Electrothermally Tuned and Electrostatically Actuated MEMS

Resonators: Dynamics and Applications

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To the memory of my grandmother Tatia,

_May Allah give her His mercy._

To my lovely son Haroun,

_Thank you for making my research journey more joyful and challenging._
ABSTRACT

Electrothermally Tuned and Electrostatically Actuated MEMS Resonators: Dynamics and Applications

The objective of this thesis is to present a theoretical and experimental investigation of the dynamics of micro and nano-electromechanical systems electrothermally tuned and electrostatically actuated, and explore their potential for practical applications.

The first part of the dissertation presents the tuning of the frequency of clamped-clamped micro and nano-resonators, straight and curved. These resonators are electrothermally or electrostatically tuned. The effect of geometric parameters on the frequency variation is investigated experimentally and theoretically using a reduced order model based on the Euler-Bernoulli beam theory. High tunability is demonstrated for micro and nano beams, straight and initially curved.

The second part discusses the dynamical behavior of a curved (arch) beam electrothermally tuned and electrostatically actuated. We show that the first resonance frequency increases up to twice its fundamental value and the third resonance frequency decreases until getting very close to the first resonance frequency triggering the veering phenomenon. We study experimentally and analytically, using the Galerkin procedure, the dynamic behavior of the arch beam. Next, upon changing the electrothermal voltage, the second symmetric natural frequency of the arch is adjusted to near twice, three times, and
four times the fundamental natural frequency. This gives rise to a potential two-to-one, three-to-one, and four-to-one autoparametric resonances between the two modes. These resonances are demonstrated experimentally and theoretically.

The third part of the dissertation is concerned with the incorporation of the electrothermally tuned and electrostatically actuated microresonators into potential applications: filtering and sensing. First, we experimentally prove an exploitation of the nonlinear softening, hardening, and veering phenomena of an arch beam, to demonstrate a flat, wide, and tunable bandwidth and center frequency by controlling the electrothermal actuation voltage. Second, a pressure sensor based on the convective cooling of the air surrounding an electrothermally heated resonant bridge is demonstrated experimentally. The concept is demonstrated using both straight and arch microbeam resonators driven and sensed electrostatically. The change in the surrounding pressure is shown to be accurately tracked by monitoring the change in the resonance frequency of the structure.
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Chapter 1 Introduction

1.1. Motivation

Micro and nano-electromechanical systems (MEMS and NEMS) have increasingly attracted the attention of researchers due to their small size/mass, mass-fabrication, reliability, and energy efficiency. For these reasons, MEMS and NEMS have been proposed, in the past few decades, for numerous potential applications including gas/mass sensing, filters, memory devices, logic devices, and gyroscopes.

On-chip actuation of micro and nano-systems has been particularly a challenging aspect of MEMS development. Electrostatic actuation is one approach that has been widely used for actuation of microsystems. While electrostatic actuation is suitable for many applications, some systems require either lower voltages or higher output forces. Then other actuation mechanisms have been used like electromagnetic and thermal actuation. These MEMS and NEMS structures are driven nowadays using different transduction mechanisms, such as electrothermal, piezoelectric, electrostatic, electromagnetic, and optical.

The basic structures that are used numerously to build MEMS and NEMS devices are principally made of cantilever or clamped-clamped beams. These devices have been drawing significant attention recently for their interesting advantages in several applications, such as energy harvesting [1], filtering [2], mass sensing [3], bio and gas sensing [4, 5] and RF micro-switches [6].
The most known class of MEMS and NEMS structures are resonators. Resonators can be bulk-mode, in which a thick block of microstructure is excited through wave propagation; or it can be based on thin-film technology, in which a thin compliant microstructure is driven into bending-mode vibrations. The first micro-resonator was revealed since the early work of Nathanson (1967)[7].

MEMS and NEMS resonators have significantly been attractive thanks to their capability to respond significantly when excited near specific frequencies. This phenomenon is known as the resonance of the structure. Tunability of MEMS resonators refers to the ability to control or tune the resonance frequency by increasing or decreasing it to the desired value. This is a highly desirable feature for various applications, including communications, filtering, gyroscopes, energy harvesting, and signal processing.

The implementation of MEMS and NEMS resonators into integrated circuit urges researchers and designers to understand the performance and the mechanical behavior of the associated mechanical components that are mainly operated dynamically near their resonance frequencies. These resonators-based MEMS and NEMS, which are basically simple in shapes like clamped-clamped bridge resonators, cantilevers, and plates, are operated either on the linear or nonlinear regime. They are mostly monitored by the change in frequency or amplitude resulted by an alteration of one or more control parameter(s), such as mass, applied force, viscous damping, surrounding gas/pressure or temperature, etc. There has been an increasing growth of theoretical and experimental works that investigate and focus on the nonlinear dynamics of MEMS and NEMS and their exploitation in different applications.
Different fundamental phenomena were studied using MEMS and NEMS; thanks to the capability to precisely monitor different physical parameters like the boundary conditions, stiffness, damping, and shapes; to open new windows to either implement them in potential applications or to avoid them while designing. Linear and nonlinear mode couplings were among the fundamental physical phenomena well investigated that attracted researchers in the literature.

Mainly, the phenomena of frequency veering and internal resonance have attracted considerable attention from solid state physics to structural dynamics. In general, when two frequencies approach each other as a control parameter is varied, as the variation of compressive stress or geometric parameter, they either intersect transversely or veer away from each other with high local curvature that leads in most cases to the transfer of the energy of one mode to the other. Internal resonances are known to enhance modal coupling and may produce a strongly coupled nonlinear response. Motivated by the results of the structural dynamics, researchers start to investigate the linear and nonlinear coupling among different modes in MEMS and NEMS and implement them in different applications.

Having a micro/nano resonator with tunable resonance frequencies is highly desirable to adopt a single micro/nano system to various applications just by tuning its frequencies. Also tuning different resonance frequencies of a resonator might lead to linear and nonlinear coupling among different modes of vibration.
1.2. Literature survey

1.2.1. Bistable structures

Bistable MEMS and NEMS structures have been drawing significant attention recently for their interesting advantages in application, such as sensors [8], actuators [9] and MEMS/NEMS based memory elements [10]. Bistable microstructures are characterized by a double-well potential, and hence at least two stable states, and commonly a third one encircles the two local wells. The motion resulting from this third state is large compared with the other two in-well motion. A well-known example of this is the snap-through motion in buckled beams. This is a highly desirable feature for many application including mirrors [11], switching [12] and actuators and sensors [8, 9]. Bistable microstructure can be realized in many configurations, such as beams sandwiched between two magnets, imperfect beams, and shallow arches and buckled beams. Our research is concerned with the two latest categories: buckling of straight microbeams and shallow arches.

1.2.1.1 Buckled microbeams

A buckled beam is mainly achieved by an axial compressive load that can be induced by several methods, such as applying a direct axial load [13] or by using thermal actuation [14]. Buckled beams have received considerable attention in the last decades. Bisplingho and Pian [15] investigated theoretically and experimentally linear vibrations of pinned-pinned buckled beams, including temperature effects and initial geometric imperfections. Nayfeh *et al.* [16] and Kreider [17] obtained an exact solution for linear vibrations of a beam around its \( n \)th buckling mode shape and experimentally validated the calculated mode shapes and natural frequencies of a clamped-clamped beam around its first
buckling mode over a wide range of static buckling deflections. Eisley [18], Eisley and Bennett [19], and Tseng and Dugundji [20] studied the problem of forced nonlinear vibrations and asymmetric and symmetric vibrations of buckled beams. Amibili and coauthors [21-23] investigated numerically the nonlinear dynamics of axially moved buckled beam. Abou-Rayan et al. [24] studied the single-mode discretized dynamics of a parametrically excited, pinned-pinned buckled beam. They [24] showed interesting nonlinear phenomena, such as period-multiplying bifurcations and chaotic motions. Chin et al. [25] developed approximate solutions for single-mode responses to a primary resonant uniform distributed excitation. They presented experimental data for nonlinear asymmetric single-mode responses for various levels of buckling. Kreider and Nayfeh [26] used the direct approach to analyze the vibrations of buckled beams with two-to-one internal resonance about the static buckling deflection, taking into account quadratic nonlinearities only. Pellicano and Vestrioni [27] addressed the nonlinear dynamic behavior of a hinged-hinged, axially moving beam in the pre- and post-critical ranges. They used the Galerkin method to discretize the system and retained the first eight modes in the expansion. Lacarbonara et al. [28] studied experimentally and theoretically the suitability of reduction methods for studying nonlinear vibrations of distributed-parameter systems. They analyzed the nonlinear planar vibrations of a clamped-clamped buckled beam about its first post-buckling configuration.

1.2.1.2 Shallow arches

Extensive research has been done on either shallow or non-shallow arches. Shallow arches have been the mostly used bi-stable structures and have received high interest in recent decades for their advantages in various MEMS applications.
At the macro-scale, the dynamic behavior of shallow arches has been studied in the nonlinear mechanics’ literature. Hsu et al. [29] investigated the problem of a clamped-clamped shallow arch under step loads and derived conditions for the stability of the structure. Hung [30] studied the dynamic buckling of some elastic hinged-hinged shallow arches under sinusoidal load. Bi and Dai [31] investigated the dynamical behavior of a shallow arch subjected to periodic excitation. They showed that this kind of structure exhibits internal resonance and period doubling cascade bifurcations leading to chaos. Lacarbonora [32] investigated the static and the dynamic behavior of an imperfect beam, initially curved, under electrostatic force and compressive load. He showed the variation of the first three symmetric natural frequencies as varying the compressive load for straight and imperfect beams using the exact solution.

At the microscale, the dynamic and the static behaviors of clamped-clamped shallow arches electrostatically actuated have been investigated extensively in the literature. Casals-Terré and Shkel [33] studied theoretically and experimentally the possibility of triggering the snap-through motion of a bi-stable electrically actuated beam driven dynamically by means of mechanical resonance. Das and Batra [34] have shown the softening effect of the MEMS arch may be dominant before it experiences its snap-through motion using coupled finite-element and boundary-element methods. Poon et al. [35] studied the dynamic buckling response to sinusoidal excitation of a clamped-clamped arch using the Runge–Kutta numerical integration method. Ouakad and Younis [36] investigated the static and dynamic behavior of clamped-clamped micro-machined arches when actuated by a small DC electrostatic load superimposed to an AC harmonic load. A Galerkin-based reduced-order model has been derived and utilized to simulate the static
behavior and the eigenvalue problem under the DC load actuation. Akhabarebch and Younis [37] investigated the effect of axial forces on the static behavior and the fundamental natural frequency of electrostatically actuated MEMS arches.

At the nanoscale, many studies have investigated the static and dynamic behavior of clamped-clamped CNTs, which are initially curved (slack). Sazonova et al. [38] reported experimental investigations showing the importance of slack on estimating the vibrational behavior of CNTs accurately. Mayoof and Hawwa [39] utilized a nonlinear curved beam model to describe the nonlinear dynamics of a slacked single-walled carbon nanotube under linear harmonic excitation. Ouakad and Younis [40] investigated the dynamics of electrically actuated single-walled carbon nanotube resonators including the effect of their initial curvature due to fabrication (slack). They showed that the quadratic nonlinearity due to slack has a dominant effect on the dynamic behavior of the CNT.

1.2.2. Tunable resonators

The static and dynamic behavior of microbeams has been under intensive interest for many researchers all over the world. Among the extensively used methods to excite the microbeams are the electrostatic and the electrothermal actuation. This class of microbeams is composed of an elastic clamped-clamped straight microbeam hanging above a stationary electrode; actuated electrothermally by passing a DC current through it and electrostatically by applying a DC voltage between it and the stationary electrode.

1.2.2.1 Electrostatically tunable resonators

Tunability of MEMS and NEMS resonators is a highly desirable feature for various applications, including communications [41], filtering [42], gyroscopes [43], energy
harvesting [44], and signal processing [45]. The tunability of MEMS resonators using electrostatic forces has been the most used technique since the early work of Nathanson [7]. However, it suffers a major drawback due to the pull-in instability. Particularly high DC voltages can lead to the collapse of the resonating microstructures compromising the device and limiting its tunability [46]. Typically, the electrostatic force is used to induce a softening effect in straight clamped-clamped beams, resulting in a reduction of their resonance frequency [47].

On the other hand, at the nanoscale, there has been experimental evidence demonstrating the tunability, mainly the increase of the resonance frequencies or slacked carbon nanotubes (CNTs) as varying the applied voltage [38, 40]. This behavior has been attributed to the high stiffness of CNTs, and hence, was thought to be a special feature of these small structures [38].

