Analyzing and Manipulating Wave Propagation in Complex Structures

Dissertation by

Rasha Al Jahdali

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

King Abdullah University of Science and Technology

Thuwal, Kingdom of Saudi Arabia

June, 2019
The dissertation of Rasha Al Jahdali is approved by the examination committee.

Committee Chairperson: Ying Wu
Committee Members: David Keyes, Gerard Schuster, Badreddine Assouar
Dedication

It is with sincerity that I dedicate this work to the soul of King Abdullah bin Abdulaziz Al Saud
ABSTRACT

Analyzing and Manipulating Wave Propagation in Complex Structures

Rasha Al Jahdali

The focus of this dissertation is analyzing and manipulating acoustic wave propagation in metamaterials, which can be used to assist the design of acoustic devices. Metamaterials are artificial materials, which are arranged in certain patterns at a scale smaller than the wavelength and can exhibit properties beyond those naturally occurring materials. With metamaterials, novel phenomena, such as focusing, super absorption, cloaking and localization of ultrasound, are theoretically proposed and experimentally verified. In recent years, a planar version of metamaterials, often called meta-surfaces, has attracted a great deal of attention. Meta-surfaces can control and manipulate the amplitude, phase, and directions of waves. In this dissertation, we conducted a systematic study by deriving the effective medium theories (EMTs), and developing the theoretical and numerical models for our proposed designed metamaterial.

Very recently, acoustic meta-surfaces have been used in the design of acoustic lenses, which can achieve various functionalities such as focusing and collimation. In the designs of acoustic lenses, impedance is an important issue because it is usually difficult to make the impedance of the lens equal to that of the environment, and mismatched impedance is detrimental to the performance of the acoustic lens. We developed an EMT based on a coupled-mode theory and transfer matrix method to characterize the propagation behavior and, based on these models, we report two designs of acoustic lenses in water and air, respectively. They are rigid thin plates decorated with periodically distributed sub-wavelength slits. The building block of the acoustic lens in water is
constructed from coiling-up spaces, and that of the acoustic lens in air is made of layered structures. We demonstrate that the impedances of the lenses are indeed matched to those of the background media. With these impedance-matched acoustic lenses, we demonstrate acoustic focusing and collimation, and redirection of transmitted acoustic energy by finite-element simulations.

In the framework of the hidden source of the volume principle, an EMT for a coupled resonator structure is derived, which shows that coupled resonators are characterized by a negative value of the effective bulk modulus near the resonance frequency and induce flat bands that give rise to the confinement of the incoming wave inside the resonators. The leakage of sound waves in a resonance-based rainbow trapping device prevents the sound wave from being trapped at a specific location. Based on our EMT, we report a sound trapping device design based on coupled Helmholtz resonators, loaded to an air waveguide, to effectively tackle the wave leakage issue. We show that a coupled resonators structure can generate dips in the transmission spectrum by an analytical model derived from Newton’s second law and a numerical analysis based on the finite-element method. We compute the transmission spectra and band diagram from the effective medium theory, which are consistent with the simulation results. Trapping and the high absorption of sound wave energy are demonstrated with our designed device.
ACKNOWLEDGEMENTS

My passion and dreams of creating a meaningful and beneficial contribution to my beautiful country lead me to embark on my graduate studies at this wonderful house of wisdom that we know as KAUST. This work is the accumulation of several years of study and research, during which time I have been inspired, motivated and supported by a number of people throughout the journey. I would like to express my gratitude to each and every one of them.

I would like to wholeheartedly express my gratitude and appreciation to my advisor Professor Ying Wu, for her unwavering support, encouragement, and invaluable guidance during my PhD journey. With her intellectual inspiration, profound knowledge, and meticulous attitude towards research, Professor Wu has been a tower of strength in my doctoral studies journey.

I would like to express my deepest gratitude and thank my committee members, Professor David Keyes and Professor Gerard Schuster from King Abdullah University of Science and Technology as well as Professor Badreddine Assouar from University of Lorraine, for generously agreeing to serve on my dissertation committee and for their invaluable suggestions. I am deeply indebted to Professor Keyes for his support and knowledge, and also for his heartfelt warmth and sincerity in all communication I have had with him.

I would like to express my immense thanks to the other members in the research group, Pai Peng, Xiujuan Zhang, Zeguo Chen, Jiajun Zhao, Changqing Xu, and Lixin Ge, for the indispensable discussions and sharing invaluable knowledge.
During my years at this wonderful house of learning, I have grown to love and appreciate the diverse and stimulating research environment and I cannot think of a better place for scientists to thrive. KAUST is unique in the Kingdom and indeed the world. I would like to pay tribute to the soul of his majesty King Abdullah bin Abdulaziz Al Saud, may Allah bless his pure soul, as it is thanks to his vision that this inspiring house of wisdom was established in my hometown, Thuwal. Having visited Thuwal as a child, I have witnessed the rapid transformation of this quiet fishing village to the thriving, vibrant community that KAUST and the surrounding area has become today. From a humble community of local people, I have seen KAUST welcome such inspiring iconic figures as world-class professors, researchers and even Nobel Laureates and world leaders. I am honored and privileged to now benefit from the highest level of education and research that an institution can provide. I could not have imagined, when I visited Thuwal during my childhood, that I would return to this place as a graduate student and be fortunate enough to follow my dreams. I would like to express my immense appreciation to KAUST, this wonderful beacon of knowledge, for shining a light of peace, hope, and reconciliation and serving the people of the Kingdom and indeed having an impact on people around the world. I believe that KAUST will continue to thrive under the vision of the Custodian of the Two Holy Mosques, King Salman bin Abdulaziz Al Saud, may Allah protect him, and the Crown Prince Mohammed bin Salman bin Abdulaziz Al Saud, may Allah preserve his highness, in order to grow and meet the expansion of future generations, and to continue to support the goals of the Kingdom’s Vision 2030, with the guidance of God Almighty.
To my friends and others that made a positive impact on my graduate study and personal life, thank you for your patience, advice, and support through this journey. I have benefitted greatly from the support and advice of KAUST staff members Lucy Vick and Elisabeth Lutanie. Special thanks go to my family, whose unwavering belief in me has been truly appreciated. I cannot adequately express my gratitude to them.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXAMINATION COMMITTEE PAGE</td>
<td>2</td>
</tr>
<tr>
<td>COPYRIGHT PAGE</td>
<td>3</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>5</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>7</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>10</td>
</tr>
<tr>
<td>LIST OF ABBREVIATIONS</td>
<td>12</td>
</tr>
<tr>
<td>LIST OF ILLUSTRATIONS</td>
<td>13</td>
</tr>
<tr>
<td>LIST OF TABLE</td>
<td>13</td>
</tr>
<tr>
<td>Chapter 1</td>
<td>17</td>
</tr>
<tr>
<td>1.1 Introduction to Metamaterials and Meta-surface</td>
<td>17</td>
</tr>
<tr>
<td>1.2 Non-resonant and Resonant Acoustic Metamaterials</td>
<td>17</td>
</tr>
<tr>
<td>1.3 Acoustic Impedance</td>
<td>20</td>
</tr>
<tr>
<td>1.4 Introduction to the EMT</td>
<td>22</td>
</tr>
<tr>
<td>1.5 Wave Propagation in Layer Medium</td>
<td>26</td>
</tr>
<tr>
<td>1.6 Arrangement of Dissertation and Overview</td>
<td>31</td>
</tr>
<tr>
<td>Chapter 2</td>
<td>35</td>
</tr>
<tr>
<td>2.1 Acoustic Meta-surface in Water</td>
<td>35</td>
</tr>
<tr>
<td>2.1.1 Coiling-up Spaces Structure</td>
<td>35</td>
</tr>
<tr>
<td>2.1.2 Unit Cell and the Microstructure</td>
<td>36</td>
</tr>
<tr>
<td>2.1.3 Theoretical Analysis</td>
<td>37</td>
</tr>
<tr>
<td>2.1.4 Validity and Limitations of EMT</td>
<td>42</td>
</tr>
<tr>
<td>2.2 Acoustic Meta-surface in Air</td>
<td>48</td>
</tr>
<tr>
<td>2.2.1 Layered Media Structure</td>
<td>48</td>
</tr>
<tr>
<td>2.2.2 Material and Geometric Parameters</td>
<td>49</td>
</tr>
<tr>
<td>2.2.3 Theoretical Analysis</td>
<td>50</td>
</tr>
<tr>
<td>2.2.4 Validity and Limitations of the Theory</td>
<td>53</td>
</tr>
<tr>
<td>2.3 Summary</td>
<td>55</td>
</tr>
<tr>
<td>Chapter 3</td>
<td>56</td>
</tr>
<tr>
<td>3.1 Motivation for the Design</td>
<td>56</td>
</tr>
</tbody>
</table>
Chapter 4 .............................................................................................................. 75

4.1 Helmholtz Resonators .................................................................................. 75
4.2 Unit Cell and the Microstructure ................................................................. 77
4.3 Theoretical Analysis ...................................................................................... 79
4.3.1 Analytic Formula Based on Newton’s Second Law .................................. 79
4.3.2 EMT: Derivation and Verification .......................................................... 81
4.3.3 Validity and Limitations of EMT .............................................................. 84
4.4 Summary ......................................................................................................... 89

Chapter 5 .............................................................................................................. 90

5.1 Motivation for the Design ............................................................................. 90
5.2 Applications of Coupled Acoustic Resonators ............................................. 92
5.2.1 Acoustic Trapping .................................................................................... 92
5.2.2 Perfect Sound Absorption ...................................................................... 96
5.3 Summary ......................................................................................................... 99

Chapter 6 .............................................................................................................. 100

6.1 Finite Elements Method .............................................................................. 100
6.2 Numerical Simulation ................................................................................... 100
6.2.1 Numerical Scheme for Non-resonant Type Metamaterial ...................... 102
6.2.1.1 Coiling-up Space Structure ............................................................... 102
6.2.1.2 Layered Media Structure ................................................................. 113
6.2.2 Numerical Scheme for Resonant Type Metamaterial .............................. 122
6.3 Summary ......................................................................................................... 130

Chapter 7 .............................................................................................................. 131

7.1 Conclusion and Discussion .......................................................................... 131
7.2 Current and Future Work ............................................................................ 135

BIBLIOGRAPHY .................................................................................................. 142

APPENDICES ...................................................................................................... 154

Publication List .................................................................................................... 164

Conference List ................................................................................................... 165
### LIST OF ABBREVIATIONS

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>SRR</td>
<td>Split-ring resonator</td>
</tr>
<tr>
<td>FP</td>
<td>Fabry-Perot resonances</td>
</tr>
<tr>
<td>1D</td>
<td>One-dimensional</td>
</tr>
<tr>
<td>2D</td>
<td>Two-dimensional</td>
</tr>
<tr>
<td>EMT</td>
<td>Effective medium theory</td>
</tr>
<tr>
<td>Arg</td>
<td>Argon</td>
</tr>
<tr>
<td>Xen</td>
<td>Xenon</td>
</tr>
<tr>
<td>GRIN</td>
<td>Gradient refraction index</td>
</tr>
<tr>
<td>HR</td>
<td>Helmholtz resonator</td>
</tr>
<tr>
<td>RT</td>
<td>Rainbow trapping</td>
</tr>
<tr>
<td>FEM</td>
<td>Finite element method</td>
</tr>
<tr>
<td>NDOF</td>
<td>The numbers of degrees of freedom</td>
</tr>
</tbody>
</table>
LIST OF ILLUSTRATIONS

Figure 1.1 Schematic the mechanism of parameter retrieval method. ............................ 24
Figure 1.2 Scattering of a plane wave normally incident on single interface and single layer
........................................................................................................................................... 28
Figure 2.1 Schematic of the building block of the meta-surface and validation of the EMT.
........................................................................................................................................... 37
Figure 2.2 Comparing the transmission spectra obtained by COMSOL simulations and
theoretical predictions of two slabs with isotropic and anisotropic effective medium
parameters.............................................................................................................................. 42
Figure 2.3 Validation of the EMT with varying the width of the slit................................. 43
Figure 2.4 The value of when the high order of diffraction taking into account. .......... 44
Figure 2.5 The values of sinc function with varying the width of the slit......................... 45
Figure 2.6 Verification of the discrepancy between the numerical simulation and the EMT is
attributed to the high diffraction orders............................................................................... 46
Figure 2.7 The influences of varying the horizontal segment of the slit on the predictions of the
derived EMT.......................................................................................................................... 47
Figure 2.8 Validation of the EMT with changing the thickness of slab on EMT. .......... 48
Figure 2.9 Schematic of acoustic meta-surface with q layers of gases immersed in air.. 50
Figure 2.10 Illustration of scattering behavior when a plane wave pass through single layer and
multilayer slab......................................................................................................................... 53
Figure 3.1 A schematic of the designed acoustic lens in water and the distributions of the
pressure field of a Gaussian beam incident on the acoustic lens .................................... 59
Figure 3.2 The pressure field distribution of a Gaussian beam incident on an acoustic lens whose
slits are filled with isopentane............................................................................................. 61
Figure 3.3 The transmission of a normally incident plane wave onto two acoustic metasurfaces
with slit is filled by isopentane with and without considering the viscosity effect......... 62
Figure 3.4 A sketch of the optimized acoustic lens in air and demonstration of high acoustic
focusing effect after the incident waves pass through the lens........................................ 64
Figure 3.5 The distribution of the pressure amplitude of a point source transmitted through
impedance-matched acoustic lenses in water and in air ................................................. 65
Figure 3.6 The input impedance and the reflection coefficient of acoustic meta-surface in air computed at wavelength 0.7d and 0.8d where the inset represent the unit cell of the lens and their pressure distributions ................................................................. 66

Figure 3.7 The variation of the impedance matching conditions satisfied at wavelength d and 1.3d, where the inset show the unit cell of the optimized acoustic meta-surface in air and their pressure distributions ................................................................. 67

Figure 3.8 Diagram of supercell structure of coiling-up space acoustic meta-surface in water and their distributions of the pressure field at the incident wavelength 0.7d, and 0.8d ...... 69

Figure 3.9 The distributions of the pressure field at the incident wavelength d,and 1.7d.... ........................................................................................................................................................................... 70

Figure 3.10 The distributions of the pressure field of the plane wave after pass through acoustic meta-surface in water, where the filling material inside the slits is isopentane at the incident wavelength 0.7d, 0.8d, d, and 1.7d. ................................................................. 73

Figure 4.1 Schematic of Helmholtz resonator shows where the correction terms of the neck should be taken into account. ...................................................................................................................................................... 76

Figure 4.2 Schematic of the building block of single HR and three coupled HRs metamaterial, and Validation of the EMT ...................................................................................................................................................... 78

Figure 4.3 Band structure calculations and eigenfield distribution for single HR and three coupled HRs ...................................................................................................................................................... 84

Figure 4.4 The influences of varying the width of the neck on the predictions of theory.... ........................................................................................................................................................................... 86

Figure 4.5 The influences of varying the length of the neck on the predictions of theory.... ........................................................................................................................................................................... 87

Figure 4.6 The influences of varying the width and the length of the cavity on the predictions of theory...................................................................................................................................................... 88

Figure 5.1 Conceptual design of the trapping device, and the contour plot of the transmission spectrum ...................................................................................................................................................... 94

Figure 5.2 The pressure intensity at the resonances frequency marked in Fig.5.1 ...... 95

Figure 5.3 Simulated absorption, transmission, and reflection spectrum with the considered loss for a single HR and coupled HRs ...................................................................................................................................................... 98
Figure 6.1 Schematic of acoustic meta-surface in water, and boundary conditions and materials that are imposed in numerical simulation ................................................................. 103

Figure 6.2 Mesh structure of coiling-up space slit with varying number of degrees of freedom from one to five per wavelength ........................................................................... 106

Figure 6.3 Mesh quality elements plot with one and four degrees of freedom per wavelength. .......................................................................................................................... 108

Figure 6.4 The transmission spectrums are computed with a default setting of a finer mesh element size, and with mesh settings with one to five degrees of freedom per wavelength. .................................................................................. 109

Figure 6.5 The errors in $L_2$-norm plotted against $h$ on a log-log scale. .................... 111

Figure 6.6 The second silt on the meta-surface with the optimized layers of two gases..... 116

Figure 6.7 Mesh structure for each fluid layer of the meta-surface .............................. 117

Figure 6.8 Mesh quality elements plot with one and four degrees of freedom per wavelength. .......................................................................................................................... 119

Figure 6.9 The transmission spectrums are computed with varying mesh element sizes.... ............................................................................................................................. 120

Figure 6.10 The errors in $L_2$-norm plotted against $h$ on a log-log scale ..................... 121

Figure 6.11 Schematic of the unit cell structure and the boundary condition setting .. 124

Figure 6.12 Mesh structure for air waveguide connected with a single HR ............ 125

Figure 6.13 Mesh quality elements plot with one and four degrees of freedom per wavelength ...................................................................................................................... 126

Figure 6.14 The transmission spectrums of a normally incident plane wave onto the meta-surface for different boundary layer thicknesses .................................................................. 128

Figure 6.15 The errors in $L_2$-norm plotted against $h$ on a log-log scale ................... 129

Figure 7.1 Schematic of the meta-surface building block, and the corresponding homogenous slab that possesses an effective liquid medium for the shallow water wave .......... 141
LIST OF TABLE

Table 3.1 The reflection and transmission coefficients of 1st order diffraction waves. ... 72
Table 5.1 Geometrical parameters for trapping device. ......................................................... 93
Table 5.2 Geometrical parameters for absorption device. ....................................................... 96
Table 6.1 Details of the mesh elements for acoustic meta-surface in water ...................... 106
Table 6.2 The error and convergence order of the numerical simulation ......................... 113
Table 6.3 The thicknesses of the gases, the input impedance, and the reflection coefficient for each slit of acoustic meta-surface in air ................................................................. 114
Table 6.4 Details of the mesh elements for acoustic meta-surface in air ......................... 118
Table 6.5 The error and convergence order of the numerical simulation ....................... 121
Table 6.6 Details of the mesh elements for single Helmholtz resonator ....................... 125
Table 6.7 The error and convergence order of the numerical simulation .................... 129
Chapter 1

Introduction

1.1 Introduction to Metamaterials and Meta-surfaces

Metamaterials are kinds of composite materials that have been extensively studied over the past two decades. “Meta” is a Greek prefix meaning beyond. A metamaterial usually possesses anomalous material parameters, indicating it is capable of achieving exotic physical properties beyond naturally occurring materials, such as negative refraction. In 1968, Victor Veselago theoretically analyzed the electromagnetic wave propagation behavior inside a material whose permittivity and permeability are both negative [1]. Since such a combination of material parameters cannot be achieved by any naturally occurring materials, his study did not gain much attention at that time until the seminal work proposed by Sir John Pendry in 1999 [2], in which negative effective permeability is realized by a practical design of split-ring resonators (SRR). Since then, metamaterials has attracted a great deal of attention in both science and engineering due to its fascinating functionalities, such as a perfect lens induced by negative refraction [2] and cloaking based on a coordinate transform [3].

Metamaterials usually gain their special properties from their structure, which is different from conventional materials whose properties are obtained from the chemical properties of atoms and molecules. Therefore, incorporating resonances into the structure of a metamaterial will give rise to unusual response functions that appear in the governing equations. Here, the response functions refer to the effective medium parameters, which are mass density and bulk modulus in
the acoustic waves, or permeability and permittivity in electromagnetic waves. During the last
decade, various types of exotic effective medium parameters, such as zero refraction index [4-7],
negative mass density [8-11], negative bulk modulus [6, 12-16], have been achieved with
metamaterials and further lead to the realization of novel phenomena, including super
absorption [17], focusing and sub-wavelength imaging [18-21], cloaking [22-31], ultrasound
localization [32], unidirectional transmission [33-35], as well as near-field amplification [36].

Within recent years, increasing attention has been drawn to meta-surfaces, which are planarized
metamaterials that consist of carefully designed sub-wavelength building blocks. Similar to the
bulk metamaterial, a meta-surface can also achieve intriguing properties by manipulating wave
propagation like tuning the shape of the wave front. However, its ultra-thin thickness makes it
easier to be integrated into devices than bulk metamaterial. In acoustics, a meta-surface is
capable of controlling sound amplitude and phase. Designing the subwavelength unit cell of the
acoustic meta-surface, including Helmholtz resonators [37], membranes [38], and coiling-up
space structures [39-41], leads to designing and realizing new physical mechanisms and
fascinating functionalities. Various functionalities, such as perfect or near-perfect absorption
[17, 38, 42, 43], collimation [44, 45], sub-wavelength imaging [18], sub-wavelength sound
diffusers [46, 47], acoustic self-bending [48, 49], a twisted wave-front [50, 51], asymmetric sound
transmission [52], nearfield acoustic holography [53, 54], and the intense artificial Mie
resonances [55], can be achieved by acoustic meta-surfaces.
In fact, there are two broad classes of metamaterial or meta-surfaces: the resonant and the non-resonant type, and the choice of either of these particular types depends on the functionality we want to realize.

1.2 Non-resonant and Resonant Acoustic Metamaterials

In the design of the meta-materials, choosing the type of the unit structure is important, where the unit of the meta-materials can be categorized into two types: resonant and non-resonant [56, 57]. A meta-material that exhibits a large dynamic range of material parameters near the resonant frequency is classified as a resonant type, meaning that a large change in the material parameters will be observed when the frequency changes within a narrow regime. In other words, the small change in the size of the resonance unit will shift the resonance frequency by a small amount, while the values of the effective parameters will change significantly and that is the attractive feature of this kind of material. Nevertheless, these material parameters exhibit a narrow bandwidth and a large loss near the resonance frequency, which is the drawback of a resonant meta-material element. In contrast, the effective parameters of the non-resonator type vary slightly when the frequency changes. Realizing a large dynamic range in the values of material parameters in this type of material might be impossible. However, this kind of material in general exhibits a broad bandwidth and a low loss close to the resonance frequency, which is the advantage of a non-resonator meta-material. Therefore, based on the device function we want to achieve, we choose the unit type of meta-material. In designing a circularly invisible cloak, for instance, a resonant type meta-material is chosen, owing to the need for a large range of dynamic materials in order to fabricate such a device [58]. On the other hand, in designing an
invisible carpet, a non-resonant type meta-material is used because a small range of dynamic materials is required to build such a device[59].

Broadening the unprecedented functionalities of metamaterials of both types of units from a local resonance frequency, over a wide range of frequency operations, is obviously a great challenge. In acoustic meta-surface, non-resonance-based methods have been derived and developed for many sound devices, which have the advantage of being broadband in their frequency responses with perfect performance of their applications [60-62]. However, it is difficult to find a naturally occurring material with an ultra-thin thickness to be directly used in an acoustic meta-surface for airborne sound, because sound waves usually propagate faster in solids and liquids than in air. Another common challenge of an acoustic meta-surface is the mismatch of its impedance to that of the environment.

