On the L^p-Integrability of Green's function for Elliptic Operators

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ABSTRACT

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In this thesis, we discuss some of the results that were proven by Fabes and Stroock in 1984. Our main purpose is to give a self-contained presentation of the proof of this results. The first result is on the existence of a "reverse Hölder inequality" for the Green's function. We utilize the work of Muckenhoupt on the reverse Hölder inequality and its connection to the A_∞ class to establish a comparability property for the Green's functions. Additionally, we discuss some of the underlying preliminaries. In that, we prove the Alexandrov-Bakelman-Pucci estimate, give a treatment to the A_p and A_∞ classes of Muckenhoupt, and establish two intrinsic lemmas on the behavior of Green's function.

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Thesis author,

AbdulRahman Alharbi

TABLE OF CONTENTS

NOTATION AND ABBREVIATIONS

A_p and A_∞ are the Muckenhoupt weights defined in page 32.

α_d is the volume of the unit ball in $\mathbb{R}^d$.

$C^k(\Omega)$ is the set of k-times continuously differentiable functions, $u : \Omega \longrightarrow \mathbb{R}$.

$C^k_c(\Omega)$ is the subset of $C^k(\Omega)$, consisting of compactly supported functions.

$\mathscr{D}(\Omega) = C^\infty_c(\Omega)$ is the set of infinitely-differentiable, compactly-supported functions.

$\mathscr{D}'(\Omega)$ is the dual of $\mathscr{D}(\Omega)$; that is, the set of all continuous linear functionals from $\mathscr{D}(\Omega)$ to $\mathbb{R}$.

$|E|$ is the Lebesgue measure (volume) of the set E.

∇u is the gradient of a function u (which we consider as a column vector).

$int(S)$ and S° is the interior of a given set, S.

L is an elliptic operator defined on page 9

L_z is the operator L, applied with respect to z

Ω is an open subset of $\mathbb{R}^d$. Ω is a *domain* if it is also connected.

$\mathrm{Tr}\,(A)$ is the trace of matrix A; that is, the sum of its diagonal entries.

$W^{k,p}(\Omega)$ is the Sobolev space of functions, $u : \Omega \longrightarrow \mathbb{R}$, which are (at least) k-times weakly differentiable and have $D^\alpha u \in L^p$ for every multi-index $|\alpha| \leq k$.

a.e. is an abbreviation for *almost everywhere*.

l.s.c. is an abbreviation for *lower-semi continuous*.

ABP is an abbreviation for *Alexandroff-Bakelman-Pucci* estimate.

LIST OF FIGURES

Chapter 1

Introduction

In December 1984, Eugene Fabes and Daniel Stroock published a paper titled *The L^p-Integrability of Green's Functions and Fundamental Solutions for Elliptic and Parabolic Equations*. Their main objective was to provide new treatments to (second-order) differential elliptic operators of the form

$$Lu = \sum_{i,j=1}^{d} a^{ij}(x)\partial_{x_i x_j} u(x), \tag{1.1}$$

and the associated Green's functions and fundamental solutions. Such elliptic differential operators often appear in the study of stochastic models and the optimal control of diffusion processes (see Lions [9]), making them an essential component of contemporary mathematics.

When talking about elliptic operators, there are two main forms they usually appear in: (i) *the divergence form*, an example of which, is

$$\tilde{L}u = \sum_{i,j=1}^{d} \partial_{x_i} \left(a^{ij}(x)\partial_{x_j} u(x) \right),$$

and (ii) *the non-divergence form*, an example of which is the operator in (1.1). The standard assumption on L and $\tilde{L}$ is that the coefficient matrix, $\mathbf{a} := (a^{ij}(x))$, is measurable on $\Omega \subset \mathbb{R}^d$, and that, for some fixed $\lambda \in (0, 1)$,

$$\lambda I \leq \mathbf{a}(x),$$

in the sense of nonnegative definiteness. The *formal adjoint* of L, in (1.1), is

$$L^* v = \sum_{i,j=1}^{d} \partial^2_{y_i y_j} \left(a^{ij}(y) v(y) \right). \tag{1.2}$$

This operator is also called a uniformly elliptic operator, in *double-divergence form.* However, it is rarely studied alone, and, in the literature, is often discussed simultaneously with its non-divergence counterpart.

Under the additional assumption that $\leq \mathbf{a}(x) \leq \lambda^{-1} I$, we are going to discuss two main results from [5]. First, we establish an estimate on the measure induced by the Green's function, $G(x, y)$, and prove the existence of a "reverse Hölder inequality" for G; that is,

$$\left[\frac{1}{|B_r|} \int_{B_r} G(x,y)^{d/(d-1)} \, dy \right]^{(d-1)/d} \leq \frac{c}{|B_r|} \int_{B_r} G(x,y) \, dy. \tag{1.3}$$

We prove that the measure (induced by G) is comparable to Lebesgue measure, in a sense that, for a ball $B \subset \Omega$, satisfying certain conditions, and measurable $E \subset B$, we have

$$\frac{\int_E G(x,y) \, dy}{\int_B G(x,y) \, dy} \geq c \left(\frac{|E|}{|B|} \right)^{\tau}, \tag{1.4}$$

where τ and c depend solely on d and λ.

Unfortunately, the proofs of the widely-cited results in [5] are not accessible for students and those whom are new to the subject. Our purpose in this thesis is to review these results, and to present their full proofs in a self-contained manner. Accordingly, the thesis starts with a chapter on preliminary theorems and concepts, which are needed in proving the results. In that chapter (Chapter 2), we prove the Alexndroff-Bakelman-Pucci (ABP) estimate, which is an apriori estimate for the strong solutions of $Lu \geq f$ in a bounded, open, domain, $\Omega \in \mathbb{R}^d$.

Next, we investigate the A_p-classes and their properties. Those classes of functions were first introduced by Muckenhoupt in [11], while studying the maximal function. The techniques Muckenhoupt used are useful to establishing some of our results. A locally integrable function ω is said to be in the A_p-class if, for some constant $A > 0$,

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int\limits_B \omega(x)^{-p'/p}dx\right)^{p/p'} \leq A.$$

It is easy to notice that the inequality is related to Hölder's inequality, in someway or another. What is not obvious, however, is the fact that this is a sort of "reversed" Hölder inequality. We will prove this among other results. The proofs in that section (Section 2.2) are based on the book of Stein [13]. The main difficulty we faced, in this part of the thesis, was to adapt the proofs from the case of $\mathbb{R}^d$ to the case of a bounded subset $\Omega \subset \mathbb{R}^d$, which required some attention to the behavior near the boundary and to the change in conditions for each of our results.

The last section of Chapter 2 is concerned with Green's function. In that section, we introduce Green's function and prove two of the properties that are necessary for the completeness of the main chapter. Particularly, we prove that, for two sets $A \subset B$, the corresponding Green's functions G_A and G_B satisfy that $G_A(x,y) \leq G_B(x,y)$ for all $x, y \in A$. We also establish a lower bound for the integral $\int_A G_B(x,y)\,dy$.

In the main chapter, Chapter 3, we proceed with writing the proofs of the results of Fabes and Stroock. We start by establishing the doubling property for (weak) adjoint solutions (i.e. solutions of $L^*v = 0$); that is, there is some constant c that depends only on λ and d such that

$$\int_{B_{2r}(z)} v(x)\,dx \,\leq\, c\int_{B_r(z)} v(x)\,dx$$

for all $r > 0$, as long as $B_{4r} \subset \Omega$, the domain of the problem. Afterwards, we show that such functions also satisfy a reverse Hölder inequality, similar to (1.3). In the following section, we extend the existence of a reverse Hölder inequality to the Green's function, G, and establish (1.3). We conclude the chapter by proving inequality (1.4).

The focus in the writing this chapter is on writing the proofs fully and explicitly. There are some gaps and assertions that were left unproven by Fabes and Stroock, in that paper. We also adjust the constants, in the conditions of theorems to make the theorems consistent. asserts (1.4). Most of the input we have on this chapter appear in the form of the preliminaries of Chapter 2. We reserve appendix for minor results that are necessary but might affect the flow of the presentation.

Chapter 2

Preliminary Concepts and Results

In this chapter, we present and prove a number of preliminary concepts and technical tools that are needed for establishing the results of Fabes and Stroock [5]. The first of these tools is the *Alexandroff-Bakelman-Pucci (ABP) estimate*, which we prove in Section 2.1. In Section 2.2, we discuss a class of functions, characterized by the A_∞ (doubling) property, which is crucial to the main chapter, Chapter 3. In Section 2.3, we define Green's function and discuss some of its elementary properties, in preparation of the main chapter. This self-contained presentation intended to make the material accessible to readers whom are unfamiliar with the topic.

2.1 Alexandroff-Bakelman-Pucci (ABP) Estimate

Before we introduce the estimate, let's discuss the following example, which illustrates the motivation behind the estimate.

Example 2.1.1. Let a, r be nonzero real numbers, with $r > 0$, and consider the differential equation

$$\begin{cases} u_{xx}(x) = a \quad \text{for all } x \in (0, r), \\ u(0) = 0 \;\text{ and }\; u(r) = 1. \end{cases}$$

Direct computations give $u(x) = \frac{1}{2}ax^2 + (r^{-1} - \frac{1}{2}ar)x$. We know that u is a parabola; hence, when $u_{xx} = a > 0$, u is convex. Then, the maximum of u must lie on the boundary of the domain, $\{0, r\}$. Alternatively, when $u_{xx} = a < 0$, u is concave and the maximum may lie in the interior or on the boundary of the domain.

This idea of convexity and concavity is the main mechanism that ABP estimate aims to capture. It does this with a general (second-order) elliptic operator. To give an apriori bound, the key idea is to search for areas of concavity (and convexity), and extracts the possible contributions they may add to the maximum (or minimum) value of u in Ω. For this example, we can use a special case of the ABP estimate (Theorem 2.1.15) to establish the following *apriori* estimate.

$$\sup_{x\in(0,r)} u(x) \leq \sup\{u(0), u(r)\} + \frac{r}{2} \times \left(\int_0^r \left| \frac{a}{1} \right|^1 \right)^1 = 1 + \frac{r^2|a|}{2}.$$

We can easily verify that, for $x \in [0, r]$,

$$u(x) = x\big(\frac{1}{2}a(x-r) + r^{-1}\big) \leq r\big(\frac{1}{2}\,|a|\,|x-r| + r^{-1}\big) \leq \frac{r^2|a|}{2} + 1,$$

which illustrate the validity of the ABP estimate. ▲

With this example in mind, we aim to establish an estimate of the same nature, but in a more general setting. So, let $\Omega \subset \mathbb{R}^d$ be bounded, open and connected. Let $\mathbf{a}(x) = (a^{ij}(x))$ be a matrix-valued function with values in $\mathbb{R}^{d\times d}$, let $\mathbf{b}(x) = (b^i(x))$ be vector-valued function with values in $\mathbb{R}^d$, and let $c(x)$ be real-valued scalar function, all defined on Ω. We set $\mathcal{D}(x) := (\det \mathbf{a}(x))^{1/d}$, and impose, further, the following assumptions:

Assumption 2.1.2.

- The functions $\mathbf{a}$, $\mathbf{b}$, and c are measurable.

- The matrix function $\mathbf{a}(\cdot)$ is uniformly bounded and positive definite. More precisely, there are positive constants, λ and Λ, with $\lambda \leq \Lambda$, such that
  $\lambda\,|\eta|^2 \;\leq\; \eta^T\mathbf{a}(x)\,\eta \;\leq\; \Lambda\,|\eta|^2$ for every $x \in \Omega$ and every $\eta \in \mathbb{R}^d$.

- $|\mathbf{b}|/\mathcal{D} \in L^d(\Omega)$.
- $f/\mathcal{D} \in L^d(\Omega)$.
- $c \leq 0$ in Ω.

Under Assumption 2.1.2, we can define a *(uniformly) elliptic operator*, which is an operator that has the form

$$Lu := \sum_{i,j} a^{ij}(x) D_{ij} u + \sum_i b^i(x) D_i u + c(x) u. \tag{2.1}$$

Remark 2.1.3. Because the eigenvalues of $\mathbf{a}(x)$ are bounded by λ and Λ, we also have that $0 < \lambda \leq (\det \mathbf{a}(x))^{1/d} = \mathcal{D}(x) \leq \Lambda$ for all $x \in \Omega$.

Given this frame of work, our goal for the rest of the section is to establish the ABP estimate, stated below, for twice-weakly-differentiable functions, u, that satisfy $Lu \geq f$, almost everywhere (a.e.) in Ω. Such solutions are called *strong solutions*.

Note that we base our proofs, in this section, on Gilbarg and Trudinger [6], and adapt few definitions and remarks from Braga, Figalli, and Moreira [2].

2.1.1 Contact Sets and Normal Mappings

Upper contact sets and normal mappings are the main technical tools we need for the proof of the ABP estimate below. To clarify the definitions, we start by discussing affine functions and concave envelopes.

Definition 2.1.4. Let V be a real vector space. A real-valued function on V, f, is *affine* if $f(\theta x + (1-\theta)y) = \theta f(x) + (1-\theta) f(y)$ for every $x, y \in V$ and all $\theta \in (0,1)$.

Let $g : \Omega \longrightarrow \mathbb{R}$ be a concave function. We know that, at every point $x \in \Omega$, there exists an affine function, $f_x : \mathbb{R}^d \longrightarrow \mathbb{R}$, such that $f_x(x) = g(x)$ and $f_x(y) \geq g(y)$ for every $y \in \Omega$; i.e. f supports g from above. The drawback is that this is specific for concave functions, and it is not true in general. To address this issue, we introduce the concept of concave envelopes.

Definition 2.1.5. Let $u : \Omega \longrightarrow \mathbb{R}$ be a continuous function. The *concave envelope* of u, $\mathcal{C}_u$, is the function

$$\mathcal{C}_u(y) := \inf \big\{ v(y) \,:\, v \text{ is affine in } \mathbb{R}^d \text{ and } u(x) \le v(x) \,\forall\, x \in \Omega \big\} \tag{2.2}$$

for every $y \in \Omega$.

Remark 2.1.6.

i. The concave envelope is concave because it is an infimum of concave functions.

ii. A function, u, is concave if and only if $\mathcal{C}_u(x) = u(x)$ for every $x \in \Omega$.

Clearly, $\mathcal{C}_u(x) \ge u(x)$. However, the set of points where equality holds is of primary interest to us.

Definition 2.1.7. Let $u : \Omega \longrightarrow \mathbb{R}$ be a continuous function. The *upper contact set* of u, Γ_u^+ or simply Γ^+, is

$$\Gamma^+ = \big\{ y \in \Omega \,:\, u(y) = \mathcal{C}_u(y) \big\}. \tag{2.3}$$

Remark 2.1.8.

i. Because of Remark 2.1.6, a function u is concave if and only if $\Gamma_u^+ = \Omega$.

ii. We can view Γ^+ as the set of points of Ω where the graph of u can be placed below some support hyperplane in $\mathbb{R}^{d+1}$; that is,

$$\Gamma^+ = \big\{ y \in \Omega \,:\, \exists\, p \in \mathbb{R}^d,\;\; u(x) \le u(y) + p \cdot (x - y) \;\forall\, x \in \Omega \big\}. \tag{2.4}$$

iii. Suppose $u \in C^1(\Omega)$, and let $y \in \Gamma^+$. Then, $u(x) \leq u(y) + Du(y)\cdot(x-y)$ for every $x \in \Omega$ since any support hyperplane is tangent to the graph of u. Moreover, by a continuity argument, we can readily see that Γ^+ is closed.

iv. As asserted by Lemma A.2, in the appendix, when $u \in C^2(\Omega)$, the Hessian matrix $D^2u(y) = [D_{ij}u(y)]$ is negative semi-definite for every $y \in \Gamma^+$.