Tajaddodianfar et al. [48] investigated using the strain gradient theory the tunability of size-dependent nano-arches actuated electrostatically. Kosinsky et al. [49] demonstrated experimentally the ability to increase and decrease the resonance frequency of nanomechanical resonators electrostatically. As will be shown in this work, such behavior can be actually exploited at the micro scale as well with a proper design of certain parameters of the resonators. Several studies have investigated the effect of the size effect on electrostatically actuated micro and nano beams and its impact on the buckling and pull-in instabilities [48, 50, 51].

Controlling the DC voltage for electrostatically actuated microbeams is the more known approach since the early work of Nathanson [7]. However, it has been mainly used
for decreasing the resonance frequency, which is also limited by the pull-in instability. Indeed, applying a DC voltage forces the microbeam to deflect towards the stationary electrode and then the beam collapses for a relatively high value of DC voltage [52].

Several numerical techniques have been presented to describe the response of a microbeam under DC voltages, such as the Rayleigh-Ritz method [52], the reduced order method (ROM) [53], the finite element method (FEM) [54], the shooting method [47], and the quadratic differential method (DQM) [55].

Numerous studies have investigated numerically and experimentally the variation of the natural frequencies of clamped-clamped microbeams under electrostatic actuation [56-59]. Nayfeh et al. [60] compared the normalized fundamental natural frequency calculated using the ROM with results obtained by solving the eigenvalue problem of the distributed-parameter system under DC loading using a shooting method presented by Abdel-Rahman et al. [47] and the experimental results obtained by Tilmans et al. [52]. They showed that the electrostatic force, as with the case of other capacitive structures, lowers the fundamental frequency of the clamped-clamped microbeam up to zero near pull-in.

On the other hand, few studies showed, using numerical methods, such as ROM, shooting, and FEM that the natural frequency might be shifted to higher values if the effect of the mid-plane stretching is taken into consideration in the model and for relatively high values of the ratio between the air gap and the thickness of the microbeam. This possibility was first demonstrated by Abdel-Rahman et al. [47] who used a shooting method. They showed that the natural frequency shifted for higher values as increasing the DC
polarization voltage before reaching the pull-in instability for a high ratio between the air gap and the thickness of the microbeam. Batra et al. [54] obtained similar results to [47] using FEM and analytical methods accounting for electrostatic fringing fields.

We note from the aforementioned review the lack of a rigorous study that sheds light on the reason behind the unexpected increase of the resonance frequency of resonators with the DC bias. Few theoretical studies have been presented at the microscale to explain the rise of the fundamental frequency of the electrostatically actuated microbeam for high values of the ratio between the air gap and the thickness of the microbeam; however with no experimental data that demonstrate the phenomenon.

1.2.2.2 Electrothermally tunable resonators

Thermal actuation (via Joule’s heating) is a common mechanism of actuation in MEMS and NEMS thanks to its easy implementation. Joule’s heating is the conversion of the electrical current energy flowing through a structure into heat. Therefore, applying a current through a doubly-clamped microbeam raises its internal temperature, which tends to elongate the microbeam due to thermal expansion. Nonetheless, the elongation is prevented by the presence of the fixed anchors of the microbeam, which induces a compressive force. This compressive load can lead to bucking of the microbeam. This phenomenon can be analyzed from two aspects: electrothermal problem, describing the conversion of the electrical power into heat, and thermo-elastic problem, describing the conversion of the heat power into compressive stress.

Thermal actuation has been drawing significant attention since it provides large displacement for low applied voltages. Several groups have investigated achieving static
buckling of different bistable structures that are electrothermally actuated, such as U-shapes structures [61-63], V-shapes structures [14, 64, 65], clamped-clamped structures [66, 67], and relay structures [68]. Mastropaolo and Cheung [63] investigated the behavior of SiC clamped-clamped bridge resonators electrothermally actuated with U-shaped aluminum electrodes on top as a function of electrode length, width, and spacing. Chioa and Lin [66] studied theoretically and experimentally the critical current for a fixed-fixed microbeam to buckle. Chen et al. [67] reported a theoretical and experimental investigation of the post-buckling behavior of electrothermally actuated beams. Wang et al. [68] developed an electrothermally actuated lateral contact micro-relay for RF applications. They studied the required voltage for the micro-relay utilizing the parallel six-beam.

Electrothermal actuation is also widely used to excite and to tune compliant resonators. At the microscale, electrothermal actuation has been utilized to tune the resonance frequency of resonators, however for a very limited range of frequency. Remtema and Lin [69] showed experimentally and theoretically that the resonance frequency of a resistively heated microbeam could be reduced by 6.5%. Goktas and Zaghloul [70] studied the tunability of CMOS-MEMS fixed-fixed beam resonators using embedded heaters to create axial stress inside the resonator. They showed that the frequency could be decreased by 42.6%. Svilićić et al. [71] presented design, fabrication, and electrical testing of MEMS resonators actuated electrothermally and including a piezoelectric sensor to detect the resonance frequency of these resonators. They demonstrated that with the increase of the electrothermal actuation voltage a tuning range of 17 kHz could be realized for a device resonating at 1.766 MHz. At the nanoscale, the
resonance frequency of electrothermally actuated nanomechanical resonators has been tuned for lower values [72], as demonstrated at the microscale.

1.2.3. Linear and nonlinear mode coupling

In the past few decades, mode coupling among various vibration modes has increasingly been investigated in micro and nanostructures motivated by the wide-ranging studied in the classical structural dynamics performed to investigate theoretically different linear and nonlinear energy transfer and exchange between different coupled modes. This increasing interest in examining the mode coupling of MEMS and NEMS was for either better understanding of different complex fundamental phenomena or developing various applications. Chaos, Hopf bifurcations, synchronization, internal resonance, mode localization, and veering are examples of physical phenomena that were better investigated and controlled using MEMS and NEMS. The coupling of those modes can be linear; through mainly veering and or mode localization, and nonlinear; through internal resonance or the geometric nonlinearity of the structure itself.

1.2.3.1. Linear coupling: Veering

The phenomena of frequency crossing and veering [73-84] have long attracted attention in the classical structural dynamics. When two frequencies approach each other as varying a control parameter, they either crossover or veer from each other. Crossing occurs mostly between two frequencies of symmetric and antisymmetric modes. On veering (avoided-crossing), on the other hand, two frequencies get close to each other, as changing a control parameter, and then deviate away from each other. Mathematically, it is due to the linear coupling between the two involved modes. In the veering regime, both
modes get affected by the shape of each other (hybridization) and then each one continues along the path that the other would have taken if they would cross. Veering has been reported to occur among symmetric and anti-symmetric modes in cable-spring systems [76, 82, 83], plates [74], and curved beams [80]. It was also shown to occur among the symmetric modes of curved beams [73], sagged cables [82, 83], CNTS [40], and curved cylinders [78]. Also, it was demonstrated in discrete systems in spring [79] and pendulums [76].

Among the early works, Petyt and Fleischer [73] investigated using the finite element method the variation of the frequencies of circular curved beams as varying the subtended angle. They showed the crossover of symmetric and antisymmetric frequencies and the veering between the symmetric frequencies.

The veering terminology was first dubbed by Liessa [74] who indicated that veering can occur between two degenerate modes of a membrane due to numerical discretization errors. On the other hand, Perkins and Mote [75] have proven that avoid-crossing can physically occur in continuous and discrete systems. Lacarbonara et al. [80] studied the nonlinear vibrations of a hinged-hinged beam accounting for the interaction among the first two modes (first symmetric and antisymmetric). Both crossover and veering phenomena are shown as varying the stiffness of the torsional spring at the end of one boundary.

Veering has been reported recently on micro and nano curved structures [40]. The static and dynamic behavior of arch beams under electrostatic actuation is well investigated in the literature [36, 85]. At the nanoscale, many studies have investigated the static and dynamic behavior of clamped-clamped CNTs (slack). Sazonova et al. [38] reported
experimental investigations showing veering and indicated the importance of slack on the dynamical behavior of CNTs.

The variation of the axial load of an initially curved beam, which accordingly changes its curvature, can lead to coupling among the different modes of vibration, which can be linear in the case of veering or nonlinear in the case of internal resonance. However assuming idealized buckled beam configuration as the initial shape, no veering has been reported in these for these structures. One can note that despite the extensive work on the mechanics of arches, veering of arches has been rarely reported, and with no experimental presented data. There is a lack of consistent analytical and experimental work into the veering phenomenon, and particularly, on the forced vibration response of arches before, at, and after veering. Most of the theoretical studies of arch beams assume an initial cosine-wave shape (similar to the buckled configuration) except for few studies. For example, the static and dynamics of curved cables/beams of an arc shape were investigated in [86-90].

1.2.3.2. Nonlinear coupling: Internal resonance:

The nonlinear interaction among various modes of vibration of MEMS and NEMS has been increasingly investigated in recent years [91-97]. This is motivated by the fact that understanding mode coupling in MEMS and NEMS resonators is vital for their successful implementation in various applications. One major mechanism of nonlinear mode coupling that has been reported is internal resonance. These have been proposed for applications, such as filters, energy harvesters, and enhanced stable oscillators.

Internal resonance or auto-parametric resonance refers to the exchange of energy between two vibrational modes in a structure through nonlinear coupling. These modes can
be either in the same or different plane of vibration. One necessary condition to activate internal resonance is that the ratio between the two involved modes is an integer. The most common internal resonances are one-to-one [98-101], two-to-one [98, 102-107], and three-to-one [98-100, 108, 109]. Internal resonance has been investigated in different structures including curved beams [99-101], cables [104, 105], and electrically and mechanically coupled structures [110].

In the past decade, internal resonances in different MEMS and NEMS structures have been widely studied. At the nanoscale, Samanta et al. [111] examined the activation of internal resonance in 2D materials; mainly NEMS MoS$_2$ structures. At the microscale, Antonio et al. [96] showed that the oscillation frequency of a nonlinear self-sustained MEMS resonator could be stabilized by coupling different modes of vibration through internal resonance. The same group [91] has recently shown a novel technique to support a self-sustained oscillation, via internal resonance, to compensate for energy losses.

Thanks to their rich dynamical behavior, arch MEMS resonators have been proposed extensively for several potential applications. These structures have rich dynamical characteristics including bistability, snap-through motion, and inherent quadratic, from initial curvature, and cubic, from stretching, nonlinearities. Moreover, one of the interesting characteristics of the arch is that its natural frequencies are sensitive to axial loads; mainly its first natural frequency increases with compressive axial loads while its third natural frequency decreases. This enables the realization of a wide range of commensurate ratios among the various modes [99, 100]. This makes curved structures possess potentially several types of internal resonances (one-to-one, two-to-one, three-to-one, and four-to-one, etc.). The three-to-one and one-to-one internal resonances of
clamped-clamped curved structures under electrostatic forces have been studied numerically and experimentally [99, 100]. The case of two-to-one internal resonance has been investigated in micro-scanners [103], cables [102], and H-shaped structures [110].

Despite the extensive research on internal resonance on MEMS and NEMS resonators, there is a lack of integrated and comprehensive theoretical and experimental investigation on the topic.

1.2.4. Applications

1.2.4.1. Bandpass MEMS filters

The intensive development of MEMS structures has led to a new generation of filters based on MEMS resonators thanks to the high-frequency selectivity and resonance frequency tunability [112-114]. To this end, electrically and/or mechanically coupled multiple MEMS and NEMS resonators to realize bandpass filters has been the subject of intensive research [114-121]. Electrical coupling has an advantage over the mechanical coupling due to the ease of post-fabrication tuning of the filter characteristics. Hajhashemi et al. [120] presented a tunable bandpass filter made of two electrostatically coupled MEMS resonators. They were able to tune the center frequency by controlling the DC voltage of the coupling electrode and controlled the bandwidth by monitoring the applied axial stress. Other groups have investigated the ability to use two MEMS resonators that are tuned and excited independently and are electrically coupled to realize a bandpass filter. Lopez et al. [118] presented a bandpass filter based on two clamped-clamped resonators with a resonance frequency around 22 MHz. They showed a bandwidth of 100-200 kHz in air, and 17 kHz in vacuum. Zou et al. [119] investigated the coupling of four similar beams connected in a square ring to be used as tunable bandpass filter by controlling the
electrostatic bias voltage. Yan et al. [121] studied the ability to use four types of MEMS arrays for bandpass filter by internal mechanical and electrical phase inversion. Single MEMS resonator was used as notch or single-frequency pass filters with a high quality factor in the vacuum condition [122]. Ouakad and Younis [85] studied the dynamic response of an electrostatically excited arch beam resonator and proposed a filter operation based on the snap-through motion.