1.3 Acoustic Impedance

Acoustic impedance ($\xi_{\text{aco}}$), specific acoustic impedance ($\xi_{\text{spe}}$), and mechanical impedance ($\xi_{\text{mec}}$) are the most common types of impedances studied in acoustics [63]. Acoustic impedance is defined as the ratio of the pressure $p$ to volume flow of the fluid through the surface $U$ ($\xi_{\text{aco}} = p/U$), where the flow of the volume is determined by $A \times u$, where $A$ and $u$ denote the cross-sectional area and particle velocity that is generated when the particle of the medium transmits a wave. The ratio of the pressure $p$ to particle velocity $u$ is known as the specific acoustic impedance ($\xi_{\text{spe}} = p/u$) and the mechanical impedance is the ratio of the force $F$ to particle velocity $u$ ($\xi_{\text{mec}} = F/u$), where the force is given by $p \times A$. From the above definitions,
we can determine the specific acoustic impedance if the pressure and particle velocity are known. The values of the pressure and particle velocity equals $-\kappa \partial \psi / \partial z$ and $\partial \psi / \partial t$, respectively. $\psi$ represents the particle displacement and $\kappa (= \rho c^2)$ is the bulk modules, where $\rho$ is the mass density and $c$ is the speed of sound. The particle displacement $\psi$ can be obtained by solving the one-dimensional (1D) wave equation which is expressed as follows:

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}$$  (1.1)

For a plane wave, it is not difficult to obtain the specific acoustic impedance of the medium as $\xi_{spec} = \pm \rho c$. The sign represents the direction of the waves traveling along the positive or negative $x$-axes. The absolute value of this quantity gives the characteristic property of the acoustic medium, which is known as the characteristic impedance and is denoted as $\xi$. For the sake of simplicity, we will use impedance to represent characteristic impedance in the remainder of this dissertation, unless otherwise specified.

Different materials have different impedance values. For instance, the impedance of air equals $438.6 \text{ Pa s m}^{-1}$ at atmospheric pressure (101 kPa) in room temperature (20°C), and the impedance of steel equals $6147 \times 10^6 \text{ Pa s m}^{-1}$.

In the design of acoustic devices, including acoustic lenses, especially in the transmitted domain, impedance is an important issue because mismatched impedances of an acoustic lens in the air environment is detrimental to the performance of the acoustic lens. The transmission of sound in an air-solid (or air-liquid) interface is controlled by the ratio of the impedances of the materials,
i.e. $\frac{\xi_{\text{air}}}{\xi_{\text{solid}}}$ (or $\frac{\xi_{\text{air}}}{\xi_{\text{liquid}}}$). The detailed expression will be shown in Section 1.4. To achieve total transmission, we need the ratio to be equal to one. However, most of the solid and liquid materials have impedance that is much larger than that of air, which will cause the performance of the acoustic lens for airborne sound to seriously deteriorate. A similar problem also exists in acoustic lenses with underwater applications. A great deal of effort has been devoted to reducing the impact of mismatched impedances [37, 45, 64-75]. One way to do so is to utilize Fabry-Perot (FP) resonances [36,40-42,45,52], which can increase the transmission energy because of the destructive interference between the multiple reflections of acoustic waves on the input and output surfaces of the acoustic lens. Because the resonant frequency of FP resonances is sensitive to the effective thickness of the acoustic lens, it may not be able to perfectly eliminate the reflections in real applications. The performance of the acoustic lens is still affected by the mismatched impedance. Thus, we investigate and derive the impedance-matching condition for our proposed acoustic lenses by studying the effective medium theory (EMT) of space coiling structure and the scattering properties of layer medium structure.

1.4 Introduction to the EMT

It is generally difficult to find analytic solutions for acoustic wave equations in a complex heterogeneous medium. However, when the wavelength is much larger than the microstructure, the wave cannot resolve the microstructures and only passes through the medium as if it is homogeneous. This suggests that the microstructure can be modeled as a homogeneous medium which dramatically reduces the complexity of the problem. This homogenization process is known as EMT, which links the macroscopic property of a material to its microstructures. Here,
the macroscopic property is characterized by the dielectric constant $\varepsilon$, and the magnetic permeability $\mu$ for electromagnetic waves, and the mass density $\rho$, and the bulk modulus $\kappa$ for acoustic waves.

The development of EMT has gone through a long history. The relationship between the dielectric constants of two different materials was given by Mossotti in 1850. However, his work did not contain the formula for the effective dielectric constant. In 1879, Clausius derived the formula explicitly in his book [76]. Thus the formula that describes the relation between the dielectric constant and molecular polarizability is known as the Clausius-Mossotti relation, and it is also called the Lorentz-Lorenz equation if the molecular polarizability relates to the refractive index instead of the dielectric constant. Before Clausius’ work was published, Maxwell gave an analogous formula in conductivity [77]. Based on this relation, the most frequently used effective medium theories, i.e., Bruggeman [78] and Maxwell-Garnett theory [79, 80], were derived, which describe the effective parameters of a dielectric medium. The pioneering work in deriving the effective parameters of composite medium was launched in 1980 when Berryman derived the effective mass density for the elastic wave in a composite medium [81]. Thereafter, efforts have been devoted to extract the effective medium parameters of the composite medium [82, 83]. All of the above mentioned progress was made in the long wavelength regimes, i.e. when the scales of the inhomogeneous medium are much smaller than the scale of wavelength. Recently, there has been growing interest in deriving EMT which is beyond the long wavelength limit, because of the advent of metamaterials [84], in which resonance becomes important and naturally breaks
the long wavelength limit. Many new EMTs have been derived. Among them, the parameter retrieval method is the easiest and most frequently used by the scientific community [85-88].

In the parameter retrieval method, the effective medium parameters of a complex structure, such as the effective impedance $\xi_{\text{eff}}$ and effective refractive index $n_{\text{eff}}$, are retrieved from the reflection and transmission coefficients of the structure [85-88]. This method was firstly proposed in electromagnetic waves for an isotropic effective medium [88]. In this method, they computed the reflection and transmission coefficients for plane waves normally incident on a slab of heterogeneous medium as shown in Fig. 1.1(a). If the heterogeneous medium can be modeled as a homogenized medium, the transmission and reflection coefficients of a slab of the homogeneous medium with the same size as that of the heterogeneous one, should be identical to the transmission and reflection coefficients obtained earlier, as illustrated in Fig. 1.1(b). With those coefficients and the size of the slab, it is possible to obtain the effective impedance $\xi_{\text{eff}}$ and effective refractive index $n_{\text{eff}}$ as demonstrated in Fig.1.1(c).

![Figure 1.1 Schematic of the mechanism for the parameter retrieval method.](image-url)
The principle and the implementation of this method is easy. However, it is known that this method is associated with problems of ambiguities because the formula of the effective impedance and effective refractive index after the inversion process are complex with multiple branches. In this study [88] they resolved this issue by considering some requirements and constraints, for example, the materials should be passive, i.e. the real part of the impedance and imaginary part of refractive index are greater than zero. They then found that the effective medium parameters depend on frequency, and show three examples of metamaterials by utilizing this approach that exhibits negative $\varepsilon$ or/and $\mu$ at some frequency regions. Many efforts have been made to improve this approach, for example, it was extended to asymmetric slab with normal incidence waves [89]. An example of an effective anisotropic medium with normal incidence waves [90] and oblique incidence waves [91] have been demonstrated as well.

Fokin et al. extended the parameter retrieval method to acoustic materials [85]. They extracted the effective impedance and effective refractive index of acoustic metamaterials by analyzing wave scattering coefficients, and two designs of acoustic metamaterials based on this approach are exemplified. The retrieval method is also applied for acoustic gratings with periodic straight slits with a subwavelength structure, to determine the effective medium parameters [65]. In the low-frequency limit, this structure is modeled as a homogenous slab with a high refractive index. Following the same process, an EMT with anisotropic mass density has been derived and it gives a good description of acoustic gratings properties [21, 92, 93].

The grating based structures have been used to design acoustic lenses and most of them are resonance-based and bulky devices in the long wavelength. The manipulation of the effective
refractive index, by tailoring the path length of the sound wave in acoustic devices, can cope with the size limitation [94]. Coiling-up space structures offer a solution and ultra-thin devices have been designed [6] and EMT based on the parameter retrieval method has been developed to characterize the wave propagation behavior in coiling-up space structures. However, without an analytic expression for the effective medium parameters of a coiling-up space structure, it would be difficult for people to use it in the design of acoustic devices.

In this dissertation, we will carefully study the scattering properties of a coiling-up space structure by using a coupled-mode theory, and compare the analytic expressions of the transmission and reflection coefficients of the structure to that of a homogenous slab. From the comparison, we can retrieve the formulae for the effective medium parameters of the coiling-up space structure. We find that the effective impedance of a coiling-up space structure is significantly higher than that of the background. Fortunately, with the help of the analytic expression, we managed to find a solution to mitigate the impedance-mismatch problem, by introducing another material with low impedance into the coiling-up space structure, to “compensate” for the mismatched impedance such that the effective impedance of the space coiling structure is identical to that of the background. Therefore, the majority of the incident wave energy will transmit through the coiling-up space structure. Our EMT is indeed valid for a normally incident wave over a large frequency range, meaning that the bandwidth of the acoustic lens designed by such a coiling-up space structure is broad.

1.5 Wave Propagation in a Layered Medium
The strategy mentioned in the previous section cannot be simply carried over into the design of acoustic devices in a medium with air as the background, because finding a material whose impedance is lower than air is challenging. Therefore, we need to find an alternative solution to tackle the impedance mismatch problem for airborne sound. The structure of a layered medium becomes an option because we can manipulate the input impedance and match it to air to fulfill the requirement of matched impedances.

The wave generally is transmitted and reflected when it passes through an interface between two media. As illustrated in Fig.1.2 (a), a sound wave propagating along the \( z \)-direction is normally incident on the interface between medium 1 to medium 2 located at \( z = 0 \). The pressure fields of the incident, reflected, and transmitted waves can be written as \( p_I = P_I e^{i(\omega z)} \), \( p_R = P_R e^{i(\omega z)} \), and \( p_T = P_T e^{i(\omega z)} \), respectively, where \( k_j = \omega c_j \) is the wave number in each medium (\( j = 1 \) and \( 2 \)), and \( c_j \) is the speed of sound in the medium \( j \). The pressure transmission and reflection coefficients are defined as: \( T = p_T / p_I \) and \( R = p_R / p_I \), which can be evaluated by taking the boundary conditions on the interface into account, i.e. the continuity of pressure fields \( p \) and the normal velocity \( u = \left( -\rho_j^{-1} \partial p / \partial z \right) \). Here, \( \rho_j \) represent the mass density in medium \( j \) and the continuity conditions are written as \( p_I + p_R = p_T \) and \( u_I + u_R = u_T \). It is straightforward to get the formulas of transmission and refraction coefficients as follows [95]:

\[
T = \frac{2 \xi_2}{\xi_2 + \xi_1} \quad (1.2)
\]
where $\xi_1 (= \rho_1 c_1)$ and $\xi_2 (= \rho_2 c_2)$ are the acoustic impedances of medium 1 and 2, respectively.

Following a similar strategy, one can easily solve the transmission and reflection coefficients of a homogenous slab as illustrated in Fig.1.2 (b). The pressure fields of the incident and reflected waves have the forms:

$$P_t (z < 0) = p_i e^{i(\omega t - k_z z)}$$  \hspace{1cm} (1.4)

$$P_R (z < 0) = p_R e^{i(\omega t + k_z z)}$$  \hspace{1cm} (1.5)
The expressions of the pressure fields of the forward and backward traveling waves inside the homogenous slab are, respectively, written as:

\[ P_A(z) = A e^{i(\omega - k_z z)} \quad (1.6) \]

\[ P_B(z) = B e^{i(\omega + k_z z)} \]

and the expression of the pressure field of the transmitted wave is given by:

\[ P_T(z > d) = p_T e^{i(\omega + k_z (d - z))} \quad (1.7) \]

The continuity conditions at interfaces at \( z = 0 \) and \( z = d \) give the following relation:

\[
\begin{bmatrix}
A \\
B
\end{bmatrix}_{x=0} = K_0 \begin{bmatrix}
 p_I \\
p_R
\end{bmatrix}_{x=0} \quad (1.8)
\]

and

\[
\begin{bmatrix}
p_T \\
0
\end{bmatrix}_{x=d} = K_1 \begin{bmatrix}
A \\
B
\end{bmatrix}_{x=d} \quad (1.9)
\]

where \( K_0 \) and \( K_1 \) represent the propagator matrices at each interface and have the forms:

\[
K_0 = \begin{bmatrix}
a^*_0 & a_0 \\
a_0 & a^*_0
\end{bmatrix} \quad \text{and} \quad K_1 = \begin{bmatrix}
a^*_1 e^{-ik_z d} & a_1 e^{ik_z d} \\
a_1 e^{-ik_z d} & a^*_1 e^{ik_z d}
\end{bmatrix}.
\]

The entries of these matrices are defined as \( a_j^\pm = \left[ 1 \pm \left( \frac{\xi_j}{2} \right) \right] / 2 \ (j = 1,2) \). Combining Eqs. (1.8) and (1.9), we deduce
Here \( K_N \ (= K_1 K_0) \) is the cumulative overall propagator of the system. From Eq. (1.10), we obtain the transmission and reflection coefficients of a single slab as follows:

\[
T = \frac{T_1T_2e^{ik_zd}}{1-R_1R_2e^{2ik_zd}} \quad (1.11)
\]

\[
R = \frac{R_1 - R_2e^{2ik_zd}}{1-R_1R_2e^{2ik_zd}} \quad (1.12)
\]

where \( T_{i=1,2} \) and \( R_{i=1,2} \) are the transmission and reflection coefficients at the interfaces \( z = 0 \) and \( z = d \), respectively, which can be evaluated from Eqs. (1.2) and (1.3). These coefficients depend on the impedances of the materials at each interface. Thus, Eqs. (1.11) and (1.12) imply that impedance plays a very important role in determining the magnitude of the waves transmitted by an interface and a flat slab. Especially for a flat slab material, if its impedance differs too much from that of the background, the transmission coefficient would be generally low, except at certain frequencies when constructive interference of multiple-scattered waves by the two interfaces occurs. Unfortunately, such a limitation widely exists in acoustic devices for airborne sound because common solid or liquid materials have much higher impedances than air, as has been discussed earlier. A conclusion that can be drawn from Eq. (1.11) is that if the impedance of the slab material is matched to that of the background, all of the incident wave energy can be transmitted regardless of the frequency, thickness and the refractive index of the slab material. Such a condition is called an impedance matching condition and is a key condition.
in the design of acoustic devices. In this dissertation, we will show explicitly the ways to satisfy the condition in the design of our acoustic meta-surfaces.

1.6 Arrangement of Dissertation and Overview

The concept of metamaterials, originally referred to as negative refractive index materials (NRI) for some decades, was introduced in the previous section. A myriad of astonishing phenomena have occurred, such as cloaking and super-resolution imaging, owing to these materials possessing anomalous effective material parameters that do not exist in nature. The achievements of exotic physical properties via metamaterials does not violate the laws of physics, yet these new properties are increasing our ability to broaden our physical perception and mathematical intuition. The main purpose of this dissertation is to achieve and characterize new acoustic wave mechanisms which interact with the proposed metamaterial design, and this will help in obtaining valuable applications that pave the way in guiding experimental fabrications and designing acoustic devices. A brief history of metamaterials and meta-surface and pioneer works in the field, with the basic concepts and background knowledge, are presented in the current chapter.

Chapter 2 presents the theory for impedance matching of the meta-surfaces to that of the background fluid over a wide range of frequency responses, whose intrinsic impedance is very different from the one of the background medium, in order to realize the total energy transmission phenomenon. Our impedance matching approach, compared to the resonance-based technique, has the attractive feature of being characterized by a broadband frequency response. Two types of impedance-matched acoustic meta-surfaces are proposed, both of which
are embedded in a different fluid medium. Their building blocks are tailored by the coiled-up spaces and the layered medium structures immersed in water and air, respectively. The impedance-matching condition for these meta-surfaces are theoretically derived, and numerically verified. Remarkably, the impedance-matching conditions for acoustic meta-surfaces in water are fulfilled by the proper combination of the coiling-up space geometric size and its filling material, whereas for another design of an acoustic meta-surface in air the impedance-matching condition is fulfilled by optimizing the thicknesses and the arrangement of the two noble gases.

Chapter 3 demonstrates various applications of both designed impedance-matched acoustic meta-surfaces, that were developed in Chapter 2, including focusing, collimation, and the redirection of acoustic wave energy. Their performances are simulated by finite-element analysis, which indicates the high transmission property of the meta-surfaces. We further numerically analyzed samples with real material parameters for both acoustic meta-surfaces, which verified our findings and may guide experimental realizations of the designs.

Chapter 4 reveals a new mechanism to slow down sound waves, providing a strong sound dispersion at the resonance frequency, and hence, the sound energy is strongly localized in the desired location without leakage. This new approach of slowing down sound waves can tackle the issue of sound-wave leakage that is detrimental to the performance of the sound trapping devices. The building block of the metamaterial is N Helmholtz resonators coupled in an individual unit cell loaded to the air waveguide. Based on the hidden source of volume principle and Newton’s second law, the analytical formula and EMT for the designed resonant
metamaterial structure have been derived, and verified by finite element simulations. Our new method of enhancing a slow sound resonance with extremely strong sound dispersion is able to generate and tune stop-bands by manipulating the number and the size of the resonators in the unit cell. Notably, the capability of manipulating the band gaps by the designed meta-structure may offer more degrees of freedom in the band gap engineering.

Chapter 5 realizes the promising applications of EMT for coupled resonators. Since the structure of the coupled resonators causes a negative effective bulk modulus at the resonance frequencies and induces flat bands, i.e. the group velocity goes to zero and slow sound phenomenon with strong sound dispersion is observed, hence the incoming waves can be extremely confined inside the resonators, rather than being allowed to leak out. Based on this finding, we design an acoustic trapping device that has different numbers of coupled Helmholtz resonators with a gradient size of the cavities of the resonators in different unit cells to achieve a perfect trapping effect. By conducting numerical simulations for the designed device, the sound trapping functionality is perfectly demonstrated by the obviously enhanced pressure intensity in the desired unit cell at the resonance frequencies that are predicted from theory, numerical analysis, and EMT, derived in Chapter 4. One aim of trapping is to obtain the perfect absorption of sound wave energy over a broad range of frequencies from the slow sound mechanism. High and broadband sound absorption is manifested by considering the thermo-viscous loss effect in the designed acoustic trapping device.

Chapter 6 investigates numerical simulations computed by the finite-element method that validates our theoretical formulas and EMTs that we derived. In the finite element simulation,
the mesh density has a high impact on the accuracy of the numerical simulation. Therefore this chapter presents the systematic study of different mesh sizes that can capture all the physical effects inside the non-resonant and resonant type metamaterials that are developed in this dissertation. Aspects of the numerical simulation, including the physics model, the geometrical structure, the material setting, the mesh refinement and the solver, amongst others, are all demonstrated in this chapter.

Chapter 7 gives concluding remarks of the dissertation and discusses current studies and potential future research.
Chapter 2

Acoustic Meta-Surface

2.1 Acoustic Meta-Surface in Water

2.1.1 Coiling-up Spaces Structure

The coiling-up space has become a popular candidate in the design of acoustic metamaterials/meta-surfaces. The first theoretical proposal of coiling-up space structure was presented by Liang and Li, in which high refractive index, double negativity metamaterial, and a density near zero metamaterial are realized. In order to model wave propagation in a coiling-up space, various homogenization schemes have been developed. For example, an S-parameter retrieval method was used to homogenize a two-dimensional (2D) acoustic metamaterial with coiling-up spaces as its building blocks [96]. Effective medium parameters were also extracted from the transmission and reflection coefficients in a theoretical proposal [72] and an experimental work [97] about 1D wave propagation through coiling-up spaces. All these schemes rely on prior knowledge of the transmission and reflection coefficients of the heterogeneous medium. In other words, under certain circumstances, we obtain the effective parameters by substituting the numerically calculated scattering information of the heterogeneous slab into the formulas of effective impedance and effective refraction that are obtained from inverting the transmission and reflection coefficients of the homogeneous slab. However, these approaches are numerical and experimental studies, which cannot give a prediction of the mechanism of wave propagation in the heterogeneous medium. Thus, if we can develop an accurate EMT of
coiling-up spaces structure, we can predict the structure that satisfies the *impedance-matching* condition. Based on this intention, we consider a 2D coiling-up spaces structure embedded in water and derive its transmission and reflection coefficients from coupled-mode theory. We compare the expressions of transmission and reflection coefficients with those obtained from the transfer matrix method and derive the formulae of effective impedance and effective refractive index. It is shown that if we choose the proper material to fill up the coiling-up space structure, then the effective impedance of the acoustic meta-surface is matched to that of water background.

### 2.1.2 Unit Cell and the Microstructure

A schematic of the coiling-up spaces used as building blocks for the acoustic meta-surface in water is presented in Fig. 2.1(a). It is a steel slab perforated with periodic curled slits immersed in water. The geometric parameters $h, a$ and $b$ represent the thickness of the plate, the width of the slits and the length of one horizontal segment of the curled slits, respectively. $d$ represents the spatial periodicity of the structure, which is the distance between the adjacent slits. The total length of the curled slit with six times folded up satisfy $h_i = h + 6b$. Each slit is filled with a designed material that has mass density, $\rho_s$, and wave velocity, $c_s$, which are determined by $\rho_s = f \rho_0$ and $c_s = c_0$, respectively. $f (= a/d)$ is the ratio of the width of the slit to the periodicity. Here, $\rho_0$ and $c_0$ denote the mass density and wave velocity of water.
Figure 2.1 (a) Schematic of the building block of the meta-surface. The gray area indicates the rigid slab. (b) The transmission spectrum of a plane wave incident on the meta-surface with the periodically distributed building block structure illustrated in Fig. 2.1(a). The red dashed curve and the blue solid curve correspond to the cases where the slits are filled with the designed material and water, respectively. The black dotted curve corresponds to the transmission spectrum of the same plane wave incident on an effective homogenous slab of the designed meta-surface.