Definition 2.1.9. Let $u : \Omega \longrightarrow \mathbb{R}$ be a continuous function. The *normal mapping* of a point $y \in \Omega$ with respect u, denoted $\chi_u(y)$ or simply $\chi(y)$, is the set of all "slopes" of support hyper-planes at y that lie above the graph of u; that is,

$$\chi(y) = \big\{p \in \mathbb{R}^d \ : \ u(x) \leq u(y) + p\cdot(x-y)\,\forall x \in \Omega\big\}. \tag{2.5}$$

Remark 2.1.10. i. It is clear that $\chi(y) \neq \phi$ if and only if $y \in \Gamma^+$.

ii. As asserted by Lemma A.1, in the appendix, if $u \in C^1(\Omega)$ and $y \in \Gamma^+$, then $\chi(y) = \{Du(y)\}$. In this case, we identify $\chi(y)$ with the vector $Du(y)$.

iii. We use $\chi_u(\Omega)$ to denote the union of the normal mappings of all points in Ω.

iv. For a general set $S \subset \mathbb{R}^{d+1}$, which is not necessarily a graph, the normal mapping is the set of all support hyper-planes lying above S; that is,

$$\chi_S(\Omega) := \{p \in \mathbb{R}^d \ : \ \exists a \in \mathbb{R} \text{ s.t. the graph of } a + p\cdot x \text{ in } \Omega$$
$$\text{touches } S \text{ and lies above it}\}.$$

The concept of a cone with an arbitrary connected, bounded base, in $\mathbb{R}^d$, is essential for the proof. For a connected and bounded subset, $E \subset \mathbb{R}^d$, *a cone, K, of vertex $(v,s) \in E \times \mathbb{R}$ and base ∂E* is the collection of all segments connecting the point (v, s) and the set $\partial E \times \{0\}$ in $\mathbb{R}^{d+1}$; that is,

$$K = \bigcup \big\{ t(v,s) + (1-t)(z,0) \,:\, t \in [0,1] \text{ and } z \in \partial E \big\}. \tag{2.6}$$

Below, we highlight am important example, where we derive an explicit formula for the normal mapping of a cone with a spherical base.

Example 2.1.11. Let $B = B_r(z)$ be a ball, in $\mathbb{R}^d$, with radius $r > 0$ and center z, and let $\beta > 0$ be constant. Let u be a function whose graph is a cone of vertex (z, a) and base ∂B; more explicitly,

$$u(x) := a\left(1 - \frac{|x-z|}{r}\right).$$

In this case, the normal mapping can be expressed explicitly by

$$\chi(y) = \begin{cases} \frac{-a(y-z)}{r(|y-z|)} & \text{for } y \neq z, \\ B_{a/r}(0) & \text{for } y = z. \end{cases} \tag{2.7}$$

It is worth noting, as well, that $\chi(B) = \chi(z)$, which comes to aid in a later discussion.

▲

The next lemma exhibits another attribute of cones, which is needed for establishing Lemma 2.1.14. We defer the proof to the appendix (see page 64).

Lemma 2.1.12. *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a function with a positive maximum at an interior point, $y \in \Omega$. Assume further that u is nonpositive on $\partial\Omega$. Let K be a cone of vertex $(y, u(y))$ and base $\partial\Omega$. Then, $\chi_K(\Omega) \subset \chi_u(\Omega)$, where $\chi_f(\Omega) = \bigcup_{x\in\Omega} \chi_f(x)$.*

2.1.2 The ABP Estimate

In this subsection, we present a detailed proof of the ABP estimate, which is stated in the following theorem.

Theorem 2.1.13. (the ABP estimate) *Let* $\Omega \subset \mathbb{R}^d$ *be bounded, open, and connected. Suppose that* $u \in W^{2,d}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ *solves* $Lu \geq f$ *in* Ω*. Then,*

$$\sup_{x\in\Omega} u(x) \leq \sup_{x\in\partial\Omega} u^+(x) + c\,\|f/\mathcal{D}\|_{L^d(\Omega)} \tag{2.8}$$

for a constant c *that depends only on* d*,* $\operatorname{diam}(\Omega)$*, and* $\|\mathbf{b}/\mathcal{D}\|_{L^d(\Omega)}$*.*

Before presenting the proof of the theorem, we need to establish several auxiliary results. Our first result, the lemma below, provides an upper-bound for functions, $u \in C^2(\Omega)$, based solely on information about u on $\partial\Omega$ and about D^2u in Γ^+.

Lemma 2.1.14. *Let* u *be an arbitrary function in* $C^2(\Omega) \cap C^0(\overline{\Omega})$ *. Then,*

$$\sup_{x\in\Omega} u(x) \leq \sup_{x\in\partial\Omega} u(x) + \frac{\operatorname{diam}(\Omega)}{\alpha_d^{1/d}} \left(\int_{\Gamma^+} |\det D^2u|\right)^{1/d}, \tag{2.9}$$

where α_d *is the volume of the unit-ball in* $\mathbb{R}^d$*, and* $\operatorname{diam}(\Omega)$ *is the diameter of* Ω*.*

Proof. 1. Without loss of generality, we can replace u with $\tilde{u} := u - \sup_{\partial\Omega} u$. Thus, it is enough to prove the lemma under the assumption $u \leq 0$ on $\partial\Omega$. Now, due to Remarks 2.1.6 and 2.1.8, the Lebesgue measure of the normal mapping of Ω satisfies

$$|\chi(\Omega)| = \left|\chi(\Gamma^+)\right| = \left|Du(\Gamma^+)\right| = \int_{Du(\Gamma^+)} dp. \tag{2.10}$$

Now, we want to apply the classical change of variables formula. However, we are not assured that $Du(\cdot)$ is one-to-one on Γ^+. So, we can only assert the inequality

$$\int_{Du(\Gamma^+)} dp \leq \int_{\Gamma^+} |\det(D^2u)|dy. \tag{2.11}$$

For a precise proof, refer to Lemma A.3, in the appendix.

2. For the next step, we give an estimate on u in terms of $|\chi(\Omega)|$. Notice that,

if u has no positive maximum, then the conclusion of the lemma is obvious (bearing in mind the previous assumption, $\sup_{\partial\Omega} u = 0$). So, we may assume that u attains a positive maximum at an interior point $y \in \Omega$.

Now, let K be a cone of vertex $(y, u(y))$ and base $\partial\Omega$, and let J be another cone of vertex $(y, u(y))$ and base $\partial B_{\mathrm{diam}(\Omega)}(y)$. According to Lemma 2.1.12, $\chi_K(\Omega) \subset \chi_u(\Omega)$. Moreover, we can easily see that J lies above K, and that $\chi_J(y) \subset \chi_K(y)$. Also, as pointed in Example 2.1.11, $\chi_J(\Omega) = \chi_J(y)$; hence, $\chi_J(\Omega) \subset \chi_K(\Omega)$. Consequently,

$$|\chi_J(\Omega)| \leq |\chi_K(\Omega)| \leq |\chi_u(\Omega)|.$$

Combining this inequality with (2.7), 2.10, and (2.11), we obtain

$$\alpha_d \left(\frac{u(y)}{\mathrm{diam}(\Omega)} \right)^d \leq \int_{\Gamma^+} \left|\det D^2 u\right|.$$

By rearranging the terms, we obtain

$$\sup_{\Omega} = u(y) \leq \frac{\mathrm{diam}(\Omega)}{\alpha_d^{1/d}} \left(\int_{\Gamma^+} \left|\det D^2 u\right| \right)^{1/d},$$

which concludes the proof of the lemma. □

A special case of Theorem 2.1.13 occurs when $b^i = c = 0$ (i.e. $Lu = \sum_{i,j} a^{ij} D_{ij} u$). In this case, the estimate (2.8), for C^2 function, follows directly from Lemma 2.1.14.

Theorem 2.1.15. (A special case of the ABP Estimate). Let u be an arbitrary function in $C^2(\Omega) \cap C^0(\overline{\Omega})$, then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \frac{\mathrm{diam}(\Omega)}{\alpha_d^{1/d} d} \left\| \frac{\sum_{i,j} a^{ij} D_{ij} u}{\mathcal{D}} \right\|_{L^d(\Gamma^+)}. \tag{2.12}$$

Proof. Let A and B be positive semi-definite symmetric matrices. We recall that

$\det(AB)$ is the product of the eigenvalues of AB, and $\mathrm{Tr}(AB)$ is their sum. Accordingly, the Arithmetic Mean-Geometric Mean inequality (AM-GM) gives

$$(\det(A)\det(B))^{1/d} = \det(AB)^{1/d} \le \frac{\mathrm{Tr}(AB)}{d}. \tag{2.13}$$

Recall that D^2u is negative semi-definite in Γ^+. Thus, with $A = -D^2u$ and $B = \mathbf{a}$,

$$\left|\det D^2u(y)\right| = \det(-D^2u(y)) \ \le \frac{1}{\mathcal{D}^d}\left(\frac{-\sum_{i,j} a^{ij} D_{ij}u(y)}{d}\right)^d, \tag{2.14}$$

for every $y \in \Gamma^+$. By applying Lemma 2.1.14, we obtain

$$\begin{aligned}
\sup_{\Omega} u &\le \sup_{\partial\Omega} u + \frac{\mathrm{diam}(\Omega)}{\alpha_d^{1/d}} \left(\int_{\Gamma^+} \left|\det D^2u\right|\right)^{1/d} \\
&\le \sup_{\partial\Omega} u + \frac{\mathrm{diam}(\Omega)}{\alpha_d^{1/d}} \left[\int_{\Gamma^+} \left(\frac{-\sum_{i,j} a^{ij} D_{ij}u}{\mathcal{D}d}\right)^d\right]^{1/d} \\
&= \sup_{\partial\Omega} u + \frac{\mathrm{diam}(\Omega)}{\alpha_d^{1/d} d} \left\|\frac{\sum_{i,j} a^{ij} D_{ij}u}{\mathcal{D}}\right\|_{L^d(\Gamma^+)}.
\end{aligned}$$

This proves the special case. □

Our next proposition combines the ideas used in the proofs of the last two results with slight modifications. By doing so, we can incorporates the special case, stated in Theorem 2.1.15, into the proof of Theorem 2.1.13.

Proposition 2.1.16. *Let $g \in L^1_{loc}(\Omega)$ be non-negative, and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be arbitrary. Then,*

$$\int_{B_{\widetilde{M}}(0)} g \le \int_{\Gamma^+} g(Du)\left|\det D^2u\right| \le \int_{\Gamma^+} g(Du)\left(\frac{-\sum_{i,j} a^{ij} D_{ij}u}{\mathcal{D}\, d}\right)^d, \tag{2.15}$$

where $\widetilde{M} := (\sup_\Omega u - \sup_{\partial\Omega} u)/\,\mathrm{diam}(\Omega)$.

Proof. This is a porism that follows by combining the proofs of the last two results and using Example 2.1.11. Indeed, the right-hand-side inequality,

$$\int\limits_{\Gamma^+} g(Du)\left|\det D^2u\right| \leq \int\limits_{\Gamma^+} g(Du)\left(\frac{-\sum_{i,j} a^{ij}D_{ij}u}{\mathcal{D}\, d}\right)^d,$$

is a direct consequence of (2.14). To prove the other part of the inequality, we recall first that $B_{\widetilde{M}}(0) \subset \chi_u(\Omega)$, which we established in the proof of Lemma 2.1.14 (the case $\sup_\Omega u = \sup_{\partial\Omega} u$ is obvious). Therefore,

$$\int\limits_{B_{\widetilde{M}}(0)} g \leq \int\limits_{\chi_u(\Omega)} g.$$

Using a similar argument to the one we used to establish (2.11), we can apply the change of variables formula to obtain that

$$\int\limits_{\chi_u(\Omega)} g \leq \int\limits_{\Gamma^+} g(Du)\left|\det D^2u\right|. \tag{2.16}$$

This proves the proposition. □

Now, we are ready present the proof of Theorem 2.1.13.

Proof of Theorem 2.1.13. We divide this proof into three steps:

I. We prove the estimate for $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, with $u \geq 0$ and $\sup_{\partial\Omega} u = 0$.

II. We extend the proof for any $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

III. Lastly, we extend the proof to the case $u \in W^{2,d}_{\text{loc}}(\Omega) \cap C^0\left(\overline{\Omega}\right)$.

I. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a nonnegative function that solves $Lu \geq f$ in Ω. Suppose further that $\sup_{\partial\Omega} u = 0$, and set $\widetilde{M} = \sup_\Omega u/\operatorname{diam}(\Omega)$. Let $\mu > 0$ be a constant, which we will choose appropriately at the end of the proof. Define, also, a (weight) function,

$$g(p) := \left(|p|^{d/(d-1)} + \mu^{d/(d-1)}\right)^{1-d}.$$

To prove the theorem, we need to find an upper-bound for $\widetilde{M}$ of the form $c\,\|f/\mathcal{D}\|_{L^d(\Gamma^+)}$, which we do as follows.

1. Notice that

$$\alpha_d \log\left(\frac{\widetilde{M}^d}{\mu^d} + 1\right) = \alpha_d \int\limits_0^{\widetilde{M}} (r^d + \mu^d)^{-1}\, dr = \int\limits_{B_{\widetilde{M}}(0)} (|p|^d + \mu^d)^{-1}\, dp.$$

Additionally, by Jensen's inequality, we have

$$2^{1-d}\, g(p)^{-1} = \left(\frac{1}{2}\,|p|^{d/(d-1)} + \frac{1}{2}\,\mu^{d/(d-1)}\right)^{d-1} \leq \frac{1}{2}\,|p|^d + \frac{1}{2}\,\mu^d.$$

Therefore,

$$\alpha_d \log\left(\frac{\widetilde{M}^d}{\mu^d} + 1\right) = \int\limits_{B_{\widetilde{M}}(0)} (|p|^d + \mu^d)^{-1}\, dp \leq 2^{d-2} \int\limits_{B_{\widetilde{M}}(0)} g(p)\, dp. \tag{2.17}$$

2. Next, we want to find an adequate upper-bound for the right integral in (2.17). Recall that, by Proposition 2.1.16, we have

$$\int\limits_{B_{\widetilde{M}}(0)} g \leq \int\limits_{\Gamma^+} g(Du)\left(\frac{-\sum_{i,j} a^{ij} D_{ij}u}{\mathcal{D}\, d}\right)^d. \tag{2.18}$$

Additionally, since $Lu \geq f$, $u \geq 0$, and $c \leq 0$, we have

$$\frac{-\sum_{i,j} a^{ij} D_{ij}u}{\mathcal{D}d} \leq \frac{-\sum_{i,j} a^{ij} D_{ij}u - cu}{\mathcal{D}d} \leq \frac{\sum_i b^i D_i u - f}{\mathcal{D}d} = \frac{b \cdot Du - f}{\mathcal{D}d}.$$

Moreover, by Cauchy-Schwartz and triangle inequalities, we have

$$\frac{\sum_i b^i D_i u - f}{\mathcal{D}d} \leq \frac{|b|\,|Du| - |f|}{\mathcal{D}d}.$$

Combining these inequalities with (2.18), we obtain

$$\int\limits_{B_{\widetilde{M}}(0)} g \ \leq \ \int\limits_{\Gamma^+} g(Du) \left(\frac{|b|\,|Du| - |f|}{\mathcal{D}\, d} \right)^d . \tag{2.19}$$

Combining this inequality with (2.17), we get

$$\alpha_d \log \left(\frac{\widetilde{M}^d}{\mu^d} + 1 \right) \ \leq \ 2^{d-2} \int\limits_{\Gamma^+} g(Du) \left(\frac{|b|\,|Du| - |f|}{\mathcal{D}\, d} \right)^d . \tag{2.20}$$

3. Now, notice that, by the (discrete) Hölder inequality, we have

$$|b||p| + |f| \leq (|b|^d + (\mu^{-1}|f|)^d)^{1/d} \underbrace{(|p|^{d/(d-1)} + \mu^{d/(d-1)})^{(d-1)/d}}_{g(p)^{-1/d}}.$$

If we raise the inequality to the power d and multiply by $g(p)$, we obtain

$$g(p)\,(|b||Du| + |f|)^d \leq |b|^d + (\mu^{-1}|f|)^d.$$

In particular, when $p = Du$, we get

$$g(Du) \left(\frac{|b||Du| + |f|}{\mathcal{D}d} \right)^d \leq \frac{|b|^d + \mu^{-d}|f|^d}{\mathcal{D}^d d^d}. \tag{2.21}$$

By incorporating (2.21) into (2.20), we find that

$$\alpha_d \log \left(\frac{\widetilde{M}^d}{\mu^d} + 1 \right) \ \leq \ \int\limits_{\Gamma^+} \frac{|b|^d + \mu^{-d}|f|^d}{\mathcal{D}^d d^d}.$$

Equivalently,

$$\frac{\widetilde{M}^d}{\mu^d} + 1 \;\leq\; \exp\left[\frac{1}{\alpha_d\, d^d}\left(\mu^{-d}\int_{\Gamma^+}\frac{|f|^d}{\mathcal{D}^d} + \int_{\Gamma^+}\frac{|b|^d}{\mathcal{D}^d}\;\right)\right]. \tag{2.22}$$

4. Now, we are ready to choose the constant μ. If $f \neq 0$ almost everywhere in Γ^+, we set $\mu = \|f/\mathcal{D}\|_{L^d(\Gamma^+)} > 0$. In this case, with a little of arithmetic, (2.22) becomes

$$\widetilde{M} \;\leq\; \|f/\mathcal{D}\|_{L^d(\Gamma^+)}\left\{\exp\left[\frac{1}{\alpha_d\, d^d}\left(1 + \int_{\Gamma^+}\frac{|b|^d}{\mathcal{D}^d}\;\right)\right] - 1\right\}^{1/d}. \tag{2.23}$$

On the other hand, if $f = 0$ almost everywhere in Γ^+ we let $\mu \to 0^+$. Albeit, the estimate (2.23) is satisfied in this case, as well. By setting

$$C := \left\{\exp\left[\frac{1}{\alpha_d\, d^d}\left(1 + \int_{\Gamma^+}\frac{|b|^d}{\mathcal{D}^d}\;\right)\right] - 1\right\}^{1/d},$$

we see that

$$\sup_{\Omega} u \;\leq\; C\ \mathrm{diam}(\Omega)\ \|f/\mathcal{D}\|_{L^d(\Gamma^+)} \;\leq\; C\ \mathrm{diam}(\Omega)\ \|f/\mathcal{D}\|_{L^d(\Omega)}\,. \tag{2.24}$$

Clearly, the constant, $c := C\ \mathrm{diam}(\Omega)$, satisfies the criterion we wanted.

II. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, and set $v := u - \sup^+_{\partial\Omega}$. If $u \leq 0$, on Ω, the estimate is obvious. Thus, assume that $u > 0$ somewhere in Ω. Let $\{C_i\}_{i\in J}$ be the collection of connected components of the set, $\Omega^+ = \{x \in \Omega\ :\ v(x) > 0\}$, for some index set J. Notice that $Lv = Lu - c(x)\sup_{\partial\Omega} u^+ \geq Lu \geq f$. Also, $\sup_{\partial\Omega} v \leq 0$; hence, $v \equiv 0$ on ∂C_i for every $i \in J$. Therefore, we can apply estimate (2.24), and obtain

$$\sup_{C_i} v \;\leq\; C\ \mathrm{diam}(C_i)\ \|f/\mathcal{D}\|_{L^d(C_i)} \;\leq\; C\ \mathrm{diam}(\Omega)\ \|f/\mathcal{D}\|_{L^d(\Omega)}\,.$$

Rewriting $v = u - \sup_{\partial\Omega} u^+$, we have that

$$\sup_{C_i} u \leq \sup_{\partial\Omega} u^+ + C \ \mathrm{diam}(\Omega) \ \|f/\mathcal{D}\|_{L^d(\Omega)}$$

for every $i \in J$. Thus,

$$\sup_{\Omega} u = \sup_{i\in J} \sup_{C_i} u \leq \sup_{\partial\Omega} u^+ + C \ \mathrm{diam}(\Omega) \ \|f/\mathcal{D}\|_{L^d(\Omega)} . \tag{2.25}$$

III. At this stage, we need to extend the result to the case $u \in W^{2,d}_{\mathrm{loc}}(\Omega) \cap C^0\left(\overline{\Omega}\right)$. We achieve this by the following approximation argument.

1. Consider the case when L is uniformly elliptic in Ω and $|b|/\mathcal{D}$ is bounded. Let $\{u_m\}$ be a sequence in $C^2(\Omega) \cap C^0(\overline{\Omega})$, such that $u_m \to u$ in $W^{2,d}_{loc}(\Omega)$. For every $\varepsilon > 0$, and a suitable $\Omega_\varepsilon \subset\subset \Omega$ (where $\cup_{\varepsilon>0}\Omega_\varepsilon = \Omega$), we can assume that $u_m \to u$ in $W^{2,d}(\Omega_\varepsilon)$ and $u_m \leq \varepsilon + \sup_{\partial\Omega} u$ on $\partial\Omega_\varepsilon$.

Under these assumptions, we can apply (2.25) to u_m, and obtain that

$$\sup_{\Omega_\varepsilon} u \leq \varepsilon + \sup_{\partial\Omega} u^+ + \frac{C}{\lambda} \left\| \sum_{i,j} a^{ij} D_{ij}(u_m - u) + \sum_i b^i D_i(u_m - u) \right\|_{L^d(\Omega_\varepsilon)} + c \|f/\mathcal{D}\|_{L^d(\Omega_\varepsilon)} .$$

Since $\{u_m\}$ converges to u uniformly in Ω_ε, we can take $m \to \infty$ to establish that

$$\sup_{\Omega_\varepsilon} u \leq \varepsilon + \sup_{\partial\Omega} u^+ + c \|f/\mathcal{D}\|_{L^d(\Omega_\varepsilon)} . \tag{2.26}$$

Taking the limit as $\varepsilon \to 0$, we obtain (2.8).

2. For the general case, let $\eta > 0$ and set

$$\begin{aligned} L_\eta u &:= Lu + \eta(\Lambda + |b|)\Delta u \\ &= \sum_{i,j} \left(a^{ij} + \delta^i_j \eta(\Lambda + |b|)\right) D_{ij} u + \sum_i b^i D_i u + cu. \end{aligned}$$

Notice that L_η satisfies the earlier restriction. Thus, as in (2.26), we have

$$\sup_{\Omega_\varepsilon} u \leq \varepsilon + \sup_{\partial\Omega} u^+ + C \left[\left\| \frac{\eta(\Lambda + |b|)\Delta u}{\mathcal{D}_\eta} \right\|_{L^d(\Omega_\varepsilon)} + \left\| \frac{f}{\mathcal{D}} \right\|_{L^d(\Omega_\varepsilon)} \right].$$

Now, we let $\eta \to 0$, and use the Dominated Convergence Theorem (DCT) to arrive at inequality (2.26). There, we take $\varepsilon \to 0$, and obtain the general case of Theorem 2.1.13. □

2.2 The A_p and A_∞ Classes

In this section, we discuss a class of functions called the A_p *class*. Some authors refer to it as *Muckenhoupt weights* since Muckenhoupt was the first to introduced them and establish bounds on Hardy-Littlewood maximal function (see [14] and [11]).

There are two main results in this section; both crucial to the material of the main chapter, Chapter 3. Those results have appeared in Coifman and Fefferman [1], presented in a concise format. The detailed proofs, however, are adapted from Stein [13], where a broader investigation of the subject can be found. The aim of the following presentation is to establish the necessary tools for the main chapter, while providing an adequate notion about the A_p class.

Unless otherwise specified, in this section, Ω denotes an open subset of $\mathbb{R}^d$, B denotes a ball, and Q denotes a cube *whose sides are parallel to the coordinate-axes.* We, also, impose the following assumptions on the measures we use within this section.

Assumption 2.2.1. Let μ be a measure on a set $\Omega \subset \mathbb{R}^d$, we assume that

i. μ is a positive Borel measure.

ii. μ is a *doubling measure*; that is, there is constants $c > 1$ such that, for every $x \in \Omega$ and every positive $r < \text{dist}(x, \partial\Omega)$, we have $\mu(B_{2r}(x)) \leq c\mu(B_r(x))$.

iii. For every open $U \subset \Omega$ and $r > 0$, the mapping $x \mapsto \mu(B_r(x) \cap U)$ is continuous.

Note that we will, later, introduce a measure whose density is a function of class A_p. Our results will show that such functions satisfy the above assumptions. Thus, all the proofs we present are valid for such measures.

2.2.1 The Dyadic Maximal Function

As mentioned in the introduction of the section, the study of *maximal function* was the main motivation to explore the A_p classes. In fact, the proofs and results show that the maximal function is ingrained in the definitions and properties of those classes. However, we are also going to use a variant of the maximal function, called *the dyadic maximal function*. For that purpose we introduce the notion of dyadic cubes.

Definition 2.2.2. Let $\mathcal{Q}_k^{\Delta}$ be the family of all closed cubes that have sides of length 2^{-k} and vertices (corners) of the form $(2^{-k}m_1, ..., 2^{-k}m_d)$, where $m_1, ..., m_d$ are integers. Let $\mathcal{Q}^{\Delta} := \bigcup_{k\in\mathbb{Z}} \mathcal{Q}_k^{\Delta}$. An element of $\mathcal{Q}^{\Delta}$ is called a *dyadic cube*.

Remark 2.2.3.

1. We can obtain a family, $\mathcal{Q}_k^{\Delta}$, from $\mathcal{Q}_0^{\Delta}$ by rescaling; $\mathcal{Q}_k^{\Delta} = 2^{-k}\mathcal{Q}_0^{\Delta}$.

2. Every $Q \in \mathcal{Q}_0^{\Delta}$ has vertices with integers coordinates. Thus, $\mathcal{Q}_0^{\Delta}$ is countable, and so is $\mathcal{Q}^{\Delta}$. Therefore, any collection of dyadic cubes is at most countable.

3. Bisecting the sides of a dyadic cube $Q \in \mathcal{Q}_k^{\Delta}$ creates 2^d cubes in $\mathcal{Q}_{k+1}^{\Delta}$. Each of these cubes is a *child* of Q, and Q is called *the parent*. Figure 2.2.1 depicts the layout of the $\mathcal{Q}_0^{\Delta}$-cubes in $\mathbb{R}^2$, and shows how $\mathcal{Q}_1^{\Delta}$ is obtained by bisection.

4. Every cube $Q \in \mathcal{Q}_k^{\Delta}$ has a unique parent in $\mathcal{Q}_{k-1}^{\Delta}$ as depicted by the figure below.

5. Every open set is the union of dyadic cubes, as shown in Lemma B.1.1.

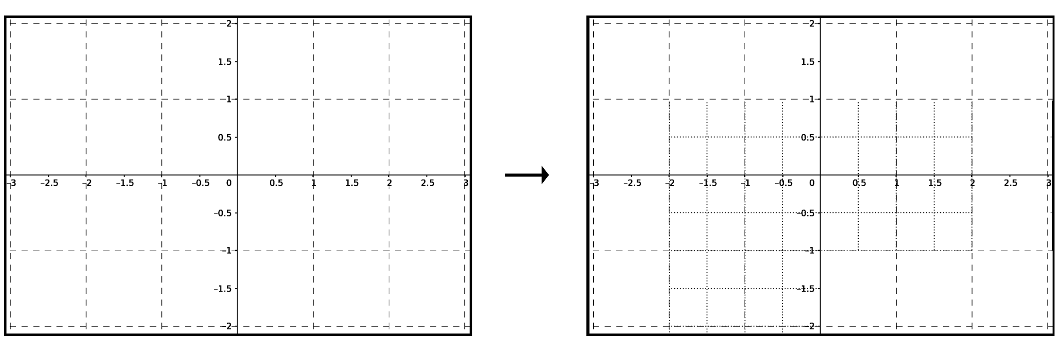

Figure 2.1: $\mathcal{Q}_0^{\Delta}$ "giving birth" to $\mathcal{Q}_1^{\Delta}$

For our next step, we are going to veer our attention to the study of maximal function with respect to measures beyond the Lebesgue measure; namely, measures that satisfy Assumption 2.2.1.

With the notions we defined above, we can proceed to discussing the maximal function and its dyadic variant.

Definition 2.2.4. Let $\Omega \subset \mathbb{R}^d$ be open, and let μ be a measure satisfying Assumption 2.2.1. For a locally integrable function, $f : \Omega \longrightarrow \mathbb{R}$, we define the *maximal function of* f *with respect to* μ, f^*_μ, by

$$f^*_\mu(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| \, d\mu, \quad \text{for all } x \in \Omega, \tag{2.27}$$

where the supremum is taken over all the balls $B \subset \Omega$ that contain x.

Definition 2.2.5. Let $\Omega \subset \mathbb{R}^d$ be open, and let μ be a measure defined on Ω, satisfying Assumption 2.2.1. For a locally integrable function $f : \Omega \longrightarrow \mathbb{R}$, we define the *dyadic maximal function of* f *with respect to* μ, f^Δ_μ, by

$$f^\Delta_\mu(x) = \sup_{\substack{Q \in \mathcal{Q}^\Delta \\ Q \ni x}} \frac{1}{\mu(Q)} \int_Q |f| \, d\mu, \quad \text{for all } x \in \Omega, \tag{2.28}$$

where the supremum is taken over all dyadic cubes $Q \subset \Omega$ such that $x \in Q$.

Remark 2.2.6.

i. The maximal functions we defined in (2.27) and (2.28) are said to have the *non-centered form*, while the *centered form* requires that the integration domain $B = B_r(x)$, for some $r > 0$.

ii. When μ is the Lebesgue measure, we use f^* and f^Δ in place of f^*_μ and f^Δ_μ.

iii. Lebesgue Differentiation Theorem (see [4], Section 1.7) tells us that, except for a set of measure zero, $\lim_{Q \to \{x\}} \ \mu(Q)^{-1} \int_Q |f| \, d\mu = |f(x)|$, where the limit is taken over dyadic cubes Q that contain x. Therefore, $f_\mu^\Delta \geq |f|$, μ-a.e.

The following proposition is an essential result for the subsequent discussion of A_p-classes.

Proposition 2.2.7. *Let $Q_\Delta \subset \mathbb{R}^d$ be a dyadic cube. Consider a measure μ that satisfies Assumption 2.2.1, and suppose that $0 < \mu(Q_\Delta) < \infty$. Let $f : Q_\Delta \longrightarrow \mathbb{R}$ be a locally integrable function, and let $\alpha \geq \mu(Q_\Delta)^{-1} \int_{Q_\Delta} |f| \, d\mu$ be a constant. Lastly, set $S_\alpha := \{x \in Q_\Delta : \ f_\mu^\Delta(x) > \alpha\}$. Then, there is a collection $\mathbf{Q} = \{\mathbf{Q_j}\}$ of dyadic cubes and a constant $\delta > 0$, depending only on μ and d, such that*

i. $\bigcup_{i=1}^\infty Q_i = S_\alpha$,

ii. the members of $\mathbf{Q}$ have disjoint interiors (i.e. $Q_i^\circ \cap Q_j^\circ = \emptyset$ for every $j \neq i$),

iii. and $\alpha < \ \frac{1}{\mu(Q_j)} \int_{Q_j} |f| \, d\mu \ \ \leq \delta^{-1} \alpha$ *for every j.*

Proof. Observe, first, that for every $x \in \Omega_\alpha$, there is a dyadic cube $Q \subset \Omega_\alpha$ that contains x. In particular, because Ω_α has a finite measure, we can select a maximal dyadic cube $Q \subset \Omega_\alpha$ that contains x and satisfies

$$\alpha < \ \frac{1}{\mu(Q)} \int_Q |f| \, d\mu.$$

Notice that, although it is a dyadic cube, $Q \neq Q_\Delta$ by the choice of α. Therefore, the immediate dyadic parent of Q is a contained in Q_Δ (possibly Q_Δ itself).

Let $\mathbf{Q} = \{Q_j\}$ be the collection of all maximal dyadic cubes. Remark 2.2.3 shows that these cubes have disjoint interiors. Then, the inequality

$$\alpha < \ \frac{1}{\mu(Q_j)} \int_{Q_j} |f| \, d\mu$$

holds for every j. This proves claims (i) and (ii), as well as the left hand side inequality of claim (iii). To complete the verification of claim (iii), we use the assumption that μ is a doubling measure; that is, there is a constant $\delta \in (0,1)$ such that, for every cube $Q \subset Q_\Delta$ and every measurable subset $E \subset Q$, we have

$$|E| \geq \frac{1}{2^d}|Q| \qquad \Longrightarrow \qquad \mu(E) \geq \delta\mu(Q).$$

Now, for every j, let Q_j' be the (unique) parent dyadic cube of Q_j, which has twice the side length of Q_j. Observe that $|Q_j| = 2^{-d}\left|Q_j'\right|$; hence, $\mu(Q_j) \geq \delta\mu(Q_j')$. Moreover, due to the maximality of Q_j and the fact that $Q_j' \subset Q_\Delta$, we have

$$\frac{1}{\mu(Q_j')}\int_{Q_j'} |f|\, d\mu \leq \alpha.$$

Combining those observations, we conclude that

$$\frac{1}{\mu(Q_j)}\int_{Q_j} |f|\, d\mu \;\leq\; \delta^{-1}\frac{1}{\mu(Q_j')}\int_{Q_j'} |f|\, d\mu \;\leq\; \delta^{-1}\alpha.$$

□

2.2.2 Defining the A_p and A_∞ Classes

In this subsection, we formulate a general definition for the A_p and A_∞ classes. However, we are mostly interested in the special case where λ is the Lebesgue measure and μ is absolutely continuous with respect to the Lebesgue measure. Additionally, note that, depending on the application, we are going to alternate between the cubes and balls in these definitions. This is valid, as our presentation will clarify.

Definition 2.2.8. Let $p > 1$ and $p' > 1$ satisfy $1/p' + 1/p = 1$. Let λ and μ be (positive) measures defined Ω. Assume further that μ is locally absolutely continuous with respect to λ (i.e. $\mu \ll \lambda$ on every bounded subset of Ω), and let $d\mu/d\lambda$ be the

Radon–Nikodym derivative. We say that μ *is of class* $A_p^\lambda(\Omega)$ if and only if there is a constant $A \in (0, +\infty)$ such that, for every ball $B \subset \Omega$,

$$\frac{\mu(B)}{\lambda(B)} \left(\frac{1}{\lambda(B)} \int\limits_B \left(\frac{d\mu}{d\lambda} \right)^{-p'/p} d\lambda \right)^{p/p'} \leq A. \tag{2.29}$$

The infimum of all such A's is called the A_p *bound*.