1.2.4.2. Pressure sensors

MEMS and NEMS demonstrate high capability for different sensing applications, such as force [123], mass/gas [124], and pressure [125]. Pressure sensors have been used to provide an accurate estimation of the surrounding pressure in both domestic and industrial applications.

The quest for miniaturized systems and low-cost deployable pressure sensors has recently sparked an interest to seek alternative approaches for pressure sensing; other than the conventional [126] and bulkier strain-gauge [127], capacitive [128], bridge resistors [129], and piezo-resistive [130] pressure sensors. Different techniques and designs have been explored to realize pressure sensors with improved sensitivity based on micro-sized diaphragms [131], carbon nanotubes [132], micromechanical drumhead resonators [133], microcantilever [134], and bridge resonators [135, 136].

A recent study [135] presented a new pressure-sensing technique based on monitoring the variation of the resonance frequency of a locally heated beam resonator by an external laser source. Although a better sensitivity was demonstrated compared to conventional sensors, the performance of such devices depends on the resonator geometry
and the external laser wavelength. Moreover, the resonant structure can only be miniaturized up to a certain level depending on the spot-size of the laser source used for the local heating.

Based on the cooling effect of the air surrounding an electrothermally heated bridge resonator, we propose an alternative pressure sensor that offers the flexibility of being scalable to small or large sizes. The operation principle of the sensor relies on tracking the change in resonance frequency of the resonant structure with pressure while actuated electrothermally.

1.3. Objectives and contributions

This thesis aims to design and characterize MEMS resonators that are highly tunable using electrostatic and electrothermal transduction mechanisms and implement them in different applications. In this work, we will focus on studying the static and dynamic behavior of in-plane MEMS resonator, straight and initially curved clamped-clamped microbeams, electrothermally tuned and electrostatically actuated. The electrostatic actuation is used to either tune the resonance frequency of the MEMS resonator due to its static part or to make the resonator vibrates in the linear and nonlinear regime. The electrothermal actuation is used as a way to tune the stiffness of the MEMS resonators and hence their resonance frequency.

In the first part of the thesis, we aim to investigate, theoretically and experimentally, the tunability of NEMS and MEMS resonators either electrothermally or electrostatically. Electrothermal tuning is performed by passing a current flowing through the silicon microbeam that generates heat and hence controls its stiffness. Electrostatic tuning is performed by applying a potential voltage between the resonant beam and a
stationary electrode. The effect of geometric parameters (thickness, initial curvature, and transduction gap) on the resonance frequency are investigated experimentally and theoretically using the reduced order model based on Euler-Bernoulli beam theory. A high tunability is demonstrated for NEMS and MEMS beams, straight and initially curved as shown in Figure 1.1, using either electrostatic or electrothermal tuning.

Figure 1.1: (a)-(b) Schematic of the in-plane straight and initially curved beams, respectively, sandwiched between two stationary electrodes.

In the second part of the thesis, we aim to investigate the linear and nonlinear coupling between the first two symmetric modes of an initially curved microbeam electrothermally tuned and electrostatically driven. The natural frequencies of the arch beam are highly tuned as tuning its stiffness electrothermally. Hence, different ratios between different modes could be obtained. In Chapters 3 and 4, we investigate the activation of linear coupling via veering phenomenon and the nonlinear coupling via internal resonance. We experimentally investigate the two-to-one, three-to-one, and four-to-one internal resonance among the first two symmetric modes of vibration. Theoretically,
we investigate in-depth the two-to-one internal resonance as a case study using a reduced order model.

The last part of the thesis will focus on implementing the frequency tuning due to the electrothermal actuation in different applications: filtering and sensing. First, we experimentally demonstrate the exploitation of the nonlinear softening, hardening, and veering phenomena on an arch beam, where the frequencies of two vibration modes get close to each other, to realize a bandpass filter with a sharp roll off from the passband to the stopband. Second, a pressure sensor based on the convective cooling of the air surrounding an electrothermally heated resonant bridge. The concept is demonstrated using both straight and arch microbeam resonators driven and sensed electrostatically. The change in the surrounding pressure is shown to be accurately tracked by monitoring the change in the resonance frequency of the structure.
Chapter 2  Tunability of electrostatically actuated clamped-clamped micro and nano resonators

In this chapter, we investigate the effect of geometric parameters on the variation of the resonance frequency of micro and nano resonators, straight and initially curved, as tuning the DC electrostatic voltage.

2.1 Straight beam

2.1.1 Problem formulation

The device under consideration, Figure 2.1, consists of a clamped-clamped microbeam actuated by a DC polarization voltage $V_{DC}$ and an AC harmonic voltage of amplitude $V_{AC}$ and of frequency $\hat{\Omega}$, and is subjected to viscous damping of coefficient $\hat{c}$.

![Figure 2.1: Schematic of an electrostatically actuated clamped-clamped microbeam.](image)

The equation of motion governing the transverse deflection of the beam $\hat{w}(\hat{x}, \hat{t})$ is written as follows [137]:

$$
\rho bh \frac{\partial^2 \hat{w}}{\partial \hat{t}^2} + c \frac{\partial \hat{w}}{\partial \hat{t}} + EI \frac{\partial^4 \hat{w}}{\partial \hat{x}^4} = \left[ \hat{N} + \frac{EA}{2l} \int_0^1 \left( \frac{\partial \hat{w}}{\partial \hat{x}} \right)^2 d\hat{x} \right] \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \frac{1}{2} \varepsilon \hat{b} \left( \frac{V_{DC} + V_{AC} \cos(\hat{\Omega} \hat{t})}{d - \hat{w}} \right)^2
$$

(2.1)
where $\hat{x}$ is the position along the microbeam, $\hat{t}$ is time, $l$, $b$ and $h$ are, respectively, the length, width, and thickness of the microbeam, $\rho$ is the material density, $I = bh^3/12$ is the moment of inertia of the rectangular cross-section of area $A = bh$, $E$ is Young’s modulus, $\hat{N}$ is the external axial force and in this work it presents the residual axial load, $d$ is the gap width between the microbeam and the stationary electrode, and $\varepsilon$ is the dielectric constant of the medium. The microbeam is subjected to the following boundary conditions:

$$\hat{w}(0,\hat{t}) = \hat{w}(l,\hat{t}) = 0 \quad \text{and} \quad \frac{\partial \hat{w}}{\partial \hat{x}}(0,\hat{t}) = \frac{\partial \hat{w}}{\partial \hat{x}}(l,\hat{t}) = 0 \quad (2.2)$$

For convenience, we introduce the nondimensional variables as below:

$$w = \frac{\hat{w}}{d} ; \quad x = \frac{\hat{x}}{l} \quad \text{and} \quad t = \frac{\hat{t}}{T} \quad (2.3)$$

where $T = \sqrt{\frac{\rho bh^4}{EI}}$ is the time scale chosen for convenience. Substituting Equation (2.3) into Equations (2.1) and (2.2), we obtain the nondimensional equation of motion of the beam:

$$\frac{\partial^2 w}{\partial t^2} + c \frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial x^4} = N + \alpha_1 \left[ \int_0^1 \left( \frac{\partial w}{\partial \hat{x}} \right)^2 d\hat{x} \right] \frac{\partial^2 w}{\partial x^2} + \alpha_2 \left( \frac{V_{dc} + V_{ac} \cos(\Omega t)}{1-w} \right)^2 \quad (2.4)$$

Subjected to the nondimensional boundary conditions:

$$w(0,t) = w(l,t) = 0 \quad ; \quad \frac{\partial w}{\partial \hat{x}}(0,\hat{t}) = \frac{\partial w}{\partial \hat{x}}(l,\hat{t}) = 0 \quad (2.5)$$
The nondimensional, except \( \alpha_2 \), parameters appearing in Equation (2.4) are defined as below:

\[
\alpha_1 = 6\left(\frac{d}{h}\right)^2; \quad \alpha_2 = \frac{6\varepsilon l^4}{Eh^4d^4}; \quad N = \frac{l^2}{EI} \hat{N}; \quad c = \frac{l^4}{EIT} \hat{c} \quad \text{and} \quad \Omega = T\hat{\Omega}
\]  

(2.6)

The deflection of the microbeam under the electrostatic force is split into a static deflection \( w_s(x) \) due to the DC component and a small dynamic deflection \( w_d(x,t) \)

\[
w(x,t) = w_s(x) + w_d(x,t)
\]  

(2.7)

To determine the static deflection \( w_s(x) \) we set the time derivatives as well as the AC force equal to zero:

\[
\frac{d^4w_s}{dx^4} = \left[ N + \alpha_1 \int_0^1 \left( \frac{dw_s}{dx} \right)^2 dx \right] \frac{d^2w_s}{dx^2} + \alpha_2 \frac{V_{DC}^2}{(1-w_s)^2}
\]  

(2.8)

with the associated boundary conditions

\[
w_s(0) = w_s(1) = 0 \quad \text{and} \quad \left. \frac{dw_s}{dx} \right|_{x=0} = \left. \frac{dw_s}{dx} \right|_{x=1} = 0
\]  

(2.9)

The linearized equation of motion describing the small dynamic behavior of the microbeam around the deflected shape is derived by substituting Equation (2.7) into equation (2.4) and dropping the terms representing the equilibrium position, equations (2.8), and the nonlinear terms. This yields
\[
\frac{\partial^2 w_d}{\partial t^2} + \frac{\partial^4 w_d}{\partial x^4} = \left[ N + \alpha_1 \int_0^1 \left( \frac{dw_x}{dx} \right)^2 \right] \frac{\partial^2 w_d}{\partial x^2} + \left[ 2\alpha_2 \int_0^1 \frac{dw_x}{dx} \frac{\partial w_d}{\partial x} \right] \frac{d^2 w_s}{dx^2} + 2\alpha_2 \frac{\left( V_{dc} + V_{ac} \cos(\Omega t) \right)^2}{(1 - w_s)^3} w_d
\]

(2.10)

with the associated boundary conditions:

\[
w_d(0,t) = w_d(1,t) = 0 \text{ and } \left. \frac{\partial w_d}{\partial x} \right|_{x=0,t} = \left. \frac{\partial w_d}{\partial x} \right|_{x=1,t} = 0
\]

(2.11)

### 2.1.2 Static response

Equations (2.8) and (2.9) describe, respectively, the static deflection of the microbeam and its associated boundary conditions. To solve Equation (2.8), we refer to the Galerkin procedure in which we use the undamped linear mode shapes of a straight microbeam as basis functions [137]. Therefore the deflection is expressed as

\[
w_s(x) = \sum_{i=0}^n u_i \phi_i(x)
\]

(2.12)

where \( u_i \) (i=0, 1, 2...n) denotes the nondimensional modal coordinates and \( \phi_i(x) \) (i=0, 1, 2...n) denotes the undamped mode shapes of the straight unactuated beam governed by

\[
\frac{d^4 \phi_i}{dx^4} - N \frac{d^2 \phi_i}{dx^2} - \omega_i^2 \phi_i = 0
\]

(2.13)

with the associated boundary conditions:
\( \phi_i(0) = \phi_i(1) = 0 \quad \text{and} \quad \left. \frac{d\phi}{dx} \right|_{x=0} = \left. \frac{d\phi}{dx} \right|_{x=1} = 0 \)  

where \( \omega_i \) represents the \( i^{th} \) natural frequency of the microbeam. We solve equation (2.8) for a range of DC voltage from zero to pull-in for different values of the ratio between the air gap and the thickness of the beam, which is represented by the nondimensional parameter \( \alpha_1 \) and assuming that the residual axial load is equal to zero. The geometrical and mechanical parameters adopted for the simulations are as shown in Table 2.1:

Table 2-1: Geometrical, mechanical properties of the microbeam.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Quantity</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l )</td>
<td>Length</td>
<td>600 ( \mu )m</td>
</tr>
<tr>
<td>( h )</td>
<td>Thickness</td>
<td>2 ( \mu )m</td>
</tr>
<tr>
<td>( b )</td>
<td>Width</td>
<td>25 ( \mu )m</td>
</tr>
<tr>
<td>( d )</td>
<td>Gap</td>
<td>1-10 ( \mu )m</td>
</tr>
<tr>
<td>( E )</td>
<td>Young’s Modulus</td>
<td>154 GPa</td>
</tr>
<tr>
<td>( \rho )</td>
<td>Density</td>
<td>2332 kg/m(^3)</td>
</tr>
</tbody>
</table>

In Figure 2.2 we plot the maximum nondimensional static deflection \( W_{\text{max}}=w_s(0.5) \) as a function of the DC voltage for different values of the gap width \( d \). The maximum deflection increases as the electrostatic force represented by the DC voltage increases. We found that the maximum deflection decreases for high values of \( \alpha_1 \) at the same level of DC voltage. On the other hand, in Figure 2.2, the pull-in instability is shifted for higher values of DC voltage as \( \alpha_1 \) increases. The reason for this is that as \( \alpha_1 \) increases, by increasing \( d \) for a fixed value of thickness \( h \), the electrostatic force effect weakens with respect to mid-
plane stretching. As reported by Younis and coworkers in [47, 53], Tilmans et al. [52], and Hung et al. [59] mid-plane stretching is crucial in the model to avoid underestimation of the pull-in instability limits.