2.1.3 Theoretical Analysis

In this section, we will analytically explore the effective medium properties of the coiling-up space from coupled-mode theory [98, 99]. The transmission and reflection coefficients of the structure shown in Fig. 2.1(a) are derived as follows. The expression of the pressure field below and above the slab can be respectively written as
\[ p_l = \sum_{\tau} \left( \delta_{0,\tau} e^{ia_{\tau,z}} + r_{\tau} e^{-ia_{\tau,z}} \right) e^{iG_{\tau,x}} \quad (2.1) \]

and

\[ p_{ll} = \sum_{\tau} t_{\tau} e^{ia_{\tau,(z-h)}} e^{iG_{\tau,x}} \quad , \quad (2.2) \]

where \( \delta_{0,\tau} \) is the Kronecker delta, and \( G_{\tau} = 2\pi \tau / d \) and \( \alpha_{\tau} = \sqrt{k_0^2 - G_{\tau}^2} \) are the momentum of the \( \tau \)th diffraction order along the \( x \) and \( z \) directions, respectively. \( k_0 \) \((= \omega / c_0)\) is the wavevector of the incident wave in water \((\omega \) is the angular frequency). \( r_{\tau} \) and \( t_{\tau} \) denote the normalized pressure field amplitudes of the \( \tau \)th diffraction order of reflected and transmitted waves, respectively. Inside the slit, the pressure field is expanded in terms of the fundamental waveguide modes, because the width of the slit is much smaller than the wavelength and only the zero-order propagation mode is supported. Thus, the pressure field inside the slit is defined as

\[ p_{ss} = Ae^{ik_s z'} + Be^{-ik_s z'} \quad , \quad (2.3) \]

where \( A \) and \( B \) are the corresponding amplitudes of pressure fields of the upward and downward propagating waves, respectively. \( k_s = \omega / c_s \) is the wave-number inside the curled slits and \( z' \) indicates the distance from the inlet of the slit to a point inside the slit. At the outlet of the slit, \( z' = h_1 \). The continuity condition requires that the pressure fields and normal velocity at the interfaces between the slits and the water \((\text{i.e., at } z = 0 \quad \text{or} \quad z' = 0) \) and \( z = h \quad \text{(or} \quad z' = h_1 \))
be continuous. Combining the continuity conditions with Eqs. (2.1) - (2.3), we can solve for the coefficients \( r_\tau \) and \( t_\tau \), which are expressed as:

\[
r_\tau = \delta_{0, \tau} - f \frac{k_\tau \rho_0}{\rho_\tau \alpha_\tau} \text{sinc}(G_\tau a/2) e^{-i G_\tau md} \left( \frac{\psi_2 (1 - e^{2ik_\tau h}) (\psi_1 - 1) + \psi_1}{(\psi_1 + 1)^2 - e^{2ik_\tau h} (\psi_1 - 1)^2} \right) \tag{2.4}
\]

\[
t_\tau = f \frac{k_\tau \rho_0}{\rho_\tau \alpha_\tau} \text{sinc}(G_\tau a/2) e^{-i G_\tau md} \left( \frac{2\psi_2 e^{ik_\tau h}}{(\psi_1 + 1)^2 - e^{2ik_\tau h} (\psi_1 - 1)^2} \right),
\]

where \( md \) denotes the \( z \)-coordinate at the center of the slit, \( \psi_1 = f (k_\tau \rho_0 / \rho_\tau) \sum_{\tau = -\infty}^{\infty} \text{sinc}^2 (G_\tau a/2) / \alpha_\tau \), and \( \psi_2 = 2 \sum_{\tau = -\infty}^{\infty} \delta_{0, \tau} e^{i G_\tau md} \text{sinc}(G_\tau a/2) \). The detailed derivations for Eq. (2.4) are presented in Appendix A. In the low-frequency or long-wavelength regime \( (\lambda > d) \), only the zero-order diffracted wave \( (\tau = 0) \) is the propagating mode if the width of the slit is much smaller than the periodicity \( (a \ll d) \). All the other higher-order modes are evanescent, because \( \alpha_\tau \) is imaginary. Then, \( \tau = 0 \) is dominant in \( \psi_1 \) and \( \psi_2 \), which yields \( \psi_1 \approx f (\xi / \xi_0) \) and \( \psi_2 \approx 2 \).

Thus, the far-field transmission and reflection coefficients can be expressed as:

\[
T_0 = -\frac{4 f (\xi / \xi_0) e^{ik_\tau h}}{[f (\xi / \xi_0) + 1]^2 - e^{2ik_\tau h} [f (\xi / \xi_0) - 1]^2} \tag{2.5}
\]

\[
R_0 = \frac{1 - f^2 (\xi / \xi_0) \psi_0 (\xi / \xi_0)^2 + e^{2ik_\tau h} [f^2 (\xi / \xi_0) \psi_0 (\xi / \xi_0)^2 - 1]}{[f (\xi / \xi_0) + 1]^2 - e^{2ik_\tau h} [f (\xi / \xi_0) - 1]^2},
\]
On the other hand, the transmission and reflection amplitudes of a plane wave normally incident on a homogenous slab with thickness $h$ were discussed in Chapter 1 and they are written respectively as

$$T(\omega) = \frac{4\left(\frac{\xi}{\xi_0}/\xi_{\text{eff}}\right)e^{ik_{\text{eff}}h}}{\left(\frac{\xi}{\xi_0}/\xi_{\text{eff}} + 1\right)^2 - e^{2ik_{\text{eff}}h}\left(\frac{\xi}{\xi_0}/\xi_{\text{eff}} - 1\right)^2}$$

(2.6)

$$R(\omega) = \frac{\left(1 - \left(\frac{\xi}{\xi_0}/\xi_{\text{eff}}\right)^2\right) + e^{2ik_{\text{eff}}h}\left(\frac{\xi}{\xi_0}/\xi_{\text{eff}}\right)^2 - 1}{\left(\frac{\xi}{\xi_0}/\xi_{\text{eff}} + 1\right)^2 - e^{2ik_{\text{eff}}h}\left(\frac{\xi}{\xi_0}/\xi_{\text{eff}} - 1\right)^2}.$$

Comparing Eq. (2.5) and Eq. (2.6), we notice a certain correspondence between the acoustic properties of the meta-surface and the homogenous slab. If we treat the meta-surface as a homogeneous slab, its effective refractive index and effective impedance is given as

$n_{\text{eff}} = \left(h/\xi_0\right)n_s$ and $\xi_{\text{eff}} = \left(1/f\right)\xi_s$, respectively. Interestingly, the impedance of the material inside the slits is given by $\xi_s = f \xi_0$, which means that the effective impedance exactly matches the impedance of the water background, $\xi_0$. Total transmission is therefore expected even though the width of the slit is narrow.

In Fig. 2.1(b), we plot the transmission coefficients of a meta-surface with $a = 0.19d$, $h = 6d$, $b = 0.65d$ chosen as the geometric parameters (so that the total length of the slit, $h$, is $9.9d$).

Since this design does not rely on resonance, nearly total transmission is observed over the frequency range of $0.01c_0/d \text{ to } 0.142c_0/d$. The transmission spectrum is compared with that of a homogeneous slab whose effective refractive index is $1.65n_0$ (where $n_0$ is the refractive index of
the water background) and whose effective impedance equals \( \xi_0 \). Despite some small deviations, which may arise from higher-order diffracted waves, the transmission spectra in general agree with each other. These results verify the effective medium model we used and indicate that the impedance of the meta-surface is indeed matched to that of the water background. In comparison, we also plot in Fig. 2.1(b) the transmission spectrum of the meta-surface when the filling material in the slit is water. Except for a few peaks, which are attributed to the FP effect, the transmission is significantly lower than the transmission of the other two cases due to the huge impedance mismatch whose effective impedance is equal to \( 7.8421 \times 10^6 \text{ Pa s m}^{-1} \). Here, we would like to point out that the effective medium of the structure shown in Fig. 2.1(a) is actually anisotropic. The rigid walls of the slits prevent the wave from propagating along the horizontal direction and the effective mass density along the horizontal direction is infinity. However, using an isotropic effective medium description would not affect the results here because we only consider the normal incidence and the transmission and reflection coefficients of a slab with anisotropic effective medium parameters by changing \( \xi_{\text{eff}} \) and \( k_{\text{eff}} \) into \( \xi_{\text{eff}, z} \) and \( k_{\text{eff}, z} \), respectively. The transmission spectrum of two slabs with isotropic material and anisotropic material for the same normally incident wave shown in Fig. 2.2, which shows excellent agreement between the two cases.
Figure 2.2 The transmission of a normally incident plane wave onto two slabs with isotropic and anisotropic effective medium parameters. Black color indicates the effective transmission coefficients of the slab with anisotropic effective medium parameters. Orange color represents the effective transmission coefficients of the slab with isotropic effective medium parameters, which is identical to the homogeneous anisotropic slab. The symbols are obtained from theory and the curves are calculated from full-wave simulations.

2.1.4 Validity and Limitations of EMT

In this section, we will show how the geometric parameters affect the accuracy of the EMT. To verify our EMT, we calculate the transmission spectrum with finite-element simulations for different cases. In Fig.2.3, we demonstrate the influence of the filling ratio of the slit $f$ on the accuracy of the EMT, by changing the width of the slit and fixing the other parameters. As the filling ratio decreases, the deviations from the EMT result increase. This influence leads to inaccurate predictions because in the theoretical derivation of an effective medium we consider only the zeroth-order diffracted wave. These deviations indeed occur due to higher-order
diffracted waves. The diffraction occurs when the acoustic wave passes through the slit and some sound waves are diffracted by the rigid wall.

![Transmission Spectrum](image)

Figure 2.3 The transmission spectrum of the normally incident plane wave on the meta-surface computed with fixed $h = 6d$, $b = 0.65d$, and various sizes of the width of the slit $a$.

To verify that the discrepancy between the numerical simulation and the EMT shown in Fig. 2.3 is attributed to the high diffraction orders which were ignored in deriving the EMT, we include more diffraction orders into the theory. In particular, we consider two cases, i.e., $a = 0.05d$ and $a = 0.19d$ to study the dependence of the width of the slit on the diffraction order. We compute the value of $t_0$ by using Eq. (2.4) when more diffraction orders are taken into account at $0.1c_0/d$.

The results are shown in Fig. 2.4, exhibiting that $t_0$ indeed changes as the order of diffraction $\chi$ increases. It implies that the $t_0$ we used in deriving Eq. (2.5) is not accurate. To have a more
accurate $t_0$, we need to include more higher-order diffracted waves. Here we truncate $\chi$ when $|t_{0,z+1} - t_{0,z}| < 10^{-3} (|\tau| < \chi)$. For the cases $a = 0.05d$ and $a = 0.19d$, we truncate $\chi$ at 13 and 9, respectively. It is noticed that the $t_0$ exhibits an “oscillatory” behavior as $\chi$ increases, which is caused by the oscillatory behavior of the sinc function as shown in Fig. 2.5. Based on these results, we conclude that the narrower the width, higher-order modes of diffraction are needed to get accurate $t_0$.

Figure 2.4 The value of $t_0$ when the high-order of modes diffraction are taken into account at $0.1d/\lambda_0$ with $h = 6d$, $b = 0.65d$, where the width of the slit (a) $a = 0.05d$ and (b) $a = 0.19d$. The insets show the enlarged scales.
Figure 2.5 The sinc functions when the width of the slit takes on the values (a) $a = 0.05d$ and (b) $a = 0.19d$. The blue dashed lines represent the zero values for the two axes.

As discussed above, for $a = 0.19d$ we keep nine diffraction orders ($|\tau| \leq 9$) in the calculation of the transmission and reflection coefficients from Eq. (2.4). The transmission spectrum results are plotted in Fig. 2.6 in black rhomboid. Good agreements between the numerical simulations and the theoretical predictions are seen. As mentioned earlier, more orders of the diffracted waves must be taken into account if the slit is becoming narrower. For the case of $a = 0.05d$, we need to consider $|\tau| \leq 13$ diffraction orders in the theoretical modeling. The results are plotted in Fig. 2.6 as the red squares, which again agree well with the numerical simulations shown by the red solid curve. We would like to point out that the analytical results are very close to the numerical results in the long-wavelength (low-frequency) region, whereas there is a small
discrepancy between them at short-wavelengths because the order of diffraction also depends on the frequency.

![Transmission graph](image)

Figure 2.6 The transmission coefficient of the plane wave normally incident on the meta-surface computed with fixed $h = 6d$, $b = 0.65d$, where the width of the slit $a = 0.05d$ in red color and $a = 0.19d$ in black color. The symbols are obtained from theory and curves are calculated by COMSOL.

In all the cases mentioned in Fig. 2.3, the transmission coefficients are approximately one over the entire frequency regime, implying that the impedance of coiling-up space is indeed matched to that of water. In the following, we set the width of the slit equal to $0.19d$, which would benefit the designs with real materials. We also computed the transmission spectrum of a plane wave normally incident onto the meta-surface with (I) different sizes of the horizontal segment $b$ and (II) various values of the plate thickness $h$ in Figs. 2.7 and 2.8, respectively. Both figures show
that the majority of the incident wave energy is transmitted. As the total length $h$, increases in both cases, the total propagation time ("acoustic path") increases which effectively leads to low sound velocity or high refraction index. Figures 2.7 and 2.8 indicate the EMT is valid and the matched-impedance gives rise to almost complete transmission. The slight deviation of the simulated results to that of the EMT prediction is again attributed to the insufficient diffraction orders that are taken in EMT.

Figure 2.7 The transmission spectrum of a normally incident plane wave onto the meta-surface was obtained from COMSOL simulations with fixed $a = 0.19d$ and $h = 6d$, and different sizes of the horizontal segment of the slit $b$. The inset shows the enlarged scale.
Figure 2.8 The transmission spectrum of a plane wave normally incident onto the meta-surface was obtained from COMSOL simulations with fixed $a = 0.19d$ and $b = 0.65d$, and various sizes of the thickness of the plate $h$. The inset shows the enlarged scale.

### 2.2 Acoustic Meta-Surface in Air

#### 2.2.1 Layered Media Structure

Properly choosing the filling material inside the slits such that the effective impedance matches the impedance of the background allows the majority of the incident wave energy to be transmitted. However, it would be very difficult to find a proper filling material that could result in an impedance-matched acoustic meta-surface in air because the impedance of air is very low. In what follows, we will introduce another design of acoustic meta-surface to achieve impedance-matching condition in air. Here, instead of making the characteristic impedance of the meta-
surface matched to the air, we make the input impedance of the meta-surface matched to the air. The meta-surface is tailored by layered media as the building blocks. The input impedance of a layer of a medium can be determined from the impedance at the back surface of the layer.

Wave propagation in multilayered media is a very important problem due to its wide applications in optics, medicine, and applied geophysics [100]. The scaling factor in layered media problem is the key parameter to control all the effects in the layers, which is the ratio between the thickness of the layer and the wavelength. In the low-frequency limit, when the thickness of the individual layers is much smaller than the wavelength, Backus averaging (Backus, 1962) can solve for the effective medium parameters for the layered medium. In the high-frequency limit, when the thickness of the layers is much greater than the wavelength, we can apply ray theory to derive transmission and reflection responses.

2.2.2 Material and Geometric Parameters

The building block of the acoustic meta-surface in air is illustrated in Fig.2.9, which is again slits separated by rigid walls. Each slit forms a layered medium, comprising layers of two noble gases, argon (Arg) and xenon (Xen) [101]. The width of the slit is $a = 0.9d$, and the thickness of the plate is $h = 6d$. The material parameters of these gases are as follows: $c_{\text{Arg}} = 323\, m/s$, $c_{\text{Xen}} = 169\, m/s$, $\rho_{\text{Arg}} = 1.78\, kg/m^3$, and $\rho_{\text{Xen}} = 5.89\, kg/m^3$. In each slit, the sum of the thicknesses of the two gases should be equal to the thickness of the acoustic meta-surface, i.e., $h = h_{\text{Arg}} + h_{\text{Xen}}$. Its effective refractive index follows the relation $n_{\text{eff}}h = n_{\text{Arg}}h_{\text{Arg}} + n_{\text{Xen}}h_{\text{Xen}}$, which is the phase accumulation in each slit, where $n_{\text{Arg}}$ and $n_{\text{Xen}}$ are the refractive indices of Arg and Xen,
respectively. We alter the thickness and sequence of the Arg and Xen layers to achieve the required effective refractive index profile, with the impedance matched to the impedance of the air background. This task indeed can be accomplished by using a transfer matrix method.

Figure 2.9 A schematic of acoustic meta-surface with $q$ layers of two gases inside each slit with $a = 0.9d$ and $h = 6d$. The labels show the materials that were used in the system.

### 2.2.3 Theoretical Analysis

The acoustic impedance of the two gases Arg and Xen ($\xi_{\text{Arg}} = 576.2 \text{ Pa s m}^{-1}$, and $\xi_{\text{Xen}} = 996.1 \text{ Pa s m}^{-1}$) inside the slit have a huge impedance mismatch with the impedance of air. While it is difficult to match the characteristic impedance with that of air, we strategically make the input impedance identical to air by tuning the thickness and sequence of two gases. The input impedance of a layer is the impedance at the interface between the incident domain and layer medium. If the input impedance is identical to the impedance of the background medium, then total transmission can be obtained, i.e., the impedance-matching condition can be achieved.
Before we derive the input impedance for \( q \) layers, it is important to derive the input impedance for a single layer. Consider a normally plane wave incident upon the boundary \( z = h \) as shown in Fig.2.10 (a), the reflection coefficient, as derived in Chapter 1, is written as

\[
R = \frac{\xi_{in}^{(2)} - \xi_3}{\xi_{in}^{(2)} + \xi_3} \quad (2.7)
\]

Here, \( \xi_{in}^{(2)} \) is the input impedance at the boundary \( z = h \), while \( \xi_3 \) \( (\rho_3 c_3) \) is the impedance of medium 3. The pressure field and normal velocity inside the layer can be expressed as

\[
p_2 = (Ae^{-ik_z z} + Be^{ik_z z})e^{ik_z x}, \quad v_2 = \frac{1}{i\omega\rho_2} \frac{\partial p_2}{\partial z} \quad (2.8)
\]

where \( A \) and \( B \) are the corresponding amplitudes of pressure fields of the upward and downward propagating waves, while \( k_z \) \( (= \omega/c_2) \) satisfies the following relation \( k_z^2 = k_{c,2}^2 + k_{c,2}^2 \).

On the interface between the layer and the host medium we consider the boundary conditions, which are continuities of pressure field and normal velocity, in order to determine the value of \( \xi_{in}^{(2)} \). The boundary conditions are written as

\[
\xi_1 \mid_{z=0} = -\left(\frac{p_2}{v_2}\right)_{z=0}
\]

\[
\xi_{in}^{(2)} \mid_{z=h} = -\left(\frac{p_2}{v_2}\right)_{z=h}, \quad (2.9)
\]

By applying the boundary conditions at the interfaces \( z = 0 \) and \( z = h \), we obtain the formula of the input impedance:
\[
\xi^{(2)}_{in} = \xi_2 \left[ \frac{\xi^{(1)}_{in} - i \xi_2 \tan(k_2 h)}{\xi_2 - i \xi^{(1)}_{in} \tan(k_2 h)} \right], \quad \xi^{(1)}_{in} = \rho_1 c_1 \tag{2.10}
\]

We can extend the above analysis to derive the input impedance of a medium with \( q \) layers system as shown in Fig.2.10 (b), which satisfies the following relation:

\[
\xi^{(q)}_{in} = \frac{\xi^{(q-1)}_{in} - i \xi_q \tan(k_q h_q)}{\xi_q - i \xi^{(q-1)}_{in} \tan(k_q h_q)} \xi_q, \tag{2.11}
\]

where \( \xi^{(q-1)}_{in} \) is the input impedance of the same layered medium without the \( q \)th layer. \( h_q, \xi_q \), and \( k_q \) respectively correspond to the thickness, impedance and wave vector of the \( q \)th layer.

The reflection coefficient of such a layered medium can be determined by

\[
R = \frac{\xi^{(q)}_{in} - \xi_0}{\xi^{(q)}_{in} + \xi_0}, \quad \text{where} \quad \xi_0 \quad \text{is the impedance of the host medium. If the input impedance,} \quad \xi^{(q)}_{in}, \quad \text{is identical to} \quad \xi_0, \quad \text{the impedance-matching condition is satisfied and the reflection coefficient equals zero. Eq. (2.11) implies that the input impedance is a function of the thickness and the arrangement of the layers, meaning that altering the sequence of the layers would result in different input impedances. In the experiment realization, very thin polyethylene films (which are sufficiently thin to be considered transparent to sound waves) can be used to separate and confine the two gases inside the meta-surface in an air environment, as had been adopted in the previous experimental work [34].}
2.2.4 Validity and Limitations of the Theory

We used the concept of input impedance, which depends on the numbers, the thickness, the sequence and the material parameters of the layers medium [102]. It can be seen from Eq. (2.11) that if the sequence of the layers changes, the input impedance will also change. In this sense, the “effective” impedance can be treated as that of the background because there is no reflection and a similar strategy has been adopted by Li et al. [72]. Thus, changing the sequence of the two gases will change the input impedance because the reflection will not be zero. It should be mentioned that the traditional quasi-static effective medium for a layered medium does not apply here, nevertheless, one can always retrieve the effective medium parameters from the transmission and reflection coefficients. This is because the effective medium condition requires the thickness of each layer to be very small and the number of layers to be very large so that one
can apply the law of large numbers to obtain the effective medium parameters [103]. For the case presented in the next chapter (in Fig.3.4), for instance, the wavelengths inside Arg and Xen are roughly $6.5d$ and $3.4d$, which are comparable to the thicknesses of their respective layers in the slits. Thus, we would not expect the traditional quasi-static effective medium work here. However, the fundamental mode assumption is still valid because the width of the slit is very small to support higher-order waveguide modes. At the frequency of $0.147c_0/d$, to support a propagating first-order waveguide mode in Arg (Xen), the width of the slit should be $3.25d$ ($1.7d$), which is much larger than the width of the slit that we have used, i.e., $0.9d$. Thus, the assumption is valid and the input impedance is applicable. But if the frequency becomes higher, the first-order mode would appear firstly in the Xen layer when the frequency is close to $0.278c_0/d$. 
2.3 Summary

To summarize, we have derived the impedance-matching condition for both acoustic meta-surfaces in water and air. For coiling-up space meta-surface, we utilized the couple-mode theory to develop an EMT and further derive the impedance-matching condition. It is found that such a method of impedance matching, compared to the resonance-based technique, has the feature of being broadband in frequency response. The validity of the theory is verified by transmission spectra calculations and we find that the theory is valid and the transmission spectra for microstructures and effective homogeneous slabs agree with each other. In other acoustic meta-surface in air, a transfer matrix method has been used and we found that the value of the input impedance relies on the thickness and the arrangement of the layers and the impedance-matching condition is satisfied if the input impedance is identical to the impedance of air. Therefore, we have to change the thickness and the sequence of the layers to obtain the desired input impedance and refractive index.
Chapter 3

Impedance-Matched Acoustic Lenses and Applications

3.1 Motivation for the Design

The acoustic lens is a device that can control acoustic wave propagation in liquids or solids, such as focusing and imaging. They have attracted interest due to their broad applications in various domains like biomedical imaging and surgery [104-106]. The functionality of an acoustic lens is made possible by engineering its material parameters, which are in general different from those of the ambient medium. For example, phononic crystal-based acoustic lenses have been designed [107] and fabricated [108] to focus incident acoustic waves. Gradient refraction index (GRIN) techniques are used in both optic and acoustic lenses to effectively control the light and sound rays. In 2006, John Pendry succeeded in designing an exotic optic lens with GRIN, which is used in the invisibility cloak [3]. Very recently, acoustic meta-surfaces have been used in the design of acoustic lenses. Space-coiling structures, for instance, have been utilized in the design of gradient acoustic meta-lenses [64, 72, 73, 75, 109, 110]. With gradient acoustic lenses, acoustic radiation patterns, such as focusing [64, 72, 73, 75, 109], tunable transmission [39, 101, 110], reflection [109] and cylindrical-to-plane wave conversion [64] can be manipulated. For an acoustic lens focused on the transmission functionality like focusing, the more energy transmitted the better the performance of the acoustic lens. In order to enhance the transmission, the impedance of the acoustic lens should be equal to or near to that of the background medium and viscous losses must be minimal. In this chapter, we designed two types
of flat acoustic lenses to focus, collimate, and redirect acoustic waves. We utilize the building blocks studied in the previous chapter, i.e., the coiling-up space and multi-layered media, in the design of acoustic lenses in water and air, respectively. These lenses have matched impedance to their surrounding media, and they possess the desired refractive indices to achieve the required functionality.

3.2 Applications of Acoustic Meta-surfaces in Water and Air

3.2.1 Focusing

To achieve the functionality of excellent focusing without any aberration of the focal spot, the refractive index of the flat acoustic lens should satisfy the following hyperbolic refractive index profile [67, 72, 111, 112]:

\[
n_{\text{eff}}(x_i) = \frac{n_0}{h} \left( \sqrt{\eta^2 + x_i^2} - \sqrt{\eta^2 + x_{10}^2} \right) + n_0, \quad i = 0, 1, ..., 10, \quad (3.1)
\]

where \( \eta \) is the focal length, and we let it equal 8\( d \). \( n_0 \) is the refractive index of the ambient medium and \( x_i \) denotes the location of the \( i \)-th slit in the lens. For simplicity but without loss of generality, we fix the total number of slits as 21. The highest value of \( n_{\text{eff}} \) occurs in the center of the lens and the lowest value occurs at the left-most and right-most slits, i.e., \( n_{\text{eff}} (\pm x_{10}) = n_0 \). For the focusing lens in water, we also fix the number of folds of each slit to six and change the length of the horizontal segment, i.e., \( b \), so that the total length, \( h_i (x_i) = h + 6b(x_i) \), is changed accordingly to satisfy the refractive index profile given in Eq. (3.1). The relation between the value
of \( b \) and the location of the slit is plotted in Fig. 3.1 (a) together with a schematic of the designed lens, demonstrating that the maximum length of the slits is reached at the center of the lens.