Remark 2.2.9. In the special case, for which λ is the Lebesgue measure, we use the notation $A_p(\Omega)$. Also, by setting $\omega = d\mu/d\lambda$, the condition in (2.29) becomes

$$\left(\frac{1}{|B|} \int_B \omega(x)\, dx \right) \left(\frac{1}{|B|} \int\limits_B \omega(x)^{-p'/p} dx \right)^{p/p'} \leq A. \tag{2.30}$$

In Theorem 2.2.17, we show that the union of all A_p classes (for a finite p) is totally characterized by a single property. This is called the A_∞ property.

Definition 2.2.10. We say that a non-negative function ω is *of class* $A_\infty(\Omega)$ if and only if $\omega \in L^1_{\text{loc}}(\Omega)$ and, for any $\alpha \in (0, 1)$, there exist a $\beta \in (0, 1)$ such that for all balls $B \subset \Omega$ and all measurable subsets $F \subset B$, we have that

$$|F| \geq \alpha |B| \quad \Longrightarrow \quad \mu_\omega(F) \geq \beta \mu_\omega(B), \tag{2.31}$$

where $\mu_\omega(S) := \int_S \omega(x)\, dx$.

Remark 2.2.11.

i. Applying Hölder's inequality on the left hand side of (2.29) gives us that $A \geq 1$.

ii. For historical reasons, (2.31) is usually called the *doubling property* (see [13]).

iii. Taking $E = B \setminus F$, the statement in 2.31 becomes

$$\gamma|B| \geq |E| \quad \Longrightarrow \quad \delta\mu_\omega(B) \geq \mu_\omega(E), \tag{2.32}$$

where $\gamma = (1-\alpha)$ and $\delta = (1-\beta)$, which is an equivalent form of the definition.

iv. Every cube, Q, can be squeezed between two concentric balls, B_1 and B_2, for which $|B_1| = \alpha_d 2^{-d}|Q|$ and $|B_2| = \alpha_d 2^{-d} d^{d/2}|Q|$, where α_d is the volume of the unit ball. Therefore, when λ is a doubling measure, we can restate Definitions 2.2.8 and 2.2.10 using cubes.

v. If ω satisfies (2.31) for *fixed* α and β, we say ω satisfies the ***weak*** A_∞ property.

2.2.3 Some Properties of the A_p and A_∞ classes

The proof of the main theorem of this section requires a number of auxiliary results, which we establish in this subsection. The following observation, shows that the A_∞ property is invariant under scaling, dilation, and translation. This observation allows us to simplify some of the forthcoming proofs.

Lemma 2.2.12. *Let $\omega \in A_\infty(\Omega)$, a and b be positive real numbers, and $h \in \mathbb{R}^d$. Also, set $\hat{\omega}(x) := a\omega(bx+h)$ and $\hat{\Omega} = b^{-1}(\Omega - h)$ (Ω shifted and rescaled). Then, $\hat{\omega} \in A_\infty(\hat{\Omega})$.*

Proof. Let $\alpha \in (0,1)$, and let $F \subset Q \subset \hat{\Omega}$ satisfy $|F| \geq \alpha|Q|$. Then, $bF + h \subset bQ + h \subset \Omega$, and $|bF+h| \geq \alpha|bQ+h|$. Thus, there is a $\beta \in (0,1)$ such that

$$\int_{bF+h} \omega(y)dy \geq \beta \int_{bQ+h} \omega(y)dy.$$

Therefore, by changing variables and multiplying be the constant a, we have

$$\int_F a\omega(bx+h)dx \geq \beta \int_Q a\omega(bx+h)dx.$$

This implies that

$$\int_F \hat{\omega}(x)dx \geq \beta \int_Q \hat{\omega}(x)dx.$$

Hence, $\hat{\omega}$ satisfy the A_∞ property on $\hat{\Omega}$. □

Remark 2.2.13. Similarly, the *weak* A_∞ class is invariant under scaling and dilation.

Now, we turn our attention to average functions of the form

$$f_B = \frac{1}{|B|}\int\limits_B |f(y)|\,dy \qquad \text{and} \qquad \hat{f}_B = \frac{1}{\lambda(B)}\int\limits_B |f|\,d\lambda. \tag{2.33}$$

Such functions appear in the definition of the maximal function (Definition 2.2.4) and the definition of the A_p class (Definition 2.2.4). Thus, they reflect several of their properties.

Proposition 2.2.14. *Let $p > 1$. Let μ and λ be (positive) measures that are absolutely continuous with respect to each other's. Suppose that $\mu \in A_p^\lambda(\Omega)$. Then, there exists $c > 0$ such that, for all non-negative measurable functions f,*

$$(\hat{f}_B)^p \leq \frac{c}{\mu(B)}\left(\int_B f^p d\mu\right). \tag{2.34}$$

where $\hat{f}_B = (\lambda(B))^{-1}\int\limits_B |f|\,d\lambda$.
Moreover, the minimum c that satisfies this property is the A_p bound of μ.

Proof. We want to show that (2.29) necessitates (2.34). Let λ, μ, and $\hat{f}_B$ be as in the statement of the proposition. By Hölder's inequality,

$$(\hat{f}_B)^p = \left(\frac{1}{\lambda(B)}\int\limits_B f d\lambda\right)^p = \left(\frac{1}{\lambda(B)}\int\limits_B f\left(\frac{d\mu}{d\lambda}\right)^{1/p}\left(\frac{d\mu}{d\lambda}\right)^{-1/p} d\lambda\right)^p$$

$$\leq \frac{1}{\lambda(B)^p}\left(\int\limits_B f^p\left(\frac{d\mu}{d\lambda}\right) d\lambda\right)\left(\int\limits_B \left(\frac{d\mu}{d\lambda}\right)^{-p'/p} d\lambda\right)^{p/p'}.$$

Since $p/p + p/p' = p$ and $\left(\frac{d\mu}{d\lambda}\right) d\lambda = d\mu$, we have

$$\left(\frac{1}{\lambda(B)}\int\limits_B f^p d\mu\right)\left(\frac{1}{\lambda(B)}\int\limits_B \left(\frac{d\mu}{d\lambda}\right)^{-p'/p} d\lambda\right)^{p/p'} \underbrace{\left[\left(\frac{\mu(B)}{\lambda(B)}\right)^{-1}\frac{\mu(B)}{\lambda(B)}\right]}_{=1}$$

$$= \left(\frac{\mu(B)}{\lambda(B)}\right)^{-1}\left(\frac{1}{\lambda(B)}\int\limits_B f^p d\mu\right)\left(\frac{\mu(B)}{\lambda(B)}\right)\left(\frac{1}{\lambda(B)}\int\limits_B \left(\frac{d\mu}{d\lambda}\right)^{-p'/p} d\lambda\right)^{p/p'}$$

Then, using the definition of the A_p class, we obtain

$$(\hat{f}_B)^p \leq \left(\frac{\mu(B)}{\lambda(B)}\right)^{-1}\left(\frac{1}{\lambda(B)}\int\limits_B f^p d\mu\right) A$$

$$= A\,\mu(B)^{-1}\int_B f^p d\mu.$$

Here, A is the constant in Definition 2.2.8. Setting $c = A$, we conclude the proof. □

When λ is the Lebesgue measure and $\mu = \mu_\omega$, we have the special case stated in this corollary.

Corollary 2.2.15. *Let ω be a non-negative function. Assume that $\omega \in A_p(\Omega)$, for some $p > 1$. Then, there exist a constant $c > 0$ such that for all non-negative* *measurable functions f*

$$(f_B)^p \leq \frac{c}{\mu_\omega(B)} \left(\int_B f^p(y)\omega(y)dy \right). \tag{2.35}$$

Moreover, the minimum c that satisfies this property is the A_p-bound.

Remark 2.2.16. From Remark 2.2.11, we know that the A_p-bound is at least 1. Therefore, $c \geq 1$.

The following theorem provides one of the two technical tool we need from this section. It is, also, the first indicator of the relation between the A_∞ class and the union of all A_p classes.

Theorem 2.2.17. *Let $\omega \in A_p(\Omega)$ for some $p > 1$. Then, the following properties hold.*

I. There is a positive real number c such that, for every ball $B \subset \Omega$ and every measurable subset $E \subset B$, we have that

$$\frac{\mu_\omega(E)}{\mu_\omega(B)} \geq c \left(\frac{|E|}{|B|} \right)^p. \tag{2.36}$$

II. The measure μ_ω is a doubling measure; that is, for every $\alpha \in (0,1)$, there exits a constant $\gamma \in (0,1)$ such that, for every ball $B \subset \Omega$ and every measurable subset $E \subset B$,

$$\alpha\, |B| \leq |E| \implies \gamma\, \mu_\omega\,(B) \leq \mu_\omega\,(E)\,. \tag{2.37}$$

III. Lastly, we have that

$$\bigcup_{p' \in (1,\infty)} A_{p'}(\Omega) \subset A_\infty(\Omega).$$

Proof. *I.* To prove the first claim, we consider a ball $B \subset \Omega$ and an arbitrary measurable subset $E \subset B$. Let $f := \chi_E$ be the characteristic function of E. By virtue of Corollary 2.2.15, we know that, for some constant $c' > 0$,

$$f_B^p = \left(\frac{|E|}{|B|}\right)^p \leq \frac{c'}{\mu_\omega(B)} \mu_\omega(E).$$

Setting $c = 1/c'$, we obtain that

$$c\left(\frac{|E|}{|B|}\right)^p \leq \frac{\mu_\omega(E)}{\mu_\omega(B)},$$

proving the claim.

II. The second result is a direct observation from the previous inequality. Assume that $\alpha\,|B| \leq |E|$ for some $0 < \alpha < 1$. Then,

$$c\,\alpha^\tau \mu_\omega(B) \leq c\left(\frac{|E|}{|B|}\right)^\tau \mu_\omega(B) \leq \frac{\mu_\omega(E)}{\mu_\omega(B)} \mu_\omega(B) = \mu_\omega(E).$$

Setting $\gamma := \alpha^\tau c$, we prove the claim (2.37).

III. The previous properties show that $\omega \in A_\infty(\Omega)$, as in Definition 2.2.10. Since $\omega \in A_p$ was chosen arbitrarily, $A_p \subset A_\infty$. Moreover, $p > 1$ was also chosen arbitrarily; hence,

$$\bigcup_{p' \in (1,\infty)} A_{p'}(\Omega) \subset A_\infty(\Omega).$$

□

Another essential characteristic of the A_p classes is the (so-called) *reverse Hölder inequality*, which controls the fluctuations of $A_p(\Omega)$ weights. The following proposition describes this inequality and assures that every A_∞ weight retains some variant of this inequality. This proposition is not essential for the purpose of this thesis. However, for the sake of completeness, we state it without a proof. (Refer to Stein [13], for a proof).

Proposition 2.2.18. *Suppose that $\omega \in A_\infty(\Omega)$. Then, for some constants $c > 1$ and $r > 1$ and for every cube $Q \subset \Omega$, we have that*

$$\left[\frac{1}{|Q|}\int_Q \omega^r dx\right]^{1/r} \leq \frac{c}{|Q|}\int_Q \omega dx. \tag{2.38}$$

Remark 2.2.19. Such an inequality is called the *reverse Hölder* inequality. Without the constant c, the inequality becomes Hölder's inequality with a reversed comparison sign.

2.2.4 Equivalence of the A_∞ Property and Reverse Hölder Inequality

Here, we discuss the main result of this section, which is stated in the following theorem.

Theorem 2.2.20. *Let ω be a non-negative function. Then, ω is of class A_∞ if and only if ω satisfies a reverse Hölder inequality for some $r \in (1, \infty)$. Moreover, $\omega \in A_p(\Omega)$ for some $p > 1$.*

Proof. 1. Proposition 2.2.18 stated above asserts that if $\omega \in A_\infty(\Omega)$, then it satisfies a reverse Hölder inequality for some $r > 0$. So, we treat the case where ω is a nonnegative function that satisfies a reverse Hölder inequality of the form

$$\left(\frac{1}{|Q|}\int_Q \omega^r\,dx\right)^{1/r} \le c\,\frac{1}{|Q|}\int_Q \omega\,dx \tag{2.39}$$

for some $r \in (1,\infty)$ and all cubes $Q \subset \Omega$. Notice that, for every $1 < s < r$, the following (Lyapunov) inequality holds

$$\left(\frac{1}{|Q|}\int_Q \omega^s\,dx\right)^{1/s} \le \left(\frac{1}{|Q|}\int_Q \omega^r\,dx\right)^{1/r}$$

for all cubes $Q \subset \Omega$.

2. Our first claim is that Lebesgue measure is an A_∞^ω weight; that is, for every $\gamma \in (0,1)$, there is a $\delta \in (0,1)$ such that, for every cube $Q \subset \Omega$ and every measurable subset $E \subset Q$, we have

$$\mu_\omega(E) \le \gamma\mu_\omega(Q) \implies |E| \le \delta\,|Q|\,. \tag{2.40}$$

To show this, suppose that $\mu_\omega(E) \le \gamma\,\mu_\omega(Q)$, for some E, Q, and γ as above. Let r' be the dual exponent of r and set $F := Q \setminus E$. We know from Hölder's inequality that

$$\mu_\omega(F) = \int_F w\,dx = \int_Q \chi_F\omega\,dx \le \left(\int_Q \chi_F^{r'}\,dx\right)^{1/r'}\left(\int_Q \omega^r\,dx\right)^{1/r}.$$

Combining this with inequality (2.39), we get

$$\mu_\omega(F) \le \left(\int_Q \chi_F^{r'}\,dx\right)^{1/r'}\left(c\,|Q|^{1/r}\,\frac{1}{|Q|}\int_Q \omega\,dx\right) = |F|^{1/r'}\,c\,\frac{|Q|^{1/r}}{|Q|}\,\mu_\omega(Q).$$

Since $1/r + 1/r' = 1$ and $F \cup E = Q$, we obtain

$$1 - \frac{\mu_\omega(E)}{\mu_\omega(Q)} \;=\; \frac{\mu_\omega(F)}{\mu_\omega(Q)} \;\leq\; c\left(\frac{|F|}{|Q|}\right)^{1/r'} \;=\; c\left(1 - \frac{|E|}{|Q|}\right)^{1/r'}.$$

Therefore,

$$1 - \gamma \leq c\left(1 - \frac{|E|}{|Q|}\right)^{1/r'} \quad \text{and} \quad \frac{|E|}{|Q|} \leq 1 - \left(\frac{1-\gamma}{c}\right)^{r'}.$$

Set $\delta := 1 - [(1-\gamma)/c]^{r'}$ and notice that $c \geq 1$. Thus, $\delta \in (0,1)$, which confirms the claim.

