r(x, t) = \bar{r}(x, t) \in L^\infty(Q_T)$ and that*

$$\text{supp } \nu \subset \mathbb{R}^{19} \times \mathbb{R}^3 \times [\underline{\delta}, \infty),$$

for some small positive constant $\underline{\delta}$. Then there exists a constant $C_5 > 0$ such that

$$\left| \left\langle \nu, \left(\frac{r}{\lambda_\theta} - \frac{\bar{r}}{\bar{\theta}} \right) (\lambda_\theta - \bar{\theta}) \right\rangle \right| \leq C_5 \langle \nu, I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \rangle \quad (5.11)$$

for all $(\bar{F}, \bar{v}, \bar{\theta}) \in \Gamma_{M, \delta}$.

Proof. Assume first that $|F|^p + \theta^\ell + |v|^2 > R$. Then

$$\begin{aligned} \left| \left\langle \nu, \left(\frac{r}{\lambda_\theta} - \frac{\bar{r}}{\bar{\theta}} \right) (\lambda_\theta - \bar{\theta}) \right\rangle \right| &= \left| \int \left(\frac{\bar{r}}{\lambda_\theta} - \frac{\bar{r}}{\bar{\theta}} \right) (\lambda_\theta - \bar{\theta}) d\nu \right| \\ &\leq \|\bar{r}\|_{L^\infty} \left| \int \frac{(\lambda_\theta - \bar{\theta})^2}{\lambda_\theta \bar{\theta}} d\nu \right| \\ &= \|\bar{r}\|_{L^\infty} \left| \int \left(\frac{\lambda_\theta}{\bar{\theta}} - 2 + \frac{\bar{\theta}}{\lambda_\theta} \right) d\nu \right| \\ &\leq \|\bar{r}\|_{L^\infty} \left(\left| \left\langle \nu, \frac{\lambda_\theta}{\bar{\theta}} \right\rangle \right| + \left| \left\langle \nu, \frac{\bar{\theta}}{\lambda_\theta} \right\rangle \right| + c_1 \right) \\ &\leq c_2 |\langle \nu, \lambda_\theta \rangle| + c_3 |\langle \nu, 1 \rangle| + c_4 \\ &\leq c_5 (|\theta| + 1). \end{aligned}$$

Choosing R sufficiently large, we get for $\ell > 1$

$$\begin{aligned} \left| \left\langle \nu, \left(\frac{r}{\lambda_\theta} - \frac{\bar{r}}{\bar{\theta}} \right) (\lambda_\theta - \bar{\theta}) \right\rangle \right| &\leq c_6 (|F|^p + |\theta|^\ell + |v|^2) \\ &\leq C'_5 \langle \nu, I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \rangle, \end{aligned}$$

where, the last inequality holds because of Lemma 5.1 and the constant C'_5 depends on \bar{r} , δ , M and $\underline{\delta}$.

Now, similarly, if $|F|^p + \theta^\ell + |v|^2 \leq R$, we have

$$\begin{aligned} \left| \left\langle \nu, \left(\frac{r}{\lambda_\theta} - \frac{\bar{r}}{\bar{\theta}} \right) (\lambda_\theta - \bar{\theta}) \right\rangle \right| &\leq \|\bar{r}\|_{L^\infty} \int \frac{|\lambda_\theta - \bar{\theta}|^2}{|\lambda_\theta \bar{\theta}|} d\nu \\ &\leq C_1 \int |\lambda_\theta - \bar{\theta}|^2 d\nu \\ &\leq C''_5 \langle \nu, I(\Phi(F), v, \theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \rangle, \end{aligned}$$

again by estimate (5.2). By choosing

$$C_5 = \max\{C'_5(\bar{r}, \delta, M, \underline{\delta}), C''_5(\bar{r}, \delta, \underline{\delta})\},$$

the proof is complete. \square

Then Theorem 5.1 extends to :

Theorem 5.2. *Let \bar{U} be a Lipschitz bounded solution of (2.7),(2.8) with initial data \bar{U}^0 and (ν, γ, U) be a dissipative measure-valued solution satisfying (2.16),(2.17), with initial data U^0 , both under the constitutive assumptions (1.3) and such that $r(x, t) = \bar{r}(x, t)$. Assume also that there exists a small constant $\underline{\delta} > 0$ such that (5.10) holds true. Suppose that $\nabla_\xi^2 \hat{\psi}(\Phi(F), \theta) > 0$ and $\hat{\eta}_\theta(\Phi(F), \theta) > 0$ and the growth conditions (3.1), (3.2), (3.3), (3.4), (5.3) hold for $p \geq 4$, and*

$\ell > 1$. If $\bar{U} \in \Gamma_{M,\delta}$, for some positive constants M, δ and $\bar{U} \in W^{1,\infty}(Q_T)$, whenever $\boldsymbol{\nu}_{(0,x)} = \delta_{\bar{U}^0}(x)$, and $\boldsymbol{\gamma}_0 = 0$ we have that $\boldsymbol{\nu} = \delta_{\bar{U}}$ and $U = \bar{U}$ a.e. on Q_T .