To examine the performance of the acoustic lens, we conducted numerical simulations of a Gaussian beam normally incident on an acoustic lens from the bottom. We plot in Fig. 3.1 (b) the field patterns when the frequency of the incident wave is chosen as \( 0.141c_0/d \). Clearly seen is a focal spot after the beam transmits though the acoustic lens. The distance from the upper surface of the acoustic lens to the focal spot is \( 7.76d \), which is about 0.97 times the predicted focal length. For comparison, we also plot in Fig. 3.1 (c) the field distribution of the same wave incident on an acoustic lens with the same structure but with water as the filling material in the slit. In addition to the focusing effect, significant reflections are observed because the effective impedance of the acoustic lens is \( (1/f)\xi_0 \) [73], which greatly differs from the impedance of water. The normalized pressure field distribution along the horizontal and vertical directions across the center of the focal spot for both lenses are plotted respectively in Figs. 3.1 (d) and 3.1 (e). Similar patterns in the pressure field are observed for the two lenses. However, the impedance-matched lens exhibits higher intensity in the pressure field than does the other, implying that much more energy is transmitted. This focal spot with high transmission in fact appears over a frequency range of \( 0.107c_0/d - 0.142c_0/d \).
Figure 3.1 (a) The upper panel is a schematic of the designed acoustic lens in water with 21 slits. The size of the horizontal segment as a function of the position of the slit is shown in the lower panel. (b) The distributions of the pressure field of a Gaussian beam at $0.141c_0/d$ frequency incident on the designed impedance-matched acoustic lens. (c) The same as (a) but the filling material inside the slits is water and the impedance does not match that of water. (c) Range and (d) cross-range distributions of the pressure fields shown in Figs. 3.1(b) and 3.1(c), respectively.

The foregoing analysis describes the design of a matched-impedance acoustic lens. In reality, it is difficult to find a material whose mass density and sound velocity satisfy the conditions of $\rho_s = (a/d)\rho_0$ and $c_s = c_0$ simultaneously to fill up the slits. However, as long as the ratio of the impedances of the material inside the slits and water is $1/f$, the impedance-matching condition is satisfied. One can adjust the length of the curled slits to meet the requirements on the effective refractive index. In the following, we describe a real sample of an acoustic lens in water. We
choose isopentane \((C_5H_{12})\) to fill up the slits. The mass density of water and isopentane is 
\[
\rho_{\text{water}} = 1000 \text{ kg/m}^3 \quad \text{and} \quad \rho_{\text{isopentane}} = 616 \text{ kg/m}^3 .
\]
The speed of sound in water and isopentane is 
\[
c_{\text{water}} = 1490 \text{ m/s} \quad \text{and} \quad c_{\text{isopentane}} = 980 \text{ m/s} \quad \text{[113].}
\]
The width of the slit is \(a = 0.4d\). The effective impedance of the acoustic lens is thus \(1.5 \times 10^6 \text{ Pa s m}^-1\), which is very close to the impedance of water \((1.46 \times 10^6 \text{ Pa s m}^-1\)). The speed of sound in isopentane is very different from the speed of sound in water. To satisfy the hyperbolic refractive index profile given by Eq. (3.1), we redesigned the coiling-up space by changing the length of the horizontal segment of each slit. In Fig. 3.2(a), we plot the pressure amplitude of a Gaussian beam at the frequency \(0.154c_0/d\) incident on this acoustic lens. The focusing effect is clearly observed. The pressure field distribution along the incident direction across the center of the focal spot is shown in Fig. 3.2 (b), where the maximum value of the pressure field occurs at \(7.24d\), close to the predicted position of the focal point \((\eta = 8d\)).
It is worth mentioning that we consider the ideal fluids in our model but in reality all fluids are viscous. Here, the viscosity of isopentane is around $0.2 \text{ cP}$, which is much smaller than that of water (around $1 \text{ cP}$), indicating that using isopentane helps in reducing the viscous loss compared to water. We can estimate the thickness of the viscous layer by $\ell = \sqrt{\eta / \rho \omega}$. Therefore, to make our result valid, the width of the slit must be much larger than $2\ell$. In typical ultrasound experiments in water with frequency around $0.1 \text{ MHz}$, the thickness of the viscous layer is negligible compared to the width of the slit [114]. The wavelength in water is around $15 \text{ mm}$ and the width of the slit for the example presented in Fig. 3.2 is around $1 \text{ mm}$. In this case, the thickness of the viscous layer $\ell$ is about $0.07 \text{ mm}$. Thus, the performance of the lens should not be seriously affected. To verify this, the transmission spectrum of two acoustic
metasurfaces in water, illustrated in Fig. 2.1(a), where the filling materials in the slit are isopentane with and without considering the viscosity effect for the same normally incident wave, are computed and shown in Fig. 3.3. The results show that the high transmission energy is obtained for the two cases (above 92%), thus the viscous loss effect will not have a significant impact on the performance of the metasurface.

Figure 3.3 The transmission of a normally incident plane wave onto two acoustic metasurfaces with slit is filled by isopentane with and without considering the viscosity effect. Blue solid curve indicates the transmission coefficient with considering the viscosity effect. Green solid curve indicates the transmission coefficient without considering the viscosity effect. The inset shows the enlarged scale.
For acoustic focusing lens in air, we used the multi-layered structure as discussed previously in chapter 2. The refractive index profile also satisfies Eq. (3.1) with a focal point chosen at $8d$, which imposes constraints on the respective thicknesses of the Arg and Xen layers. We have to optimize the arrangement of these two gases to achieve the best input impedance. The inset of Fig. 3.4 (a) shows an optimized structure, where the width of the slit is $a = 0.9d$ and the thickness of the plate is $h = 6d$. The calculated input impedances and the reflection coefficients for an incident wave at the frequency $0.147c_0/d$ in each slit are plotted in Fig. 3.4 (a) with red and blue dots, respectively. The results indicate that the impedance is indeed matched to that of the background. A successful acoustic focusing effect is achieved with the optimized design of the acoustic lens. The distributions of the amplitude of the pressure field of a Gaussian beam at the frequency $0.147c_0/d$ incident normally from the bottom of the lens are shown in Fig.3.4 (b). A focal point is observed in the transmission domain and the reflection is weak. The location of the focal spot agrees well with the prediction.
Figure 3.4 (a) The input impedance and the reflection coefficient for each slit at a chosen frequency of $0.147 c_0/d$. A sketch of the optimized acoustic lens in air is shown in the inset in which different colors represent different gases. (b) The pressure-field distribution of a Gaussian beam at the frequency $0.147 c_0/d$ incident normally from the bottom of the lens.

### 3.2.2 Collimation

Collimation is the inverse process of focusing, which transforms the energy radiated by a point source into a collimated Gaussian beam. The high performance of acoustic collimation can be achieved by utilizing the two matched-impedance acoustic lenses in water and in air designed in the previous section. Figures 3.5 (a) and (b) present the pressure-field distribution of a point source radiating through two real acoustic lenses in water and air, respectively. The point source is located at $8d$ away from the bottom surface of the acoustic lens, with a frequency at $0.154 c_0/d$ (Fig. 3.5(a)) and $0.147 c_0/d$ (Fig. 3.5(b)). Obvious collimation effects are observed for both cases.
3.2.3 Redirection of Beams

In this section we redesign the acoustic meta-surface to redirect the transmitted acoustic waves. According to previous analysis, the impedance matching condition can be satisfied by filling each slit with proper materials for both meta-surfaces. We need to engineer the effective refractive index profile to redirect the incident acoustic wave. It has been shown that an acoustic impedance-matching meta-surface can redirect the transmitted plane waves in air at a given wavelength [101]. We redesigned the structure for other wavelengths. Figures 3.6 (b) and 3.6(d) show such a meta-surface can redirect acoustic beams for a normally incident plane wave with wavelengths $0.7d$, and $0.8d$, respectively.
Figure 3.6 The input impedances and the reflection coefficients for each slit at wavelengths (a) 0.7$d$ and (c) 0.8$d$. A sketch of the optimized acoustic meta-surface in air is shown in the inset, in which different colors represent different gases. The distribution pressure field of a Gaussian beam at incident normally from the bottom of the meta-surface at wavelengths (b) 0.7$d$ and (d) 0.8$d$.

When $\lambda_0 = d$ the wave propagates only in the $x$-direction with wave vectors equals $\pm 2\pi/d$ and refractive angle equals 90° owing to $r_0$ and $t_{\pm 1}$ have large values and $k_{\pm 1}$ equal to zero at this critical point. However, as observed in Fig.3.7 (d) this meta-surface reflects the incident waves even though the impedance is matched to the host medium at $\lambda_0 = 1.3d$. 
Figure 3.7 (a) The input impedances and the reflection coefficients for each slit at wavelengths (a) $d$ and (c) $1.3d$. A sketch of the optimized acoustic meta-surface in air is shown in the inset in which different colors represent different gases. The distribution pressure fields of a Gaussian beam normally incident from the bottom of the meta-surface at wavelengths (b) $d$ and (d) $1.3d$.

In addition, the input impedances and the reflection coefficients for all the cases in each slit are plotted in Figs. 3.6(a,c) and 3.7(a,c) with red and blue dots, respectively, while the insets illustrate the supercell structures for acoustic meta-surfaces in air. The results show that the impedance
matching condition is satisfied by tuning the sequence and the thickness of the layers at
wavelengths $0.7d$, $0.8d$, $d$, and $1.3d$. Details of this work can be found in reference [101].

For acoustic meta-surfaces in water, we kept the filling material the same as before to satisfy the
impedance matching condition. However, our EMT theory works at a finite frequency $[0.01c_0/d$
- $0.142c_0/d]$, i.e., at the wavelength range $[7.0423d, 100d]$, which is beyond the desired
wavelength that we are interested in. Thus, we need to redesign our acoustic meta-surface so
that the normalized wavelength $(\lambda_0/p)$ is within the wavelength range where the acoustic
impedance of the meta-surface matches that of air. The schematic of the acoustic meta-surface
is shown in Fig.3.8 (a). It is a steel slab perforated with periodically repeated supercells. Each
supercell contains 12 curled slits. Due to the length of the horizontal segments $b(x_i)$ gradually
increasing in the supercell we folded the slit twenty times to avoid overlapping slits. The width
of the slit, the period of the slit, and the thickness of the slab are denoted as $a$, $p$, and $h$,
respectively. $d$ represents the spatial periodicity of the supercell and is given by $d = \nu p$, where
$\nu$ is the number of the slits in one supercell. The effective refractive index, $n(x_i) = h_i(x_i)/h$, as
derived in Chapter 2, is changed by varying the total length $h_i(x_i)$ of each slit. Here, $h_i(x_i)$ equal
to $h + 20 b(x_i)$, where $b(x_i)$ is the horizontal segment and is given by $(i-1)\lambda_0/20 \nu$. To
examine the performance of the meta-surfaces, we conducted numerical simulations of a
Gaussian beam normally incident onto an acoustic meta-surface from the bottom with
$a = (0.19/12) d$, $h = 0.5d$, $p = (1/12) d$ chosen as the geometric parameters. In Figs.3.8 (b) and
3.8 (c) we plot the pressure field of a Gaussian beam incident on this acoustic meta-surfaces at
wavelengths $\lambda_0 = 0.7d$ and $\lambda_0 = 0.8d$, respectively. We notice that the incident plane waves can bend in various directions after they pass through the meta-surface. The amplitude of the transmitted wave is similar to that of the incident wave because of the matched impedance. The underlying mechanisms can be explained by utilizing the generalized Snell’s law of refraction [115], which is given by $2\pi/\lambda_0 \left[ \sin \theta_r - \sin \theta_i \right] = d\varphi/dx$ where $\theta_r$ (incidence), and $d\varphi/dx$ is the transverse phase gradient and equal to $2\pi/d$. The refraction angle $\theta_r$ is, therefore, obtained by $\arcsin(\theta_0)$ where $\theta_0 = \lambda_0/d$, i.e., $\theta_r \big|_{\lambda_0=0.7d} = 44.42^\circ$ and $\theta_r \big|_{\lambda_0=0.8d} = 53.13^\circ$.

Figure 3.8 (a) Diagram of supercell structure of coiling-up space acoustic meta-surface in water. The pressure-field distributions for a plane wave normally incident from the bottom of the meta-surface where the filling material inside the slits is the designed material at the incident wavelengths (b) $0.7d$ and (c) $0.8d$. 
Figure 3.9 The pressure-field distributions for a plane wave normally incident from the bottom of the meta-surface where the filling material inside the slits is the designed material at the incident wavelengths (a) $d$ and (b) $1.7d$.

We redesign the supercells by decreasing the number of slits to be 8 in one supercell to achieve better results for $\lambda_0 = d$ and $\lambda_0 = 1.7d$, where the width of the slits $a$ and the period of the slit $p$ equal to $(0.19/8)d$ and $(1/8)d$. As illustrated in Fig. 3.9(a), when $\lambda_0 = d$, the transmitted wave seems to propagate on the upper surface of the meta-surface and the transmission is weak. And in Fig. 3.9(b), the incident wave is almost reflected even though the impedance of the meta-surface is matched to the host medium at $\lambda_0 = 1.7d$. In order to understand the physical mechanism of this phenomena we develop a theoretical model based on coupled-mode theory for the supercell. The pressure field below and above the meta-surface can be respectively written as Eqs.(2.1) and (2.2), while the pressure field inside the slit is defined as Eq.(2.3), where $z'$ at the outlet of the slit is equal to $h_i(x_i)$, $i$ is the number of the slit. By applying the
continuous conditions for the pressure fields and normal velocity at the interfaces (i.e., at \( z = 0 \) (or \( z' = 0 \)) and \( z = h \) (or \( z' = h_1(x) \))), we will obtain the following equations:

\[
1 + r_0 \left( r_1 e^{iG_{11} \beta} + r_{-1} e^{iG_{11} \beta} \right) \text{sinc} \left( G_{11} a/2 \right) = A_i + B_i \quad i = 1, 2, \ldots, 12
\]

\[
t_0 + (t_1 e^{iG_{11} \beta} + t_{-1} e^{iG_{11} \beta}) \text{sinc} \left( G_{11} a/2 \right) = A_i e^{ik_h(x)} + B_i e^{-ik_h(x)} \quad i = 1, 2, \ldots, 12
\]

\[
(1-r_0) \frac{k_0 d}{\rho_0} = \frac{k_0 a}{\rho_x} \sum_{i=1}^{12} (A_i - B_i)
\]

\[
t_0 \frac{k_0 d}{\rho_0} = \frac{k_0 a}{\rho_x} \sum_{i=1}^{12} (A_i e^{ik_h(x)} - B_i e^{-ik_h(x)})
\]

(3.2)

\[
-r_{1\pm} \frac{\alpha_{1\pm} d}{\rho_0} = \frac{k_0 a}{\rho_x} \sum_{i=1}^{12} [A_i - B_i] e^{-iG_{11} \beta} \text{sinc} \left( G_{11} a/2 \right)
\]

\[
-t_{1\pm} \frac{\alpha_{1\pm} d}{\rho_0} = \frac{k_0 a}{\rho_x} \sum_{i=1}^{12} [A_i e^{ik_h(x)} - B_i e^{-ik_h(x)}] e^{-iG_{11} \beta} \text{sinc} \left( G_{11} a/2 \right)
\]

We only consider the first-order diffracted waves in order to understand the behavior of the propagated waves and the other higher-order diffracted waves are negligible. Thus, the reflection and transmission coefficients can be obtained by solving Eq. (3.2). The three coefficients of the reflection and transmission are listed in Table 3.1 for all of the cases. When \( \lambda_0 = 0.7d \) and \( \lambda_0 = 0.8d \), most of the incident waves are transmitted through the meta-surfaces, owing to the magnitude of \( r_0 \) being small and the interference between the transmission coefficients \( t_{+1} \) and \( t_{-1} \) yield to an oblique transmitted wave with refraction angle equal to
arcsin(θ₀). The interference is dominated by t⁺⁺ because |t⁺⁺| ≫ |t⁻⁻| as shown in Table 3.1. At

\[ \lambda_0 = d \], \( r_0 \) and \( t_± \) are large. Because \( k_{z,±} \) is equal to zero, the waves will not propagate along
the \( z \)-direction and we obtain a propagating mode in the \( x \)-direction with \( k_{x,±} = ±2\pi/d \) and
refraction angle 90° as shown in Fig.3.9 (a). For \( \lambda_0 = 1.7d \), the incident wave is almost reflected
because \( r_0 ≈ 1 \) and \( t_± \approx 0 \). In addition, \( t_± \) are large but they correspond to evanescent waves
along \( z \)-direction owing to \( k_{z,±} \) being purely imaginary and the interference between these
evanescent transmitted waves causes the surface-bound mode as shown in Fig.3.9 (b).

<table>
<thead>
<tr>
<th>Table 3.1</th>
<th>The reflection and transmission coefficients of first order diffraction waves</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wave-length</td>
<td>Transmission coefficients</td>
</tr>
<tr>
<td></td>
<td>( t_0 )</td>
</tr>
<tr>
<td>0.7</td>
<td>(-0.528+6.47i) x10⁻¹⁵</td>
</tr>
<tr>
<td>0.8</td>
<td>(-0.698+3.106i) x10⁻¹⁵</td>
</tr>
<tr>
<td>1</td>
<td>(-6.71-4.28i) x10⁻¹⁶</td>
</tr>
<tr>
<td>1.7</td>
<td>(2.63+1.84i) x10⁻¹⁵</td>
</tr>
</tbody>
</table>

To realize the functionality of redirection transmitted waves, we design an acoustic meta-surface
in water, whose slits are filled with isopentane, and compute the pressure field of the plane wave
incident normally from the bottom of the meta-surface, at the wavelengths \( 0.7d, 0.8d, d \), and
Perfect results were obtained and agree well with the theoretical model as shown in Fig. 3.10.

Figure 3.10 The pressure-field distributions for a plane wave normally incident from the bottom of the meta-surface where the filling material inside the slits is isopentane at the incident wavelengths (a) 0.7d, (b) 0.8d, (c) d, and (d) 1.7d.
3.3 Summary

In this chapter, we proposed two types of impedance-matched acoustic meta-surfaces to focus, collimate, and redirect acoustic waves. One is designed from the coiling-up spaces in water, and the other is built from multi-layered media in air. We utilized the conclusion of Chapter 2 to satisfy impedance matching conditions and achieved the desired refractive index profile by changing the length of the coiling-up space and the thickness of the alternating layers in two types of meta-surfaces. Numerical simulations demonstrate the realizations of focusing, collimation and redirection of acoustic waves in air and water respectively with our designed meta-surfaces. These meta-surfaces are experimentally feasible.
Chapter 4

Coupled Helmholtz Resonators Meta-Materials

4.1 Helmholtz Resonators

In acoustic metamaterials, designing and engineering the unit structures enable the realization of anomalous material parameters at resonance frequencies [61]. Many of these structures can be characterized by classical mechanics. For example, Helmholtz resonators (HR), are a widely used element in the design of acoustic metamaterials which can be modeled by a classical mass-spring system: the compressible fluid inside the cavity acts as a spring, while the fluid in the neck acts as the mass.

The German physicist Hermann Ludwig Ferdinand von Helmholtz derived the mathematical formula of the resonance frequencies of Helmholtz resonators in 1860 [116]. His formula of resonance frequency relies on the cross-sectional area of the neck and the volume of the cavity, and it is given by \( c_0 \sqrt{2a/V} \), where \( a \) represents the radius of the neck, \( V \) and \( c_0 \) represent the volume of the cavity and the speed of sound in air, respectively. However, the particles of the fluid produce some motion, which cause additional mass on both sides of the neck, meaning that the short neck becomes longer. In order to accurately predict the resonance frequency, the correction terms at the inner (\( \delta_{\text{inner}} \)) and outer (\( \delta_{\text{outer}} \)) opening neck should be added to the actual length of the neck [117-119]. This is illustrated in Fig. 4.1.
Figure 4.1 Schematic of Helmholtz resonator shows where the correction terms of the neck should be taken into account.

Then, the expression of the resonance frequency of HR was developed by considering the effective length of the neck \( l_{\text{eff}} \) under the assumptions that the particles of the fluid oscillate at the same velocity and in the same phase in the short neck, in which the length of the neck must be shorter than the wavelength. The formula of the resonance frequency becomes

\[
\left( \frac{c_0}{2\pi} \right) \sqrt{\left( \frac{\pi a^2}{l_{\text{eff}} V} \right)} \quad [120].
\]

The length correction of the resonator has been derived based on the radiation of a circular piston model from an infinite baffle. The exterior and the interior end-correction terms of the neck are given by \( 8a/3\pi \). However, the interior end-correction term is valid if the dimensions of the neck are much smaller than the dimensions of the cavity. Ingard [121] found another mathematical formula for the interior end-correction term that relied on a piston radiating into the cavity which can be approximately equal to \( (8a/3\pi)(1-1.25a/R) \). \( R \) is the radius of the cavity, and this term would be valid when \( a/R < 0.4 \).
Recently, at the resonance frequency, Fang et al. [12] realized the resonance-induced negative effective bulk modulus for ultrasound by a waveguide connected to a series of Helmholtz resonators. Due to the frequency dispersion of the local resonances, negative bulk modulus was observed, which causes a flat band in the dispersion relation. Thus, the group velocity goes to zero and a slow sound phenomenon was observed. As a result, the sound waves are trapped inside the resonator at the resonance frequency. This fascinating phenomena can be utilized to realize the rainbow trapping effect. However, the leakage of sound waves in a resonance-based rainbow trapping device prevents the sound wave being trapped at a specific location.

In this work, we propose a new mechanism to slow down the sound wave by coupling N Helmholtz resonators in one unit cell. This approach offers flexibility in engineering the bandgaps induced by the HRs as each HR causes a stop band. We derive an analytic formula for the proposed design from Newton’s second law. In addition, we developed an effective medium model by using the principle of the hidden source of volume. The results show that coupled resonators generate a negative effective bulk modulus near the resonance frequency and induce flat bands that give rise to the confinement of the incoming wave inside the resonators. This phenomenon can effectively eliminate the energy leakage issue and perfectly trap sound energy at a precise location.

### 4.2 Unit Cell and the Microstructure

We start with a single Helmholtz resonator in two dimensions, composed of a cavity and short neck on one side, loaded to a non-closed air waveguide, as illustrated in Fig. 4.2(a). The width and length of the short neck are denoted as $a_1$ and $\chi_1$, the width and length of the cavity are $b_1$
and $h_i$, and the width and length of the waveguide are $w$ and $h$. The geometry parameters utilized for the single HR case are $a_i = 0.5$ [mm], $\chi_1 = 1$ [mm], $h_i = 1.6$ [mm], $b_1 = 5.5$ [mm], and with $h = 4$ [mm] and $w = 7$ [mm]. For the triple coupled HRs case, the three resonators are identical in shape and have the same geometry parameters as those used in the single HR case.