![Figure 2.2](image-url)

Figure 2.2: The maximum static deflection of the microbeam versus the DC polarization voltage for different values of $\alpha_1$.

### 2.1.3 Eigenvalue problem

To determine the variation of the natural frequency of the microbeam under the DC polarization voltage, we resort to the Galerkin discretization to represent the dynamic deflection $w_d(x,t)$ and to solve the eigenvalue problem [137]. Toward this, we let:

$$w_d(x,t) = \sum_{i=0}^{n} u_i(t) \phi_i(x)$$  \hspace{1cm} (2.15)

where $u_i(t) (i=0,1,2..n)$ denotes the nondimensional modal coordinates and $\phi_i(x) (i=0,1,2..n)$ denotes the undamped mode shapes of the straight unactuated beam.
Then, we substitute Equation (2.15) into Equation (2.10), multiplying the outcome by the mode shape $\phi_j$, applying the orthogonality condition of the mode shapes, and integrating over the beam domain (from 0 to 1), which yields the below equation:

$$
\ddot{u}_j + u_j \phi_{j, non, 1}^2 = \left[ 2\alpha \int_0^1 \phi_j \frac{d^2 w_s}{dx^2} dx \int_0^1 \left( \sum_{i=0}^n u_i \phi_i \frac{dw_s}{dx} \right) dx \right] + \left[ \alpha \int_0^1 \phi_j \left( \sum_{i=0}^n u_i \phi_i \right) dx \right] + \left[ \alpha \int_0^1 \phi_j \left( \frac{d^2 w_s}{dx^2} \right) dx \right] + \left[ \alpha \int_0^1 \phi_j \phi_{j, non, 1}^2 \left( \sum_{i=0}^n u_i \phi_i \right) dx \right]
$$

(2.16)

Using three symmetric mode shapes, the above system can be expanded into the following three modal equations:

$$
u_1 + u_1 \phi_{1, non, 1}^2 = \left[ 2\alpha \int_0^1 \phi_1 \frac{d^2 w_s}{dx^2} dx \int_0^1 \left( \sum_{i=0}^3 u_i \phi_i \frac{dw_s}{dx} \right) dx \right] + \left[ \alpha \int_0^1 \phi_1 \left( \sum_{i=0}^3 u_i \phi_i \right) dx \right] + \left[ \alpha \int_0^1 \phi_1 \left( \frac{d^2 w_s}{dx^2} \right) dx \right] + \left[ \alpha \int_0^1 \phi_1 \phi_{1, non, 1}^2 \left( \sum_{i=0}^3 u_i \phi_i \right) dx \right]
$$

(2.17)

$$
u_2 + u_2 \phi_{2, non, 2}^2 = \left[ 2\alpha \int_0^1 \phi_2 \frac{d^2 w_s}{dx^2} dx \int_0^1 \left( \sum_{i=0}^3 u_i \phi_i \frac{dw_s}{dx} \right) dx \right] + \left[ \alpha \int_0^1 \phi_2 \left( \sum_{i=0}^3 u_i \phi_i \right) dx \right] + \left[ \alpha \int_0^1 \phi_2 \left( \frac{d^2 w_s}{dx^2} \right) dx \right] + \left[ \alpha \int_0^1 \phi_2 \phi_{2, non, 2}^2 \left( \sum_{i=0}^3 u_i \phi_i \right) dx \right]
$$

(2.18)

$$
u_3 + u_3 \phi_{3, non, 3}^2 = \left[ 2\alpha \int_0^1 \phi_3 \frac{d^2 w_s}{dx^2} dx \int_0^1 \left( \sum_{i=0}^3 u_i \phi_i \frac{dw_s}{dx} \right) dx \right] + \left[ \alpha \int_0^1 \phi_3 \left( \sum_{i=0}^3 u_i \phi_i \right) dx \right] + \left[ \alpha \int_0^1 \phi_3 \left( \frac{d^2 w_s}{dx^2} \right) dx \right] + \left[ \alpha \int_0^1 \phi_3 \phi_{3, non, 3}^2 \left( \sum_{i=0}^3 u_i \phi_i \right) dx \right]
$$

(2.19)

Equations (2.17), (2.18) and (2.19) represent a discretized system of three ordinary differential equations. For a given $V_{DC}$, we compute the Jacobean of the system of Equations (2.17), (2.18) and (2.19) and find the corresponding eigenvalues. Then, by taking
the square root of these eigenvalues, we find the natural frequencies of the resonators under a DC voltage. In Figure 2.3, we show the variation of the fundamental frequency as a function of the DC loading for different values of $\alpha_1$ for the same parameters used in Section 2.3. As demonstrated in Section 2.3, the pull-in instability shifts for higher values as $\alpha_1$ is increased. We demonstrate in Figures 2.3 that the variation of the first natural frequency is highly dependent to the variation of the electrostatic force as well as the variation of the ratio between the air gap and the thickness of the microbeam.

For low values of $\alpha_1$ that are represented in Figure 2.3a by gaps comparable to the thickness of the microbeam, the fundamental frequency decreases as the DC voltage is increased, as traditionally known in the literature. In this case, the effect of mid-plane stretching is negligible. However, for low DC voltage, regardless of the value of $\alpha_1$, the effect of the electrostatic force becomes dominant, which leads to the decrease of the natural frequency. For $20 \leq \alpha_1 \leq 30$, the effect of the mid-plane stretching is comparable to the effect of the electrostatic force. Then, the natural frequency increases slightly with the increase of the DC voltage as shown in Figure 2.3b. For $\alpha_1 \geq 30$, Figure 2.3c, the effect of the mid-plane stretching dominates the effect of the electrostatic force. As sweeping the DC voltage in this case, the natural frequency reaches more than 100% increase with respect to its zero-voltage value. For example, for $\alpha_1 = 150$, the natural frequency can increase from 40 kHz for $V_{DC}=0$ V to 60 kHz at $V_{DC}=100$ V, 80 kHz at $V_{DC}=160$ V and 96 kHz at $V_{DC}=220$ V. Therefore, we prove here that one can increase the frequency to more than twice its original value by increasing the gap between the microbeam and the stationary electrode and by increasing the DC voltage that is far from the pull-in instability.
Figure 2.3: The fundamental natural frequency versus the DC polarization voltage for different values of $\alpha_1$, for low, medium and high values of $\alpha_1$ (a), (b) and (c) respectively. In the legends, $d$ has the unit of micrometers ($\mu$m).

Figure 2.4 shows the variation of the fundamental frequency as a function of DC voltage for various values of axial load for different values of $\alpha_1$. For low values of $\alpha_1$, the axial force only shifts $\omega_1$ and the pull-in instability when varying from compressive to tensile loading and it is softening with DC as shown in Figure 2.4a. At low DC voltages, there is tunability in frequency and pull-in instability for medium and high values of $\alpha_1$ when there is compressive stress as well as no stress conditions shown in Figures 2.4b and 2.4c. At high
values of DC voltages, the resonator is highly tunable for medium and high $\alpha_1$. However, the resonator is highly unstable for medium values $\alpha_1$. These shifts in $\omega_1$ and pull-in instability proved in [47] for the same values of DC voltages and for low values of $\alpha_1$.

Figure 2.4: Fundamental natural frequency versus DC voltage for different values of axial load for $\alpha_1$ = 6; (a); $\alpha_1$ = 37.5; (b); and $\alpha_1$ = 121.5; (c).

To further understand the frequency shift due to the DC voltage under various $\alpha_1$, we solve Equation (2.16) employing one symmetric mode in the Galerkin discretization and assuming a harmonic motion in the first mode shape $u_1(t) = e^{j\omega t}$. Then, Equation (2.16) can be written as
\[-\omega_i^2 + \omega_{non,1}^2 = \left[2\alpha_1 \int_0^1 \phi_1 \frac{d^2 w_1}{dx^2} \, dx \int_0^1 \phi_1', \frac{dw_1}{dx} \, dx \right] + \left[2\alpha_2 \int_0^1 \left(\frac{dw_1}{dx}\right)^2 \, dx \right] \int_0^1 \phi_1 \phi_1' \, dx + \int_0^1 \frac{2\alpha_2 V_{DC}^2}{(1-w_s)^3} \phi_1^2 \, dx\]

(2.20)

The new fundamental frequency under the DC loading can be written as follows:

\[\omega_i^2 = \omega_{non,1}^2 + \text{Shift}\]

(2.21)

where

\[\text{Shift} = \left[-\alpha_1 \left[2\int_0^1 \phi_1 \frac{d^2 w_1}{dx^2} \, dx \int_0^1 \phi_1', \frac{dw_1}{dx} \, dx + \int_0^1 \left(\frac{dw_1}{dx}\right)^2 \, dx \int_0^1 \phi_1 \phi_1' \, dx \right] \right] + \left[-\int_0^1 \frac{2\alpha_2 V_{DC}^2}{(1-w_s)^3} \phi_1^2 \, dx \right]\]

(2.22)

The right-hand side of Equation (2.22) is composed of the summation of the shift due to the mid-plane stretching and the shift due to the electrostatic force, respectively. We show in Figure 2.5a the nondimensional shift of the natural frequency, \(\text{Shift}\), as a function of the DC voltage for the same beam described in Section 2.3 for a fixed \(d = 7 \mu m\). As shown in Figure 2.5b, the shift due to the electrostatic force is always a negative quantity, thus leading as expected to the softening effect. Hence, it acts to reduce the natural frequency of the beam since and decreases its stiffness. On the other hand, the shift due to mid-plane stretching, Figure 2.5c is always a positive quantity, which tends to raise the natural frequency. Adding both opposing effects leads to the total shift in Figure 2.4a, in which case the effect of midplane stretching due to the deformation of the beam from the DC bias happens to overcome the softening effect of the DC bias.
Figure 2.5: Frequency shift versus DC polarization voltage for $\alpha_1=62.5$ (a) total shift, (b) shift due to electrostatic force, and (c) shift due to mid-plane stretching.

Next, we study the effect of the electrostatic force as well as the ratio between the air gap and the thickness of the microbeam on the higher modes of vibrations. Figure 2.6 shows the variation of the second and third symmetric dimensional natural frequencies as a function of the DC loading for different values of $\alpha_1$ for the parameters of Table 2.1. We demonstrate that the effect of the electrostatic force is not strong for higher order modes. The shift is more influenced by the midplane stretching due to the static deformation of the
beam from the DC bias that leads to an increase of the second and third symmetric natural frequencies as increasing the DC load.

![Graph showing second and third symmetric dimensional natural frequencies versus the DC polarization voltage for different values of \( \alpha \).](image)

Figure 2.6: Second and Third symmetric dimensional natural frequencies versus the DC polarization voltage for different values of \( \alpha \).

2.1.4 Experimental setup

We conducted the experimental investigation on a polysilicon chip containing clamped-clamped straight microbeams with different length, thickness, and gap width dimensions. These microbeams were fabricated from SOI wafers with highly conductive 25 \( \mu \)m Si device layer by MEMSCAP [138]. We excite them electrostatically to get the resonance frequency. To get the resonance frequency \( \omega_1 \), we sweep the electrostatic voltage around \( \omega_1 \) while maintaining small AC voltage values.

For this work, we consider two beams to conduct the experiments. The length of the beams is 600 \( \mu \)m, their width is 25 \( \mu \)m, and the gaps separating them from the substrate have nominal values of 7 and 8 \( \mu \)m. A top view of one of the beams is depicted in Figure 2.7a. In the experimental setup, shown in Figure 2.7b, we use the planner motion analyzer.
(PMA) [139] that is designed for in-plane microstructure vibration and motion analysis using stroboscopic video microscopy.

Figure 2.7: (a) Top view of the microbeam. (b) The experimental set-up.