3. Now, fix $\gamma \in (0,1)$ and set $\delta = 1 - [(1-\gamma)/c]^{r'}$. Let $Q \subset \Omega$ be a given cube (with side parallel to the coordinates), and let $Q_0 \in \mathcal{Q}_0^\Delta$ be a unit-volume dyadic cube. Select $a, b > 0$ and $h \in \mathbb{R}^d$ such that $Q = bQ_0 + h$ and $\mu_{\hat{\omega}}(Q_0) = 1$, where $\hat{\omega}(x) := a\omega(bx+h)$. By the classical change of variables formula and because $|bQ' + h| = b^d\,|Q'|$, we have that

$$\frac{1}{|Q'|}\int_{Q'} \hat{\omega}^s(x)\,dx = \frac{a^s}{|bQ'+h|}\int_{bQ'+h} \omega^s(y)\,dy \tag{2.41}$$

for every cube $Q' \subset \hat{\Omega} := b^{-1}(\Omega - h)$ and every $s \in \mathbb{R}$. Applying this to inequality (2.39), we see that $\hat{\omega}$ satisfies the same reverse Hölder inequality of ω. Namely, for every cube $Q' \subset \hat{\Omega}$,

$$\left(\frac{1}{|Q'|}\int_{Q'} \hat{\omega}^r\,dx\right)^{1/r} \leq c\,\frac{1}{|Q'|}\int_{Q'} \hat{\omega}\,dx. \tag{2.42}$$

Moreover, identity (2.41) guarantees that, for every $E \subset Q'$,

$$\mu_{\hat{\omega}}(E) \leq \gamma\mu_{\hat{\omega}}(Q') \implies |E| \leq \delta\,|Q'|. \tag{2.43}$$

We claim that, there is a constant $C_{\delta,\gamma}$, depending only on λ and δ, such that

$$\int_{Q_0} \hat{\omega}^{1-p'}\, dx \le C_{\delta,\gamma}.$$

Set $f = \hat{\omega}^{-1}$ and $\eta = (\delta\gamma)^{-1}$. Consider the sets $S_k := \{x \in Q_0 : f^{\Delta}_{\hat{\omega}}(x) > \eta^k\}$ for every $k \ge 0$, and notice that $\eta^k \ge 1 = \mu_{\hat{\omega}}(Q_0)^{-1} \int_{Q_0} f\hat{\omega}\, dx$. By Proposition 2.2.7, we can write $S_k = \bigcup_j Q_j^k$, where $\{Q_j^k\}$ are dyadic cubes with disjoint interiors. Moreover, for each Q_j^k,

$$\eta^k\, \mu_{\hat{\omega}}(Q_j^k) < \int_{Q_j^k} f(x)\, \hat{\omega}(x)\, dx \le \delta^{-1}\, \eta^k\, \mu_{\hat{\omega}}(Q_j^k). \tag{2.44}$$

By summing over all j's and substituting $f = \hat{\omega}^{-1}$, we obtain

$$\eta^k\, \mu_{\hat{\omega}}(S_k) < |S_k| \le \delta^{-1}\, \eta^k\, \mu_{\hat{\omega}}(S_k).$$

Notice that $S_k \subset S_{k-1}$; hence, $|S_k| \le |S_{k-1}|$, and

$$\eta^k\, \mu_{\hat{\omega}}(S_k) < |S_k| \le |S_{k-1}| \le \delta^{-1}\, \eta^{k-1}\, \mu_{\hat{\omega}}(S_{k-1}).$$

In particular, $\mu_{\hat{\omega}}(S_k) < \delta^{-1}\, \eta^{-1}\, \mu_{\hat{\omega}}(S_{k-1}) = \gamma\, \mu_{\hat{\omega}}(S_{k-1})$.

Now, property (2.43) asserts that $|S_k| \le \delta\, |S_{k-1}|$; hence, $|S_k| \le \delta^k\, |S_0| \le \delta^k$. Moreover, as in Remark 2.2.6, we have that $f \le f^{\Delta}_{\hat{\omega}}$ a.e. Therefore, for every real number $p > 1$ and its dual p',

$$\begin{aligned}\int_{Q_0} \hat{\omega}^{1-p'}\, dx \;=\; \int_{Q_0} f^{\,p'-1}\, dx \le \int_{Q_0} (f^{\Delta}_{\hat{\omega}})^{p'-1}\, dx & \\ = \int_{Q_0\setminus S_0} (f^{\Delta}_{\hat{\omega}}(x))^{p'-1}\, dx + \sum_{k=0}^{\infty} \int_{S_k\setminus S_{k+1}} (f^{\Delta}_{\hat{\omega}}(x))^{p'-1}\, dx & \\ \le \int_{Q_0\setminus S_0} 1\, dx + \sum_{k=0}^{\infty} \int_{S_k\setminus S_{k+1}} \left(\eta^{k+1}\right)^{p'-1}\, dx & \\ \le |Q_0| + \sum_{k=0}^{\infty} |S_k| \left(\eta^{k+1}\right)^{p'-1} \le 1 + \eta^{p'-1}/(1-\delta\,\eta^{p'-1}). & \end{aligned} \tag{2.45}$$

Select $p' > 1$ small enough such that $1 > \delta\,\eta^{p'-1}$; for example, $p' = 1 + \frac{1}{2\ln(\eta)}\ln(\delta^{-1})$. Since η depends only on δ and γ, the value $p = (1 - 1/p')^{-1}$ also depends only on δ and γ. Therefore, by setting $C_{\delta,\gamma} = 1 + \eta^{p'-1}/(1-\delta\,\eta^{p'-1})$ we verify the claim.

4. Finally, we observe that the identity in (2.41) and our choice of $\int_{Q_0} \hat{\omega}(x)\, dx = 1$ give us

$$\left(\frac{1}{|Q|}\int_Q \omega\, dy\right)^{p'/p} \frac{1}{|Q|}\int_Q \omega^{-p'/p}\, dy \;=\; \frac{a^{1-p'}}{|bQ_0+h|}\int_{bQ_0+h} \omega^{1-p'}(y)\, dy \;=\; \int_{Q_0} \hat{\omega}^{1-p'}\, dx \le C_{\delta,\gamma}.$$

Equivalently,

$$\left(\frac{1}{|Q|}\int_Q \omega\, dy\right)\left(\frac{1}{|Q|}\int_Q \omega^{-p'/p}\, dy\right)^{p/p'} \le C_{\delta,\gamma}^{p/p'}. \tag{2.46}$$

Here, $C_{\gamma,\delta}$ and p are independent of the choice of cube Q. Thus, we conclude that w is an A_p measure. $\square$

2.3 Green's Function

There are multiple approaches in the literature to defining Green's function. Krylov [7] and Rudin [12], for example, view Green's function from a functional analytic perspective, and define it as a distribution. To make the presentation accessible, we describe, below, the definition of *Green's function* for an elliptic operator that we follow in this Thesis.

Let Ω be a smooth open subset of $\mathbb{R}^d$. Let λ be a real number in $(0,1)$, and let $\mathbf{a} := (a^{ij}(\cdot))$ be a smooth, symmetric, $d \times d$ matrix–valued function on Ω, such that

$$\lambda I \leq \mathbf{a}(x) \leq \frac{1}{\lambda} I, \tag{2.47}$$

in the sense of positive definiteness. We are interested in properties of the operator

$$Lu := \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x), \tag{2.48}$$

which, in this case, is described as an *elliptic operator in non-divergence form.* We also, consider its formal adjoint

$$L^* v := \sum_{i,j=1}^{d} \frac{\partial^2}{\partial y_i \partial y_j} \left(a^{ij}(y) v(y) \right). \tag{2.49}$$

In our context, a weak solution of $L^* v = 0$ in Ω is a function a locally integrable that satisfies $\int_\Omega v\, Lu\; dy \;=\; 0$ for every nonnegative test function $u \in C_c^\infty(\Omega)$. Equivalent notion is assumed for $L^* v \;\leq\; 0$ and $L^* v \;\geq\; 0$.

2.3.1 Defining Green's Function

The intuition behind the definition of Green's function comes from the well-known Poisson representation formula for Laplace equation and the heat equation.

Definition 2.3.1. Let L be an elliptic operator, and let Ω be a subset of $\mathbb{R}^d$. Consider a map, $x \longmapsto G(x, \cdot)$, which maps $x \in \Omega$ to a distribution $G(x, \cdot) \in \mathscr{D}'(\Omega)$. Suppose, further, that G solves

$$\begin{cases} L_y^* G(x,y) = -\delta(x-y) & \text{for every } y \in \Omega \\ G(x,y) = 0 & \text{for every } y \in \partial\Omega, \end{cases}$$

in the sense of distributions, where δ is Dirac's delta; that is,

$$v(x) = -\int_\Omega L_y^* G(x,y)\ v(y)\, dy = -\int_\Omega G(x,y)\ Lv(y)\, dy \tag{2.50}$$

for every $v \in C_c^\infty(\Omega)$. We say G is *Green's function of L corresponding to Ω.*

Remark 2.3.2. By applying integration by parts *formally*, we have

$$\begin{aligned} u(x) &= -\int_\Omega L_y^* G(x,y)\ u(y)\, dy \\ &= -\int_\Omega G(x,y)\, Lu(y)\, dy + \int_{\partial\Omega} \sum_{i,j=1}^d \underbrace{G(x,y)}_{=0} a^{ij}(y) u_{y_i}(y) \nu_j\, dS(y) \\ &\qquad - \int_{\partial\Omega} \sum_{i,j=1}^d \left(a^{ij}(y) G(x,y)\right)_{y_j} u(y) \nu_i\, dS(y). \end{aligned}$$

In particular, if u solves

$$\begin{cases} Lu = \varphi & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{2.51}$$

we have the representation formula

$$u(x) = -\int_\Omega G(x,y)\, \varphi(y)\, dy - \int_{\partial\Omega} \sum_{i,j=1}^d \left(a^{ij}(y) G(x,y)\right)_{y_j} g(y) \nu_i\, dS(y). \tag{2.52}$$

This identity holds whenever $\partial\Omega$ is of class C^2. However, this is a mere heuristic approach to derive it. For a precise proof, refer to Miranda [10].

2.3.2 Intrinsic Properties of Green's Function

Our first lemma establishes an intrinsic inequality between Green's functions for the same operator, L, but that correspond to two distinct nested sets.

Lemma 2.3.3. *Let $B_R \subset \mathbb{R}^d$ be a ball of radius $R > 0$, and let $B_r \subset \mathbb{R}^d$ be another ball, concentric with B_R, such that $R > r > 0$. Let L be an elliptic operator defined on B_R as above. Let G_R be Green's function of L, corresponding to B_R, and let G_r be that, corresponding to B_r. Then, $G_R(x, y) \geq G_r(x, y)$ for every $x, y \in B_r$.*

Proof. 1. Suppose the hypotheses above hold. Let $u \in C_c^\infty(B_R)$ be a function supported in B_r (i.e. $\operatorname{supp} u \subset B_r$). Then, by the definition of Green's function, we have

$$u(x) = -\int_{B_R} L^* G_R(x, y)\ u(y)\, dy = -\int_{B_r} L^* G_R(x, y)\ u(y)\, dy,$$

and

$$u(x) = -\int_{B_r} L^* G_r(x, y)\ u(y)\, dy.$$

Therefore, for a fixed $x \in B_r$ and for every $u \in C_c^\infty(B_r)$,

$$\int_{B_r} \left(L^* G_R(x, y) - L^* G_r(x, y) \right)\ u(y)\, dy = 0.$$

This means that $W(y) = G_R(x, y) - G_r(x, y)$ is a weak solution of

$$\begin{cases} L^* W(y) = 0 & \text{in } B_r \\ W(y) = G_R(x, y) & \text{on } \partial B_r. \end{cases}$$

Let $\varphi \in C_c^\infty(B_r)$ be a nonnegative function, $\varphi \not\equiv 0$, and let u be a solution of

$$\begin{cases} Lu = \varphi & \text{in } B_r \\ u = 0 & \text{on } \partial B_r. \end{cases}$$

By the definition of Green's function, we have that

$$0 = \int_{B_r} W(y)\, Lu(y)\, dy - \int_{\partial B_r} \sum_{i,j} W(y) a^{ij}(y) u_{y_i}(y) \nu_j \, dS(y).$$

This can be rewritten as

$$\int_{B_r} W(y)\, \varphi(y)\, dy = \int_{\partial B_r} G_R(x,y) \Big(Du(y)\, \mathbf{a}(y) \nu \Big)\, dS(y). \tag{2.53}$$

2. Our next step is to show that $Du(y)\, \mathbf{a}(y)\nu \geq 0$ for every $y \in \partial B_r$. (This directional derivative is sometimes called *conormal derivative*; e.g. Miranda [10]). To simplify the proof, fix $y \in \partial B_r$, and consider an arbitrary symmetric, positive-definite matrix, A. Let α and β be positive real numbers such that $\alpha I \leq A \leq \beta I$. Let $\varepsilon > 0$ be small enough so that $y - \varepsilon A \nu \in B_r$ (e.g. $\varepsilon = R\alpha\beta^{-2}$), and set $g(t) := u\left(y - \varepsilon(1-t)A\nu\right)$ for $t \in [0,1]$. Notice that, by the maximum principle, $u \leq \sup_{\partial B_r} u = 0$; hence, $g(t) \leq g(1) = 0$. Therefore,

$$Du(y)\varepsilon A\nu = g'(1) = \lim_{\delta \to 0^+} \frac{g(1-\delta)}{-\delta} \geq 0.$$

By dividing by ε, we obtain that $Du(y)A\nu \geq 0$. Since A is arbitrary and y is fixed, we can set $A = \mathbf{a}(y)$. Then, we obtain $Du(y)\mathbf{a}(y)\nu \geq 0$. Combining this with (2.53), we get

$$\int_{B_r} W(y)\, \varphi(y)\, dy \geq 0$$

for every $\varphi \in C_c^\infty(B_r)$. Thus, $W(y) \geq 0$ and $G_R(x,y) \geq G_r(x,y)$. □

For a fixed $x \in \Omega$, the Green's function, $G(x,y)$, decays, as y approaches the boundary, in a controlled and moderate manner. The following lemma captures this behavior.

Lemma 2.3.4. *Let $r > 0$ and $\delta \in (0, 1)$ be arbitrary. Consider an open ball, B_r, of radius r. Let L be an elliptic operator defined as in (1.1). Let G_r be Green's function for L with respect to B_r. Then, for every $x \in B_{\delta r}$, we have*

$$\int_{B_{\delta r}} G_r(x,y)\, dy \geq \lambda \frac{(\delta r)^2 - |x|^2}{2d},$$

where, $B_{\delta r}$ is a ball of radius δr, concentric with B_r. In particular,

$$\inf_{x \in B_{2r/3}} \int_{B_{5r/6}} G_r(x,y)\, dy \geq \frac{\lambda r^2}{8d}.$$

Proof. Let G_δ be the Green's function for L with respect to $B_{\delta r}$. Then, we fix $x \in B_{\delta r}$, and notice that, since G_δ, as a distribution, solves $L_y^* G_\delta(x,y) = -\delta(x-y)$ in $B_{\delta r}$, we have

$$1 = -\int_{B_{\delta r}} L_y^* G_\delta(x,y)\, dy = -\int_{\partial B_{\delta r}} \sum_{i,j=1}^{d} \partial_{y_i} \left(a^{ij}(y) G_\delta(x,y) \right) \nu_j \, dS(y). \tag{2.54}$$

On the other hand, if we use the test function $|y|^2 - |x|^2$, we obtain

$$\begin{aligned} 0 &= -\int_{B_{\delta r}} L_y^* G_\delta(x,y) \left(|y|^2 - |x|^2 \right) dy \\ &= -\int_{B_{\delta r}} G_\delta(x,y) L_y \left(|y|^2 - |x|^2 \right) dy \\ &\qquad\qquad - \int_{\partial B_{\delta r}} \sum_{i,j=1}^{d} \partial_{y_i} \left(a^{ij}(y) G_\delta(x,y) \right) \left(|y|^2 - |x|^2 \right) \nu_j \, dS(y) \\ &= -2 \int_{B_{\delta r}} G_\delta(x,y) \mathrm{Tr}(\mathbf{a}(y))\, dy \\ &\qquad\qquad - \left(\delta^2 r^2 - |x|^2 \right) \int_{\partial B_{\delta r}} \sum_{i,j=1}^{d} \partial_{y_i} \left(a^{ij}(y) G_\delta(x,y) \right) \nu_j \, dS(y). \end{aligned}$$

Combining this with (2.54), we obtain

$$\int_{B_{\delta r}} G_\delta(x,y)\, 2\,\mathrm{Tr}(\mathbf{a}(y))\, dy = \delta^2 r^2 - |x|^2 .$$

By our assumptions on $\mathbf{a}(y)$, we have that

$$\delta^2 r^2 - |x|^2 = \int_{B_{\delta r}} G_\delta(x,y)\, 2\,\mathrm{Tr}(\mathbf{a}(y))\, dy \ \leq \ 2(d\lambda^{-1}) \int_{B_{\delta r}} G_\delta(x,y)\, dy.$$

Moreover, by Lemma 2.3.3, $G_r(x,y) \geq G_\delta(x,y)$ for every $x, y \in B_{\delta r}$. Therefore,

$$\int_{B_{\delta r}} G_r(x,y)\, dy \geq \int_{B_{\delta r}} G_\delta(x,y)\, dy = \lambda \frac{\delta^2 r^2 - |x|^2}{2d}.$$

This proves the main part of the lemma. To prove the second part, set $\delta = 5/6$ and take $x \in B_{2r/3}$. Then,

$$\begin{aligned} \int_{B_{5r/6}} G_r(x,y)\, dy &\geq \lambda \frac{(5r/6)^2 - |x|^2}{2d} \\ &\geq \lambda \frac{(5r/6)^2 - (2r/3)^2}{2d} = \frac{\lambda r^2}{8d}. \end{aligned}$$

□

Chapter 3

Fabes and Stroock Estimates

In this chapter, we present some results from Fabes and Stroock [5]. To make the presentation accessible, we follow Fabes and Stroock's notation. In particular,

- Ω denotes an open subset of $\mathbb{R}^d$.
- B_r denotes an open ball of radius r (with a center specified as needed), and
- B_{kr} denotes a ball of radius kr that is concentric with B_r.
- Q_s denotes a cube with side-length s, whose sides are parallel to the axes, and
- Q_{ks} denotes another such cube, which is concentric with Q_s; that is, Q_{ks} is a homothetized (rescaled) version of Q_s with respect to its center.