Figure 4.2 Schematic of the building block of (a) single HR and (b) three coupled HRs metamaterial, and the corresponding mechanical mass-spring system. (c) and (d) The transmission coefficient of a plane wave incidence from the left side of the unit cell is given by the theoretical prediction (black solid curve), and the numerical simulation is given by the red
dashed curve. In comparison, the transmission coefficient of the same plane wave incident on an effective homogenous slab is given by the blue dotted curve for the unit cell illustrated in Figs. 4.2(a) and (b), respectively.

4.3 Theoretical Analysis

4.3.1 Analytic Formula Based on Newton’s Second Law

Consider the air in the neck, by employing Newton’s second law, we have the following expression:

\[ \rho_0 \alpha_i \chi_i' \dot{\eta}_i = a_i \left( p - \Delta p_i \right) \]  

(4.3.1)

where \( \rho_0 \) is the mass density of air, \( \chi_i' = \chi_i \alpha \left[ \left( 0.514 \ 0.318 \ln \frac{l_i}{a} \right) a_i \right] \) is the effective length of the neck of the \( i \)th resonator [122], \( p \) is the pressure that is applied to the resonator, and \( \eta_i \) is the displacement of the air. \( \Delta p_i \) is the variation of the pressure inside the resonator, and is given by \(-\kappa_0 a_i \left( \eta_1 / \nu_1 \right)\) where \( \kappa_0 \) is the bulk modulus of air, and \( \nu_1 ( = b_1 h_1 ) \) is the area of the cavity. In what follows, we use the harmonic expression for the pressure and the displacement at angular frequency \( \omega \), which take the following expressions \( p = p e^{i \omega t} \) and \( \eta_1 = \eta_1 e^{i \omega t} \), respectively. Equation (4.3.1) is converted into:

\[ \eta_1 = -\frac{a_i p}{\omega^2 \rho_0 \alpha_i \chi_i' + \alpha_i^2 \kappa_0 / \nu_1} \]  

(4.3.2)
The velocity of a particle \( u_i \) is determined as \( u_i = \eta_i' = i\omega \eta_i \), which yields the normalized specific impedance of the resonator, and can be written in the form

\[
\frac{\xi}{\xi_0 u_i} = \frac{-1}{i\omega \xi_0} \left[ \omega^2 \rho_0 \chi_i' + \frac{\kappa_0 a_i}{v_1} \right]
\]  

(4.3.3)

The transmission coefficient of the system can be calculated from the following relation [123]:

\[
T = \frac{2h\xi}{(2h\xi + a_i)}
\]  

(4.3.4)

The general case of the previous analysis can be demonstrated by increasing the number of resonators in each unit cell attached to the waveguide. The transmission coefficient of N HRs can be calculated by applying the equation of motion for each resonator, which is written in the form

\[
\rho_0 a_i \chi_i' \Delta \eta_i = a_i \left( \Delta p_i - \Delta p_{i-1} \right), \quad i = 2, \ldots, N
\]  

(4.3.5)

The variation of the pressure inside the \( i \)th resonator is expressed as \( \Delta p_i = (\kappa_0/v_i)\left(a_i \eta_i - a_{i-1} \eta_{i-1} \right) \) where \( v_i \) (\( = bh_i \)) is the area of the \( i \)th cavity. Coupling these resonators leads to the following equation:

\[
q \left( \omega, a_j, \chi_j', h_j, b_j \right) \eta_j - \delta_{i,j} \left( pa_{j=1} - \kappa_0 a_{j=1}^2 \right) = 0, \quad j = 1, \ldots, N.
\]  

(4.3.6)

\( q \) represents a tridiagonal stiffness matrix, where the subdiagonal and superdiagonal terms are given by \( q_{i,j-1} = -\kappa_0 a_j a_{j-1}/v_{j-1}, \quad q_{j,j+1} = -\kappa_0 a_j a_{j+1}/v_{j+1} \), and the main diagonal terms are expressed as \( q_{j,j} = -\rho_0 a_j \chi_j' \omega^2 + \kappa_0 a_j^2 \left( v_j^{-1} + v_{j-1}^{-1} \right) \). We consider a 3x3 matrix since the system
is composed of three coupled HRs in a unit cell as shown in Fig. 4.2 (b), and the displacement of air in the neck of the first resonator is given by:

$$\eta_1 = a_1 p \left( \frac{q_{22}q_{32} - q_{22}q_{33}}{q_{11}q_{23}q_{32} + q_{12}q_{21}q_{33} - q_{11}q_{22}q_{33}} \right)$$  \hspace{1cm} (4.3.7)

Then the normalized specific impedance of three coupled resonators is given by:

$$\zeta = \frac{p}{\varepsilon_0 a_1} = \frac{1}{i \omega \varepsilon_0 a_1} \left[ \frac{q_{11}q_{23}q_{32} + q_{12}q_{21}q_{33} - q_{11}q_{22}q_{33}}{q_{23}q_{32} - q_{22}q_{33}} \right]$$  \hspace{1cm} (4.3.8)

The detailed derivations for $\eta_1$ and $\zeta$ are presented in Appendix B. Substituting $\zeta$ into Eq.(4.3.4), we obtain the transmission coefficient of the unit. To verify the theory and give a clear physical picture, a numerical simulation was carried out using COMSOL Multiphysics. Figures 4.2(c) and (d) show the transmission coefficients for two structures with different numbers of resonators, indicating the number of transmission dips is equal to the number of resonators. The red solid curve represents the simulated transmission spectrum of a plane wave normally incident from the left of the waveguide, where the black solid curve is obtained from Eq.(4.3.4). Good agreement between the theoretical prediction and numerical simulation is seen from Figs.4.2(c) and (d).

**4.3.2 EMT: Derivation and Verification**

In this section, we developed an effective medium theory based on the principle of the hidden source of volume [124, 125]. The hidden source of volume for a single resonator or multiple...
resonators can be obtained by $\Delta \varphi_h = a_i \eta_1$. Therefore, the effective bulk modulus of a single resonator and coupled resonators system can be rewritten as

$$\kappa_{\text{eff}} = \kappa_0 \left( 1 + \frac{\Delta \varphi_h}{\Delta \varphi} \right)$$  \hspace{1cm} (4.3.9)

where $\Delta \varphi$ represents the change in the waveguide volume. Using Eq. (4.3.9), we write $\kappa_{\text{eff}}$ for a single resonator as

$$\kappa_{\text{eff}} = \frac{\kappa_0}{1 + \frac{a_i^2 \kappa_0}{v} \left( \kappa_0 a_i - \omega^2 \rho_0 \chi' \right)}$$  \hspace{1cm} (4.3.10)

where $v (= w \times h)$ denotes the volume of the waveguide, and $\kappa_{\text{eff}}$ for three coupled resonators as

$$\kappa_{\text{eff}} = \frac{\kappa_0}{1 + \frac{a_i^2 \kappa_0}{v} \left( \frac{q_{23}q_{32} - q_{22}q_{33}}{q_{11}q_{23}q_{32} + q_{12}q_{23}q_{33} - q_{11}q_{22}q_{33}} \right)}$$  \hspace{1cm} (4.3.11)

The transmission spectra of slabs with effective moduli given by Eqs. (4.3.10) and (4.3.11) and effective mass density of $\rho_0$, are presented in the dotted blue curves in Figs. 4.2(c) and (d). Good agreements among the theory, numerical simulation and the effective medium predictions are observed. These results verify the effective medium prediction and indicate that the dips in transmission spectra correspond to the coupling effect of the resonators, and hence the sound energy is strongly localized in the resonators at the resonance frequencies where the coupling between each resonator determines the resonance frequency. We also compute the band
structure of a single HR and a triply coupled HRs using COMSOL and plot the results in red dots as illustrated in Figs.4.3 (a) and (b). For comparison, the band structures predicted by the EMT are plotted in solid black curves. Figures 4.3 (c) and (d) show the corresponding effective bulk modulus. Typical resonance behavior is observed and the more the HRs, the greater the number of resonant frequencies. The occurrence of negative modulus is attributed to the instantaneous displacement of the mass in each resonator which flips the phase from in-phase to out-of-phase at the resonance frequencies [12]. To further understand the new mechanism of slowing down sound waves, we plot the eigenfield patterns at points labeled on the band structure at the flat branch in Figs.4.3 (e) and (f). It should be noted from the field distribution at each eigenfrequency on the flat band that the pressure intensity is confined in the unit cell with a high concentration of pressure intensity on one resonator, and owing to the coupling effect of the resonators the pressure intensity spreads out significantly into other resonators. These results provide us with a clear evidence that by adding a resonator to the unit cell, a new flat band is attainable, and hence the group velocity goes to zero as well. Based on this mechanism, it is easy to see that the width and the number of the band gaps with flat branches can be tuned by changing the number and the size of the resonators at desired frequency domains to achieve trapping effects.
Figure 4.3 The verification of the derived EMT. Band structure calculations of (a) single HR and (b) three coupled HRs using COMSOL (red dots), compared with EMT (black dots). (c) and (d) Corresponding effective bulk modulus calculated from Eqs.(4.3.10) and (4.3.11). Eigenfield distribution for points (e) “A” marked in (a), (f) “A”, “B”, and “C” marked in (b). Dark blue and dark red represent the zero and maximum values of the pressure intensity.

4.3.3 Validity and Limitations of EMT

The classic mass-spring model is based on the lumped element method, which requires the dimensions of the structure of the acoustic device to be sub-wavelength. In the lumped element method, the motion of the fluid through the structure is an analogous to mechanical model, where this model contains the following lumped mechanical elements: mass, stiffness, and resistance. In the HRs, sound waves are radiated by the neck’s open end, which generates
radiation resistance and a radiation mass in the system. The fluid in the neck of the HRs generates another mass element, and due to the thermos-viscous losses at the neck walls of the HRs, additional resistance is created. Furthermore, the fluid compression in the cavity generates stiffness [95]. Here, the analysis approach in deriving the theoretical model of the single and coupled HRs is based on the lumped element method, and we assumed that the dimensions of HRs are significantly smaller than the dimensions of the wavelength. Thus only the fundamental waveguide mode propagates inside the cavity, which has uniform pressure distribution. In Fig. 4.3 (f), the wavelengths at the resonance frequencies (4.5 [kHz], 13.5 [kHz], and 20 [kHz]) are about 9.8, 3.3, and 2.2 times the total length of the unit cell, respectively. The pressure distribution inside the cavities at the second and third resonance frequencies is not uniform, indicating the higher-order modes. However, the lumped-element approximation is still valid for the higher-order modes since the length of the structure is smaller than the wavelengths at resonance frequencies. The calculations of band structure and transmission spectra verify our conclusion, as demonstrated in Figs. 4.2(c) and (d) and Figs. 4.3(a) and (b). In order to analyze the influence of the dimensions of the structure on the accuracy of the theoretical model, we calculate the transmission spectrum with finite-element simulations for different cases. In Fig. 4.4, we plot the simulated results of the transmission spectrum of the incident plane wave with various neck widths. The black solid curves correspond to the numerical solutions, while the colored dashed curves correspond to the theoretical prediction. When the width of the neck becomes small, the theoretical prediction is identical to the numerical solution.
Figure 4.4 The transmission spectrum of the incident plane wave computed with fixed $\chi_1 = 1$ [mm], $h_i = 1.6$ [mm], $b_i = 5.5$ [mm], and various sizes of the width of the neck $a_1$. The dots curves are obtained from theory and the black solid curves are calculated by COMSOL.

Similarly to Fig. 4.4, we compute the transmission spectrum of a normally incident plane wave with various sizes of the neck length. Although the lumped approach requires the dimensions of the structure to be smaller than the wavelength, when the length of the neck becomes too small the theoretical results deviate from the numerical results. This is a consequence of the radiation loss of the waveguide, which becomes higher as the neck becomes shorter and this effect is not considered in the theory.
The transmission spectrum of the incident plane wave computed with fixed $a_i = 0.5\, [\text{mm}]$, $h_i = 1.6\, [\text{mm}]$, $b_i = 5.5\, [\text{mm}]$, and various sizes of the length of the neck $\chi_i$. The dots curves are obtained from the theory and the black solid curves are calculated by COMSOL.

We also calculated the transmission spectrum of a plane wave normally incident with (I) different widths of the cavity and (II) various values of the cavity length in Figs. 4.6(a) and (b), respectively. Both figures show that as the length or the width of the cavity become larger the results of the theoretical and numerical models are in good agreement with each other. These results lead to inaccurate predictions which contradict the assumption of the theoretical model, because the smaller dimensions of the structure should give us more accurate results. It is well known that as the size of the cavity increases the resonance frequency becomes lower, i.e. the resonance frequency occurs in the long-wavelength regime. However, we would like to emphasize that the
longest dimension of the HR in all numerical simulations presented herein is small compared to
the wavelengths at resonance frequencies, so that the lumped element approximation that we
assumed in our derivation is still valid.

Figure 4.6 The transmission spectrum computed with fixed \(a_i = 0.5 \text{ [mm]}, \chi_i = 1 \text{ [mm]}\), and (a)
various sizes of the width of the cavity \(b_i\) and (b) various sizes of the length of the cavity \(h_i\). The
dots curves are obtained from the theory and the black solid curves are calculated by COMSOL.
4.4 Summary

The study presented in this chapter reveals that the coupling of HRs’ modes generates dips in the transmission spectrum, and more dips appear when more resonators are included in each unit cell. We also find in the band structure diagram that the flat bands at the resonance frequencies, which correspond to slow sound wave propagation with small group velocities, are induced by the negative value of the effective modulus that can be excited by the coupling of the resonators’ modes. Such a design, supporting and enhancing a slow sound resonance with extremely strong sound dispersion, is able to perfectly trap sound waves at the desired location rather than allow them to leak out.
Chapter 5

Coupled Resonators for Sound Trapping and Absorption

5.1 Motivation for the Design

Slowing down acoustic waves introduces a delay to the sound and allows temporary storage of sound waves in the resonator structure, resulting in strong coupled sound waves and leading to novel sound applications such as super-absorption [126-129] and rainbow trapping [130-132]. To demonstrate perfect acoustic rainbow trapping, strong acoustic dispersion by local resonances is desired. Many structures have been proposed to achieve a rainbow trapping effect, including chirped sonic crystal [133], periodically grooved rigid metawires [134], coiling-up space metamaterials [131, 135], waveguides composed of numerous phonon barriers [136], phoxonic crystal slabs with air holes drilled on solid membranes [137], and an array of gradient grooves perforated on a rigid panel [132]. Meanwhile, considering the energy loss, rainbow trapping with perfect absorption is observed [127].

To achieve perfect absorption, the impedance of the structure should match that of the background medium. However, it is in general difficult to satisfy this condition for airborne sound. Broadband perfect sound absorption has been realized by slow sound phenomena in closed [11, 38, 43, 129, 138-147] and non-closed [126-128, 148-152] sub-wavelength waveguides. If the leakage of the energy and the inherent loss of the device at the resonance frequency are balanced, critical coupling between the system and the environment is achieved [153, 154]. In particular, the problem of achieving perfect absorption in a non-closed
waveguide, with the reflection and transmission taken into account, becomes complicated. This is because the scattering matrix exhibits two different eigenvalues that rely on the values of the transmission and reflection coefficients of the structure, and zero eigenvalues are required to turn the transmission and reflection into perfect absorption at the resonance frequency \[128\]. This is referred to as the critical coupling condition. By a careful design which fulfils the critical coupling condition, perfect sound absorption is demonstrated \[151\]. When the symmetric and antisymmetric modes are critically coupled, implying that the eigenvalues of the scattering matrix vanish, the incident wave is perfectly absorbed within the structure, and this is the underlying mechanism of the critical coupling condition. Experiments show that critical coupling condition can be satisfied by two HRs loaded to a waveguide and perfect absorption is realized at a specific frequency \[128\]. Unfortunately, simultaneous critical coupling of the symmetric and antisymmetric modes is not always attainable. Quasi-perfect absorption is observed by implementing strong-sound dispersion, which occurs when many layers of \(N\) identical HRs are loaded to the waveguide, and the symmetric and antisymmetric modes are approaching in very close resonance frequency \[126\]. Recently, the same structure but with gradient geometry was designed to slow down the sound waves with different frequencies at different locations so that rainbow trapping and perfect absorption were realized \[127\]. Owing to the overlapping of the bandgaps of the neighboring resonators, however, the desired functionality of rainbow trapping at specific locations is not perfectly achieved.

In this chapter, based on the coupled mass-spring model, we propose a design for acoustic metamaterials based on coupled HRs unit cell, which is studied in the previous chapter, to trap
sound waves with different frequencies at different locations and to absorb broadband sound wave energy. Trapping and high absorption of sound wave energy are demonstrated with our designed device.

5.2 Applications of Coupled Acoustic Resonators

5.2.1 Acoustic Trapping

From the previous analysis in Chapter 4, we note that coupled HRs offer more degrees of freedom, such as the number and the size of the resonators, in designing the acoustic trapping devices. Figure 5.1 (a) shows an example of an acoustic metamaterial with different numbers of coupled HRs attached to an air waveguide. In this design, we consider 7 unit cells loaded to the air waveguide where an incident plane wave is coming from the left. To obtain an excellent trapping effect, a gradient size of the cavities of the resonators in different unit cells has been used, whereas the size of the necks are fixed for all the HRs as demonstrated in Fig.5.1 (a). For simplicity but without loss of generality, identical resonators in one building block are utilized. $a_j$, $\chi_j$, $b_j$, and $h_j$ correspond, respectively, to the width and length of the short neck, and the width and length of the cavity in the $j$th unit cell. We optimize the geometric parameters by gradually decreasing the number and the size of the resonators in different unit cells to obtain the trapping effect. The values of the geometric parameters are given in Table 5.1.
### Table 5.1 Geometrical parameters for trapping device.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$a_j$ [mm]</th>
<th>$x_j$ [mm]</th>
<th>$h_j$ [mm]</th>
<th>$b_j$ [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>1</td>
<td>1.66</td>
<td>5.5</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>1</td>
<td>1.54</td>
<td>5.5</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>1</td>
<td>1.42</td>
<td>5.5</td>
</tr>
<tr>
<td>4</td>
<td>0.5</td>
<td>1</td>
<td>1.20</td>
<td>5.5</td>
</tr>
<tr>
<td>5</td>
<td>0.5</td>
<td>1</td>
<td>1.06</td>
<td>5.5</td>
</tr>
<tr>
<td>6</td>
<td>0.5</td>
<td>1</td>
<td>0.94</td>
<td>5.5</td>
</tr>
<tr>
<td>7</td>
<td>0.5</td>
<td>1</td>
<td>0.65</td>
<td>5.5</td>
</tr>
</tbody>
</table>

The transmission spectra for the seven units are calculated separately and plotted collectively in Figs. 5.1(b) (theoretically results) and 5.1(c) (numerical results). The dark area indicates high transmission while bright regions correspond to low transmission. It is worth mentioning that there are multiple low-frequency resonances but they are not suitable for precise trapping. The sound-wave energy will leak out rather than being trapped in the desired unit, due to the couplings between those low-frequency resonance modes. However, for resonances at high frequencies, the Q-factor is high, and the coupling between neighboring modes are weak, leading to perfect sound trapping. The frequencies for sound wave trapping by each unit are chosen and indicated by green dashed ovals.
Figure 5.1 (a) Conceptual design of the trapping device, composed of air waveguide attached with 7 unit cells of coupled HRs with various numbers and sizes of cavities of the resonators. The contour plot of the transmission spectrum is calculated by the (b) theoretical prediction and (c) numerical simulation for each unit illustrated in Fig. 3(a). Green dash ovals indicate the resonance frequency of the perfect sound trapping of each unit.

It should be noted that although the first bandgap of one unit is much wider than the others, it overlaps with the first band gaps of its neighboring units and thus the coupling between them is unavoidable so the wave will not be trapped in the desired unit. Indeed, those first band gaps can be obtained by utilizing a single HR, which is not a good candidate to realize perfect sound trapping [127, 149]. In this study, we provide an optimal solution via coupled HRs, producing stopbands with various widths, many of which have flat band edges and do not interact with
band gaps of their neighboring units. Therefore, the waves are trapped in the desired unit rather than leak out. To examine the performance of our design, we measure the pressure intensity for each unit cell at the resonances marked in Fig. 5.2. The trapping effect is manifested by obviously enhanced pressure intensity in the desired unit cell (1, 2, 3, 4, 5, 6, and 7) at the resonance frequencies (2[kHz], 18.51[kHz], 20.9[kHz], 21.3[kHz], 24.32[kHz], 24.38[kHz], and 24.9[kHz]) that are predicted from theory, numerical analysis, and EMT. Since we do not consider loss here, the sound wave will eventually be reflected back [7, 9, 10], despite the fact that the group velocity at the resonance frequencies is zero.

Figure 5.2 The pressure intensity at the resonances frequency marked in Fig. 3 (2[kHz], 18.51[kHz], 20.9[kHz], 21.3[kHz], 24.32[kHz], 24.38[kHz], and 24.9[kHz]) for units (1, 2, 3, 4, 5, 6, and 7),
respectively.

### 5.2.2 Perfect Sound Absorption

One purpose of trapping is to achieve perfect absorption of energy over a broad frequency range from the slow sound propagation. We redesigned our proposed acoustic metamaterial by increasing the number of unit cells to 8, which is the same structure as the rainbow-trapping device but we just added one more unit cell with a single HR, to achieve better results since more resonators lead to more peaks in the absorption spectrum, where the geometry parameters are given in Table 5.2.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$a_j$ [mm]</th>
<th>$\chi_j$ [mm]</th>
<th>$h_j$ [mm]</th>
<th>$b_j$ [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>1</td>
<td>1.66</td>
<td>5.5</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>1</td>
<td>1.54</td>
<td>5.5</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>1</td>
<td>1.42</td>
<td>5.5</td>
</tr>
<tr>
<td>4</td>
<td>0.5</td>
<td>1</td>
<td>1.30</td>
<td>5.5</td>
</tr>
<tr>
<td>5</td>
<td>0.5</td>
<td>1</td>
<td>1.06</td>
<td>5.5</td>
</tr>
<tr>
<td>6</td>
<td>0.5</td>
<td>1</td>
<td>0.94</td>
<td>5.5</td>
</tr>
<tr>
<td>7</td>
<td>0.5</td>
<td>1</td>
<td>0.82</td>
<td>5.5</td>
</tr>
<tr>
<td>8</td>
<td>0.5</td>
<td>1</td>
<td>0.70</td>
<td>5.5</td>
</tr>
</tbody>
</table>

Here, we introduce thermo-viscous losses into the device. Mathematically, the loss is introduced by adding an imaginary part into the wave number $k = \frac{2\pi}{\lambda_0} - i\tau$, where $\lambda_0$ being the sound wavelength, and $\tau$ is the attenuation coefficient. In our model, the attenuation coefficient is given by $2\pi f / c_0 \times \eta$, where $\eta$ represents the energy loss in air. Here, we set $\eta = 0.01$. This is the convention adopted in the literature [130]. Figure 5.3(b) exhibits broadband and perfect energy absorption of coupled Helmholtz resonators metamaterial calculated by finite-element simulations. Over 90% of the incoming sound energy is absorbed over a broad frequency range
from 14.28 [kHz] to 18.02 [kHz], and perfect absorption is observed over the frequency range from 14.44 [kHz] to 14.55 [kHz] and from 17.62 [kHz] to 17.85 [kHz]. In Fig. 5.3 (d), we also plot the pressure intensity distribution at various wavelengths of 23.3 [mm] and 19.3 [mm] (14.7 [kHz], and 17.77 [kHz]), showing that the sound waves are trapped and absorbed at different locations. For comparison, we calculate the reflection, absorption and transmission spectra of the same structure but only keep one Helmholtz resonator in each unit. The results are plotted in Fig. 5.3(a). High absorption is also observed, but the frequency range is narrower compared to the coupled ones. The sound pressure intensity distributions of different wavelengths are plotted in Fig. 5.3 (c), which shows the sound waves are trapped in many units rather than a single unit.
Figure 5.3 Simulated absorption, transmission, and reflection spectrum with the considered loss for (a) a single resonator and (b) coupled resonators in each unit. (c) and (d) The pressure intensity distribution at different frequency resonances for both designs, respectively.
5.3 Summary

In this chapter, we report a design of sound trapping device based on coupled Helmholtz resonators, loaded to an air waveguide, which can effectively tackle the wave leakage issue. Perfect absorption of energy is achieved over a broad frequency range, based on the proposed slow sound propagation mechanism. Trapping and absorption functionalities are demonstrated by finite-element simulations. Our findings may have applications in acoustic device designs, such as acoustic insulation, acoustic filters, and broadband perfect absorbers.
Chapter 6

Numerical Scheme

6.1 Finite Element Method

This chapter introduces the numerical model stages, used to verify our theoretical predictions. Different numerical methods have been developed to obtain the numerical model of the wave equations, such as the finite difference method, and the finite element method. The finite element method (FEM) is utilized to compute an approximation solution of the exact solution of the wave equation at discrete points over the domain. This numerical approach can accurately represent any complex geometry with dissimilar materials and it has a robust ability to capture all the local physical effects in the system. The main steps of this approach are transforming the original equation into the weak form, followed by a discretization process. After the discretization scheme is employed in the model, the computational domain is divided into subdomains with a finite number of elements. By assembling these elements into a system of algebraic equations, we obtain the approximation solution. In this dissertation, all the numerical studies presented are simulated by COMSOL Multiphysics, which is a commercial finite element method software package [155].