### 2.1.5 Results and discussion

Next, we demonstrate experimentally the increase of the natural frequency of the microbeam as increasing the DC voltage for high values of $\alpha_I$. To investigate the resonance frequency of the studied beams, we proceed by conducting frequency sweep tests for different DC voltages and $V_{AC}=3$ V as shown in Figures 2.8b and 2.8d. To guarantee linear frequency-response curves, and hence avoid any softening or hardening behavior in these curves, we limit the DC voltage by $V_{DC}=90$ V for the beam with the nominal gap 7 $\mu$m (beam 1) and $V_{DC}=95$ V for the beam with nominal gap 8$\mu$m (beam 2).
Figures 2.8a and 2.8c show that for the small DC loads, the natural frequency of the beam decreases as demonstrated in Section 2.4. Increasing the DC voltage further leads to the raise of the natural frequency as well. We reach for beam 1 a raise of 25% and for beam 2 a raise of 35% of the natural frequency. We found that the thickness of the beam is not uniform along the beam. Hence, in our simulation we try to estimate an effective $\alpha_1$, which takes into consideration the non-uniformity of the thickness. For beam 1, we found that the effective $\alpha_1 = 38$ and for beam 2 the effective $\alpha_1 = 60$. The Figures show good agreement between the experimental data and the analytical results.

![Graphs showing natural frequency versus DC polarization voltage](image)

Figure 2.8: First natural frequency versus DC polarization voltage and the frequency response curve of different DC voltage with $V_{AC} = 3$ V respectively for beam 1 (a) and (b) and beam 2 (c) and (d).

2.2 Arch beam

2.2.1 Problem formulation

The device under consideration, Figure 2.9, consists of an initially curved doubly-clamped nanobeam, which is curved due to residual stresses from fabrication. This effect
can be controlled to some degree through the choices of the beam length and thickness.

The nanobeam is actuated by a dc polarization voltage $V_{DC}$ and an ac harmonic voltage of amplitude $V_{AC}$ and frequency $\Omega$, and is subjected to viscous damping of coefficient $c$. The initial shape $w_0$ along the beam length $x$ is approximated to be of the buckled form [16] expressed as

$$w_0(x) = -\frac{1}{2} b_0 \left( 1 - \cos(2\pi \frac{x}{l}) \right)$$

(2.23)

Figure 2.9: Schematic of an electrostatically actuated doubly-clamped curved beam.

The nonlinear Euler-Bernoulli equation of motion governing the transverse deflection of the beam $w(x,t)$ in space and time $t$ is written as [137, 140]:

$$\rho bh \frac{\partial^2 w}{\partial t^2} + c \frac{\partial w}{\partial t} + E I_s \frac{\partial^4 w}{\partial x^4} = \left( \frac{\partial^2 w}{\partial x^2} + \frac{d^2 w_0}{dx^2} \right) \left[ N + \frac{EA}{2l} \int_0^l \left( \frac{\partial w}{\partial x} \right)^2 + 2 \frac{\partial w}{\partial x} \frac{d w_0}{dx} \right] dx + \frac{1}{2} \varepsilon b \frac{(V_{DC} - V_{AC} \cos(\Omega t))^2}{(d-w-w_0)^2} - \frac{1}{2} \varepsilon b \frac{V_{DC}^2}{(d+w+w_0)^2}$$

(2.24)

The nanobeam is subjected to the fixed-fixed boundary conditions

$$w(0,t) = w(l,t) = 0 \text{ and } \left. \frac{\partial w}{\partial x} \right|_{(0,t)} = \left. \frac{\partial w}{\partial x} \right|_{(l,t)} = 0$$

(2.25)
where $l$, $b$, and $h$ are, respectively, the length, depth, and thickness of the beam, $d$ is the gap between the beam and the stationary electrodes, $\rho$ is the material density, $I_s=bh^3/12$ is the moment of inertia of the cross-section of area $A=bh$, $E$ is Young’s modulus, and $\varepsilon$ is the dielectric constant of the gap medium. $N$ denotes the residual axial force originated from fabrication. The last term in Equation (2.24) represents the electrostatic force from the sensing electrode.

In this work, a multi-mode Galerkin procedure is adopted to solve for the variation of the resonance frequency of the nanobeam with $V_{DC}$ [137]. To show more clearly the contribution of the various parameters on the resonance frequency, a one mode discretization is shown next, which provides a closed form approximate solution of the first resonance frequency as below

$$\omega^2 = \frac{EI_s}{\rho bh^3} \left\{ \left[ \frac{1}{0} \left( \phi_1 \phi_1^{(iv)} \right) d\bar{x} \right] - \left[ \frac{1}{0} \phi_1 \left( \frac{d^2 \tilde{w}_s + d^2 \tilde{w}_0}{d\bar{x}^2} \right) d\bar{x} \right] \left[ \frac{1}{0} \left( \frac{d \tilde{w}_s + d \tilde{w}_0}{d\bar{x}} \right) d\bar{x} \right] \right\}$$

$$- \left[ \frac{1}{0} \left( \frac{d \tilde{w}_s}{d\bar{x}} \right)^2 + 2 \left( \frac{d \tilde{w}_s}{d\bar{x}} \right) \left( \frac{d \tilde{w}_0}{d\bar{x}} \right) d\bar{x} \right] \left[ \frac{1}{0} \phi_1^{(iv)} d\bar{x} \right]$$

$$+ \frac{\tilde{N} + \alpha_1}{0} \left[ \left( \frac{d \tilde{w}_s}{d\bar{x}} \right)^2 + 2 \left( \frac{d \tilde{w}_s}{d\bar{x}} \right) \left( \frac{d \tilde{w}_0}{d\bar{x}} \right) d\bar{x} \right] \left[ \frac{1}{0} \phi_1^{(iv)} d\bar{x} \right]$$

$$+ 2\alpha_2 \left[ \frac{1}{0} \phi_1^2 \frac{V_{DC}^2}{(1-\tilde{w}_s - \tilde{w}_0)^3} d\bar{x} + 2 \alpha_2 \left[ \frac{1}{0} \phi_1^2 \frac{V_{DC}^2}{(1+\tilde{w}_s + \tilde{w}_0)^3} d\bar{x} \right] \right]$$

where

$$\bar{x} = \frac{x}{l}, \quad \tilde{w}_s = \frac{w_s}{d}, \quad \tilde{w}_0 = \frac{w_0}{d}, \quad \alpha_1 = 6 \left( \frac{d}{h} \right)^2, \quad \alpha_2 = \frac{6 \varepsilon l^4}{E h d^3}, \quad \tilde{N} = \frac{2}{EI_s} N$$

(2.27)

where $\phi_l(x)$ denotes the first mode shape of the unactuated shallow arch, $\omega$ denotes the new natural frequency of the nanobeam under the DC bias voltage and $w_s(x)$ is the static
deflection of the nanobeam due to $V_{DC}$ governed by

$$EI_s \frac{d^4 w_s}{dx^4} = \left( \frac{d^2 w_s}{dx^2} + \frac{d^2 w_0}{dx^2} \right) \left[ N + \frac{EA}{2l} \int_0^l \left( \frac{dw_s}{dx} \right)^2 + 2 \frac{dw_s dw_0}{dx dx} \right] dx$$

$$+ \frac{1}{2} \frac{V_{DC}^2}{\left(d-w_s-w_0\right)^2} - \frac{1}{2} \frac{V_{DC}^2}{\left(d+w_s+w_0\right)^2}$$

(2.28)

The first term in the right-hand side of Equation (2.26) is positive and represents the bending contribution that implicitly describes the fundamental frequency of the nanobeam at zero $V_{DC}$. The second term is negative and represents the effect of the midplane stretching, axial load, and initial curvature. The last term is positive and represents the effect of the electrostatic force induced by the two-side electrodes. One can note that for a perfectly straight beam $w_0=0$ and $w_S=0$ (because of the two equal electrostatic forces of opposite signs), and thus the second term cancels out. Hence, the resonance frequency is only affected by the softening effect of the electrostatic force.

Equation (2.26) demonstrates that the eigenvalue problem, and hence the resonance frequency of the resonators, depends strongly on the thickness $h$, gap $d$, and the mid-beam initial rise $b_0$ (through $w_0$). It is possible to strengthen the stretching effect by minimizing $h$ and maximizing $d$, and hence maximizing $\alpha_1$, which tends to increase the resonance frequency to overcome the softening effect of the electrostatic force. Also, note that the presence of $w_0$ is essential to create non-zero $w_S$, and hence induces mid-plane stretching.

We study analytically the combined effect of the mid-beam initial rise and the gap-to-thickness ratio on the resonance frequency of silicon-based nanobeams. All the analytical results are obtained considering a Young’s modulus of 169 GPa and a Poisson’s
ratio of 0.27 for nanobeams of 15 µm length, 850 nm depth, and 150 nm thickness. The results are obtained using five modes in the Galerkin procedure.

Figures 2.10a-2.10d show the variation of the beam midpoint deflection and the corresponding resonance frequency for various initial curvature rises for a 250 nm and a 850 nm transduction gap between the beam and sensing/driving electrodes. For \( b_0 = 0 \) nm, Figure 2.10a indicates zero static deflection while Figure 2.10b shows a continuous decrease in the resonance frequency. A slight increase in \( b_0 \) induces mid-plane stretching and curvature effects, which impact considerably the static deformation and hence the resonance frequency. For the cases of 850 nm gap, Figures 2.10a and 2.10b, increasing the voltage always increases the curvature of the already curved beams resulting in continuous increase in the resonance frequency until reaching finally pull-in. Because of the relatively large gap compared to the thickness (gap-to-thickness ratio= 5.67) and initial rise, the increase in stiffness always dominates the electrostatic force effect, despite the fact that the beam deforms toward one of the electrodes as increasing \( V_{DC} \). For the cases of 250 nm gap, Figures 2.10c and 2.10d, for small values of \( b_0 \) increasing \( V_{DC} \) increases the curvature and the resonance frequency until pull-in, similar to the previous cases, in which the electrostatic force is dominated by the mid-plane stretching and curvature effects. For larger values of \( b_0 \), the beam is curved toward and very close to one of the electrostatic electrode. This maximizes the electrostatic force effect that dominates the curvature and stretching effects. This can be also noticed from the small gap-to-thickness ratio of 1.67.
Figure 2.10: The simulated static deflection of the beam midpoint and resonance frequency for various initial rises versus the DC bias voltages for initially curved nanobeams of 15 µm length, 850 nm depth, 150 nm thickness, and the transduction gaps of $d$ = 850 nm (a,b) and $d$ = 250 nm (c,d).

Next, we investigate the effect of the gap $d$ considering a similar nanobeam of Figure 2.10 while fixing the curvature. Figures 2.11a and 2.11b show the variation of the resonance frequency with $d$. Figure 2.11a shows that for $d$ = 150 nm (gap-to-thickness ratio = 1) the softening effect due to electrostatic force is dominant over the mid-plane stretching; thus resulting in a decrease in the resonance frequency with increasing $V_{DC}$. On the other
hand, for \( d = 250 \text{ nm}-450 \text{ nm} \) (gap-to-thickness ratio =1.67-3) the frequency increases with the voltages, showing the dominant effect of mid-plane stretching over the softening effect of the electrostatic force. However, the tunability is still limited due to the relatively small pull-in voltages. For beams with larger transduction gaps, Figure 11b, higher tunability ranges are achieved. This is clearly attributed to the dominant mid-plane stretching effect as indicated from the larger gap-to-thickness ratio and the initial curvature over the softening effect of the electrostatic force. Figures 2.11a and 2.11b show similar results for another case study of a beam with higher initial curvature of \( b_0 = 130 \text{ nm} \). The results indicate similar trend to that of Figure 2.11 however with higher tunability range. As noted for large values of \( d \), very high pull-in voltages are predicted, which practically eliminate the possibility of pull-in.

![Graphs showing simulated resonance frequency versus dc voltage for beams of 15 \( \mu \text{m} \) length, 850 nm depth, 150 nm thickness, and 45 nm initial rise for various values of gap of (a) \( d = 150 \text{ nm}-450 \text{ nm} \), (b) \( d = 0.65 \mu \text{m}-1.3 \mu \text{m} \).](image-url)

Figure 2.11: The simulated resonance frequency versus dc voltage for beams of 15 \( \mu \text{m} \) length, 850 nm depth, 150 nm thickness, and 45 nm initial rise for various values of gap of (a) \( d = 150 \text{ nm}-450 \text{ nm} \), (b) \( d = 0.65 \mu \text{m}-1.3 \mu \text{m} \).
2.2.2 Measurement

To demonstrate the frequency tunability under the DC bias, doubly-clamped beams are fabricated from a highly-conductive boron doped Si device layer of silicon-on-insulator (SOI) wafer with gap-to-thickness ratio of more than four. The fabrication has been done by Dr. S. Kazim at the KAUST cleanroom. The fabrication process is detailed in [141]. Due to residual stresses, the beams are curved as shallow arches. Table 2.2 summarizes the dimensions of fabricated nanobeams.