3.1 Introduction to the Problem

Let λ be a positive real number such that $0 < \lambda < 1$, and let $\mathbf{a} := (a^{ij}(\cdot))$ be a smooth, symmetric, $d \times d$ matrix–valued function on $\mathbb{R}^d$, that satisfies

$$\lambda I \leq \mathbf{a}(x) \leq \frac{1}{\lambda} I, \tag{3.1}$$

in the sense of positive definiteness. Under the prior assumptions on $\mathbf{a}(\cdot)$, we define the (*uniformly*) *elliptic operator*, L, and its formal adjoint, L^*, by (1.1) and (1.2), respectively.

The operator L^*, in (1.2), often appears in differential equations that model probability measures arising from stochastic processes (see [9]). In particular, the elliptic equation (3.2), below, models the steady state of some "value-functions" in the stochastic setting. Its adjoint equation, (3.3), models the probability distribution of the driving stochastic process, at the steady state.

Intuitively, when particles are moving according to a (stochastic) diffusion process, concentrations at few points are not expected—because things are "diffusing". Additionally, the coefficients, a^{ij}, which describe the diffusivity of the medium, are smooth and bounded by (3.1). Virtually, this should prevent particles from migrating in a single direction, and fairly apportions the movement of particles between directions, based on the diffusion coefficients.

In this chapter, we study the operators L and L^*, and quantitatively investigate these intuitions. In particular, we establish bounds on (weak) solutions of

$$Lu(x) = 0 \qquad \text{for } x \in \Omega, \tag{3.2}$$

and the, so-called, *adjoint problem*

$$L^*v(y) = 0 \qquad \text{for } y \in \Omega. \tag{3.3}$$

We also investigate the associated Green's function, $G(x, y)$, which, as discussed before in Section 2.3, satisfies $L_y^*G(x, y) = -\delta(x - y)$, in the sense of distributions. In our context, a function $v \in L^1_{\text{loc}}(\Omega)$ is called a *weak solution* of (3.3) if it satisfies $\int_\Omega v\, Lu\; dy = 0$ for every nonnegative (test function) $u \in C_c^\infty(\Omega)$. An equivalent notion is assumed for weak solutions of $L^*v \leq 0$ and $L^*v \geq 0$. Note, also, that a solution of (3.3) is often called an *adjoint solution*, as we will often do in this chapter.

3.2 A Reverse Hölder Inequality for Adjoint Solutions

In this part of the thesis, we examine the ratio between the measures of concentric balls and other nested sets (e.g. $m(B_2)/m(B_1)$), where m is a measure whose density solves the adjoint problem (3.3). The following lemma, concerned with this question, establishes what we call the *doubling property* for a measure.

Lemma 3.2.1. *There exists a constant $c = c_{\lambda,d}$, depending only λ and d, such that for every nonnegative weak solution, v, of the problem $L^*v \leq 0$, in Ω, and for every ball B_r for which $B_{\frac{4}{3}r} \subset \Omega$, we have*

$$\int_{B_r} v(y)\, dy \leq c \int_{B_{r/2}} v(y)\, dy. \tag{3.4}$$

Proof. Let $B_r \subset \Omega$ and v be as in the statement. Without loss of generality, we assume that B_r is centered at the origin. Fix $\delta \in (0,1)$ and define

$$h(x) = \begin{cases} \left[(1+\delta)^2 r^2 - |x|^2 \right]^2 & \text{for } x \in B_{(1+\delta)r} \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $x \in B_{(1+\delta)r}$, we have that

$$Lh(x) = 8 \sum_{i,j=1}^{d} a^{ij}(x) x_i x_j \; - 4\left((1+\delta)^2 r^2 - |x|^2 \right) \sum_{i=1}^{d} a^{ii}(x). \tag{3.5}$$

Since $\lambda \leq \mathbf{a}(x) \leq 1/\lambda$ and $x \in B_{(1+\delta)r}$, we obtain

$$Lh(x) \;\leq\; \frac{8}{\lambda} |x|^2 - 4\left((1+\delta)^2 r^2 - |x|^2 \right) \lambda d \;\leq\; \frac{8}{\lambda} (1+\delta)^2 r^2,$$

and

$$Lh(x) \;\geq\; 8\lambda |x|^2 - 4\left((1+\delta)^2 r^2 - |x|^2 \right) \frac{d}{\lambda} \;\geq\; -4(1+\delta)^2 r^2 \frac{d}{\lambda}.$$

Additionally, when $(1+\delta)\,r > |x| \geq (1-\delta)\,r$, (3.5) gives

$$Lh(x) \geq \frac{8}{\lambda} r^2 \left(\lambda^2 (1-\delta)^2 - 2\delta\, d\right).$$

Due to these inequalities, we can fix a small $\delta > 0$, depending only on d and λ, and obtain positive constants c_1 and c_2, depending only on d and λ, such that

$$\begin{cases} Lh(x) \geq c_1 r^2 & \text{for } r \geq |x| \geq (1-\delta)\,r, \\ Lh(x) \geq 0 & \text{for } \ (1+\delta)\,r > |x| \geq (1-\delta)\,r, \ \text{ and} \\ |Lh(x)| \leq c_2 r^2 & \text{for } \ (1+\delta)\,r > |x|\,. \end{cases} \tag{3.6}$$

Now, according to (3.6), we have

$$\begin{aligned} c_1 \int_{B_r \setminus B_{(1-\delta)r}} v(y)\, dy &\leq \int_{B_r \setminus B_{(1-\delta)r}} v(y) L\left(h/r^2\right)\, dy \\ &\leq \int_{B_{(1+\delta)r} \setminus B_{(1-\delta)r}} v(y) L\left(h/r^2\right)\, dy. \end{aligned}$$

Using this inequality and (3.6) once more, we have

$$\begin{aligned} \int_{B_{(1+\delta)r}} v(y) L\left(h/r^2\right)\, dy &= \int_{B_{(1+\delta)r} \setminus B_{(1-\delta)r}} v(y) L\left(h/r^2\right)\, dy + \int_{B_{(1-\delta)r}} v(y) L\left(h/r^2\right)\, dy \\ &\geq c_1 \int_{B_r \setminus B_{(1-\delta)r}} v(y)\, dy + \int_{B_{(1-\delta)r}} v(y)(-c_2)\, dy \\ &= c_1 \int_{B_r} v(y)\, dy - (c_1 + c_2) \int_{B_{(1-\delta)r}} v(y)\, dy. \end{aligned} \tag{3.7}$$

On the other hand, since v is a weak solution of $L^* v \leq 0$, we have

$$0 \geq \int_{B_{(1+\delta)r}} v(y) L\left(h/r^2\right)\, dy. \tag{3.8}$$

Therefore, by combining (3.7) and (3.8), we obtain

$$\frac{c_1 + c_2}{c_1} \int_{B_{(1-\delta)r}} v(y)\, dy \geq \int_{B_r} v(y)\, dy.$$

By iteration, there is a positive integer $k = k_\delta$ such that $(1-\delta)^k \leq 1/2$ and

$$(1 + c_2/c_1)^k \int_{B_{r/2}} v(y)\, dy \geq (1 + c_2/c_1)^k \int_{B_{(1-\delta)^k r}} v(y)\, dy \geq \int_{B_r} v(y)\, dy. \tag{3.9}$$

Setting $c = (1 + c_2/c_1)^k$ proves the lemma. □

Remark 3.2.2. i. To established the bound in (3.8), we exploited the definition of weak solutions. Unfortunately, our definition requires the test function, h, to be in $C_c^\infty(\Omega)$. Clearly, h is *not* in $C_c^\infty(\Omega)$. However, we also know that $h \in C_c^1(\Omega) \cap W^{2,\infty}(\Omega)$. And, by a smooth-approximation argument, we can extend the property of weak solution to include such test functions. (See Lemma C.2 in the appendix).

ii. This proof works, as well, when v is Green's function for L since it satisfies $L_y^* G(x, y) \leq 0$ in the weak sense.

Now, observe that every cube in $\mathbb{R}^d$, with sides of length s, is circumscribed in a ball whose radius is $\frac{1}{2}\sqrt[d]{d}\, s$. Such a cube also circumscribes a ball whose radius is $\frac{1}{2}s$. With this observation, we can extend Lemma 3.2.1 to cubes, as formulated by the following corollary.

Corollary 3.2.3. *Let $b = \sqrt[d]{d}/2$ and let $Q_s \subset \Omega$ be a cube of side length s. Let B_{bs} be the smallest ball that contains Q_s, and suppose that $B_{\frac{4}{3}bs} \subset \Omega$. Then, there exists a constant c, depending only on λ and d, such that, for every $v \geq 0$ that solves $L^* v \leq 0$ weakly, in Ω,*

$$\int_{Q_s} v(y)\, dy \leq c \int_{Q_{s/2}} v(y)\, dy. \tag{3.10}$$

Proof. Suppose Q_s satisfies the hypothesis. By Lemma 3.2.1, every $k \geq 0$ stratifies

$$\int_{B_{bs}} v(y)\,dy \leq c_1^k \int_{B_{bs/2^k}} v(y)\,dy,$$

for some $c_1 = c_1(\lambda, d)$. Since $v \geq 0$, we have that $\int_{Q_s} v(y)\,dy \leq \int_{B_{bs}} v(y)\,dy$. Furthermore, there is a $k \in \mathbb{N}$, depending only on d, such that $b \leq 2^{k-2}$. For that k, we have $B_{bs/2^k} \subseteq B_{s/4} \subset Q_{s/2}$. Therefore,

$$\int_{Q_s} v(y)\,dy \leq \int_{B_{bs}} v(y)\,dy \leq c_1^k \int_{B_{bs/2^k}} v(y)\,dy \leq c_1^k \int_{Q_{s/2}} v(y)\,dy.$$

Setting $c := c_1^k$, we conclude the proof. □

A measure that satisfies the doubling property, which we saw above, often possesses another related property, which called the *reverse Hölder inequality* (see [13]). The following theorem shows that (weak) adjoint solutions, that solve (3.3), possess such a property.

Theorem 3.2.4. *There exists a constant $c = c_{\lambda,d}$, depending only on λ and d, such that for every nonnegative weak solution of $L^*v = 0$ in Ω, and for every ball B_r, for which $B_{2r} \subset \Omega$, the following inequality holds.*

$$\left[\frac{1}{|B_r|}\int_{B_r} v(y)^{d/(d-1)}\,dy\right]^{(d-1)/d} \leq c\frac{1}{|B_r|}\int_{B_r} v(y)\,dy.$$

Proof. Since $v \geq 0$, we can (by Lemma C.1) bound the $L^{d/(d-1)}-$norm of v by

$$\|v\|_{L^{d/(d-1)}} \leq \sup\left\{\int_{B_r} v(y)\varphi(y)\,dy \;:\; \varphi \in C_c^\infty(B_r), \varphi \geq 0, \|\varphi\|_{L^d} \leq 1\right\}. \tag{3.11}$$

To exploit this fact, let $\varphi \in C_c^\infty(\Omega)$ be a nonnegative function such that $\|\varphi\|_{L^d} \leq 1$

and $supp(\varphi) \subset B_r$. Let u solve

$$\begin{cases} Lu = \varphi & \text{in } B_{2r} \\ u = 0 & \text{on } \partial B_{2r}. \end{cases}$$

Because φ is smooth, we know that u is smooth (for a proof, see [3]). Moreover, by the Alexandroff-Bakelman-Pucci (ABP) estimate (Theorem 2.1.15), we have

$$\|u\|_{L^\infty(B_{2r})} \leq \frac{4r}{\alpha_d^{1/d}\, d} \left\| \frac{\varphi}{\det\left(a(x)\right)^{1/d}} \right\|_{L^d(B_r)} \leq c_1 r \left\|\varphi\right\|_{L^d(B_r)} \leq c_1 r, \tag{3.12}$$

where α_d is the volume of the unit ball and c_1 is dependent on d and λ only.

Next, we take a cutoff function $\xi_r \in C_c^\infty(B_{2r})$ such that $\xi_r = 1$ on B_r and $supp(\xi_r) \subset B_{3r/2}$. We also require that, there is $\theta > 0$, such that $\left|\frac{\partial^\beta}{\partial x^\beta}\xi_r\right| \leq \theta r^{-|\beta|}$ for every multi-index β with $|\beta| \leq 2$. Then, because $L^* v = 0$, we have that

$$0 = \int_{B_{3r/2}} vL\left(\xi_r u\right)\, dx = \int_{B_{3r/2}} v\left(\xi_r Lu + 2\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial \xi_r}{\partial x_j}\frac{\partial u}{\partial x_i} + L\xi_r\, u\right) dx. \tag{3.13}$$

Additionally, by Cauchy-Schwartz inequality, and due to our choice of $\left|\frac{\partial}{\partial x_i}\xi_r\right| \leq \theta r^{-1}$, we have

$$\begin{aligned} \left|\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial \xi_r}{\partial x_j}\frac{\partial u}{\partial x_i}\right| &= \left|\nabla u\ \mathbf{a}(\mathbf{x})\nabla \xi_{\mathbf{r}}^{\mathbf{T}}\right| \\ &\leq \left|\nabla u\right| \left\|\mathbf{a}(\mathbf{x})\right\| \left|\nabla \xi_r^T\right| \leq \left|\nabla u\right| \lambda^{-1}\theta\sqrt{d}r^{-1}. \end{aligned}$$

Therefore, by Hölder's inequality and the fact that $3r/2 \le 2r$.

$$\int_{B_{3r/2}} v \left| \sum_{i,j=1}^{d} a^{ij} \frac{\partial \xi_r}{\partial x_j} \frac{\partial u}{\partial x_i} \right| dx \le \frac{\theta\sqrt{d}}{r\lambda} \int_{B_{3r/2}} v \, |\nabla u| \, dx$$
$$\le \frac{\theta\sqrt{d}}{r\lambda} \left(\int_{B_{3r/2}} v \, dx \right)^{1/2} \left(\int_{B_{2r}} v |\nabla u|^2 \, dx \right)^{1/2}. \qquad (3.14)$$

Additionally, Lemma C.5 (in appendix) gives us that $|L\xi_r| \le \theta\lambda^{-1} r^{-2} (d!)^{1/d}$. Thus, combining this with (3.12), we obtain

$$\int_{B_{3r/2}} v \, |u| \, |L\xi_r| \, dx \le \int_{B_{3r/2}} v \, \|u\|_{L^\infty(B_{2r})} \, \theta\lambda^{-1} r^{-2} (d!)^{1/d} \, dx \le c_1 \theta \lambda^{-1} r^{-1} (d!)^{1/d} \int_{B_{3r/2}} v \, dx. \qquad (3.15)$$

Combining all of (3.12)-(3.15), we get

$$\int_{B_r} v\varphi = \int_{B_{3r/2}} v \xi_r L u = - \int_{B_{3r/2}} v \left(2 \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial \xi_r}{\partial x_j} \frac{\partial u}{\partial x_i} + L\xi_r \, u \right) dx$$
$$\le c_2 r^{-1} \left[\int_{B_{3r/2}} v \, dx + \left(\int_{B_{3r/2}} v \, dx \right)^{1/2} \left(\int_{B_{2r}} v |\nabla u|^2 \, dx \right)^{1/2} \right] \qquad (3.16)$$

fo a suitable c_2, which depends only on λ and d.

Now, recall that $\lambda I \le \mathbf{a}(\mathbf{x})$ and $Lu = \varphi$. Thus, we have

$$\int_{B_{2r}} v \, \lambda \, |\nabla u|^2 \, dx \le \int_{B_{2r}} v \left(\sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) dx = \frac{1}{2} \int_{B_{2r}} v \left(L\left(u^2\right) - 2u\varphi \right) dx. \qquad (3.17)$$

Notice that $\nabla(u^2) = 0$ and $u^2 = 0$ on ∂B_{2r}. Therefore, because $L^* v = 0$ in B_{2r}, we have that $\int_{B_{2r}} v L(u^2) = 0$ (see Lemma C.2 in the appendix). Combining this with

(3.17) and (3.12), we obtain

$$\begin{aligned}\int_{B_{2r}} v\lambda \left|\nabla u\right|^2 dx &\leq \int_{B_{2r}} v|u|\varphi \, dx \\ &\leq \int_{B_{2r}} v \left\|u\right\|_{L^\infty(B_{2r})} \varphi \, dx \leq c_1 r \int_{B_{2r}} v\varphi \, dx = c_1 r \int_{B_r} v\varphi \, dx.\end{aligned} \tag{3.18}$$

Accordingly, bounding the right hand side of (3.16) by (3.18), we get

$$r \int_{B_r} v\varphi \, dx \leq c_2 \left[\int_{B_{3r/2}} v \, dx + \left(\int_{B_{3r/2}} v \, dx \right)^{1/2} \left(\frac{c_1 r}{\lambda} \int_{B_r} v\varphi \, dx \right)^{1/2} \right]. \tag{3.19}$$

For clarity, set $z = r \int_{B_r} v \, \varphi \, dx$ and $w = \int_{B_{3r/2}} v \, dx$. Then, (3.19) becomes

$$z \leq c_2 \, (w + \frac{c_1}{\lambda} w^{1/2} z^{1/2}).$$

We can easily deduce (see Lemma C.6) that

$$z \leq \left[\left(c_2^{-1} + \frac{c_1^2}{4\lambda^2} \right)^{1/2} - \frac{c_1}{2\lambda} \right]^{-2} w.$$

Thus,

$$r \int_{B_r} v\varphi \, dx \leq c_3 \int_{B_{3r/2}} v \, dx$$

for some c_3 that depends only on λ and d.