6.2 Numerical Simulation

In fluid and solid materials, acoustic simulation in COMSOL is mediated by a set of physics interfaces that can be commanded by controlling the geometric parameters, materials, physics settings, boundary conditions, mesh, studies, and solvers. The convergence of the finite element
mesh is one of the most important parameters in achieving a high order accurate solution to the problem that we are attempting to solve. Appropriate mesh size can be determined by knowing the wavelength in the given system. In acoustic waves, the wavelength relies on the material parameters such as the mass density and bulk modulus, which means that the wavelength changes within the solution domain accordingly with material parameters. Thus, adaptive mesh refinement, also known as multiple mesh densities, should be applied to solve the problem. Although the finer mesh (small element size) gives us a more accurate solution, the computational time and computer memory required to solve the model will increase. Therefore, we need an efficient meshing that can accurately resolve the smallest wavelength without going overboard with meshing refinement. The discretization error, which arises from the creation of the mesh, needs to be minimized when the mesh is created. The maximum element size should be set as \( \lambda_i/5 \) or smaller than that value, where \( \lambda_i \) is the incident wavelength. However, when acoustic waves propagate in a lossy medium or in a narrower dimensional structure, the waves become attenuated so that the finite element mesh must address the viscous effect. The energy losses occur in the viscous boundary layer near the walls of the structure, thus the meshing has to resolve the viscous boundary layers to capture all the local physical effects accurately. It is well known that the computation becomes expensive and contains many degrees of freedom (elements) in such a system. In order to obtain a suitable mesh setting, a good resolution of the acoustic boundary layer is required. For our two-dimensional system, we chose the element order as quadratic in the discretization and we employed triangular elements. More detail about meshing algorithms for our loss and lossless models will be given in later sessions.
The meshing of a large complex structure requires a lot of memory. As mentioned earlier, when solving any finite element problem, a system of linear equations arises after employing the discretization scheme. Assembling and storing the matrix and the vector of the linear equations require significant memory, where these equations can be solved by utilizing direct or iterative methods. MUMPS is a direct solver in COMSOL, which is based on LU decomposition and is able to solve a large linear system with symmetric positive definite matrices or any general symmetric and asymmetric matrices [156, 157]. The attractive features of MUMPS solver are the capability for parallel factorization and analysis, scaling the matrices, and forward elimination during the factorization process. The disadvantage of this solver is that as the number of degrees of freedom and the matrix density increase, the computational time and computer memory required goes up very rapidly. Of particular note, the numerical errors of solving system of linear equations can be minimized in the solver by controlling the relative tolerance, which is the maximum amount of error that we are allowed in the solution. In the following section, we demonstrate the computational setup of the simulation results of each unit cell structure, which are used as the building blocks for non-resonant and resonant type metamaterials that are developed in this dissertation. The basic aspects of the computation processes including geometric parameters, materials, physics settings, boundary conditions, mesh, and the error and convergence analysis for each structure will be presented.

6.2.1 Numerical Scheme for Non-resonant Type Metamaterial

6.2.1.1 Coiling-up Space Structure

Setting of Geometric and Material Parameters, Physics Settings, and Mesh
(I) Geometric and Material Parameters

Figure 6.1 shows the unit cell structure of 2D acoustic meta-surface, which consists of a periodic steel slab perforated with curled slits immersed in water and the slits are filled with designed material. The geometric and material parameters have the same values as those used in Section 2.1.2. The acoustic plane wave is normally incident from the bottom of the meta-surface along the $y$ direction, and the periodic boundary conditions along the $x$ direction are used due to the periodicity of the unit cell. Plane wave radiation conditions on the top and bottom of the calculation domain are applied to eliminate the reflected waves from the boundary. Since the unit cell is made of steel, hard boundaries are imposed at the internal walls of the curled slit and at the interfaces of the environment medium and the unit structure, as illustrated in Fig. 6.1.

Figure 6.1 (a) Schematic of acoustic meta-surface with periodic subwavelength coiling-up space channels embedded in water. (b) Boundary conditions and materials that are imposed in numerical simulation.
A pressure acoustics model is applied to all parts of the structure to compute the acoustic pressure field. The model is solved for two dependent variables, the pressure \( p \) and the displacement field \( u \), in frequency-domain over the frequency range of \( 0.01c_0/d - 0.142c_0/d \), where the incoming wave has amplitude equal to \( 1[\text{Pa}] \). By evaluating the pressure field on the transmission domain and normalizing by the input pressure field (source), the transmission spectrum can be obtained.

**II) Creating and Adjusting the Mesh**

As a starting point, we may utilize the default setting of the mesh with a normal element size. Then, we can easily modify individual parameters by controlling predefined parameter sets. Under element size setting, we can find five parameters to adjust and modify individual mesh elements, which are

1. **Maximum element size**: the largest size of an element
2. **Minimum element size**: the smallest size of an element
3. **Maximum element growth rate**: control the size difference of two neighboring mesh elements
4. **Curvature factor**: control how big a mesh element along a boundary can be
5. **Resolution of narrow regions**: determine the number of layers of the elements which must be taken into account in narrow domains

Thus, by adjusting these parameters we can build a good mesh that obtain us an accurate solution. In our study, the finer mesh element is selected to resolve the wavelength well. We
start with the default setting of the extremely fine element size. The maximum element size equals 0.3, which is about 23 degrees of freedom per the smallest wavelength (6.9[m]). Since the slit has narrower size and filled with another material that is different from water, we need to adjust the element sizes of the mesh around the slit in order to accurately resolve the wavelength. To do this, we change the element size at the internal boundaries of the slit. The mesh element size at the slit boundaries equals 0.15[m], which is about 48 degrees of freedom per the smallest wavelength (7.2[m]). The complete mesh consists of 830 domain elements and 252 boundary elements. Therefore, the number of degrees of freedom solved for this model equals 1909. The physical memory needed to compute the solution is 1.02[GB], and the virtual memory is 1.22[GB]. The result is plotted in Fig. 6.4 in the black circles, which runs for 5 seconds.

Following that, different mesh sizes have been utilized to study the convergence behavior of the solution. We sweep the number of degrees of freedom from one to five per wavelength (9.396[mm]), which is denser than the default setting of extremely fine element size. As illustrated in Fig. 6.2, we create the mesh solely on the domain that the waves propagate onto it in order to save time and memory. The details describing the mesh data set are listed in Table 6.1, which shows the number of degrees of freedom per wavelength, the number of degrees of freedom solved for the entire model, the domain and boundary elements, the minimum mesh quality, the physical and virtual memory, and the running time for each mesh setting. The running time taken for the whole study is 7169 seconds.
Figure 6.2 Mesh structure of coiling-up space slit with varying number of degrees of freedom from one to five per wavelength (a-e), where the mesh is created solely on propagation regions.

Table 6.1 Details of the mesh elements for acoustic meta-surface (coiling-up spaces) structure

<table>
<thead>
<tr>
<th>Number of degrees of freedom per wavelength</th>
<th>Number of degrees of freedom solved for the model</th>
<th>Domain elements</th>
<th>Boundary elements</th>
<th>Minimum mesh quality</th>
<th>The physical memory[GB]</th>
<th>The virtual memory[GB]</th>
<th>Solution time[s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>92265</td>
<td>44550</td>
<td>3190</td>
<td>0.7179</td>
<td>1.17</td>
<td>1.37</td>
<td>308</td>
</tr>
<tr>
<td>2</td>
<td>216301</td>
<td>105042</td>
<td>6254</td>
<td>0.6065</td>
<td>1.34</td>
<td>1.54</td>
<td>743</td>
</tr>
<tr>
<td>3</td>
<td>341645</td>
<td>166196</td>
<td>9296</td>
<td>0.6895</td>
<td>1.44</td>
<td>1.67</td>
<td>1256</td>
</tr>
<tr>
<td>4</td>
<td>463905</td>
<td>225804</td>
<td>12346</td>
<td>0.4487</td>
<td>1.64</td>
<td>1.88</td>
<td>1776</td>
</tr>
<tr>
<td>5</td>
<td>589449</td>
<td>287054</td>
<td>15394</td>
<td>0.4655</td>
<td>1.87</td>
<td>2.14</td>
<td>2506</td>
</tr>
</tbody>
</table>

The statistical analysis is involved in this study to assess the influence of mesh density on the quality of the mesh. The ratio of the length to width of the elements indicates the mesh quality. The mesh statistics in COMSOL bring up details of the mesh elements, which show the minimum mesh quality. The physics and solvers used in the model have various requirements of the quality
of the mesh. While the mesh setting relies on the model that we attempt to solve and our computational limits, different model requires different mesh quality. Generally, the value of the minimum mesh quality must be greater than 0.1. If the quality is below 0.01, it will be considered as poor and a warning message will appear to avoid very low quality. In Table 6.1, the minimum value of the mesh element quality is shown for each study. Low quality elements in some cases may not lead to a convergence problem, if these elements are positioned in a domain with less importance in the calculation. Further assessment of the mesh quality can be displayed in the mesh quality plot. The colored mesh quality plot helps in determining the location of the low quality elements, and in adjusting the mesh size parameters to better address the problematic region. Figs. 6.3 (a) and (b) depict the mesh quality when we have high (low) minimum mesh quality with one (four) degrees of freedom per wavelength. Where the blue color indicates low mesh quality while the red color corresponds to high mesh quality. As we can see from the figures, the overall quality is high, above 0.1 which is the minimum required to obtain good mesh quality based on the guidelines of finite element modeling that we have discussed above. The mesh quality is almost equal to one in the entire domain.
Figure 6.3 Mesh quality elements plot with (a) one and (b) four degrees of freedom per wavelength. The blue color indicates low mesh quality while the red color corresponds to high mesh quality.

**Simulation Results and Accuracy Tests**

In Fig. 6.4, the transmission spectrum results are plotted with different mesh refinements to demonstrate the influence of the mesh settings on the accuracy of the solution. The results indicate that as the number of degrees of freedom increases per wavelength from one (red solid line) to five (blue dash line), the numerical simulation results converge.
Figure 6.4 The transmission spectra are computed with a default setting of a finer mesh element size (black circle), and with mesh settings with one to five degrees of freedom per wavelength, whose mesh elements are denser than the default setting.

Despite the big difference between the number of degrees of freedom for each case, the results are very close to each other and the discrepancy between results is very small. However, the deviations increase when more higher-order diffracted waves are taken into account, implying that the finer mesh size is needed to get an accurate result. To verify this, we compute the error in the $L_2$-norm at different frequencies. The results show that the error becomes smaller when the element size becomes smaller, i.e., the number of degrees of freedom is increasing, as
depicted in Fig. 6.5. In addition, the figure shows that at the frequencies in which more diffraction orders are taken into account, the numerical error becomes bigger than the other frequencies that have less effect on the diffraction waves. It implies that a greater number of elements in the mesh density is required at these frequencies to address the effect of the high order of diffraction waves on the accuracy of the numerical solution. As demonstrated in Fig. 6.5, when more diffraction orders are exhibited in the system at the frequency $0.1101c_0/d$, the error of the numerical simulation reduces as the mesh size becomes finer. Nevertheless, the numerical error in all the cases mentioned in Fig. 6.5 is less than 1% which is acceptable in FM simulations. Based on these results, we can conclude that the mesh element size should be sufficiently smaller than the wavelength to achieve an accurate solution with less time solving and computer memory required.
Additionally, we compute the convergence order, which is quantified by the convergence of the numerical solutions; we must note that the accuracy of the numerical simulations depend on the density of the mesh. The convergence order $\sigma$ is defined by

$$
\|e\| = o(h^\sigma)
$$

(6.1)

where $e$ is the error, and $h$ is the element size. Therefore, the numerical errors for different refinement levels of the mesh must be determined to obtain the order of the convergence. For unstructured meshes, however, evaluating the mean size of the elements $h$ is difficult. Alternatively, the degrees of freedom (DOF) are utilized to determine the convergence order. The
formula of the convergence order that relies on DOF was derived by Jänicke and Kost in 1999 [158], which is written as

$$\sigma = -2 \left( \frac{\ln(\| e_n \| / \| e_{n+1} \|)}{\ln(\text{DOF}(n)/\text{DOF}(n+1))} \right)$$  \hspace{1cm} (6.2)$$

where $n$ represents the run number. However, in our numerical scheme, adaptive mesh has been employed locally, specifically, at the internal edges of the unit cell. Due to a huge difference between the numbers of degrees of freedom of two consecutive solutions, it is difficult to predict the convergence rate from Eq. (6.2). In this case, we can obtain the convergence order from the ratio of two consecutive errors to their characteristic grid spacing, which is given by

$$\sigma = \left( \frac{\ln(\| e_n \| / \| e_{n+1} \|)}{\ln(h_n / h_{n+1})} \right)$$  \hspace{1cm} (6.3)$$
Table 6.2 The error ($L_2$-norm) and convergence order according to element order at 0.0094$c_0/d$

<table>
<thead>
<tr>
<th>h [m]</th>
<th>Error</th>
<th>Convergence order</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0035308</td>
<td>8.4830830726634E-07</td>
<td>3</td>
</tr>
<tr>
<td>0.0023539</td>
<td>2.43746964834202E-07</td>
<td>3.1</td>
</tr>
<tr>
<td>0.0017654</td>
<td>1.01938131975318E-07</td>
<td>3</td>
</tr>
<tr>
<td>0.0014123</td>
<td>6.32704597664582E-08</td>
<td>2.7</td>
</tr>
<tr>
<td>0.0011769</td>
<td>3.840800943955003E-08</td>
<td>3.2</td>
</tr>
<tr>
<td>0.0010088</td>
<td>2.316978233905121E-08</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.2 shows the error in the $L_2$-norm and the values of the convergence rate for quadratic elements solved at $0.0094c_0/d$ with different grid spacing. We added two more cases of six and seven degrees of freedom per wavelength to obtain an accurate prediction for the convergence rate, which has grid spacing 0.0011769[m] and 0.0010088[m], respectively. As indicated by the table that the convergence order is 3 up to the measurement accuracy, in a good agreement with the theoretical convergence order for 2D test problem with quadratic elements [158, 159].

6.2.1.2 Layered Media Structure

We used layered media as building blocks for acoustic meta-surface in air, which is comprised of layers of two noble gases. We analytically explore how the impedance-matching condition can be achieved from a transfer matrix method. As mentioned in Chapter 2, we compute the input impedance that satisfies the impedance-matching condition for each slit, separately, and for the desired phase difference between neighboring slits at a specific frequency. Basically, we determined the amount of the two gases inside each slit by solving the equation of the phase accumulation and the geometric size required. After determining $h_{Ar}$ and $h_{Xen}$, the for-loop was performed to determine the number of layers of each gas, and their thickness and arrangement.
The iteration number of the loop is unknown, and the loop will stop when the impedance match condition is fulfilled.

Table 6.3 The thicknesses of the gases, the input impedance, and the reflection coefficient for each slit

<table>
<thead>
<tr>
<th>Number of Slit</th>
<th>1st(Arg)</th>
<th>2nd(Xen)</th>
<th>3rd(Arg)</th>
<th>4th(Xen)</th>
<th>( \frac{\xi_{\text{in}}}{\xi_0} )</th>
<th>Reflection Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3.386594</td>
<td>0.179973</td>
<td>2.257729</td>
<td>0.175704</td>
<td>1.0003</td>
<td>1.5E-5</td>
</tr>
<tr>
<td>3</td>
<td>1.446382</td>
<td>1.003861</td>
<td>3.549757</td>
<td></td>
<td>1.0001</td>
<td>9.3 E-5</td>
</tr>
<tr>
<td>4</td>
<td>3.073978</td>
<td>1.303612</td>
<td>1.317419</td>
<td>3.04991</td>
<td>1.0007</td>
<td>3.6E-5</td>
</tr>
<tr>
<td>5</td>
<td>1.550524</td>
<td>2.162069</td>
<td>2.287407</td>
<td></td>
<td>1.0002</td>
<td>1.5E-5</td>
</tr>
<tr>
<td>6</td>
<td>1.408095</td>
<td>2.655356</td>
<td>1.936549</td>
<td></td>
<td>1.0009</td>
<td>4.8E-4</td>
</tr>
<tr>
<td>7</td>
<td>1.305518</td>
<td>3.078724</td>
<td>1.615758</td>
<td></td>
<td>1.0001</td>
<td>6.5E-5</td>
</tr>
<tr>
<td>8</td>
<td>1.804547</td>
<td>1.584421</td>
<td>0.773377</td>
<td>1.837655</td>
<td>1.0004</td>
<td>2.1E-5</td>
</tr>
<tr>
<td>9</td>
<td>1.39454</td>
<td>1.641965</td>
<td>0.929693</td>
<td>2.033801</td>
<td>1.0005</td>
<td>2.7E-5</td>
</tr>
<tr>
<td>10</td>
<td>0.867336</td>
<td>0.472827</td>
<td>1.301004</td>
<td>3.35834</td>
<td>1.0006</td>
<td>3.1E-4</td>
</tr>
<tr>
<td>11</td>
<td>0.846291</td>
<td>0.526707</td>
<td>1.269436</td>
<td>3.357565</td>
<td>1.0003</td>
<td>1.9E-4</td>
</tr>
</tbody>
</table>

The thicknesses and arrangements of the two gases needed to achieve the required effective refractive index profile and the impedance matching condition, for an incident wave at \( 0.147c_0/d \) are presented in Table 6.3. This table also includes the corresponding input impedance and the reflection coefficient, for each slit obtained. These data have been used to demonstrate the acoustic focusing with this optimized design of the acoustic lens.

Similarly to the previous model, the pressure acoustics model is adopted on the whole structure to compute the acoustic pressure field in which two unknown variables, the pressure \( p \) and the displacement field \( u \), need to be solved in the frequency domain. To test the convergence of the solution, we compute the transmission coefficient, which is the normalized pressure at the transmission domain in the far-field. The study performed over the frequency range \( 0.0292c_0/d - 0.147c_0/d \) for the optimized layered media structure, that has matched input impedance to the background, with various setting of refinement mesh. As we have stated above,
the input impedance is a function of the thickness and the arrangement of the layers and the
frequency. Thus when the frequency changes, the input impedance will not be perfectly matched
to the impedance of the air background. The purpose of this computation is to study the influence
of the adaptive mesh on the convergence of the solution in the wide range of frequency to obtain
a clear picture of the accuracy of the solution. For the sake of simplicity, we performed the study
on the second slit of the meta-surface. The error analysis and the convergence of the solution
were tested with different levels of refinement mesh in the next sections.

Setting of Geometric and Material Parameters, Physics Settings, and Mesh

(I) Geometric and Material Parameters

The detail of the slit-layered medium meta-surface design is demonstrated schematically in Fig.6.6. At the boundary above and below the meta-surface, the plane wave radiation boundary
condition is implemented to eliminate the reflected waves from the boundary. Plane waves are
launched on the bottom of the meta-surface to generate source waves with pressure amplitude
equals 1[Pa], where sound hard boundaries are utilized on the side edges of the meta-surface.
The material parameters of the argon, xenon, and air are as follows: the speed of sound in these
materials are $c_{\text{Arg}} = 323[\text{m/s}]$, $c_{\text{Xen}} = 169[\text{m/s}]$, and $c_{\text{Air}} = 343[\text{m/s}]$, while the mass density of
the gases are $\rho_{\text{Arg}} = 1.78[\text{kg/m}^3]$, $\rho_{\text{Xen}} = 5.89[\text{kg/m}^3]$, and $\rho_{\text{Air}} = 1.29[\text{kg/m}^3]$. 
Figure 6.6(a) The second silt on the meta-surface with the optimized layers of two gases, argon (yellow) and xenon (purple) in air background (blue), where the plane wave incidents normally from the bottom of the meta-surface.

(II) Creating and Adjusting the Mesh

Following the same process as shown before, we created a proper mesh structure according to geometry by utilizing tools such as refinement and inflation. The proper mesh helps in obtaining accurate results. Here, adaptive mesh is required at each layer that is filled with different gases in order to resolve the wavelength in each domain and obtain an accurate solution. Practically, we need to specify the wavelength with respect to the filling fluid in each layer. The wavelengths inside air, argon, and xenon layers are as follows: $0.0343 \text{ m}$, $0.0323 \text{ m}$, and $0.0169 \text{ m}$, respectively, at a chosen frequency of $10 \text{ kHz}$. The reason for choosing a very high frequency
is to create a very fine mesh element size for the unit structure of the meta-surface, in order to study the discretization error and the accuracy of the solution compared with the default setting of the finer mesh. Figure 6.7 illustrates the mesh setting for air, argon, and xenon domains. It can be seen from Fig. 6.7 (a-c) that the number of the elements varies according to the material parameters of the filling fluid.

![Figure 6.7 Mesh structure for each fluid layer of the meta-surface.](image)

A further improvement of the mesh setting is obtained by increasing the number of degrees of freedom with respect to their wavelength in each domain. Table 6.4 shows the number of degrees of freedom, the domain and boundary elements, the minimum mesh quality, the
physical and virtual memory, and the time taken for each case. A good quality mesh element is achieved, as the value of the minimum mesh quality is higher than 0.5 in all cases, as illustrated in Table 6.4.