Proof. The proof is a simple variant of the one for Theorem 5.1. Assuming the sequence $\{\varphi_n\}$ as before, the relative entropy inequality (4.16) becomes

$$\begin{aligned} & \int \varphi_n(0) \langle \boldsymbol{\nu}, I(\lambda_{U_0} | \bar{U}_0) \rangle dx + \int_0^t \int \varphi_n'(\tau) [\langle \boldsymbol{\nu}, I(\lambda_U | \bar{U}) \rangle dx d\tau + \boldsymbol{\gamma}(dx d\tau)] \\ & \geq - \int_0^t \int \varphi_n(\tau) \left[-\partial_t \bar{\theta} \langle \boldsymbol{\nu}, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) \rangle \right. \\ & \quad + \partial_t \Phi^B(\bar{F}) \left\langle \boldsymbol{\nu}, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta | \Phi(\bar{F}), \bar{\theta}) \right\rangle \\ & \quad + \partial_\alpha \bar{v}_i \left\langle \boldsymbol{\nu}, \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) - \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \right\rangle \\ & \quad + \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\bar{F}), \bar{\theta}) \right) \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\bar{F}) \right) \langle \boldsymbol{\nu}, (\lambda_{v_i} - \bar{v}_i) \rangle \\ & \quad \left. + \left\langle \boldsymbol{\nu}, \left(\frac{r}{\lambda_\theta} - \frac{\bar{r}}{\bar{\theta}} \right) (\lambda_\theta - \bar{\theta}) \right\rangle \right] dx d\tau. \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$ and then as $\varepsilon \rightarrow 0^+$ we obtain

$$\begin{aligned} & \int \langle \boldsymbol{\nu}, I(\Phi(\lambda_F), \lambda_v, \lambda_\theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \rangle dx dt \\ & \leq C \int_0^t \int \langle \boldsymbol{\nu}, I(\Phi(\lambda_F), \lambda_v, \lambda_\theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \rangle dx d\tau \\ & \quad + \int \langle \boldsymbol{\nu}, I(\Phi(\lambda_F), \lambda_v, \lambda_\theta | \Phi(\bar{F}), \bar{v}, \bar{\theta}) \rangle(x, 0) dx \end{aligned}$$

for $t \in (0, T)$. Here, we used that $\boldsymbol{\gamma} \geq 0$ and the estimates (5.4), (5.5), (5.6), (5.7) and (5.11), so that constant C depends on the smooth bounded solution \bar{U} and $\underline{\delta}$. By virtue of Gronwall's inequality and (5.8), we conclude the proof. \square

Let us note that, as Lemma 5.3 indicates, one needs to assume (5.10) in order to be able to bound from below the averaged temperature $\langle \boldsymbol{\nu}, \lambda_\theta \rangle$ and achieve estimate (5.11). Though it could be considered as a rather mild assumption, it is interesting that all the estimates in Lemmas 5.1 and 5.2 that involve the averaged temperature, do not require (5.10) to hold. This is because the averaged temperature is involved only through the constitutive functions $\hat{\psi}$, \hat{e} and $\hat{\eta}$ which we assume to be smooth enough, i.e. $\hat{\psi} = \hat{e} - \theta \hat{\eta} \in C^3$, and therefore we avoid any loss of smoothness as the temperature approaches zero.