Now, by Lemma 3.2.1, we have

$$r \int_{B_r} v\varphi \leq c_3 \int_{B_{3r/2}} v \, dx \leq \ c_4 \int_{B_{3r/4}} v \, dx \ \ \leq c_4 \int_{B_r} v \, dx \tag{3.20}$$

for an appropriate c_4. Lastly, by combining (3.20) and (3.11), we have

$$\left[\int_{B_r} v(y)^{d/(d-1)}\,dy\right]^{(d-1)/d} \leq \sup_{\varphi} \int_{B_r} v(y)\varphi(y)\,dy \leq cr^{-1}\int_{B_r} v(y)\,dy.$$

This concludes the proof. □

By an argument similar to that of Corollary 3.2.3, we obtain the following result.

Corollary 3.2.5. *Let $b = \sqrt[d]{d}/2$ and let $Q_s \subset \Omega$ be a cube of side length s. Let B_{bs} be the smallest ball that contains Q_s, and suppose that $B_{2bs} \subset \Omega$. Then, there exists a constant c, depending only on λ and d, such that, for every $v \geq 0$ that solves $L^*v = 0$ weakly, in Ω,*

$$\left[\frac{1}{|Q_s|}\int_{Q_s} v(y)^{d/(d-1)}\,dy\right]^{(d-1)/d} \leq c\frac{1}{|Q_s|}\int_{Q_s} v(y)\,dy.$$

Proof. Suppose Q_s satisfies the hypothesis. Let k be a positive integer such that $b \leq 2^{k-1}$. Then, $B_{bs/2^k} \subseteq B_{s/2} \subset Q_s$. By Lemma 3.2.1 and Theorem 3.2.4, there are constants c_1 and c_2, depending only on λ and d, such that

$$\int_{B_{bs}} v(y)\,dy \leq c_1^k \int_{B_{bs/2^k}} v(y)\,dy \leq c_1^k \int_{Q_s} v(y)\,dy,$$

and

$$\left[\frac{1}{|B_{bs}|}\int_{B_{bs}} v(y)^{d/(d-1)}\,dy\right]^{(d-1)/d} \leq \frac{c_2}{|B_{bs}|}\int_{B_{bs}} v(y)\,dy \leq \frac{c_2}{|B_{bs}|}c_1^k \int_{Q_s} v(y)\,dy.$$

Additionally, $Q_s \subset B_{bs}$ and $v \geq 0$; hence,

$$\left[\frac{|Q_s|}{|B_{bs}|}\,\frac{1}{|Q_s|}\int_{Q_s} v(y)^{d/(d-1)}\,dy\right]^{(d-1)/d} \leq \frac{|Q_s|}{|B_{bs}|}\,\frac{c_2\,c_1^k}{|Q_s|}\int_{Q_s} v(y)\,dy.$$

Let α_d be the volume of the unit ball, and notice that

$$\frac{|Q_s|}{|B_{bs}|} = \frac{s^d}{(bs)^d \alpha_d} = b^{-d}\alpha_d^{-1} = 2^d \alpha_d / d.$$

Therefore, we conclude that

$$\left[\frac{1}{|Q_s|}\int_{Q_s} v(y)^{d/(d-1)}\,dy\right]^{(d-1)/d} \le c\frac{1}{|Q_s|}\int_{Q_s} v(y)\,dy,$$

where $c := c_2\, c_1^k b^{-1}\alpha_d^{-1/d}$. □

3.3 A Reverse Hölder Inequality for Green's Function

In the previous section, we proved that weak adjoint solutions satisfy reverse Hölder inequality. The proof, however, does not encompass the case of Green's function, which solves $L_y^* G(x,y) = -\delta_x(y)$ for $y \in \Omega$, in the sense of distributions (see Definition 2.3.1). Consequently, we need a different approach to extend the result to Green's function.

Theorem 3.3.1. *Let $G(x,y)$ be the Green's function of L corresponding to Ω* (see Definition 2.3.1). *There exists a constant $c := c_{\lambda,d}$, depending only on λ and d such that, for every ball B_r with $B_{\frac{10}{3}r} \subset \Omega$, we have*

$$\left[\frac{1}{|B_r|}\int_{B_r} G(x,y)^{d/(d-1)}\,dy\right]^{(d-1)/d} \le \frac{c}{|B_r|}\int_{B_r} G(x,y)\,dy.$$

Proof. If $x \notin B_{2r}$, the proof of Theorem 3.2.4 applies for $v = G(x,\cdot)$. So, we assume that $x \in B_{2r}$. For simplicity, we further assume that B_r is centered at the origin. Let $\widetilde{G}(x,y)$ be the Green's function corresponding to B_{3r}. Due to Lemma 2.3.3, we know that $G(x,y) \geq \widetilde{G}(x,y)$ for all $y \in B_{3r}$. Since $z^{d/(d-1)}$ is convex, we also have

$$\frac{1}{|B_r|}\int_{B_r}\left(\frac{1}{2}G(x,y)\right)^{d/(d-1)} dy \leq \frac{1}{|B_r|}\left[\int_{B_r}\frac{1}{2}\big(G-\widetilde{G}\big)^{d/(d-1)}\,dy + \int_{B_r}\frac{1}{2}\,\widetilde{G}^{\,d/(d-1)}\,dy\right].$$

Observe that, for a fixed x, $v(y) := G(x,y) - \widetilde{G}(x,y)$ is a nonnegative solution of

$$\begin{cases} L^* v = 0 & \text{in } B_{3r} \\ v(y) = G(x,y) & \text{on } \partial B_{3r}. \end{cases}$$

Thus, we can apply Theorem 3.2.4 to v and obtain that

$$\left[\frac{1}{|B_r|}\int_{B_r}\Big(G(x,y)-\widetilde{G}(x,y)\Big)^{d/(d-1)}\,dy\right]^{(d-1)/d} \leq \frac{c_1}{|B_r|}\int_{B_r}\Big(G(x,y)-\widetilde{G}(x,y)\Big)\,dy, \tag{3.21}$$

where c_1, as before, depends only on λ and d.

Next, for $x, y \in B_3$, we define the functions

$$\mathbf{a}_r(x') := \mathbf{a}(rx') \qquad \text{and} \qquad G_r(x',y') := \widetilde{G}(rx',ry')r^{d-2}.$$

Simple computations reveal that G_r is Green's function of the operator

$$L_r := \sum_{i,j=1}^{d} a_r^{ij}(x)\partial^2_{x_i x_j}$$

corresponding to B_3. Let $\varphi \in C_c^\infty(B_3)$ such that $\|\varphi\|_{L^d} = 1$, and let u solve

$$\begin{cases} L_r u = \varphi & \text{in } B_3 \\ u = 0 & \text{on } \partial B_3. \end{cases}$$

We know that $u(x) = \int_{B_3} G_r(x,y)\,\varphi(y)\,dy$, and, by the ABP estimate, we also have

$$\int_{B_3} G_r(x,y)\,\varphi(y)\,dy = u(x) \le c_2\,\|\varphi\|_{L^d} = c_2,$$

where the constant c_2 depends only on λ and d. By Lemma C.1 in the appendix,

$$\left[\int_{B_3} G_r(x,y)^{\frac{d}{d-1}}\,dy\right]^{\frac{d-1}{d}} \le \sup\left\{\int_{B_3} G_r(x,y)\varphi\,dy\ :\varphi\in C_c^\infty(B_3),\varphi\ge 0,\|\varphi\|_{L^d}=1\right\}.$$

Therefore,

$$\left[\int_{B_3} G_r(x,y)^{\frac{d}{d-1}}\,dy\right]^{\frac{d-1}{d}} \le c_2.$$

Moreover, by Lemma 2.3.4,

$$\inf_{x\in B_2}\int_{B_{5/2}} G_r(x,y)\,dy \ge c_3$$

for some $c_3 > 0$ depending only on λ and d. Combining these inequalities, we have

$$c_2^{-(d-1)/d}\left[\int_{B_3} G_r(x,y)^{\frac{d}{d-1}}\,dy\right] \le c_3^{-d/(d-1)}\left[\int_{B_{5/2}} G_r(x,y)\,dy\right]^{d/(d-1)}.$$

Since $G_r(x',y') := \widetilde{G}(rx',ry')r^{d-2}$, this implies

$$\left(\frac{1}{r^d}\int_{B_{5r/2}} G_r(x,y)^{\frac{d}{d-1}}\,dy\right) \le c_4\left[\frac{1}{r^d}\int_{B_{5r/2}} G_r(x,y)\,dy\right]^{d/(d-1)},$$

where $c_4 = c_3^{-d/(d-1)}c_2^{(d-1)/d}$. By Lemma 3.2.1, we have that

$$\int_{B_{5r/2}} G_r(x,y)\,dy \le c_5\int_{B_r} G_r(x,y)\,dy$$

for some c_5, depending only on λ and d. This concludes the proof. □

3.4 A Comparability Property for Green's Function

The following theorem, which is a product of all the previous results, is the most important theorem in this thesis. It asserts that the measure induced by Green's function is a doubling measure, and gives a bound more explicit than the one in Lemma 3.2.1.

Theorem 3.4.1. *Let $G(x,y)$ be Green's function of the operator L corresponding to Ω. Then, there are positive real numbers τ and c, depending only on λ and d, such that for every B_r, with $B_{4r} \subset \Omega$, and every measurable $E \subset B_r$, we have*

$$\frac{\int_E G(x,y)\,dy}{\int_{B_r} G(x,y)\,dy} \geq c\left(\frac{|E|}{|B_r|}\right)^{\tau}.$$

Proof. Due to Theorem 3.3.1, $G(x,\cdot)$ posses some form of a reverse Hölder inequality. So, let $c > 1$ be a constant, depending only on λ and d, such that

$$\left[\frac{1}{|B_r|}\int_{B_r} G(x,y)^{d/(d-1)}\,dy\right]^{(d-1)/d} \leq \frac{c}{|B_r|}\int_{B_r} G(x,y)\,dy, \tag{3.22}$$

for every ball B_r with $B_{10r/3} \subset \Omega$. Then, by Theorem 2.2.20, we know that $G(x,\cdot) \in A_\tau(\Omega)$ for some $\tau > 1$ that depends only on c and d. Therefore, we can apply Theorem 2.2.17, to deduce directly that

$$\frac{\int_E G(x,y)\,dy}{\int_{B_r} G(x,y)\,dy} \geq c\left(\frac{|E|}{|B_r|}\right)^{\tau}.$$

This concludes the proof. □

With this we conclude this chapter.

APPENDICES

A On the ABP Estimate

In this appendix, we prove some of the lemmas we need in Chapter 1 but are unfit to the flow of the presentation of the chapter.

We begin by the following proof of Lemma 2.1.12 from Section 2.1.

Proof of Lemma 2.1.12. Let $p \in \chi_K(\Omega)$ be arbitrary. Since Ω is bounded and $u \in C^0(\overline{\Omega})$, there exists a non-negative real number a such that

$$u(x) \leq a + p \cdot x, \quad \text{for all } x \in \overline{\Omega}.$$

In particular, we can take $a = \sup_{\Omega}\{-p \cdot x + u(x)\}$. Because $\overline{\Omega}$ is closed and bounded, there is some $z \in \overline{\Omega}$ for which $a = -p \cdot z + u(z)$. Hence,

$$u(x) \leq -p \cdot z + u(z) + p \cdot x = u(z) + p \cdot (x - z).$$

Now, we claim that we can select z to be an interior point. Indeed, we notice that K consists of lines from $(y, u(y))$ to $\partial\Omega$. Also, every hyper-plane in χ_K is tangent to K at a point on one of these lines and lies above it; hence, it is tangent to K at $(y, u(k))$, as well. That implies $\chi_K(\Omega) = \chi_K(y)$. Therefore, if $z \in \partial\Omega$, we have that

$$
\begin{aligned}
u(x) &\le u(z) + p\cdot(x-z) \\
&\le \ \ 0 \ \ + p\cdot(x-z) && \text{“since } u \le 0 \text{ on } \Omega\text{”} \\
&= K(z) + \big(p\cdot(x-y) + p\cdot(y-z)\big) && \text{“}K = 0 \text{ on } \partial\Omega\text{”} \\
&\le (K(y) + p\cdot(z-y)) + p\cdot(x-y) + p\cdot(y-z) && \text{“since } p \in \chi_K(\Omega) = \chi_K(y)\text{”} \\
&= u(y) + p\cdot(x-y).
\end{aligned}
$$

Therefore, we can replace z by the interior point y. Thus, $\chi_K(\Omega) \subset \chi_u(\Omega)$. □

The following two lemmas were used in the proofs of Lemma 2.1.14 and Theorem 2.1.15, related to the ABP estimate. Here, we present the proof.

Lemma A.1. Let Ω be an open subset of $\mathbb{R}^d$, and let $u : \Omega \longrightarrow \mathbb{R}$ be a C^1 function. Also, let Γ^+ be the contact set of u (see Definition 2.1.7). Then, for every $y \in \Gamma^+$, we have that

$$u(x) \le u(y) + Du(y)\cdot(x-y)$$

for all $x \in \Omega$. Moreover, $Du(y)$ is the only vector that satisfies this property.

Proof. Let $y \in \Gamma^+$. By the definition of a contact set, there is a $p \in \mathbb{R}^d$ such that

$$u(x) \le u(y) + p\cdot(x-y)$$

for all $x \in \Omega$. Thus, for every v, with $|v| = 1$, and for every small $\delta > 0$ we have that

$$u(y+\delta v) - u(y) \le \delta p\cdot v \qquad \text{and} \qquad \delta p\cdot v \le u(y) - u(y-\delta v).$$

Now, we subtract $\delta Du(y) \cdot v$ and divide by δ to obtain

$$\frac{u(y+\delta v) - u(y) - \delta Du(y) \cdot v}{\delta} \leq (p - Du(y)) \cdot v \leq \frac{u(y) - u(y-\delta v) - \delta Du(y) \cdot v}{\delta}.$$

By taking the limits, as $\delta \to 0$, we have

$$0 \leq (p - Du(y)) \cdot v \leq 0$$

for every v with $|v| = 1$. Thus, $p = Du(y)$. □

Lemma A.2. Let Ω be an open subset of $\mathbb{R}^d$, and let $u : \Omega \longrightarrow \mathbb{R}$ be a C^2 function. Also, let Γ^+ be the contact set of u. Then, for every $y \in \Gamma^+$, the matrix $D^2u(y)$ is negative semi-definite.

Proof. First, we show this is true when $d = 1$. Let $y \in \Gamma^+$ and let $\varepsilon > 0$ be small so that $y \pm \varepsilon \in \Omega$. Then, by Taylor's Theorem and Cauchy's Remainder form, we have

$$u(y+\varepsilon) = u(y) + u'(y)\varepsilon + \frac{1}{2}u''(y+\theta)\varepsilon^2$$

for some $\theta \in (0, \varepsilon)$. Moreover, since $y \in \Gamma^+$, Lemma A.1 implies that

$$u(y+\varepsilon) \leq u(y) + u'(y)\varepsilon.$$

Therefore,

$$\frac{1}{2}u''(y+\theta)\varepsilon^2 = u(y+\varepsilon) - (u(y) + u'(y)\varepsilon) \leq 0.$$

As $\varepsilon \to 0$, we have that $\theta \to 0$. Thus, $u''(y) \leq 0$, which is what we needed to show.

Now, we resolve the case $d \geq 1$. We want to reduce this to the case $d = 1$. So, let $y \in \Gamma^+$, and let $v \in B_1(0)$ be arbitrary. Define $f(\varepsilon) = u(y + \varepsilon v)$ for $\varepsilon \in (-1, 1)$, small enough so that $y + \varepsilon v \in \Omega$. Then, by Lemma A.1, we have

$$f(\varepsilon) = u(y + \varepsilon v) \leq u(y) + Du(y) \cdot \varepsilon v = f(0) + f'(0)\varepsilon.$$

Thus, $0 \in \Gamma_f^+$ and $f''(0) = v^t D^2 u(y) v \leq 0$. Since $v \in B_1(0)$ was arbitrary, $D^2 u(y)$ is negative semi-definite. □

The classical change of variable formula is applicable when we have injective and bijective mappings. However, when we have a non-injective mapping, we might not have an equality. The following lemma addresses this issue, as needed in the proof of Lemma 2.1.14.