Table 6.4 Details of the mesh elements for acoustic meta-surface (layered media) structure

<table>
<thead>
<tr>
<th>Number of degrees of freedom per wavelength</th>
<th>Number of degrees of freedom solved for the model</th>
<th>Domain elements</th>
<th>Boundary elements</th>
<th>Minimum mesh quality</th>
<th>The physical memory[GB]</th>
<th>The virtual memory[GB]</th>
<th>Solution time[s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>78443</td>
<td>38722</td>
<td>1265</td>
<td>0.7448</td>
<td>2.15</td>
<td>3.20</td>
<td>32</td>
</tr>
<tr>
<td>2</td>
<td>310865</td>
<td>154440</td>
<td>2522</td>
<td>0.7376</td>
<td>2.25</td>
<td>2.78</td>
<td>132</td>
</tr>
<tr>
<td>3</td>
<td>735539</td>
<td>366282</td>
<td>3779</td>
<td>0.7300</td>
<td>3.10</td>
<td>3.73</td>
<td>342</td>
</tr>
<tr>
<td>4</td>
<td>1466073</td>
<td>731054</td>
<td>5036</td>
<td>0.5215</td>
<td>4.50</td>
<td>5.30</td>
<td>748</td>
</tr>
<tr>
<td>5</td>
<td>2050249</td>
<td>1022648</td>
<td>6291</td>
<td>0.7134</td>
<td>5.82</td>
<td>6.79</td>
<td>1144</td>
</tr>
</tbody>
</table>

To demonstrate and visualize that the mesh setting is of good quality, we plot the mesh quality when we have high (low) minimum mesh quality with one (four) degrees of freedom per wavelength. As shown in Figs.6.8 (a) and (b), the quality of the mesh is high in the whole geometry.
Simulation Results and Accuracy Tests

The calculated transmission spectrums, by evaluating the normalized pressure field on the transmission domain for different mesh settings, are presented in Fig. 6.9. It can be seen that the results in general agree with each other. However, the inset of the figure shows there are very small discrepancies among the results.
The number of degrees of freedom per wavelength:

- One
- Two
- Three
- Four
- Five

Figure 6.9 The transmission spectra are computed with varying mesh element sizes from one (black solid curve) to five (gray dots curve) degrees of freedom per wavelength.

To analyze the convergence behavior of the solution, the error in $L_2$-norm of the solutions is computed with respect to the characteristic grid spacing $h$, as depicted in Fig.6.10. The error is small when the grid space $h$ becomes small. However, if the frequency becomes higher, the error increases, as presented in Fig.6.10.
Finally, it is noteworthy that the desired accuracy is obtained because the numerical error is less than 1%, as observed in Fig. 6.10. To validate that, the rate of convergence is computed from Eq. (6.3). The third order of accuracy is achieved for 2D layered medium meta-surface with quadratic elements as shown in Table 6.5, which is in good agreement with the theoretical convergence order for the 2D test problem with quadratic elements [158, 159].

<table>
<thead>
<tr>
<th>h [m]</th>
<th>Error</th>
<th>Convergence order</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01715</td>
<td>1.25804921070081E-06</td>
<td>3.5</td>
</tr>
<tr>
<td>0.011433</td>
<td>2.94600257149361E-07</td>
<td>3.9</td>
</tr>
<tr>
<td>0.008575</td>
<td>9.3686392893904E-08</td>
<td>3.4</td>
</tr>
<tr>
<td>0.00686</td>
<td>4.35120320885218E-08</td>
<td></td>
</tr>
</tbody>
</table>
6.2.2 Numerical Scheme for Resonant Type Metamaterial

In the previous models, the wave equation was simulated under the assumption that the viscosity of the fluids is neglected (set to zero). Therefore the mesh setting of these models, that can resolve the smallest wavelength and provide a high level of accuracy, is a trivial task. In reality, however, the viscous damping is exhibited in the system, owing to sound wave propagation through the narrow neck of Helmholtz resonator (HR), so that the meshing of this model must resolve the viscous boundary layers. A high quality mesh, avoiding too many mesh elements, can be obtained by controlling the mesh element size at the viscous layers. To do this, we should know the thickness of the viscous boundary layer at each step in the frequency domain [160]. In air at 20°C, for instance, the thickness of the viscous layer equals 0.22[mm] at 100[Hz]. Hence, we need to use a parametric sweep to build a good mesh size depending on the frequency. In general, the boundary layer thickness \( \ell_{vis} \) in air is approximated by

\[
\ell_{vis} = 0.22[\text{mm}] \sqrt{100[\text{Hz}]/f_0}
\]

where \( f_0 \) is selected as the desired frequency in the domain study. By sweeping \( f_0 \), we adjust the element sizes of the mesh and tune the thickness of the viscous layer around the HR. However, this increases the solution time. Since the viscous boundary layer decreases as the frequency and the fluid density increase, the thickness variations of the layer will seriously affect the accuracy of the numerical simulations, especially in the long-wavelength (low-frequency) region. Therefore, the thickness of the viscous boundary layer should be kept at a set value, in the lowest value of the frequency range to obtain the most accurate results with the least
computational time. In the following section, we investigate and analyze the numerical simulation conducted for air waveguide connected with a single HR, which are the building block of the trapping and absorption device, as shown in Chapter 4. Note that there is no significant difference between single HR and triply coupled HRs in the numerical study. The computational scheme and physical settings such as the boundary conditions, the material, and the mesh are demonstrated in the following sections. Moreover, the error and the convergence order of the solution are discussed for the HR model.

Setting of Geometric and Material Parameters, Physics Settings, and Mesh

(I) Geometric and Material Parameters

The schematic of the unit cell structure and the boundary condition setting for a single HR are illustrated in Fig.6.11. The pressure acoustics model is applied in the entire structure which is solved for two dependent variables, the pressure $p$ and the displacement field $u$, in the frequency domain over the frequency range $1[kHz]$–$50[kHz]$. The acoustic plane wave generated from the left with pressure amplitude equals $1[Pa]$, as shown in Fig. 6.11. The material parameters of the air background medium are as follows: mass density equals $1.29[kg/m^3]$ and the speed of sound in air is $343[m/s]$. The geometric parameters used for the single HR case are $a_i = 0.5[mm]$, $\chi_i = 1[mm]$, $h_i = 1.6[mm]$, $b_i = 5.5[mm]$, and with $h = 4[mm]$ and $w = 7[mm]$. 
(II) Creating and Adjusting the Mesh

In this numerical study, we set the maximum and the minimum element size equal to $0.1372\text{[mm]}$ and $0.686\text{[mm]}$, which are about $5$ and $10$ wavelength at the frequency $50\text{[kHz]}$, respectively, in all the computational domains. To capture the viscosity effect in this model, we apply an additional size to the triangular element mesh at the internal boundaries of the HR by implementing Eq.(6.4) at $50\text{[kHz]}$, which is much finer than the mesh size in the waveguide domain, as illustrated in Fig.6.11(a). Then, we set the maximum and the minimum element size equal to $\ell_{\text{vis}}(0.0098387\text{[mm]})$ and $\ell_{\text{vis}}/2(0.0049194\text{[mm]})$. After that, we applied a boundary layer mesh at the edges of the narrow neck of the HR to adjust the element sizes of the mesh and capture the effect of the viscous layer at this narrower region, with the thickness equal to $\ell_{\text{vis}}/3$, as shown in Fig.6.12 (b). The complete mesh consists of 12127 domain elements and 481 boundary elements, yielding a total of $42036$ degrees of freedom solved for this model. We compute the transmission spectrum of the system and it is depicted (the black curve) in Fig.6.14. The running time for this study is $6$ minutes and $53$ seconds. The physical memory reported is $1.11\text{[GB]}$ and the virtual memory reported is $1.25\text{[GB]}$. 

Figure 6.11 A schematic of the unit cell structure and the boundary condition setting.
Figure 6.12 (a) Mesh structure or air waveguide connected with a single HR. (b) the viscous boundary layer setting at one edge of the neck of HR, which marked by red dashed rectangle in Fig.6.12(a).

To investigate the influence of the viscous layers on the accuracy of the solution, we adjust the thickness of the boundary layer and the mesh element size at the edges of the HR by sweeping the number of degrees of freedom from one to five per $\ell_{vis}$. Reduction in the thickness of the boundary layer $\ell_{vis}$ lead to, therefore, increasing the number of elements as shown in Table 6.6.

For each case, the number of degrees of freedom, the domain and boundary elements, the minimum mesh quality, and the physical and virtual memory, are described in Table 6.6. The running time for this study is 2 hours, 16 minutes, and 38 seconds.

<table>
<thead>
<tr>
<th>Number of degrees of freedom per $\ell_{vis}$</th>
<th>Number of degrees of freedom solved for this model</th>
<th>Domain elements</th>
<th>Boundary elements</th>
<th>Minimum mesh quality</th>
<th>The physical memory [GB]</th>
<th>The virtual memory [GB]</th>
<th>Solution time [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>177636</td>
<td>87311</td>
<td>1892</td>
<td>0.1003</td>
<td>1.37</td>
<td>1.53</td>
<td>275</td>
</tr>
<tr>
<td>2</td>
<td>388625</td>
<td>191382</td>
<td>3533</td>
<td>0.1033</td>
<td>1.75</td>
<td>1.95</td>
<td>694</td>
</tr>
<tr>
<td>3</td>
<td>604221</td>
<td>297768</td>
<td>5157</td>
<td>0.1067</td>
<td>2.14</td>
<td>2.37</td>
<td>1148</td>
</tr>
<tr>
<td>4</td>
<td>819844</td>
<td>404163</td>
<td>6776</td>
<td>0.1051</td>
<td>2.55</td>
<td>2.83</td>
<td>1303</td>
</tr>
<tr>
<td>5</td>
<td>1261163</td>
<td>623416</td>
<td>8384</td>
<td>0.1001</td>
<td>3.25</td>
<td>3.62</td>
<td>2115</td>
</tr>
</tbody>
</table>
Figure 6.13 Mesh quality elements plot with (a) one and (b) four degrees of freedom per wavelength. The blue color indicates low mesh quality while the red color corresponds to high mesh quality.

Figures 6.13 (a) and (b) show the quality of the mesh at the high and low value with three and five degrees of freedom per wavelength, respectively. The high quality is demonstrated in the whole domain for both cases. However, the mesh quality reduces at the viscous boundary layer.
The values of the quality between $0.7 - 0.1$, indicate that the mesh is still in good condition at the viscous boundary layer.

**Simulation Results and Accuracy Tests**

The results of the transmission spectra with various number of degrees of freedom per $\ell_{\text{vis}}$ are illustrated in Fig. 6.14. It is seen that the transmission spectra coincides with others for all the cases of the boundary layer thickness. At the resonance frequency, however, the result of the transmission with thicker thickness of the viscous boundary layer (black curve) deviates slightly from other results, as demonstrated in the inset of Fig.6.14.
To obtain a clear picture of the influence of the viscosity of the boundary layer on the accuracy of the solution, the error in $L_2$-norm is computed at different frequencies including the resonance frequency. The errors plotted against $h$ on a log-log scale are displayed in Fig. 6.15. As can be seen, the thinner boundary layer leads to a smaller error than the thicker boundary layer. The results also show clearly that at the resonance domain the solution delivers a higher value of error than other frequency domains. Nevertheless, the error at the resonance domain is less than 1% and that is agreeable in the FM simulations. Thus, adaptive meshing refinement with a
thinner boundary layer, needs in this numerical model in order to obtain an accurate solution at the stop band (low-transmission) domain.

![Figure 6.15 The errors in L₂-norm plotted against h on a log-log scale.](image)

To examine the convergence of the solution, we computed the rate of the convergence at the frequency $9.4\,[\text{kHz}]$. The convergence order lies constantly at about 3 for all cases of mesh refinements as shown in Table 6.7.

<table>
<thead>
<tr>
<th>h [m]</th>
<th>Error</th>
<th>Convergence order</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00492</td>
<td>0.000272178807941709</td>
<td>3.4</td>
</tr>
<tr>
<td>0.00328</td>
<td>0.0000676448831587262</td>
<td>3.3</td>
</tr>
<tr>
<td>0.00246</td>
<td>0.0000262658907436658</td>
<td>3.1</td>
</tr>
<tr>
<td>0.00197</td>
<td>0.0000131278291441578</td>
<td></td>
</tr>
</tbody>
</table>
6.3 Summary

The numerical studies, based on finite-element simulations that validate our analytical solutions for the unit structure of our proposed non-resonant and resonant type metamaterials, have been extensively investigated in this chapter. The basic aspects of the computation processes including geometric parameters, materials, physics settings, boundary conditions, mesh, and the error and convergence analysis, have been shown for both types of metamaterial. We show how the accuracy of the solution depends to the mesh element size. The type of fluid materials, either viscous or non-viscous, must be taken into account when applying mesh settings. The viscous boundary layer should be introduced in the numerical model to obtain more accurate results if the fluids are viscous. Moreover, when the materials, whether viscous or not, change from one domain to another domain, the mesh should be modified to capture all the local physical effects accurately. Thus, adaptive mesh refinement should be applied through the structure of the building blocks of metamaterial. The analysis results in this chapter indicate that highly accurate solutions are achieved for all numerical models of building blocks of our designed metamaterials covered in this dissertation.
Chapter 7

Conclusions and Future Work

7.1 Conclusion and Discussion

In this dissertation, we derived EMTs for our designs of meta-materials and meta-surfaces in which their effective parameters can describe the behavior of the propagating waves through these structures. For each design, the analytic formulas and the EMTs were derived, and then the validation of the derivations was verified by conducting numerical simulations based on the finite-element method. Our theoretical findings are utilized to realize their prospective applications in designing real-application devices.

To summarize, we derived the impedance-matching condition in Chapter 2 for two types of acoustic meta-surfaces. Their respective building blocks are constructed from the coiling-up spaces immersed in water and the layered structures embedded in air. The couple-mode theory is utilized to develop an EMT for coiling-up space meta-surface in order to derive the conditions for matched impedances. We found that using an appropriate combination of the coiling-up space’s geometric size and filling material gives rise to a matched impedance to the environment.

Interestingly, our method of impedance matching has a broadband frequency response. The theory is validated by transmission spectra calculations. Practically, several numerical simulations were carried out to show the influence of the geometric parameters on the accuracy of the theoretical prediction, and we found that the transmission spectrum for the microstructure is in good agreement with the one predicted from EMT. A transfer matrix method was used to derive
the impedance-matching condition for the other acoustic meta-surface in air. We observed that the impedance-matching condition is satisfied if the input impedance is identical to the impedance of air. In order to satisfy such a condition the thickness and the sequence of the layers inside the meta-surface need to be tuned, since the value of the input impedance relies on the thickness and the arrangement of the layers. In the high frequency regime, we noticed that the fundamental mode assumption of our theoretical model is not valid. Therefore, the equation of the input impedance is not applicable at the high frequency range.

In Chapter 3, based on the impedance-matching condition that was derived in Chapter 2, we designed two types of acoustic meta-surfaces in water and in air, in which their impedance match their surrounding media, to focus, collimate, and redirect acoustic waves. The desired functionality can be obtained by manipulating the refractive indices. Numerical simulations demonstrated the realizations of focusing, collimation and the redirection of acoustic waves in water and in air with our designed acoustic meta-surfaces. The required conditions of the impedance and refractive index would not be difficult to verify numerically. However, getting the filling material that exactly satisfies these conditions is challenging. In the design of the real device, we optimized the geometry and the material parameters, and established a more general condition that is experimentally feasible. A myriad of interesting applications can be achieved by these acoustic impedance-matching lenses, including non-destructive testing techniques that inspect materials for hidden flaws, and biomedical applications that gather the particles into the focal point.
A new mechanism of slowing down sound waves with coupled Helmholtz resonators, which are able to perfectly trap sound waves, was developed and studied both theoretically and numerically in Chapter 4. The key feature of this mechanism is that it provides a strong sound dispersion, leading to strong localization of sound waves in an individual unit cell at desired resonance frequencies, which can effectively overcome the energy leakage issue. Newton’s second law, in conjunction with the hidden source of volume principle, was used to derive an analytic formula and develop an EMT for the proposed metamaterial design. The theoretical prediction and numerical simulations showed that the coupling of the resonators’ modes can generate dips and flat bands at the resonance frequencies in the transmission spectrum and band structure diagram, respectively. The proposed design offers more degrees of freedom to engineer the bandgap such as the number, the location, and the width of the band gaps by manipulating the number and the size of the resonators. In this chapter we also investigate the influence of the dimensions of the structure on the accuracy of the theoretical model by calculating the transmission spectrum with finite-element simulations for different cases.

In Chapter 5, based on our previous analysis in Chapter 4, we successfully designed an acoustic trapping device that can tackle the wave leakage issue, which was examined numerically. Simulations showed that the designed acoustic trapping metamaterial can perfectly trap the sound wave energy and enhance the pressure intensity in the desired unit at the resonance frequencies, validating our findings. To further demonstrate the applicability of the proposed designed meta-materials, we introduced thermo-viscous losses into the device to achieve perfect sound energy absorption over a broad frequency range. High and broadband sound absorption
was demonstrated numerically. Compared with the single resonator structure, the functionality of the sound wave absorption is realized in a narrower frequency range. Moreover, such a structure is not an effective design to obtain perfect sound trapping because the sound pressure intensity spreads out to the other neighboring units, rather than being trapped at a precise location due to the unavoidable coupling between neighboring modes. This gave us clear evidence of the effectiveness of our design in obtaining the desired functionality. Our findings pave the way for achieving a myriad of fascinating applications such as acoustic insulation, acoustic filters, and broadband perfect absorbers.

Finally, the numerical studies based on the finite-element method, that validate our derivation of theoretical models and EMTs for the unit structure of our proposed non-resonant and resonant type metamaterials that are developed in this dissertation, were thoroughly explored in Chapter 6. In the finite-element method, mesh density plays a dominant role in the accuracy of the numerical analysis solution. Therefore, a systematic study in finding the efficient mesh density that can accurately resolve the smallest wavelength and give us accurate numerical results for both types of metamaterial were conducted in this chapter. Statistical analysis was invoked to show the effects of mesh density in the numerical results. We observed from this analysis that the quality of the mesh was high in the entire geometry for non-resonant type metamaterials with a non-viscous fluid, demonstrated by the element quality plot. For the resonant type metamaterials with the viscous fluid finite element model, consideration of the viscous boundary layer on the edges of the resonant is required to capture all the physical effects. The mesh quality plot reveals that the values of the quality is reduced at the boundary layer.
Nevertheless, the numerical model gives us an accurate solution. This is because the quality of the mesh on the viscous boundary layer is still in good condition, based on the guidelines of attaining good mesh quality in finite element modeling as discussed in the literature review. Adaptive mesh refinement relies on the type of fluid and the geometry structure applied for all numerical models. Convergence and error analysis were performed in this numerical investigation to thoroughly analyze the influence of the mesh refinement on the simulation accuracy. The results imply that the highly accurate numerical solutions are achieved for all finite element simulations conducted in this dissertation.

7.2 Current and Future Work

We designed acoustic meta-surfaces and meta-materials for focusing, collimating, absorption, trapping and redirecting sound wave energy, which have the capability of tackling the most challenging issues in the fabrication of airborne devices. Although we have demonstrated interesting applications of the elementary unit cell of our design structures in this dissertation, further novel functionalities can be realized by improving the theories. The theoretical methods we utilized and the resulting physical phenomena are not restricted to acoustic waves, and therefore, our approach can be applied to all types of classical waves including electromagnetic and water waves. This is reasonable, since in two-dimensional structures there is mathematical mapping among these types of waves. Taking advantage of the flexibility of applying our finding of matching the dissimilar impedance materials, we have recently derived the analytical solution and EMT for water surface waves (WSWs) propagation through an ultra-thin space-coiled meta-surface to realize the impedance matching phenomena over a large frequency range underwater.
In the past decade, a great deal of attention has been devoted to elucidating the mechanism of water wave propagation in the ocean, owing to the rich energy stored in the ocean WSWs. The energy derived from these waves is considered to be one kind of clean renewable energy that can serve to defuse the approaching energy crisis, and its energy density is at least four times greater than the density from wind or solar energy [161-163]. Focusing on water waves in a broad band of frequencies could have an impact on ocean wave energy harvesting [164]. In our present work, by adopting the parameter retrieval method and couple-mode theory, we derive analytical formalism of the effective depth and the effective gravitation of the water wave. The aim of this study is to obtain broadband focusing with high intensity of the water wave, implying that significantly more ocean wave energy is transmitted. The theoretical derivation result indicates that proper tuning of the effective depth of the water and the size of the curled channel lead to matching the impedance of the meta-surface with that of the background media. To verify the impedance matching condition of WSWs that we derived, we are developing a numerical model based on the finite element method in COMSOL for shallow water waves. Our findings provide an excellent explanation of the experimental realizations, showing how water waves in a wide range of frequency domains can be highly-focused and allow the majority of the incident wave energy to be transmitted based on the impedance matching condition.

**Theoretical Model and EMT for Shallow Water Wave**

A schematic of a coiling-up space unit cell of the meta-surface with height equal to $d_c$ is illustrated in Fig.7.1 (a), which is curled slit perforated in a steel slab submerged in water. $d_0$ represents the depth of the water in the environment medium. In order for the structure not to
be inundated by water, \( d_c \) must be larger than \( d_0 \). The width and the length of the horizontal segment of the curled slit are represented by \( a \) and \( b \), whereas \( l_c \) and \( w \) denote the thickness of the plate and the periodicity of the structure, respectively, as shown in Fig.7.1 (b). Assume a plane wave incident from the bottom along the \( x \)-direction. In the incident domain, the displacement of the water surface can be expressed in terms of the sum of diffraction order as follows:

\[
\eta_i(x, y) = \sum_{\chi} \left( \delta_{0,\chi} e^{ia_{\chi}x} + r_{\chi} e^{-ia_{\chi}x} \right) e^{iU_{\chi}y}
\]

where \( \delta_{0,\chi} \) is the Kronecker delta. \( r_{\chi} \) represents the normalized displacement field amplitudes of the \( \chi \)th diffraction order of reflected waves, and \( U_{\chi} = 2\pi\chi/w \) and \( \alpha_{\chi} = \sqrt{k_0^2 - U_{\chi}^2} \) are the momentum of the \( \chi \)th diffraction order along the plate surface and propagation directions, respectively. \( k_0 = (\omega/\sqrt{g_0 d_0}) \) is the wave-vector of the incident wave (\( \omega \) is the angular frequency), \( g_0 \) is the gravitational acceleration and \( d_0 \) is the depth of the environment medium.