APPENDIX A. THE NATURAL BOUNDS FOR VISCOUS APPROXIMATIONS OF POLYCONVEX THERMOELASTICITY

Since measure-valued solutions usually occur as limits of an approximating problem, consider the system of polyconvex thermoelasticity with Newtonian viscosity and Fourier heat conduction

$$\begin{aligned}
\partial_t \Phi^B(F^{\mu,k}) &= \partial_\alpha \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F^{\mu,k}) v_i^{\mu,k} \right) \\
\partial_t v_i^{\mu,k} &= \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F^{\mu,k}), \theta^{\mu,k}) \frac{\partial \Phi^B}{F_{i\alpha}}(F^{\mu,k}) \right) \\
&\quad + \partial_\alpha (\mu \partial_\alpha v_i^{\mu,k}) \\
\partial_t \left(\frac{1}{2} |v^{\mu,k}|^2 + \hat{e}(\Phi(F^{\mu,k}), \theta^{\mu,k}) \right) &= \partial_\alpha \left(\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F^{\mu,k}), \theta^{\mu,k}) \frac{\partial \Phi^B}{F_{i\alpha}}(F^{\mu,k}) v_i^{\mu,k} \right) \\
&\quad + \partial_\alpha (\mu v_i^{\mu,k} \partial_\alpha v_i^{\mu,k} + k \partial_\alpha \theta^{\mu,k}) + r \\
\partial_t \hat{\eta}(\Phi(F^{\mu,k}), \theta^{\mu,k}) &= \partial_\alpha \left(k \frac{\nabla \theta^{\mu,k}}{\theta^{\mu,k}} \right) + k \frac{|\nabla \theta^{\mu,k}|^2}{(\theta^{\mu,k})^2} + \mu \frac{|\nabla v^{\mu,k}|^2}{\theta^{\mu,k}} + \frac{r}{\theta^{\mu,k}}
\end{aligned} \tag{A.1}$$

where

$$\begin{aligned}
\frac{\partial \hat{\psi}}{\partial F_{i\alpha}}(\Phi(F), \theta) &= \frac{\partial \psi}{\partial F_{i\alpha}}(\Phi(F), \theta) \frac{\partial \psi}{\partial \zeta^{k\gamma}}(\Phi(F), \theta) \frac{\partial (\text{cof } F)_{k\gamma}}{\partial F_{i\alpha}} \\
&\quad + \frac{\partial \psi}{\partial w}(\Phi(F), \theta) \frac{\partial (\det F)}{\partial F_{i\alpha}}.
\end{aligned}$$

To our knowledge, there is presently no available result concerning global existence of smooth solutions for the system of thermoviscoelasticity (A.1). Nevertheless, we assume the existence of a global classical solution and proceed to examine the appropriate notion of measure-valued solutions induced by uniform energy bounds on the initial data. To be specific, we set the energy radiation $r \equiv 0$ and assume that the viscosity and heat conduction coefficients satisfy the bounds

$$\begin{aligned}
|\mu(F, \theta)\theta| &< \mu_0 |e(F, \theta)|, \\
|k(F, \theta)| &< k_0 |e(F, \theta)|,
\end{aligned} \tag{A.2}$$

for some constants $\mu_0, k_0 > 0$. Then, we proceed to investigate how measure-valued solutions emerge in the limit as $\mu_0 \rightarrow 0$ and $k_0 \rightarrow 0$, in a periodic domain in space $Q_T = \mathbb{T}^d \times [0, T]$, for $T \in [0, \infty)$ and $d = 3$. We impose the growth conditions

$$c(|F|^p + \theta^\ell) - c \leq e(F, \theta) \leq c(|F|^p + \theta^\ell) + c. \tag{A.3}$$

$$|\psi(F, \theta)| \leq c(|F|^p + \theta^\ell) + c, \tag{A.4}$$

$$\lim_{|F|^p + \theta^\ell \rightarrow \infty} \frac{|\partial_\theta \psi(F, \theta)|}{|F|^p + \theta^\ell} = \lim_{|F|^p + \theta^\ell \rightarrow \infty} \frac{|\eta(F, \theta)|}{|F|^p + \theta^\ell} = 0, \tag{A.5}$$

and

$$\lim_{|F|^p + \theta^\ell \rightarrow \infty} \frac{|\partial_F \psi(F, \theta)|}{|F|^p + \theta^\ell} = \lim_{|F|^p + \theta^\ell \rightarrow \infty} \frac{|\Sigma(F, \theta)|}{|F|^p + \theta^\ell} = 0, \tag{A.6}$$

for some constant $c > 0$ and $p, \ell > 1$.