Lemma A.3. (Change of Variables for Non-Injective Functions)
Let $A \subset \mathbb{R}^d$ be a compact set. Let $F : A \longrightarrow \mathbb{R}^d$ be a continuously differentiable, and let $g : F(A) \longrightarrow [0, \infty)$ be an integrable function. Suppose that $DF(x)$ is positive semi-definite for every $x \in A$. Then,

$$\int_{F(A)} g(p)\, dp \leq \int_A g(F(x))\, |\det\, DF(x)|\, dx.$$

Proof. Define the mapping $F_\varepsilon(x) := F(x) + \varepsilon x$ for every $\varepsilon > 0$. Since $DF(x)$ is positive semi-definite, the Jacobian of $F_\varepsilon(x)$, namely $DF(x) + \varepsilon I$, is positive *definite*. Thus, F_ε is *locally* one-to-one in A. Consequently, by the classical change of variables formula, every $x \in A$ has a (relative) neighborhood, $B_\delta(x) \cap A$, such that

$$\int_{F_\varepsilon(B_\delta(x)\cap A)} g(p)\, dp = \int_{B_\delta(x)\cap A} g(F_\varepsilon(y))\, |\det\, DF_\varepsilon(y)|\, dy.$$

By our assumptions, A is compact; hence, the cover $\{B_\delta(x)\}$ has a finite subcover;

say $B_{\delta_1}(x_1), ..., B_{\delta_m}(x_m)$, which covers A. Now, set $V_m := A \cap B_{\delta_m}(x_m)$, and

$$V_i := A \cap B_{\delta_i}(x_i) \setminus \bigcup_{j=i+1}^{m} B_{\delta_j}(x_j)$$

for $j = 1, ..., m-1$. Then, those sets are disjoint, and the function F_ε is one-to-one on each of them. Thus, we have

$$\begin{aligned} \int_{F_\varepsilon(A)} g(p)\,dp &\le \sum_{j=1}^{m} \int_{F_\varepsilon(V_j)} g(p)\,dp \\ &= \sum_{j=1}^{m} \int_{V_j} g(F_\varepsilon(y))\,|\det\ DF_\varepsilon(y)|\,dy \\ &= \int_A g(F_\varepsilon(y))\,|\det\ DF_\varepsilon(y)|\,dy. \end{aligned}$$

We conclude the proofs by taking the limit $\varepsilon \to 0$. □

B On Dyadic Cubes and A_p Classes

B.1 A Lemma on Dyadic Cubes

Lemma B.1.1. *Let O be an open subset of $\mathbb{R}^d$. Then, for every $x \in O$, there exists a dyadic cube $Q \in \mathcal{Q}^{\Delta}$ such that $x \in Q$ and $Q \subset O$.*

Proof. Notice that, because O is open, there is an $r > 0$ such that $B_r(x) \subset O$. Let n be an integer such that $\sqrt{d}2^{-n} < r$. We know that $\bigcup_{Q \in \mathcal{Q}_n^{\Delta}} Q = \mathbb{R}^d$; hence, there is a dyadic cube $Q \in \mathcal{Q}_n^{\Delta}$ such that $x \in Q$. Since the longest distance between two points in Q is the diagonal whose length is $\sqrt{2^{-2n} + ... + 2^{-2n}} = \sqrt{d}2^{-n} < r$, we establish that $Q \subset B_r(x) \subset O$. □

Remark B.1.2. Since every $x \in O$ is contained in a dyadic cube $Q_x \in \mathcal{Q}^{\Delta}$, we have that $O = \bigcup_{x \in O} Q_x$— a union of dyadic cubes.

B.2 On Maximal Functions and A_p Properties

In this appendix, we discuss some details that are not essential for the presentation of the results in Section 2.2.

Lemma B.1. Let Ω be open, and let μ be a positive measure. Suppose that $f : \Omega \longrightarrow \mathbb{R}$ is locally integrable. Then, the maximal function $f_\mu^*(x)$ is lower semi-continuous (l.s.c.); that is, for every $a \in \mathbb{R}$, the set $S_a := \{x \in \Omega \; : \; f_\mu^*(x) > a\}$ is open. Additionally, f_μ^{Δ} is l.s.c..

Proof. Notice that the function f_μ^* is positive. Hence, for all $a < 0$, $S_a = \Omega$. So, assume that $a \geq 0$ and let $x \in S_a$ be arbitrary. We want to prove that there is a ball inside S_a that contains x. Thus, note that $f_\mu^*(x)$ is a limit point of the set

$$\left\{\mu(B)^{-1}\int_B |f|\; d\mu \,:\, B \subset \Omega \text{ is a ball that contains } x\right\}.$$

Therefore, if $f_\mu^*(x)$ is finite, there exists a ball B_0 that contains x and satisfies

$$\mu(B_0)^{-1}\int_{B_0} |f|\; d\mu > a.$$

Consequently, for every point $z \in B_0$,

$$f_\mu^*(z) \geq \mu(B_0)^{-1}\int_{B_0} |f|\; d\mu > a,$$

and z is also in S_a. In case $f_\mu^*(x)$ is infinite, the definition of f_μ^* gives us that there exists a ball B_1 that contains x and satisfies

$$\mu(B_1)^{-1}\int_{B_1} |f|\; d\mu > a.$$

Thus, every point $z \in B_1$ is also in S_a. Therefore, S_a is open. The conclusion for f_μ^Δ uses the same argument. $\square$

Remark B.2. The Lebesgue Differentiation Theorem (LDT), which we applied in Remark 2.2.6, is based on taking the limit over arbitrary cubes (or balls) that contain the point x and converges to $\{x\}$. However, we restricted the limit to dyadic cubes. In that case, a question might arise about points lying on boundaries of dyadic cubes. This application is valid because the set of such points is of measure zero.

Indeed, let $Q \in \mathcal{Q}_k^\Delta$ be a dyadic cube. If $x \in \partial Q$, then, there is $Q' \in \mathcal{Q}_{k'}^\Delta$ such that $x \in \partial Q'$ for every integer $k' \geq k$. This can be shown easily by induction. Now, let N^Δ be the set of all such points; that is, $N^\Delta := \{x \in \mathbb{R}^d :$

x has a coordinate of the form $2^k m$ with $k, m \in \mathbb{Z}\}$. It is clear that N^Δ is a countable union of measure-zero sets (they are $(d-1)-$dimensional). Thus, $\left|N^\Delta\right| = 0$. We can directly apply the LDT to $\Omega \backslash N^\Delta$, over dyadic cubes, and still have the "almost-everywhere" property. So, it is valid for Ω as well.

C On Reverse Hölder Inequality

Lemma C.1. For $p \geq 1$, let $v \in L^p(\Omega)$ be a nonnegative function. Let $q > 1$ be the dual of p; that is $1/p + 1/q = 1$. Then, we have that

$$\|v\|_{L^q(\Omega)} \leq \sup \left\{ \int_\Omega v(y) f(y)\, dy \ : \ f \in C_c^\infty(\Omega), f \geq 0, \|f\|_p \leq 1 \right\}.$$

Remark. The concern here is to prove the bound without the use of Riesz representation theorem or any complicated approximation theorems. Indeed, equality actually holds for every $v \in L^1_{\text{loc}}$. (For the case $v \in L^q$, see Lieb and Loss [8]).

Proof. 1. Due to Hölder inequality, v is locally integrable; more precisely,

$$\|v\|_{L^1(K)} \leq \|v\|_{L^p(K)} \|1\|_{L^q(K)} = \|v\|_{L^p(K)} |K|^{1/q} < \infty$$

for every compact $K \subset \Omega$.

2. Let $S_a = \{x \in \Omega \ : \ v(x) \leq a\}$. For every $a > 0$, set $v_a(x) = \chi_{S_a}(x) v(x)$. Now, consider the following two cases.

- **Case 1,** $q \geq p$: In this case, we have that

$$\begin{aligned}\int_\Omega v_a(x)^q\, dx &= \int_\Omega \chi_{S_a} v(x)^p v(x)^{q-p}\, dx \\ &\leq \int_\Omega 1\, v(x)^p a^{q-p}\, dx = a^{q-p} \int_\Omega v(x)^p\, dx < +\infty.\end{aligned}$$

  Hence, $v_a \in L^q(\Omega)$. Note also that $v_a \leq v$. Therefore, by consulting the lemma on dual spaces in Lieb and Loss [8], we have

$$\begin{aligned}\|v_a\|_{L^q(\Omega)} &= \sup\left\{\int_\Omega v_a(y)f(y)\,dy \ : \ f\in C_c^\infty(\Omega), \|f\|_p \le 1\right\}\\ &\le \sup\left\{\int_\Omega v(y)f(y)\,dy\right\}.\end{aligned}$$

Since it is in $L^p(\Omega)$, v is finite almost everywhere in Ω. Thus, $\lim_{a\to\infty} v_a = v$ a.e. By the Monotone Convergence Theorem,

$$\|v\|_{L^q(\Omega)} = \lim_{a\to\infty}\|v_a\|_{L^q(\Omega)} \le \left\{\int_\Omega v(y)f(y)\,dy \ : \ f\in C_c^\infty(\Omega), \|f\|_p\le 1\right\}.$$

- **Case 1,** $q<p$: Let $r = p/q \ge 1$. Then, by Hölder inequality,

$$\int_K v^q\,dx \le \left(\int_K (v^q)^r\,dx\right)^{1/r}\left(\int_K 1\,dx\right)^{1/r'} = \|v\|^q_{L^p(K)}\,|K|^{1/r'} < +\infty,$$

for every compact $K\subset\Omega$. Therefore, $v\in L^q(K)$ and

$$\begin{aligned}\|v\|_{L^q(K)} &= \sup\left\{\int_K v(y)f(y)\,dy \ : \ f\in C_c^\infty(K), \|f\|_p\le 1\right\}\\ &\le \left\{\int_\Omega v(y)f(y)\,dy\right\}.\end{aligned}$$

Finally, we complete the proof by observing that

$$\|v\|_{L^q(\Omega)} = \sup_{\substack{\text{compact}\\ K\subset\Omega}}\|v\|_{L^q(K)} \le \left\{\int_\Omega v(y)f(y)\,dy \ : \ f\in C_c^\infty(\Omega), \|f\|_p\le 1\right\}.$$

□

The Arithmetic Mean-Geometric Mean inequality is a useful tool to bound $L\xi$ by the determinant of the Hessian matrix, $D^2\xi$. The following two lemmas utilize the AM-GM inequality.

Lemma C.2. Let L and L^* be defined as in (1.1) and (1.2), respectively. Let v be a weak solution of $L^*v \leq 0$ in Ω; that is, $v \in L^1_{\text{loc}}(\Omega)$ and satisfies

$$\int_\Omega v\, L\varphi\, dx \leq 0$$

for every $\varphi \in C_c^\infty(\Omega)$. Let $h \in W_c^{2,\infty}$, then

$$\int_\Omega v\, Lh\, dx \leq 0. \tag{C.1}$$

Remark C.3. 1. The second derivative of h is taken in the weak sense.

2. We are especially interested in applying the lemma for a function $h \in C_c^1(\Omega)$, which is a piece-wise C^2.

Proof. To prove this we use the well-known smooth approximation theorems. (See Lieb and Loss [8], Theorem 2.16 and the remarks following it). Let $\{h_n\}$ be a sequence of functions in $C_c^\infty(\Omega)$ such that $h_n \to h$ in $W^{2,\infty}$. Notice that $\operatorname{supp} h_n =: K_n \to K := \operatorname{supp} h$. Due to the convergence in $W^{2,\infty}$, we have that $\det D^2(h_n - h) \to 0$ in $W^{2,\infty}$ as well. Then, using the triangle inequality and the AM-GM inequality, we obtain

$$\int_\Omega v\, Lh\, dx = \int_{K \cup K_n} v\, Lh_n\, dx + \int_{K \cup K_n} v\, L(h - h_n)\, dx$$

$$\leq 0 + \int_{K \cup K_n} |v|\; d\lambda^{-1} \left|\det\left(D^2(h_n - h)\right)\right|^{1/d}\, dx.$$

However, $\det\left(D^2(h_n - h)\right) \longrightarrow 0$ in $W^{2,\infty}$. This establishes the claim. □

Remark C.4. More details on why $L(h - h_n) \leq d\lambda^{-1} \left|\det\left(D^2(h_n - h)\right)\right|^{1/d}$ can be found in the proof of the next lemma.

Lemma C.5. Let $\xi \in C_c^\infty(B_{2r})$, and let $L\xi := \sum_{i,j=1}^d a^{ij}\partial_{ij}\xi = \operatorname{Tr}(\mathbf{a}(x)D^2\xi(x))$. Suppose that $|\partial_{ij}\xi| \leq \theta r^{-2}$ for some $\theta \in \mathbb{R}_{>0}$. Then,

$$|L\xi| \leq d\theta\lambda^{-1}r^{-2}(d!)^{1/d}.$$

Proof. Let $A := \mathbf{a}(x)D^2\xi(x)$, and let $\mu_1, ..., \mu_d \in \mathbb{C}$ be the eigenvalues of A. We know that $\operatorname{Tr}(A) = \mu_1 + ... + \mu_d$ and that $\det(A) = \mu_1 \cdots \mu_d$. Therefore, by the triangle inequality and AM-GM inequality, we have

$$\begin{aligned}|\operatorname{Tr}(A)| &\leq |\mu_1| + ... + |\mu_d| \\ &\leq d\left(|\mu_1|\cdots|\mu_d|\right)^{1/d} = d\left|\det(A)\right|^{1/d}.\end{aligned}$$

Thus,

$$\begin{aligned}|L\xi| \leq d\left|\det(A)\right|^{1/d} &= a\left|\det(\mathbf{a}(x))\right|^{1/d}\left|\det\left(D^2\xi\right)\right|^{1/d} \\ &\leq d\lambda^{-1}\left|\det\left(D^2\xi\right)\right|^{1/d}.\end{aligned} \tag{C.2}$$

To bound $\left|\det(D^2\xi)\right|^{1/d}$, notice first that

$$\det\left(D^2\xi\right) = \sum_{\sigma\in S_d}(-1)^\sigma\partial_{1\sigma(1)}\xi\cdots\partial_{d\sigma(d)}\xi,$$

where S_d is the symmetry group of order d and $(-1)^\sigma$ denotes the parity of σ. Therefore, by the triangle inequality and our assumptions,

$$\begin{aligned}\left|\det\left(D^2\xi\right)\right| &\leq \sum_{\sigma\in S_d}\left|\partial_{1\sigma(1)}\xi\right|\cdots\left|\partial_{d\sigma(d)}\xi\right| \\ &\leq \sum_{\sigma\in S_d}\left(\theta r^{-2}\right)\cdots\left(\theta r^{-2}\right) = d!\,\theta^d r^{-2d}.\end{aligned}$$

Combining this inequality and (C.2), we obtain

$$|L\xi| \leq d\lambda^{-1} \, (d!)^{1/d} \, \theta r^{-2}.$$

This concludes the proof. □

The following lemma is purely arithmetic, but we include it for completeness.

Lemma C.6. Let w, z, q, p be positive real numbers that satisfy

$$z \leq p(w + qw^{1/2}z^{1/2}). \tag{C.3}$$

Then, the also satisfy

$$z \leq \left[(p^{-1} + q^2/4)^{1/2} - q/2\right]^{-2} w. \tag{C.4}$$

Proof. Let $x = \sqrt{z}$ and $y = \sqrt{w}$. Then, (C.3) becomes

$$\begin{aligned} x^2 &\leq p(y^2 + qxy) \\ &= p\left(y^2 + 2y\left(\frac{q}{2}x\right) + \left(\frac{q}{2}x\right)^2\right) - p\left(\frac{q}{2}x\right)^2. \end{aligned}$$

Therefore,

$$\left[p^{-1} + \left(\frac{q}{2}\right)^2\right] x^2 \leq \left[y + \left(\frac{q}{2}x\right)\right]^2.$$

Taking the square-root of both sides, and subtracting $qx/2$, we have

$$\left[\left(p^{-1} + \left(\frac{q}{2}\right)^2\right)^{1/2} - \frac{q}{2}\right] x \leq y.$$

By squaring again and substituting, we have

$$\left[\left(p^{-1}+\left(\frac{q}{2}\right)^2\right)^{1/2}-\frac{q}{2}\right]^2 z \leq w.$$

Inequality (C.4) follows immediately from here. □

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