<table>
<thead>
<tr>
<th></th>
<th>Length (μm)</th>
<th>Depth (nm)</th>
<th>Thickness (nm)</th>
<th>Gap (μm)</th>
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<td>D1</td>
<td>15</td>
<td>850</td>
<td>130</td>
<td>0.85</td>
</tr>
<tr>
<td>D2</td>
<td>15</td>
<td>850</td>
<td>130</td>
<td>1</td>
</tr>
<tr>
<td>D3</td>
<td>15</td>
<td>850</td>
<td>180</td>
<td>0.85</td>
</tr>
<tr>
<td>D4</td>
<td>15</td>
<td>850</td>
<td>180</td>
<td>1</td>
</tr>
</tbody>
</table>

Figure 2.12 shows an SEM image of a curved beam fabricated. Note that the beam is slightly curved due to the residual stress in the device layer.

Figure 2.12: An SEM image of the entire device structure (bottom) along with a magnified view (top) showing the nanobeam and the transduction gap separating it from the driving/sensing electrodes.
The electrical characterization was performed by Dr. S. Kazmi in an ST-500 JANIS probe station. Figure 2.13 shows a schematic of the driving and sensing configuration of the beam in its two-port measurement configuration. The beam is biased by a DC bias voltage and driven at resonance by an ac excitation signal provided at the driving electrode from one of the ports of the network analyzer (Agilent E5071C). The motional current is capacitively sensed through the other electrode, named as sensing electrode, and is fed into a low-noise amplifier (LNA) with its output coupled to the other port of the network analyzer. All the experiments are performed at $5.2 \times 10^{-5}$ mbar pressure and at room temperature.

2.2.3 Results and discussion

Figure 2.14a and Figure 2.14b show the variation of the resonance frequency of the 130 nm and 180 nm thick nanobeams of 15 µm length, 850 nm depth, and with the initially designed transduction gaps of 850 nm and 1 µm, respectively, between the nanobeam and the driving/sensing electrodes for various dc polarization voltages. The figures compare the analytical and experimental data. The analytical results here are obtained by
considering the dimensions of Table 2.3, which indicates thickness values different from
the nominal designed ones due to the fabrication imperfections and tolerances.

Table 2-3: Dimensions used for obtaining the analytical results.

<table>
<thead>
<tr>
<th></th>
<th>Length (μm)</th>
<th>Depth (nm)</th>
<th>Thickness (nm)</th>
<th>Gap (μm)</th>
<th>Initial Rise (nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1</td>
<td>15</td>
<td>850</td>
<td>70</td>
<td>0.85</td>
<td>122</td>
</tr>
<tr>
<td>D2</td>
<td>15</td>
<td>850</td>
<td>120</td>
<td>0.85</td>
<td>116</td>
</tr>
<tr>
<td>D3</td>
<td>15</td>
<td>850</td>
<td>70</td>
<td>01</td>
<td>123</td>
</tr>
<tr>
<td>D4</td>
<td>15</td>
<td>850</td>
<td>120</td>
<td>1</td>
<td>97.3</td>
</tr>
</tbody>
</table>

The results show an increase in the resonance frequency for the beams with the
increased dc bias voltages. The beams of 130 nm thickness with 850 nm and 1 μm
transduction gap between the beam and the driving/sensing electrodes show tunability of
77.12% and 85.51% under DC biased conditions, respectively. Similarly, the resonance
frequencies of arches of 180 nm thickness are also found to increase with increasing the dc
polarization voltages with tunability of 73.05% and 108.14% for 850 nm and 1 μm
transduction gap between the beam and the driving/sensing electrodes, respectively. A
closer look towards the tunability of these beams reveals that those with 1 μm transduction
gaps have higher % tunability, i.e., 66.07% and 45.14% compared to similar beams with
850 nm transduction gaps, i.e., 56.29% and 34.71% for dc bias changes from 40 V to 110
V, respectively. This is clearly attributed to the dominating combined effect of thickness
to gap ratio and the presence of initial curvature of the nanobeam over the softening effect
of the electrostatic force for large gaps. Moreover, thicker arches can achieve higher
tunability compared to the thin arches due to the relatively higher pull-in voltages.
However, the thinner beams show higher tunability per unit increase in the dc polarization
voltage due to the combined effect of increasing arch stiffness and dominating effect of
mid-plane stretching over electrostatic force for large gaps.

Figure 2.14: The resonance frequency versus dc voltage for beams of 15 µm length, 850 nm depth, and (a) 130 nm thickness with a transduction gap of 850 nm and 1 µm, respectively, (b) 180 nm thickness with a transduction gap of 850 nm and 1 µm, respectively. In (a) and (b), simulations results are compared with the experimental data.
Chapter 3  Tunability of electrothermally and electrostatically actuated clamped-clamped microbeams

In this chapter, we investigate the effect of the stiffness tuning via Joule’s heating on the resonance frequency of micro-resonators, straight and initially curved.

3.1  Straight beam

3.1.1  Problem formulation

We consider the same microbeam of Figure 2.1. Here, the in-plane clamped-clamped microbeam is actuated electrothermally by a DC voltage $V_{Th}$ and electrostatically by a DC polarization voltage $V_{DC}$ as presented in Figure 3.1. The electrothermal voltage $V_{Th}$ is applied between the anchors of the microbeam inducing a current $I_{Th}$ passing through the microbeam that heats it up and controls its internally induced axial stress.

Figure 3.1: Schematic of an electrothermally and electrostatically actuated clamped-clamped microbeam.
a Electrothermal problem

Passing an electrical current through a conductor induces the so-called Joule heating, which is mainly caused by the interaction between the moving particles that form the electrical current and the atomic ions that make up the body of the conductor. These particles forming the electrical current give up some of their kinetic energy each time they collide with an ion. This kinetic energy induces the rise of the temperature inside the conductor, which transforms the electrical energy into thermal energy.

In the studied device of this work, applying a potential voltage across the anchors of the microbeam generates a heat flux of density $E = J^2/\sigma_e$ per volume, where $J$ represents the current density defined by $I_{Th}/A$ and $\sigma_e$ represents the electrical conductivity of the microbeam material. The current density is assumed to be uniformly distributed along the microbeam. In our study the convection and the thermal radiation of the microbeam are assumed negligible. Also, the deformation of the microbeam arising from the thermoelastic coupling induced by the electric current is neglected. For simplification, thermal conductivity and electrical conductivity are assumed to be independent of temperature.

Under all the above assumptions and referring to the heat equation; Fourier’s law; the equation governing the average temperature across the section of the microbeam induced by the current $I_{Th}$ is given as below:

$$-k \frac{d^2T}{dx^2} = \frac{J^2}{\sigma_e}$$  \hspace{1cm} (3.1)
where \( k \) is the thermal conductivity of the microbeam material. One should mention here that the above Fourier's law is not applicable for nano structures with dimensions in the same order of magnitude as the phonon mean-free-path.

The current density can be written as a function of the DC electrothermal voltage 
\[ J = \sigma_e \frac{V_{th}}{l}. \]
Therefore, the heat equation can be written as
\[ -k \frac{d^2 T}{d\hat{x}^2} = \frac{\sigma_e V_{th}^2}{l^2} \] (3.2)

Solving Equation (3.2), assuming that the temperature at the ends of the microbeam is equal to the ambient temperature \( T_a \), gives a closed form solution of the distribution of the temperature along the microbeam, which has a parabolic shape and given by the below equation:
\[ T(\hat{x}) = \frac{\sigma_e V_{th}^2}{2k} \left( \frac{\hat{x}}{l} - \frac{\hat{x}^2}{l^2} \right) + T_a \] (3.3)

The variation of the temperature along the microbeam induces the thermal load given by
\[ \hat{S}_{th} = \alpha EA \int_0^l \frac{(T(\hat{x}) - T_a)}{l} d\hat{x} \] (3.4)

where \( \alpha \) is the coefficient of thermal expansion, which is assumed here to be independent of temperature.

b Equation of motion
The governing equation of motion of the microbeam under consideration describing its transverse deflection \( \hat{w}(x,t) \), Figure 3.1, and its subjected boundary condition are given by Equations (2.1) and (2.2), respectively. The term \( \hat{N} = \hat{N}_0 - \hat{S}_{Th} \), in Equation (2.1), represents, in this case, the axial load due to the residual axial load, where \( \hat{N}_0 \) arising from the fabrication process and the compressive axial load and \( \hat{S}_{Th} \) originated from the thermal stress induced by the electrical current \( I_{Th} \) given by Equation (3.4).

The nondimensional equation of motion and boundary condition are described in Section 2.1 of Chapter 2. In this case, the nondimensional stress is defined as below:

\[
\begin{align*}
N & = N_0 + S_{Th} ; N_0 = \frac{l^2}{EI} \hat{N}_0 ; S_{Th} = \frac{l^2}{EI} \hat{S}_{Th} \\
(3.5)
\end{align*}
\]

Since the microbeam is subjected to a compressive load that increases as much as we increase the electrothermal voltage, it is expected that the microbeam encounters a pitchfork bifurcation near a critical load; below which the microbeam remained straight and above which the microbeam buckles. Therefore, we split the problem into two parts: the pre-buckling behavior and the post-buckling behavior of the microbeam.

i. Pre-buckling study

Here, the microbeam is modeled as a straight microbeam under a compressive axial load and electrostatic force. A reduced-order model is derived to compute the static deflection as well as the variation of the fundamental natural frequency while varying the electrothermal voltage \( V_{Th} \) and for fixed values of the electrostatic voltage \( V_{DC} \) [60, 137].
In this part, we adopt the same procedure developed in the previous chapter. For a given $V_{Th}$ and $V_{DC}$, we compute the Jacobin of the system of equations Equation (2.17), Equation (2.18), and Equation (2.19) (Section 2.3 in Chapter 2)) and we find the corresponding eigenvalues. Then, by taking the square root of these eigenvalues, we find the natural frequencies of the resonators under $V_{Th}$ and $V_{DC}$.

ii. Post-buckling study:

The term “Post-buckling” here refers to the fact that the applied axial force exceeds the critical load of the case without electrostatic force. Essentially, the electrostatic force biases the beam, and hence the pitchfork bifurcation of the buckling instability becomes a perturbed pitchfork bifurcation. Here we use the buckled mode shapes and frequencies in the Galerkin discretization as well as the first buckled configuration, as developed by Nayfah et al. [16, 142].

Here, we consider the buckling problem of the beam under the compressive load without electrostatic forces. We follow in the derivation here the work of Nayfah et al. [16, 142]. The equation governing the static configuration in the buckled position $\psi(x)$ can be written as

$$\psi'' + P\psi'' - \alpha_1 \psi \int_0^1 \psi'^2 \, dx = 0$$  \hspace{1cm} (3.6)

where $P = S_{Th} - N_0$ represents the total axial load. Equation (3.6) is subjected to the following boundary conditions
\[ \psi(0) = \psi(1) = 0 \text{ and } \left. \frac{d\psi}{dx} \right|_{x=0} = \left. \frac{d\psi}{dx} \right|_{x=1} = 0 \]  

(3.7)

The analytical solution of the first buckling configuration is given by

\[ \psi(x) = \frac{1}{2} b_1 \left[ 1 - \cos(2\pi x) \right] \]  

(3.8)

where \( b_1 \) is the rise at midpoint of the microbeam of the first buckling mode of the clamped-clamped microbeam and is given by the following expression:

\[ b_1 = \sqrt{\frac{2(p-4\pi^2)}{\alpha^2\pi^2}} \]  

(3.9)

To compute the natural frequencies and the mode shape of the buckled beam, we solve the eigenvalue problem governing the mode shape of the buckled unactuated beam governed by [16, 142]:

\[ \phi'' + 4\pi^2 \phi'' - 4\alpha_1 \pi^2 \cos(2\pi x) \int_0^1 \phi' \sin(2\pi x) dx = \omega^2 \phi'' \]  

(3.10)

where \( \omega \) is the nondimensional frequency. Equation (3.10) is subjected to the below boundary condition:

\[ \phi(0) = \phi(1) = 0 \text{ and } \left. \frac{d\phi}{dx} \right|_{x=0} = \left. \frac{d\phi}{dx} \right|_{x=1} = 0 \]  

(3.11)

To solve Equation (3.10), we set \( \phi \) equal to the superposition of a homogeneous solution \( \phi_h \) and a particular solution \( \phi_p \). Hence we let

\[ \phi(x) = \phi_h(x) + \phi_p(x) \]  

(3.12)
The general solution of Equation (3.10) that represents the mode shapes of the buckled beam is given by:

\[ \varphi(x) = c_1 \cos(s_1x) + c_2 \sin(s_1x) + c_3 \cosh(s_2x) + c_4 \sinh(s_2x) + c_5 \cos(2\pi x) \]  

(3.13)

where \( s_1 \) and \( s_2 \) are defined as below

\[ s_{1,2} = \sqrt{\pm 2\pi^2 + \sqrt{4\pi^2 + \omega^2}} \]  

(3.14)

The \( c_i \) are constants determined by substituting Equation (3.13) into the boundary conditions (Equation (3.11)) and Equation (3.12) into Equation (3.10) using the fact that \( \varphi_p(x) = c_5 \cos(2\pi x) \). This yields five algebraic equations for the coefficients \( c_i \) and the natural frequency \( \omega \). For a fixed level of \( \beta_1 \), the symmetric natural frequencies and their corresponding mode shapes are found by setting the determinant of the coefficient matrix equal to zero.