Integrating the energy equation (A.1)₃ in Q_T we get

$$\begin{aligned}
\int \left(\frac{1}{2} |v^{\mu,k}|^2 + \hat{e}(\Phi(F^{\mu,k}), \theta^{\mu,k}) \right) dx &\leq \\
&\leq \int \left(\frac{1}{2} |v_0^{\mu,k}|^2 + \hat{e}(\Phi(F_0^{\mu,k}), \theta_0^{\mu,k}) \right) dx \leq C_1
\end{aligned} \tag{A.7}$$

so that

$$\sup_{0 < t < T} \int \left(\frac{1}{2} |v^{\mu,k}|^2 + \hat{e}(\Phi(F^{\mu,k}), \theta^{\mu,k}) \right) dx \leq C < \infty.$$

Therefore (A.3) implies that

$$\int \left(|F^{\mu,k}|^p + \frac{1}{2} |v^{\mu,k}|^2 + |\theta^{\mu,k}|^\ell \right) dx \leq C < \infty \quad (\text{A.8})$$

and then the functions $(F^{\mu,k}, v^{\mu,k}, \theta^{\mu,k})$ are all bounded in the spaces:

$$F^{\mu,k} \in L^\infty(L^p), \quad v^{\mu,k} \in L^\infty(L^2), \quad \theta^{\mu,k} \in L^\infty(L^\ell) \quad (\text{A.9})$$

and converge weakly to the averages

$$\begin{aligned} F^{\mu,k} &\rightharpoonup \langle \nu, \lambda_F \rangle =: F, & \text{weak-* in } L^\infty(L^p), \\ v^{\mu,k} &\rightharpoonup \langle \nu, \lambda_v \rangle =: v, & \text{weak-* in } L^\infty(L^2), \\ \theta^{\mu,k} &\rightharpoonup \langle \nu, \lambda_\theta \rangle =: \theta, & \text{weak-* in } L^\infty(L^\ell). \end{aligned}$$

Integrating now (A.1)₄ ($r \equiv 0$) in Q_T we obtain

$$\begin{aligned} \int \hat{\eta}(\Phi(F^{\mu,k}), \theta^{\mu,k}) dx - \int \hat{\eta}(\Phi(F^{\mu,k}), \theta^{\mu,k})(x, 0) dx & \quad (\text{A.10}) \\ &= \int_0^T \int \left(k \frac{|\nabla \theta^{\mu,k}|^2}{(\theta^{\mu,k})^2} + \mu \frac{|\nabla v^{\mu,k}|^2}{\theta^{\mu,k}} \right) dx dt. \end{aligned}$$

Then (A.5) and (A.9) immediately imply that $\hat{\eta}(\Phi(F^{\mu,k}), \theta^{\mu,k}) \in L^\infty(L^1)$ while

$$0 < \int_0^T \int \left(k \frac{|\nabla \theta^{\mu,k}|^2}{(\theta^{\mu,k})^2} + \mu \frac{|\nabla v^{\mu,k}|^2}{\theta^{\mu,k}} \right) dx dt \leq C. \quad (\text{A.11})$$

Now, let us consider the first equation in (A.1). We employ Lemma 2.1, in order to pass to the limit in the minors and the identities (2.18). It follows that (A.1)₁ holds in the classical weak sense for motions with regularity as in (2.19) and

$$(\Phi(F), v, \theta) = (F, \zeta, w, v, \theta)$$

in the space $L^\infty(L^p) \times L^\infty(L^q) \times L^\infty(L^\rho) \times L^\infty(L^2) \times L^\infty(L^\ell)$ with $p \geq 4, q \geq 2, \rho, \ell > 1$.

To pass to the limit in the second equation (A.1)₂, we use the Theorem of Ball [2] on representation via Young measures in the L^p setting:

Lemma A.1. *Let $f^\epsilon : \bar{Q}_T \rightarrow \mathbb{R}^m$ be a bounded function in L^p . Then for all $F : \mathbb{R}^m \rightarrow \mathbb{R}$ which are continuous and such that $F(f^\epsilon)$ is L^1 weakly precompact, there holds (along a subsequence)*

$$F(f^\epsilon) \rightharpoonup \langle \nu, F \rangle, \quad \text{weakly in } L^1(Q_T).$$

If $f^\epsilon : \bar{Q}_T \rightarrow \mathbb{R}^m$ is uniformly bounded in Q_T , then for all continuous $F : \mathbb{R}^m \rightarrow \mathbb{R}$ there holds (along a subsequence)

$$F(f^\epsilon) \rightharpoonup \langle \nu, F \rangle, \quad \text{weak-* in } L^\infty(Q_T).$$