In the transmission domain, the displacement of the water surface can be

\[
\eta_{\text{trans}}(x \geq l_c, y) = \sum_{\chi} t_{\chi} e^{ia_{\chi}(x-l_c)} e^{iU_{\chi}y}
\]

where \( t_{\chi} \) denotes the normalized displacement field amplitudes of the \( \chi \)th diffraction order of transmitted waves. In the meta-surface domain, the water wave equation is

\[
\Delta \eta_{\text{meta}} + \frac{\omega^2}{g} \eta_{\text{meta}} = 0
\]

(7.1)

where \( k_c \) satisfies the following dispersion relation: \( \omega^2 = g_c k_c \tan(k_c \vartheta_c) \), while \( \vartheta_c \) (\( = \tan(k_c d_c)/k_c \)) is the reduced wave depth. For a shallow water case, we have \( \tan(k_c d_c) \approx k_c d_c \).
Then $\vartheta \approx d_c$ and the linear dispersion relation is given by $\omega = k_c \sqrt{g_c d_c}$. The displacement of the water surface $\eta_{II}$ is determined by considering the boundary conditions, which is the zero flow through the rigid walls of the meta-surface, namely

$$\frac{\partial \eta_{II}}{\partial n} = 0$$

(7.2)

where $n$ is the normal direction to the surface of the unit structure. Applying the boundary condition, the displacement of the water surface in the curled channel can be written as:

$$\eta_{II} (0 \leq x \leq l_c, y) = \sum_p \cos(k_{y,p} y) \left( A_p e^{i k_{x,p} x} + B_p e^{-i k_{x,p} x} \right)$$

(7.3)

with $k_{y,p} = 2 p \pi / a$ and $k_{x,p} = \sqrt{k_c^2 - k_{y,p}^2}$. Since the width of the slit is significantly smaller than the wavelength, only the fundamental waveguide mode propagates inside the curled channel. Therefore the Eq. (7.3) becomes

$$\eta_{II} (0 \leq x \leq l_c, y) = A e^{i k_{x} x} + B e^{-i k_{x} x}$$

(7.4)

where $A$ and $B$ are the corresponding amplitudes of the displacement of the water surface of the upward and downward propagating waves, respectively. The displacement velocity of the particles in the shallow water wave is given by $v_j = -g_{0,c} \nabla \eta_j$ where $j(= I, II, III)$ indicates the propagation domain. Matching the boundary conditions at the interfaces between the curled slit and the host medium in the inlet and the outlet of the slit (i.e., at $x = 0$ and $x = l_c$), in which the displacement of the water surface and the velocity must be continuous, the coefficients $t_x$ and
can be obtained. Since the meta-surface has a narrow curled channel, in the long-wavelength regime \( \lambda \gg w \) the propagation of water waves can be approximated by retaining only zero-order diffracted wave mode, while the other higher-diffraction-order modes are evanescent. Consequently, the transmission and reflection coefficients of the shallow water waves in the far-field can be obtained as:

\[
t_0 = \frac{4(a/w)(g_c k_c/g_0 k_0) e^{ik_l}}{[(a/w)(g_c k_c/g_0 k_0) + 1] - [(a/w)(g_c k_c/g_0 k_0) - 1] e^{2ik_l}} \tag{7.5}
\]

\[
r_0 = \frac{\left[1 - (a/w)^2 (g_c k_c/g_0 k_0)^2\right] + [(a/w)(g_c k_c/g_0 k_0) - 1] e^{2ik_l}}{[(a/w)(g_c k_c/g_0 k_0) + 1] - [(a/w)(g_c k_c/g_0 k_0) - 1] e^{2ik_l}}.
\]

Comparing these coefficients of a real structure with the transmission and reflection coefficients of a plane water wave normally incident on a slab of a homogenous medium with thickness \( l \), which are given respectively by

\[
T(\omega) = \frac{4(g_{\text{eff}} k_{\text{eff}}/g_0 k_0) e^{ik_{\text{eff}} l}}{[(g_{\text{eff}} k_{\text{eff}}/g_0 k_0) + 1]^2 - [(g_{\text{eff}} k_{\text{eff}}/g_0 k_0) - 1]^2 e^{2ik_{\text{eff}} l}} \tag{7.6}
\]

\[
R(\omega) = \frac{\left[1 - (g_{\text{eff}} k_{\text{eff}}/g_0 k_0)^2\right] + [(g_{\text{eff}} k_{\text{eff}}/g_0 k_0) - 1] e^{2ik_{\text{eff}} l}}{[(g_{\text{eff}} k_{\text{eff}}/g_0 k_0) + 1]^2 - [(g_{\text{eff}} k_{\text{eff}}/g_0 k_0) - 1]^2 e^{2ik_{\text{eff}} l}}.
\]

The detailed derivations for Eq. (7.6) are presented in Appendix C. We found certain correspondence between the transmission and reflection coefficients of the meta-surface and the homogenous slab. Using \( k_c = \omega/\sqrt{g_c d_c} \), \( k_{\text{eff}} = \omega/\sqrt{g_{\text{eff}} d_{\text{eff}}} \) and \( k_0 = \omega/\sqrt{g_0 d_0} \), we can
approximate the space-coiling meta-surface as a homogenous slab with effective impedance and effective refractive index, which are given respectively by $\psi_{\text{eff}} = q^{-1}\psi_c$ and $n_{\text{eff}} = (l_c/l)n_c$, where $q = a/w$, $\psi_c = \sqrt{d_c/g_c}$ and $n_c$ are the ratio of the width of the curled channel to the periodicity, the impedance, and refractive index of the meta-surface. Furthermore, the effective liquid medium of the homogenous slab for the shallow water wave possesses an effective gravitational acceleration $g_{\text{eff}}$ and effective depth $d_{\text{eff}}$, which are obtained by $g_{\text{eff}} = q(l/l_c)g_c$ and $d_{\text{eff}} = q^{-1}(l/l_c)d_c$, respectively, as illustrated in Fig. 7.1(c). In reality, the gravitational acceleration of the meta-surface $g_c$ has the same value of the environment gravitational acceleration $g_0$, i.e., $g_c = g_0$. Thus, we have only one degree of freedom to manipulate the amplitude of surface water waves, which is the depth of the water inside meta-surface $d_c$. Significantly, by choosing $d_c$ to be equal to $q^2d_0$, the effective impedance of the meta-surface exactly matches the impedance of the host medium. The shallow water waves therefore are totally transmitted, although the width of the curled channel is very narrow.
Figure 7.1 Schematic of the meta-surface building block with periodic subwavelength coiling-up space, where the purple area indicates the rigid slab. (a) Side view of the unit. (b) Top view of the unit. (c) The corresponding homogenous slab that possesses an effective gravitational acceleration $g_{\text{eff}}$ and effective depth $d_{\text{eff}}$.

Based on these efforts, we will design a new lens that is able to transmit highly-focused water surface waves and achieve a perfect water wave cloak effect based on flat focusing in broad frequency response. The application and realization of this study will be demonstrated by finite-element simulations. The new finding could be readily applied to many important applications on controlling and harvesting ocean energy.


90. F.-J. Hsieh and W.-C. Wang, “Full extraction methods to retrieve effective refractive index and parameters of a bianisotropic metamaterial based on material dispersion models,” Journal of Applied Physics, 112(6), 064907, 2012.


152


Appendix A Derivation of the Transmission and Reflection Coefficients of the Fluid-filled Coiling-up spaces Meta-surface in Section 2.1.3

In section 2.1.3, the transmission and reflection coefficients of the fluid-filled coiling-up spaces meta-surface have been used to derive the effective medium theory based on the parameter retrieval method. Detailed mathematical derivation of these coefficients will be presented in this appendix.

The pressure field below and above the meta-surface can be written as the sum of different diffraction orders, which can be respectively expressed as Eqs.(2.1) and (2.2). Inside the fluid-filled coiling-up spaces meta-surface, we solved the acoustic wave equation with hard wall boundary conditions at the interfaces between the slits and the water to obtain the pressure field within this region. The hard wall boundary condition is defined as the zero normal velocity at the wall, i.e. \( \rho \frac{\partial p}{\partial z} = 0 \). \( \rho \) represents the mass density of the ambient medium \( (\rho_0) \) or the mass density of the material inside the slits \( (\rho_s) \). Then the pressure field can be expressed as:

\[
p_{II} = \sum_{n=0} \cos(k_{zn}z)(Ae^{ik_{zn}z} + Be^{-ik_{zn}z}) \quad (A.1)
\]

Here, \( n \) denotes the order of the waveguide modes, while \( k_{xn} = 2n\pi/a \) and \( k_{zn} = \sqrt{k_s^2 - k_{xn}^2} \) are wave vectors in \( x \) and \( z \)-directions, respectively. Owing to the width of the slit being much smaller than the wavelength, only the zero-order propagation mode is supported and other modes are evanescent in the \( z \) direction, as discussed in section 2.1.3. Therefore, the pressure
field inside the fluid-filled coiling-up spaces is defined as Eq.(2.3). The normal velocity in the
ambient medium and fluid-filled curled slit are as follows:

\[
v_j = -\frac{i}{\rho_0} \sum \alpha_e \left( \delta_{0, z} e^{i\alpha e z} - r_e e^{-i\alpha e z} \right) e^{iG_x} \quad (A.2)
\]

\[
v_{ll} = -\frac{ik_s}{\rho_s} \left( Ae^{ik_s z} - Be^{-ik_s z} \right) \quad (A.3)
\]

\[
v_{lll} = -\frac{i}{\rho_0} \sum \alpha_{r e} \left( e^{i\alpha_r (z-h)} \right) e^{iG_x} \quad (A.4)
\]

To solve the expansion coefficients, the continuity boundary condition of the pressure fields and
normal velocity at the interfaces should be applied. At the inlet of the fluid-filled coiling-up
spaces, the continuity condition of the pressure fields and normal velocity are imposed, and given
by

\[
\sum \alpha_{r e} (\delta_{0, z} + r_e) e^{iG_x} = A + B \quad (A.5)
\]

\[
\frac{1}{\rho_0} \sum \alpha_{r e} (\delta_{0, z} - r_e) e^{iG_x} = \frac{k_s}{\rho_s} (A - B) \quad (A.6)
\]

Integrating Eq.(A.5) with respect to \( x \) from \( md + a/2 \) to \( md - a/2 \), where \( md \) denotes the
position of the center of the slit, then gives

\[
\int_{md + a/2}^{md - a/2} \sum \alpha_{r e} (\delta_{0, z} + r_e) e^{iG_x} dx = \int_{md + a/2}^{md - a/2} (A + B) dx \quad (A.7)
\]
\[
\sum_\tau (\delta_{0,\tau} + r_\tau) \frac{e^{iG_{md} \delta_{0,\tau}}}{G_\tau (a/2)} \sin \left( G_\tau \frac{a}{2} \right) = A + B \quad (A.8)
\]

\[
\sum_\tau (\delta_{0,\tau} + r_\tau) e^{iG_{md} \delta_{0,\tau} \sin} \left( G_\tau \frac{a}{2} \right) = A + B \quad (A.9)
\]

Owing to the normal velocity being defined all over the surface, i.e. rigid and non-rigid, we have to integrate from \( md + d/2 \) to \( md - d/2 \)

\[
\int_{md - d/2}^{md + d/2} \frac{1}{\rho_0} \sum_\tau \alpha_\tau \left( \delta_{0,\tau} - r_\tau \right) e^{iG_{x,\tau} \delta_{0,\tau}} dx = \int_{md - d/2}^{md + d/2} \frac{k}{\rho_0} (A - B) \alpha_\tau e^{-iG_{x,\tau}} dx \quad (A.10)
\]

Multiplying both sides by \( e^{-iG_{0,x}} \), obtains

\[
\int_{md - d/2}^{md + d/2} \frac{1}{\rho_0} \sum_\tau \alpha_\tau \left( \delta_{0,\tau} - r_\tau \right) e^{iG_{x,\tau} e^{-iG_{0,x}}} dx = \int_{md - d/2}^{md + d/2} \frac{k}{\rho_0} (A - B) e^{-iG_{0,x}} dx \quad (A.11)
\]

Due to the orthogonality property, the infinite series of diffraction orders in the left-hand side of Eq.(A.11) collapses to the single term, i.e. \( \sum_\tau \alpha_\tau \left( \delta_{0,\tau} - r_\tau \right) e^{iG_{x,\tau} e^{-iG_{0,x}}} = \alpha_\tau \left( \delta_{0,\tau} - r_\tau \right) \). Then by using the orthogonality and integrating over the period, we obtain the following expression:

\[
\frac{d}{\rho_0} \alpha_\tau \left( \delta_{0,\tau} - r_\tau \right) = a \frac{k}{\rho_0} (A - B) e^{-iG_{md} \alpha_\tau \sin} \left( G_\tau \frac{a}{2} \right) \quad (A.12)
\]

On the other hand, the continuity condition of the pressure fields and normal velocity at the outlet of the fluid-filled coiling-up spaces are satisfied by the following:

\[
\sum_\tau t_\tau e^{iG_{x,\tau} e^{-ik h_{\tau}}} = A e^{i k h_1} + B e^{-i k h_1} \quad (A.13)
\]
\[
\frac{1}{\rho_0} \sum_{\tau} \alpha_{\tau} t_{\tau} e^{iG_{\tau}x} = \frac{k_s}{\rho_s} (A e^{ik_{\tau}h} - B e^{-ik_{\tau}h}) \quad (A.14)
\]

In the same manner as the inlet case, integrating Eqs.(A.13) and (A.14) with respect to \( x \) gives

\[
\sum_{\tau} t_{\tau} e^{iG_{\tau}md} \text{sinc} \left( G_{\tau} \frac{a}{2} \right) = A e^{ik_{\tau}h} + B e^{-ik_{\tau}h} \quad (A.15)
\]

\[
\frac{d}{\rho_0} \alpha_{\tau} t_{\tau} = a \frac{k_s}{\rho_s} (A e^{ik_{\tau}h} - B e^{-ik_{\tau}h}) e^{-iG_{\tau}md} \text{sinc} \left( G_{\tau} \frac{a}{2} \right) \quad (A.16)
\]

Equations (A.12) and (A.16) give us the analytical expressions of the coefficients \( r_{\tau} \) and \( t_{\tau} \), which rely on the upward and downward propagating waves. Substituting (A.12) into (A.9) and (A.16) into (A.15), we obtain the corresponding amplitudes of pressure fields of the upward and downward propagating waves as:

\[
\psi_2 - \psi_1 (A - B) = A + B \quad (A.17)
\]

\[
\psi_1 (A e^{ik_{\tau}h} - B e^{-ik_{\tau}h}) = A e^{ik_{\tau}h} + B e^{-ik_{\tau}h} \quad (A.18)
\]

where \( \psi_1 = f \left( k_s, \rho_0 / \rho_s \right) \sum_{\tau=-\infty}^{\infty} \text{sinc}^2 \left( G_{\tau} a/2 \right) / \alpha_{\tau} \), and \( \psi_2 = 2 \sum_{\tau=-\infty}^{\infty} \delta_{\tau} e^{iG_{\tau}md} \text{sinc} \left( G_{\tau} a/2 \right) \). \( f = a/d \)

is the ratio of the width of the slit to the periodicity. Solving Eqs.(A.17) and (A.18), we get the corresponding amplitudes of pressure fields of the upward and downward propagating waves. Then we will have Eq.(2.4).
Appendix B Derivation of the Displacement of the Air and the Normalized Specific Impedance of Three Coupled Resonators in Section 4.3.1

In this appendix, we derive the displacement at the first neck and the normalized specific impedance for three coupled resonators, which were utilized in Section 4.3.1.

We consider a $3 \times 3$ matrix since the system is composed of three coupled HRs in a unit cell as shown in Fig.4.2 (b). Applying Newton’s second law of motion to the first, second and third resonators, obtains

\[ m_i \ddot{h}_i = a_i \left( p - \Delta p_i \right) \quad \text{(B.1)} \]
\[ m_2 \ddot{h}_2 = a_2 \left( \Delta p_2 - \Delta p_1 \right) \quad \text{(B.2)} \]
\[ m_3 \ddot{h}_3 = a_3 \left( \Delta p_3 - \Delta p_2 \right) \quad \text{(B.3)} \]

\( m_i \left( = \rho_i a_i \chi'_i \right) \) is the mass of the fluid in the neck at the \( i \)th HRs, where \( \rho \), \( a_i \), and \( \chi'_i \) denote the mass density of the fluid, and the width and the effective length of the neck of the \( i \)th resonator, respectively. From the definition of bulk modulus \( \kappa_0 \), the change in pressure \( \Delta p_i \) inside the resonator can be defined as \( \Delta p_i = -\kappa_0 \frac{\Delta v_i}{v_i} \), \( v_i \) is the area of the cavity of \( i \)th HR, whereas the change of the area \( \Delta v_i \) is equal to \( a_i \eta_i - a_{i+1} \eta_{i+1} \) \( (i = 1,2, \text{ and } 3) \). Taking the harmonic expression for the pressure and the displacement at angular frequency \( \omega \), leads to the following equations:
Rewriting Eqs. (B.4-B.6) in matrix form yields

\[
\begin{pmatrix}
q_{11} & q_{12} & q_{13} \\
q_{21} & q_{22} & q_{23} \\
q_{31} & q_{32} & q_{33}
\end{pmatrix}
\begin{pmatrix}
\eta_1 \\
\eta_2 \\
\eta_3
\end{pmatrix}
= 
\begin{pmatrix}
a_1 p \\
0 \\
0
\end{pmatrix}
\]

(B.7)

The elements of a tridiagonal stiffness matrix are defined as

\[
\begin{align*}
q_{11} &= -\rho_0 a_1 \chi_1^' \omega^2 + \kappa_0 \left( a_1^2 / v_1 \right) \\
q_{12} &= -\kappa_0 \left( a_1 a_2 / v_1 \right) \\
q_{13} &= 0 \\
q_{22} &= -\rho_0 a_2 \chi_2^' \omega^2 + \kappa_0 a_2^2 \left( v_1^{-1} + v_2^{-1} \right) \\
q_{23} &= -\kappa_0 \left( a_2 a_3 / v_2 \right) \\
q_{33} &= -\rho_0 a_3 \chi_3^' \omega^2 + \kappa_0 a_3^2 \left( v_2^{-1} + v_3^{-1} \right)
\end{align*}
\]

Solving Eq.(B.7), we can obtain the displacement of air in the neck of the first resonator, which is given by:

\[
\eta_1 = a_1 p \left( \frac{q_{23} q_{32} - q_{22} q_{33}}{q_{11} q_{23} q_{32} + q_{12} q_{23} q_{32} - q_{11} q_{22} q_{33}} \right)
\]

(B.8)
Then the normalized specific impedance of the three coupled resonators is given by \( \xi = \frac{p}{\xi_0 u_1} \), where \( \xi_0 \) is the impedance of the air and \( u_1 (= \eta'_1 = i\omega \eta_1) \) is the velocity of a particle in the first neck. Then substituting \( \xi \) into Eq. (4.3.4), we obtain the transmission coefficient of the three coupled HRs.
Appendix C Derivation of the Transmission and Reflection Coefficients of an Effective Homogeneous Slab for Shallow Water Waves in Section 7.2

In Section 7.2, based on the parameter retrieval method, we have utilized the transmission and reflection coefficients of the equivalently ultra-thin space-coiled meta-surface structure to derive its effective parameters. In this appendix, we derive the analytical mathematical formalism of these coefficients.

Figure C.1 Illustration of the homogenous slab that possesses effective material parameters $g_{\text{eff}}$ and $d_{\text{eff}}$, and thickness $l$.

As illustrated in Fig. C.1, a plane water wave propagating along the $x$-direction is normally incident on the slab of a homogenous medium with thickness $l$. The displacement of the incident ($\eta_i$), reflected ($\eta_r$), and transmitted ($\eta_t$) water waves can be written as

$$\eta_i(x \leq 0, -w/2 \leq y \leq w/2) = \eta_i e^{i(\omega t - k \cdot x)}$$

(C.1)
\[ \eta_R (x \leq 0, -w/2 \leq y \leq w/2) = \eta_R e^{i(\omega t - k_0 x)} \quad (C.2) \]
\[ \eta_L (x \geq l, -w/2 \leq y \leq w/2) = \eta_L e^{i(\omega t + k_0 (x-l))} \quad (C.3) \]

The expressions of the displacement of the forward and backward traveling the water waves inside the homogenous slab are, respectively, given by:

\[ \eta_A (0 \leq x \leq l, -w/2 \leq y \leq w/2) = Ae^{i(\omega t + k_{0,eff} x)} \quad (C.4) \]
\[ \eta_B (0 \leq x \leq l, -w/2 \leq y \leq w/2) = Be^{i(\omega t - k_{0,eff} x)} \quad (C.5) \]

\[ k_0 (= \omega/\sqrt{g_0 d_0}) \text{ and } k_{0,eff} (= \omega/\sqrt{g_{eff} d_{eff}}) \] are the wave-vector in the environment medium and the homogenous slab, respectively, where \( \omega \) is the angular frequency. \( g_0 \) is the gravitational acceleration and \( d_0 \) is the depth of the environment medium, whereas \( g_{eff} \) is the effective gravitational acceleration and \( d_{eff} \) is the effective depth of the effective liquid medium of the homogenous slab. The displacement velocity of the particles in the shallow water wave is given by \( v_i = -g_{0,eff} \nabla \eta_i \), where \( i (= I, II, III) \) indicates the propagation domain. Then the expressions of the normal velocity of water wave in each domain are written as:

\[ u_I (x \leq 0, -w/2 \leq y \leq w/2) = -ik_0 g_0 \eta_I e^{i(\omega t + k_{0} x)} \quad (C.6) \]
\[ u_R (x \leq 0, -w/2 \leq y \leq w/2) = ik_0 g_0 \eta_R e^{i(\omega t - k_{0} x)} \quad (C.7) \]
\[ u_A (0 \leq x \leq l, -w/2 \leq y \leq w/2) = -ik_{0,eff} g_{eff} A e^{i(\omega t + k_{0,eff} x)} \quad (C.8) \]
\[ u_B \left( 0 \leq x \leq l, -w/2 \leq y \leq w/2 \right) = i k_{\text{eff}} g_{\text{eff}} B e^{i(\omega t-k_{\text{eff}}y)} \quad (C.9) \]

\[ u_T \left( x \geq l, -w/2 \leq y \leq w/2 \right) = -ik_0 g_0 \eta_T e^{i(\omega t-k_0 (x-l))} \quad (C.10) \]

On the interface between the slab and the environment medium at \( x=0 \) and \( x=l \), we apply the boundary conditions, which are continuities of the displacement of the water surface and the velocity, in order to obtain the transmission and reflection coefficients of the homogenous slab.

Then we obtain the following equations:

\[ \eta_l + \eta_R = A + B \quad (C.11) \]

\[ k_0 g_0 \left( \eta_l - \eta_R \right) = k_{\text{eff}} g_{\text{eff}} \left( A - B \right) \quad (C.12) \]

\[ Ae^{ik_{\text{eff}}l} + Be^{-ik_{\text{eff}}l} = \eta_T \quad (C.13) \]

\[ k_{\text{eff}} g_{\text{eff}} \left( Ae^{ik_{\text{eff}}l} - Be^{-ik_{\text{eff}}l} \right) = k_0 g_0 \eta_T \quad (C.14) \]

Solving Eqs. (C.11-C.14) yield the expressions for the transmission \( \hat{T}(\omega) \) and reflection \( \hat{R}(\omega) \) coefficients, taking the forms of

\[ T(\omega) = \frac{\eta_T}{\eta_l} = \frac{4 \left( g_{\text{eff}} k_{\text{eff}} / g_0 k_0 \right) e^{ik_{\text{eff}}l}}{\left[ \left( g_{\text{eff}} k_{\text{eff}} / g_0 k_0 \right) + 1 \right]^2 - \left[ \left( g_{\text{eff}} k_{\text{eff}} / g_0 k_0 \right) - 1 \right]^2 e^{2ik_{\text{eff}}l}} \quad (C.15) \]

\[ R(\omega) = \frac{\eta_R}{\eta_l} = \frac{\left[ 1 - \left( g_{\text{eff}} k_{\text{eff}} / g_0 k_0 \right)^2 \right] + \left[ \left( g_{\text{eff}} k_{\text{eff}} / g_0 k_0 \right)^2 - 1 \right] e^{2ik_{\text{eff}}l}}{\left[ \left( g_{\text{eff}} k_{\text{eff}} / g_0 k_0 \right) + 1 \right]^2 - \left[ \left( g_{\text{eff}} k_{\text{eff}} / g_0 k_0 \right) - 1 \right]^2 e^{2ik_{\text{eff}}l}} \quad (C.16) \]
Publication List


5). Ze-Guo Chen, Changqing Xu, Rasha Al Jahdali, Jun Mei, and Ying Wu “Corner States in a Second-order Acoustic Topological Insulator as Bound States in the Continuum " Physical Review B 7, 100 (2019).

6). Rasha Al Jahdali, Lixin Ge, and Ying Wu" Ocean Waves Focusing by Impedance-matched Subwavelength Meta-surfaces ". In preparation
Conference List