In this section, we study the transverse vibration induced by the electrothermal and/or the electrostatic force around the static buckled configuration. To do so, we split the static deflection induced by both thermal stress and electrostatic force, \( w_{s,b}(x) \), into two components as follows

\[ w_{s,b}(x) = \psi(x) + \chi(x) \]  

(3.15)

where \( \chi(x) \) is deflection induced by the electrostatic force and \( \psi(x) \) is the buckled configuration given by Equation (3.8). Substituting Equation (3.15) into Equation (2.8) and subtracting the static equation for \( \psi \), Equation (3.6), we obtain the governing equation of \( \chi(x) \).
\[ \chi'' = \left[ N_0 - S_n + \alpha_1 \int_0^1 \chi'^2 \, dx + \alpha_1 \int_0^1 \psi'' \chi' \, dx + 2 \alpha_1 \int_0^1 \psi' \chi' \, dx \right] \chi'' \]

\[ + \left[ \alpha_1 \int_0^1 \chi'^2 \, dx + 2 \alpha_1 \int_0^1 \psi' \chi' \, dx \right] \psi'' + \alpha_2 \frac{V_{DC}^2}{(1 - \psi - \chi)^2} \]

with the associated boundary condition:

\[ \chi(0) = \chi(1) = 0 \quad \text{and} \quad \frac{d\chi}{dx} \bigg|_{x=0} = \frac{d\chi}{dx} \bigg|_{x=1} = 0 \]

To solve Equation (3.16), we use the Galerkin procedure with the undamped linear mode shapes of an unactuated (zero DC voltage) buckled beam, given by Equation (3.10), as basis functions.

Next, we determine the variation of the natural frequency of the buckled microbeam under the DC polarization voltage and as varying the electrothermal voltage. Toward this, we solve the eigenvalue problem obtained by perturbing the deflection around the static configuration. Thus,

\[ w(x, t) = w_{s,b}(x) + y(x, t) \]

Then, the Galerkin discretization is used to represent the dynamic amplitude \( y(x, t) \) as

\[ y(x, t) = \sum_{i=0}^{n} z_i(t) \varphi_i(x) \]

where \( z_i(t)(i=0,1,2..n) \) denotes the nondimensional modal coordinates and \( \varphi_i(x) \) is as defined in (3.10).
Then, we substitute Equations (3.18) and (3.19) into Equation (2.10), multiplying the outcome by the mode shape $\varphi_j$, applying the orthogonality condition of the mode shapes, and integrating over the beam domain (from 0 to 1), which yields the below equation [137]:

$$
\ddot{z}_j + z_j\omega_j^2 = \left[ 2\alpha\int_0^1 \left( \varphi_j \frac{\partial^2 w}{\partial x^2} \right) dx \right] \int_0^1 \left( \sum_{i=0}^n u_i \varphi_i \frac{\partial w}{\partial x} \right) dx + \alpha^2 \int_0^1 \left( \varphi_j \frac{\partial w}{\partial x} \right)^2 dx + \frac{1}{2} \int_0^1 \left( \varphi_j \left( \sum_{i=0}^n u_i \varphi_i \right) \right) dx + \frac{1}{2} \int_0^1 \left[ \varphi_j \frac{2\alpha V_{DC}^2}{(1-w_s^3)} \sum_{i=0}^n u_i \varphi_i \right) dx
$$

(3.20)

Three symmetric mode shapes are used to compute the Jacobin of the system and find the corresponding eigenvalues by taking the square root of these eigenvalues. Then, we find the natural frequencies of the resonators under $V_{Th}$ and $V_{DC}$.

### 3.1.2 Experimental setup

The experimental investigation was conducted on an in-plane 600 μm long clamped-clamped microbeam fabricated from SOI wafers with highly conductive 25 μm Si device layer by MEMSCAP[138]. We used the same techniques in Section 2.4.

The assumed geometrical, mechanical, thermal and electrical parameters of the studied microbeam are shown in Table 3.1.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Quantity</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l$</td>
<td>Length</td>
<td>600 μm</td>
</tr>
<tr>
<td>$h$</td>
<td>Thickness</td>
<td>2 μm</td>
</tr>
<tr>
<td>$b$</td>
<td>Width</td>
<td>25 μm</td>
</tr>
</tbody>
</table>

Table 3-1: Geometrical, mechanical, thermal and electrical properties of the microbeam.
<table>
<thead>
<tr>
<th></th>
<th>Gap</th>
<th>μm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td>Young’s Modulus</td>
<td>7</td>
</tr>
<tr>
<td>$E$</td>
<td>Density</td>
<td>120</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Coefficient of Thermal Expansion</td>
<td>2.6 $10^{-6}$ K$^{-1}$</td>
</tr>
<tr>
<td>$k$</td>
<td>Thermal Conductivity</td>
<td>165 W/(m K)</td>
</tr>
<tr>
<td>$\sigma_e$</td>
<td>Electrical Conductivity</td>
<td>0.78 $10^4$ S/m</td>
</tr>
</tbody>
</table>

At the beginning, the microbeam under consideration is actuated electrothermally by passing a DC current through it. These DC current heats up the beam and controls its internally induced axial stress. A topographical characterization is used to determine the static deflection of the midpoint of the microbeam as varying the electrothermal voltage. The resonance frequencies of the microbeam are measured, as varying the DC electrothermal voltage, using ring down measurement and getting the fast Fourier transform (FFT). The FFT for a zero electrothermal voltage of the microbeam under consideration is depicted in Figure 3.2. The first resonance frequency of the unactuated microbeam is found at 57 kHz.

![Figure 3.2: FFT of the unactuated in-plane clamped-clamped microbeam.](image-url)
Then, the microbeam is actuated, in addition to the electrothermal actuation, electrostatically by adding an external circuit as shown in Figure 3.3. We applied a constant DC bias voltage to actuate the microbeam electrostatically and as varying the DC electrothermal voltage we measured the resonance frequency using the ring down measurement.

![Figure 3.3](image)

Figure 3.3: An electrothermally and electrostatically actuated clamped-clamped microbeam in the buckled configuration. Top view SEM picture with a schematic showing the actuation method.

### 3.1.3 Results and discussion

Next, we investigate theoretically and experimentally the tunability, decreasing and increasing the frequency, of this bridge resonator as varying the electrothermal voltage and for fixed DC polarization voltages.

The microbeam is first actuated electrothermally without including the electrostatic force. The electrothermal voltage $V_{Th}$ is increased from small values, and therefore the
compressive stress inside the microbeam increases. The exact solution of the static deflection due to the electrothermal actuation is given by the first buckling configuration (Equation (3.8)). Figure 3.4 shows the measured and the simulated midpoint deflection of the microbeam due to electrothermal actuation without the electrostatic force demonstrating the critical buckling limit. After this limit, the microbeam is no longer straight and it is buckled. Figure 3.4 demonstrates that the microbeam encounters a pitchfork bifurcation at the critical buckling limit that describes the required voltage to buckle the microbeam. A good agreement is shown among the analytical and experimental results.

![Graph showing static deflection vs. voltage](image)

**Figure 3.4:** The midpoint deflection of the microbeam under electrothermal actuation.

The variation of the first resonance frequency of the studied resonator as varying the electrothermal actuation voltage without including the electrostatic force is shown in Figure 3.5. The pre-buckling eigenvalue is determining by solving the eigenvalue problem of the straight microbeam under axial load presented by the system of Equations (2.17), (2.18) and (2.19). For the buckling problem, the eigenvalue problem of buckled beam is solved. As shown in Figure 3.5, the frequency decreases from the initial resonance...
frequency to almost zero before buckling. After buckling, the fundamental frequency increases from zero to higher values, which can be as high as double the original resonance frequency. In that case, the behavior of the beam is transformed from straight to buckle beam. A good agreement is shown among the analytical and experimental results.

Figure 3.5: The first resonance frequency of the microbeam under electrothermal actuation with no DC electrostatic bias voltage.

Next, we combined the electrothermal actuation with the electrostatic force. We applied a constant DC polarization voltage, far from the pull-in voltage, between the microbeam and the stationary electrode, and then we varied the electrothermal voltage. The analytical results of the static deflection of the midpoint of the microbeam as varying the electrothermal voltage for a fixed DC polarization voltage is shown in Figure 3.6. The static deflection is obtained by solving Equation (2.8), before the critical load, and Equation (3.14), after exceeding the critical load. Figure 3.5 shows that the microbeam encounters a perturbed pitchfork bifurcation, where the discontinuity in the deflection curve observed in Figure 3.4 disappears.
Figure 3.6: The static deflection versus electrothermal actuation voltage for a constant DC electrostatic bias voltage.

Then, solving the system of algebraic Equations, (2.17), (2.18) and (2.19), for the pre-buckling behavior and the system of algebraic Equation (3.20) for the post-buckling behavior, we compute the first natural frequency of the microbeam under constant $V_{DC}$ and as varying $V_{Th}$. Figures 3.7a, 3.7b and 3.7c display the first resonance frequency of the microbeam versus the electrothermal actuation for a DC electrostatic bias voltage equal to 45 V, 55 V, and 65 V, respectively. A good agreement is shown among the analytical and experimental results.
Figure 3.7: The first resonance frequency of the microbeam versus electrothermal actuation voltage for a constant DC electrostatic bias voltage. (a) $V_{DC} = 45$ V. (b) $V_{DC} = 55$ V. (c) $V_{DC} = 65$ V.

One can note that adding a DC electrostatic bias changes the qualitative nature of the tunability both before and after buckling, which adds another independent way of tuning. Figure 3.8 further clarifies this aspect. Figure 3.8 shows the resonance frequency, computed analytically while varying the electrothermal voltage for different values of DC polarization voltages. Adding a DC electrostatic bias reduces the dip in the resonance frequency before buckling, and can eliminate it if desired with increasing the DC polarization voltage, and further enhances the increase in the resonance frequency after buckling. Additionally, Figure 3.8 displays that the initial natural frequency, at zero electrothermal voltage, increases as the DC voltage exceeds 65 V. This fact is due to the dominating effect of mid-plane stretching over electrostatic force for large gaps, as shown in Chapter 2.
Figure 3.8: The first resonance frequency of the microbeam versus electrothermal actuation voltage for different constant DC electrostatic bias voltages.

3.2 Arch beam

Next, we explore experimentally the tunability of initially curved clamped-clamped microbeam actuated electrothermally. First, we study the static behavior under electrothermal load. Then, we demonstrate the tunability of the resonator under consideration as varying the electrothermal voltage. We show the theoretical model that will be done as future work.

3.2.1 Problem formulation

The device under consideration consists of a silicon in-plane clamped-clamped arch beam, Figure 3.9, of an initial shape \( \hat{w}_0(\hat{x}) \) governed by:

\[
\hat{w}_0(\hat{x}) = \frac{1}{2} \hat{b}_0 \left( 1 - \cos \left( \frac{2\pi \hat{x}}{l} \right) \right)
\]  

(3.21)

where \( \hat{b}_0 \) and \( l \) represent the rise at the midpoint and the length of the arch beam, respectively. The studied arch is considered shallow since the ratio \( \partial \hat{w}_0 / \partial \hat{x} \ll 1 \) [36]. The arch is actuated electrothermally by a DC voltage \( V_{Th} \). It has a Young’s modulus \( E \), material
density $\rho$, width $b$, and thickness $h$. It is assumed to have a rectangular cross-section area $A = bh$ and a moment of inertia $I = bh^3/12$. The arch is separated from a stationary electrode with a gap width $d$. The electrothermal voltage $V_{Th}$ is applied between the anchors of the arch inducing a current $I_{Th}$ that heats up it and controls its internally induced axial stress.