Note that the assumption $\{F(f^\epsilon)\}$ is L^1 weakly precompact cannot be dropped or just replaced by L^1 boundness, because in such case concentrations might develop. We are going to examine each term separately, in order to obtain (along a non-relabeled subsequence)

$$\partial_t \langle \nu, \lambda_{v_i} \rangle = \partial_\alpha \left\langle \nu, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) \right\rangle$$

in the sense of distributions, which means

$$\begin{aligned} & \int \langle \boldsymbol{\nu}, \lambda_{v_i} \rangle (x, 0) \phi(x, 0) dx + \int_0^T \int \langle \boldsymbol{\nu}, \lambda_{v_i} \rangle \partial_t \phi dx dt \\ &= \int_0^T \int \left\langle \boldsymbol{\nu}, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) \right\rangle \partial_\alpha \phi dx dt \end{aligned}$$

for $\phi(x, t) \in C_c^1(Q_T)$. Observe that (A.8), (A.6) combined with (2.4) yield that the term

$$\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F^{\mu,k}), \theta^{\mu,k}) \frac{\partial \Phi^B}{F_{i\alpha}}(F^{\mu,k})$$

is representable with respect to the Young measure $\boldsymbol{\nu}_{(x,t) \in \bar{Q}_T}$ so that

$$\frac{\partial \hat{\psi}}{\partial F_{i\alpha}}(\Phi(F^{\mu,k}), \theta^{\mu,k}) \rightharpoonup \left\langle \boldsymbol{\nu}, \frac{\partial \hat{\psi}}{\partial F_{i\alpha}}(\Phi(\lambda_F), \lambda_\theta) \right\rangle, \quad \text{weakly in } L^1.$$

Now, we examine the limit of the diffusion term $\text{div}(\mu \nabla v^{\mu,k})$ as $\mu_0 \rightarrow 0$. First, we write

$$\lim_{\mu_0 \rightarrow 0} \left| \int_0^T \int \text{div}(\mu \nabla v^{\mu,k}) \phi dx dt \right| = \lim_{\mu_0 \rightarrow 0} \int_0^T \int |\mu \nabla v^{\mu,k} \cdot \nabla \phi| dx dt,$$

while using Holder's inequality:

$$\begin{aligned} & \lim_{\mu_0 \rightarrow 0} \int_0^T \int |\mu \nabla v^{\mu,k} \cdot \nabla \phi| dx dt \\ & \leq \lim_{\mu_0 \rightarrow 0} \left[\left(\int_0^T \int |\mu| \frac{|\nabla v^{\mu,k}|^2}{\theta^{\mu,k}} dx dt \right)^{1/2} \left(\int_0^T \int |\mu \theta^{\mu,k}| |\nabla \phi|^2 dx dt \right)^{1/2} \right] \\ & \leq \lim_{\mu_0 \rightarrow 0} C \mu_0 = 0 \end{aligned} \tag{A.12}$$

for $\phi(x, t) \in C_c^1(Q_T)$, by virtue of (A.2)₁, (A.3) and (A.11).

Next, we move to the entropy identity (A.1)₄, with $r \equiv 0$. In the limit, we aim to have

$$- \int \langle \boldsymbol{\nu}, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta) \rangle (x, 0) \phi(x, 0) dx - \int_0^T \int \langle \boldsymbol{\nu}, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta) \rangle \partial_t \phi dx dt \geq 0,$$

for $\phi(x, t) \in C_c^1(Q_T)$, $\phi \geq 0$. Growth condition (A.5) combined with (2.5)₁ allows to use again the Fundamental Lemma on Young measures to get

$$\hat{\eta}(\Phi(F^{\mu,k}), \theta^{\mu,k}) \rightharpoonup \langle \boldsymbol{\nu}, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta) \rangle, \quad \text{weakly in } L^1,$$

The diffusion term $\text{div} \left(k \frac{\nabla \theta^{\mu,k}}{\theta^{\mu,k}} \right)$ can be treated exactly as in (A.12), using again Hölder's inequality, (A.2)₂, (A.3) and (A.11):

$$\begin{aligned} & \lim_{k_0 \rightarrow 0} \left| \int_0^T \int \text{div} \left(k \frac{\nabla \theta^{\mu,k}}{\theta^{\mu,k}} \right) \phi dx dt \right| \\ &= \lim_{k_0 \rightarrow 0} \left| \int_0^T \int k \frac{\nabla \theta^{\mu,k}}{\theta^{\mu,k}} \cdot \nabla \phi dx dt \right| \\ & \leq \lim_{k_0 \rightarrow 0} \left[\left(\int_0^T \int |k| \frac{|\nabla \theta^{\mu,k}|^2}{(\theta^{\mu,k})^2} dx dt \right)^{1/2} \left(\int_0^T \int |k| |\nabla \phi|^2 dx dt \right)^{1/2} \right] \\ & \leq \lim_{k_0 \rightarrow 0} C k_0 = 0 \end{aligned} \tag{A.13}$$

where $\phi(x, t) \in C_c^1(Q_T)$, $\phi \geq 0$. On account of (A.11), we apply the Banach-Alaoglu Theorem to the sequence $\left| \sqrt{k} \frac{\nabla \theta^{\mu, k}}{\theta^{\mu, k}} \right|^2$ to obtain a sequential weak-* limit $\sigma_k \in \mathcal{M}^+(\bar{Q}_T)$

$$\sigma_k(\phi) = \int_0^T \int \phi d\sigma_k = \lim_{k_0 \rightarrow 0} \int_0^T \int \phi \left| \sqrt{k} \frac{\nabla \theta^{\mu, k}}{\theta^{\mu, k}} \right|^2 dx dt,$$

$\forall \phi \in C(\bar{Q}_T)$, $\phi \geq 0$. Similarly, the term

$$\mu \frac{|\nabla v^{\mu, k}|^2}{\theta^{\mu, k}} \rightharpoonup \sigma_\mu, \quad \text{weak-* in } \mathcal{M}^+(\bar{Q}_T),$$

as $\mu_0 \rightarrow 0$, because of (A.11). In summary, in the limit as $k_0, \mu_0 \rightarrow 0$, it holds (along a subsequence)

$$\partial_t \langle \nu, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta) \rangle = \sigma_\mu + \sigma_k \geq 0 \quad (\text{A.14})$$

in the sense of distributions.

Next we consider the energy equation (A.1)₃. The function

$$(x, t) \mapsto \left(\frac{1}{2} |v^{\mu, k}|^2 + \hat{e}(\Phi(F^{\mu, k}), \theta^{\mu, k}) \right) dx dt$$

is weakly precompact in the space of nonnegative Radon measures $\mathcal{M}^+(\bar{Q}_T)$, but not weakly precompact in L^1 , therefore the Young measure representation fails. To capture the resulting formation of concentrations, we introduce the concentration measure $\gamma(dx, dt)$, which is a nonnegative Radon measure in Q_T , for a subsequence of $\frac{1}{2} |v^{\mu, k}|^2 + \hat{e}(\Phi(F^{\mu, k}), \theta^{\mu, k})$, according to the analysis in Section 3. As we aim to construct dissipative solutions, in the limit, we consider an integrated averaged energy identity, tested against $\varphi = \varphi(t)$ in $C_c^1[0, T]$, that does not depend on the spatial variable; as a result, all the flux terms in (A.1)₃ vanish. Having this in mind, the energy equation becomes

$$\begin{aligned} & \int \varphi(0) \left(\langle \nu, \frac{1}{2} |\lambda_v|^2 + \hat{e}(\Phi(\lambda_F), \lambda_\theta) \rangle(x, 0) dx + \gamma_0(dx) \right) \\ & + \int_0^T \int \varphi'(t) \left(\left\langle \nu, \frac{1}{2} |\lambda_v|^2 + \hat{e}(\Phi(\lambda_F), \lambda_\theta) \right\rangle dx dt + \gamma(dx dt) \right) = 0 \end{aligned}$$

for all $\varphi = \varphi(t)$ in $C_c^1[0, T]$. Altogether, we conclude:

$$\begin{aligned} \partial_t \Phi^B(F) &= \partial_\alpha \left(\frac{\partial \Phi^B}{\partial F_{i\alpha}}(F) v_i \right) \\ \partial_t \langle \nu, \lambda_{v_i} \rangle &= \partial_\alpha \left\langle \nu, \frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(\lambda_F), \lambda_\theta) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(\lambda_F) \right\rangle \\ \partial_t \langle \nu, \hat{\eta}(\Phi(\lambda_F), \lambda_\theta) \rangle &\geq 0 \end{aligned}$$

and

$$\begin{aligned} & \int \varphi(0) \left(\langle \nu, \frac{1}{2} |\lambda_v|^2 + \hat{e}(\Phi(\lambda_F), \lambda_\theta) \rangle(x, 0) dx + \gamma_0(dx) \right) \\ & + \int_0^T \int \varphi'(t) \left(\left\langle \nu, \frac{1}{2} |\lambda_v|^2 + \hat{e}(\Phi(\lambda_F), \lambda_\theta) \right\rangle dx dt + \gamma(dx dt) \right) = 0. \end{aligned}$$

The above analysis indicates the relevance of Definition 2.1 of *dissipative measure valued solution* stated in Section 2.

Remark A.1. Testing the energy equation (A.1)₃ against a test function $\varphi = \varphi(x, t) \in C_c^1(Q_T)$, yields a different notion of measure-valued solution in the limit, the so-called *entropy measure-valued* solution. In that case, additional assumptions are required. First, one should represent the term

$$\frac{\partial \hat{\psi}}{\partial \xi^B}(\Phi(F^{\mu, k}), \theta^{\mu, k}) \frac{\partial \Phi^B}{\partial F_{i\alpha}}(F^{\mu, k}) v_i^{\mu, k}$$

in the flux, which requires growth conditions on $\Sigma_{i\alpha}(\Phi(F^{\mu,k}), \theta^{\mu,k})v_i^{\mu,k}$. Second, to treat the terms $\operatorname{div}(\mu v_i^{\mu,k} \partial_\alpha v_i^{\mu,k})$ and $\operatorname{div}(k \partial_\alpha \theta^{\mu,k})$ the additional uniform bounds

$$|\mu(F, \theta)\theta| \leq C\mu_0 \quad \text{and} \quad |k(F, \theta)\theta| \leq k_0|e(F, \theta)|,$$

on the diffusion coefficients are needed to pass to the limit.

Remark A.2. Let us return to the notion of *dissipative measure-valued* solution but now with $r \neq 0$. To establish the limit as $\mu_0, k_0 \rightarrow 0^+$, we need to represent the terms r and $\frac{r}{\theta^{\mu,k}}$, as they appear on the right-hand side of (A.1)₃ and (A.1)₄ respectively. Then (A.7) and (A.10) become

$$\begin{aligned} \int \left(\frac{1}{2} |v^{\mu,k}|^2 + \hat{e}(\Phi(F^{\mu,k}), \theta^{\mu,k}) \right) dx &\leq \int_0^T \int |r| dx dt \\ &+ \int \left(\frac{1}{2} |v_0^{\mu,k}|^2 + \hat{e}(\Phi(F_0^{\mu,k}), \theta_0^{\mu,k}) \right) dx \end{aligned}$$

and

$$\begin{aligned} \int \hat{\eta}(\Phi(F^{\mu,k}), \theta^{\mu,k}) dx - \int \hat{\eta}(\Phi(F^{\mu,k}), \theta^{\mu,k})(x, 0) dx \\ = \int_0^T \int \left(k \frac{|\nabla \theta^{\mu,k}|^2}{(\theta^{\mu,k})^2} + \mu \frac{|\nabla v^{\mu,k}|^2}{\theta^{\mu,k}} \right) dx dt + \int_0^T \int \frac{r}{\theta^{\mu,k}} dx dt \end{aligned}$$

respectively. In turn, to have the bounds (A.8) and (A.11), the previous analysis suggests to require that in limit $(\mu_0, k_0) \rightarrow (0, 0)$

$$r \in L^\infty(Q_T) \quad \text{and} \quad 0 < \bar{\delta} \leq \theta^{\mu,k} \tag{A.15}$$

for some $\bar{\delta} > 0$, so that both r and $\frac{r}{\theta^{\mu,k}} \in L^1(Q_T)$ uniformly in μ and k . Hence, as $\mu_0, k_0 \rightarrow 0$, we get

$$\begin{aligned} r &\rightharpoonup r \langle \boldsymbol{\nu}, 1 \rangle, \\ \frac{r}{\theta^{\mu,k}} &\rightharpoonup \left\langle \boldsymbol{\nu}, \frac{r}{\lambda_\theta} \right\rangle = r \left\langle \boldsymbol{\nu}, \frac{1}{\lambda_\theta} \right\rangle, \end{aligned}$$

weak-* in $L^\infty(\bar{Q}_T)$. In summary, the viscosity limit produces a dissipative measure valued solution assuming (A.2) and the growth estimates (A.3)–(A.6) for $r \equiv 0$ and in addition, (A.15) for $r \neq 0$.

APPENDIX B. LEGENDRE TRANSFORMS AND THERMODYNAMIC POTENTIALS.

Next, we highlight certain connections between the Legendre transform and the various thermodynamic potentials (internal energy, Helmholtz free energy, enthalpy, and Gibbs free energy). The reader may refer for detailed presentations to classical books on thermodynamics (*e.g* [7, Ch. 5]) or to the lecture notes [14, Ch. I, II].

The Legendre transform of a function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\phi^*(y) = \sup_{x \in \mathbb{R}} (xy - \phi(x)), \quad y \in \mathbb{R}.$$

The definition applies to any function ϕ and even for functions $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$, but for the purposes here we restrict to functions $\phi \in C^2(\mathbb{R})$ satisfying hypotheses of convexity and coercivity

$$\phi''(x) > 0, \quad \lim_{|x| \rightarrow \infty} \frac{\phi(x)}{|x|} = \infty.$$

Then, given $y \in \mathbb{R}$, there exists a unique x that maximizes

$$x \mapsto xy - \phi(x)$$

i.e. there exists a unique $x^* = x^*(y)$ such that $y = \phi'(x^*)$. The Legendre transform is then realized through the formula

$$\begin{aligned}\phi^*(y) &= \max_{x \in \mathbb{R}} (xy - \phi(x)) \\ &= x^*y - \phi(x^*) \quad \text{where } y = \phi'(x^*)\end{aligned}$$

Moreover, note the simple calculations

$$\begin{aligned}(\phi^*(y))' &= x^*(y) + (y - \phi'(x^*(y)))(x^*(y))' \\ &= x^*(y)\end{aligned}$$

and

$$(\phi^*(y))'' = (x^*(y))' = (\phi''(x^*))^{-1}.$$

Thus the convexity of $\phi(x)$ implies that $\phi^*(y)$ is also convex. Additionally, the convex conjugate of ϕ^* is ϕ *i.e.*

$$\phi^{**} = \phi.$$

The Legendre transform finds notable applications when switching from Lagrangian to Hamiltonian formulations in classical mechanics, or when switching thermodynamic potentials in classical thermodynamics.

To illustrate the latter, recall that the constitutive theories describing adiabatic thermoelastic nonconductors of heat (or more generally thermoviscoelasticity) [10] may be expressed through different thermodynamic potentials. For instance in (1.3), the constitutive theory of adiabatic thermoelasticity is derived using the Helmholtz free energy $\psi = \psi(F, \theta)$ and all thermodynamic quantities are functions of the deformation gradient and temperature, F and θ , viewed as independent variables. An alternative is to consider the deformation gradient F and specific entropy η , as independent variables and to base the theory on the internal energy $e = e(F, \eta)$ as thermodynamic potential through the relations

$$e = e(F, \eta), \quad \Sigma = \frac{\partial e}{\partial F}(F, \eta), \quad \theta = \frac{\partial e}{\partial \eta}(F, \eta). \quad (\text{B.1})$$

To see the relation between the two theories, consider (B.1) as the starting point and impose that $e(F, \eta)$ for $\eta > 0$, is convex and coercive

$$\frac{\partial^2 e}{\partial \eta^2}(F, \eta) > 0, \quad \lim_{\eta \rightarrow \infty} \frac{e(F, \eta)}{\eta} = \infty. \quad (\text{B.2})$$

The assumption of convexity of internal energy (in η) is intrinsic in the derivation of thermodynamics (see [14][Ch I]) and if we assume that the temperature at zero entropy is zero namely $\theta(F, 0) = 0$, then we have the implication

$$\theta(F, \eta) = \frac{\partial e(F, \eta)}{\partial \eta} > 0. \quad (\text{B.3})$$

Also note that the coercivity hypothesis (B.2) is consistent with the technical hypothesis (3.1) employed throughout this paper.

We define the negative of the free-energy ψ as the Legendre transform of $e(F, \eta)$,

$$-\psi(F, \theta) = \sup_{\eta \in \mathbb{R}} (\theta\eta - e(F, \eta)), \quad \theta \in \mathbb{R}^+.$$

and note that by (B.2) the unique maximizer η^* satisfies

$$-\psi(F, \theta) = \theta\eta^* - e(F, \eta^*), \quad \text{and} \quad \eta^* = -\frac{\partial \psi}{\partial \theta}(F, \theta) \quad (\text{B.4})$$

which is exactly (1.3)₃. Since $e(F, \eta)$ is convex in η , $\psi(F, \theta)$ has to be concave in θ , which justifies condition (1.5)₂ and yields

$$\eta_\theta(F, \theta) = -\psi_{\theta\theta}(F, \theta) > 0. \quad (\text{B.5})$$

The behavior in F is not restricted from the above considerations and the assumption of polyconvexity in F does not contradict the remaining thermodynamical structure. All the above formulas can be directly transferred to the constitutive functions $\hat{\psi}(\xi, \theta)$ and the corresponding $\hat{e}(\xi, \eta)$.

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