Figure 3.9: Schematic of an electrothermally actuated clamped-clamped shallow arch.

\section*{a Electrothermal problem}

Applying a potential voltage across the anchors of the arch beam generates a heat flux of density $E = J^2/\rho_e(T)$ per volume, where $J$ represents the current density defined by $I_{Th}/A$ and $\rho_e(T)$ represents the electrical resistivity of the microbeam material. The current density is assumed to be uniformly distributed along the arch beam. In this study, the convection and the thermal radiation of the microbeam are assumed negligible as was found in [14]. Under these assumptions and referring to the heat equation; Fourier’s law; the equation governing the average temperature across the cross-section of the microbeam induced by the current $I_{Th}$ is given by

$$-\frac{d}{d\tilde{x}} \left( k(T) \frac{dT}{d\tilde{x}} \right) = J^2 \rho_e(T)$$

(3.22)
where \( k(T) \) is the thermal conductivity of the microbeam material. The variation of the electrical resistivity of doped silicon is assumed to be linear with temperature [38] since the doping of the silicon is assumed to be uniform. The variation of the thermal conductivity and the electrical resistivity with the temperature are governed by the below functions [143]:

\[
k(T) = \frac{2.7}{\left[ (-2.2 \times 10^{-11} T^3) + (9 \times 10^{-8} T^2) - (10^{-5} T) + 0.014 \right]}
\]

\[
\rho_e(T) = \rho_0 \left[ 1 - \eta (T - T_a) \right]
\]

where \( \rho_0 \) is the electrical resistivity at room temperature \( T_a \) and \( \eta \) is the resistivity temperature coefficient. The parameters \( \rho_0 \) and \( \eta \) are determined experimentally. For this, the beam is placed on a hot plate with controllable temperatures.

The temperature at the ends of the microbeam is assumed equal to the ambient temperature \( T_a \). The current density can be written as a function of the DC electrothermal voltage \( J = \frac{V_{th}}{\rho_e(T) l} \).

Consequently, Fourier’s law equation can be written as

\[
-\frac{d}{dx} \left( k(T) \frac{dT}{dx} \right) = \frac{V_{th}^2}{\rho_e(T) l^2}
\]

Equation (3.25) is solved analytically, by using a boundary-value algorithm in the software Mathematica [144], to obtain an expression of the variation of the temperature along the arch beam. The variation of the temperature along the microbeam induces a compressive load given by
\[
\hat{S}_{Th} = EA \int_0^l \frac{\alpha(T)(T(\hat{x})-T_0)}{l} \, d\hat{x}
\]  
(3.26)

where \( \alpha(T) \) is the coefficient of thermal expansion, expressed as below [145]

\[
\alpha(T) = \left(3.75 \times (1-e^{-5.88 \times 10^{-4}(T-125)}) + 5.548 \times 10^{-4} T\right) \times 10^{-6}
\]  
(3.27)

\section*{b Equation of motion}

The governing equation of motion of the microbeam under consideration, Figure 3.9, describing its transverse deflection \( \hat{w}(\hat{x},\hat{t}) \) assuming no damping is written as [36, 137]:

\[
\rho bh \frac{\partial^2 \hat{w}}{\partial \hat{t}^2} + E I \frac{\partial^4 \hat{w}}{\partial \hat{x}^4} + \left[ \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \frac{d^2 \hat{w}_0}{d\hat{x}^2} \right] \left[ \hat{N} - \frac{EA}{2l} \int_0^l \left( \frac{\partial \hat{w}}{\partial \hat{x}} \right)^2 + 2 \frac{\partial \hat{w}}{\partial \hat{x}} \frac{d \hat{w}_0}{d\hat{x}} \right] \, d\hat{x} = 0
\]  
(3.28)

The microbeam is subjected to the following boundary conditions:

\[
\hat{w}(0,\hat{t}) = \hat{w}(l,\hat{t}) = 0 \text{ and } \left. \frac{\partial \hat{w}}{\partial \hat{x}} \right|_{(0,\hat{t})} = \left. \frac{\partial \hat{w}}{\partial \hat{x}} \right|_{(l,\hat{t})} = 0
\]  
(3.29)

where \( \hat{x} \) is the position along the microbeam and \( \hat{t} \) is time. The term \( \hat{N} = \hat{S}_{Th} - \hat{N}_0 \) represents the axial load due to the residual axial load, where \( \hat{N}_0 \) arising from the fabrication process and the compressive axial load and \( \hat{S}_{Th} \) is originated from the thermal stress as given by Equation (3.26).

One should note that Equation (3.28) accounts for both the transversal and the longitudinal deformation effect in the arch. The longitudinal deformation is already accounted for through the integral term (which comes from dropping the longitudinal inertia), since it is much higher than the transversal one, integrating the resulting
longitudinal equation (averaging the axial stress), and substituting the outcome into the transversal bending equation [146].

For convenience, we introduce the nondimensional variables as below:

\[
\begin{align*}
\hat{w} &= \frac{w}{r} ; \hat{x} = \frac{x}{l} ; T_s = \frac{\hat{t}}{b_0} ; \hat{w}_0(\hat{x}) = \frac{\hat{w}_0(\hat{x})}{r} = \frac{1}{2} b_0 \left(1 - \cos(2\pi \hat{x})\right) \quad \text{and} \quad b_0 = \frac{\hat{b}_0}{r} \quad (3.30)
\end{align*}
\]

where \( r = \sqrt{I/A} \) denotes the radius of gyration of the cross-section and \( T_s = \sqrt{\rho bh^4 / EI} \) is a time scale. Substituting Equation (3.30) into Equations (3.28) and (3.29), we obtain the nondimensional equation of motion

\[
\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + \left[ \frac{\partial^2 w}{\partial x^2} + \frac{d^2 w_0}{dx^2} \right] \left[ N - \frac{1}{2} \int_0^1 \left( \frac{\partial w}{\partial x} \right)^2 + 2 \frac{\partial w}{\partial x} \frac{d w_0}{dx} \right] \, dx \right] = 0 \quad (3.31)
\]

subjected to the nondimensional boundary conditions

\[
w(0, t) = w(1, t) = 0 \quad \text{and} \quad \left. \frac{\partial w}{\partial x} \right|_{0,t} = \left. \frac{\partial w}{\partial x} \right|_{1,t} = 0 \quad (3.32)
\]

The nondimensional parameters appearing in Equation (3.31) are defined as

\[
N = S_{th} - N_0 ; N_0 = \frac{l^2}{EI} \hat{N}_0 ; S_{th} = \frac{l^2}{EI} \hat{S}_{th} \quad (3.33)
\]

The deflection of the arch under the electrothermal force is split into a static deflection \( w_s(x) \) due to the static electrothermal voltage \( V_{th} \) and a small dynamic deflection \( w_d(x,t) \), that is

\[
w(x,t) = w_s(x) + w_d(x,t) \quad (3.34)
\]

To determine the static deflection \( w_s(x) \), the time derivatives in Equation (3.31) are set equal to zero and \( w \) is replaced with \( w_s(x) \), which yields
\[
\frac{d^4w}{dx^4} + \left( \frac{d^2w}{dx^2} \right)^2 \left[ N = \frac{1}{2} \int_0^1 \left( \frac{dw}{dx} \right)^2 + 2 \frac{dw}{dx} \frac{dw_0}{dx} \right] \right] dx = 0
\] (3.35)

with the associated boundary conditions

\[
w_s(0) = w_s(1) = 0 \text{ and } \frac{dw_s}{dx} \bigg|_{x=0} = \frac{dw_s}{dx} \bigg|_{x=1} = 0
\] (3.36)

The linearized equation of motion describing the small dynamic behavior of the shallow arch around the deflected shape is derived by substituting Equation (3.34) into Equation (3.31) and dropping the terms representing the equilibrium position, Equation (3.35), and the nonlinear terms. This yields

\[
\frac{\partial^2 w_d}{\partial t^2} + \frac{\partial^4 w_d}{\partial x^4} + \frac{\partial^2 w_d}{\partial x^2} \left[ N - \frac{1}{2} \int_0^1 \left( \frac{dw}{dx} \right)^2 + 2 \frac{dw}{dx} \frac{dw_0}{dx} \right] dx = 0
\] (3.37)

with the associated boundary conditions

\[
w_d(0,t) = w_d(1,t) = 0 \text{ and } \frac{\partial w_d}{\partial x} \bigg|_{x=0,t} = \frac{\partial w_d}{\partial x} \bigg|_{x=1,t} = 0
\] (3.38)

We follow Nayfeh et al. [16] and derive analytical solutions for the static problem and the eigenvalue problem of an arch under an axial load. Toward this, we let

\[
\psi(x) = w_d(x) + w_0(x)
\] (3.39)

Taking into account Equation (3.39), Equations (3.35) and (3.36) become
\[ \psi'' + \psi'''' + \left[ N + \frac{1}{2} \int_{0}^{1} \left( (w_0')^2 - (\psi')^2 \right) dx \right] = w_0'''' \]  \hspace{1cm} (3.40)

\[ \psi(0) = \psi(1) = 0 \hspace{0.5cm} \text{and} \hspace{0.5cm} \left. \frac{d\psi}{dx} \right|_{x=0} = \left. \frac{d\psi}{dx} \right|_{x=1} = 0 \]  \hspace{1cm} (3.41)

The integral term appearing in Equation (3.40) can be treated as a constant term [16]. Hence, we let

\[ \lambda^2 = N + \frac{1}{2} \int_{0}^{1} \left( (w_0')^2 - (\psi')^2 \right) dx \]  \hspace{1cm} (3.42)

and we write Equation (3.40) as

\[ \psi'''' + \lambda^2 \psi'' = w_0'''' \]  \hspace{1cm} (3.43)

To solve Equation (3.42), we set \( \psi \) equal to the superposition of a homogeneous solution \( \psi_h \) and a particular solution \( \psi_p \), hence

\[ \psi(x) = \psi_h(x) + \psi_p(x) \]  \hspace{1cm} (3.44)

Taking into consideration the initial curvature of Equation (3.21), the general solution of Equation (3.43) that represents the new static configuration is written as:

\[ \psi(x) = c_1 + c_2 x + c_3 \cos(\lambda x) + c_4 \sin(\lambda x) + c_5 \cos(2\pi x) \]  \hspace{1cm} (3.45)

Substituting Equation (3.45) into Equation (3.41) and solving the resulting equation, we end up with the exact solution of the static configuration given as

\[ \psi(x) = \frac{1}{2} q(1 - \cos(2\pi x)) \]  \hspace{1cm} (3.46)
where \( q \) is given by:

\[
q = \frac{4\pi^2b_0}{4\pi^2 - \lambda^2}
\]  

(3.47)

Substituting Equations (3.45), (3.46), and (3.47) into Equation (3.42), we obtain the characteristic equation for \( q \):

\[
q^3 - \left( \frac{4}{\pi} N + b_0 \right) q - 16b_0 = 0
\]  

(3.48)

To solve the eigenvalue problem of the arch around the obtained static position, we substitute Equation (3.39) into Equation (3.37). Then, the equation governing the linearized dynamic equation becomes

\[
\frac{\partial^2 w_d}{\partial t^2} + \frac{\partial^4 w_d}{\partial x^4} + \frac{\partial^2 w_d}{\partial x^2} \left[ N + \frac{1}{2} \left( (w'_0)^2 - (\psi')^2 \right) \right] - \psi'' \int_0^1 \psi' \frac{\partial w_d}{\partial x} \ dx = 0
\]  

(3.49)

To compute the eigenvalue problem as varying the axial load, we let

\[
w_d(x,t) = e^{i\omega t} \phi(x)
\]  

(3.50)

Substituting Equation (3.50) into Equation (3.49) we obtain

\[
\phi'' + \zeta \phi''' - 2q^2 \pi^3 \cos(2\pi x) \left[ \int_0^1 \phi' \sin(2\pi x) \ dx - \omega^2 \phi = 0
\]  

(3.51)

where \( \omega \) is the natural frequency and \( \zeta = 4\pi^2 \left( 1 - \frac{b_0}{q} \right) \).
In order to solve Equation (3.51), \( \varphi \) is assumed to be composed of a homogeneous solution \( \varphi_h \) and a particular solution \( \varphi_p \), hence

\[
\varphi(x) = \varphi_h(x) + \varphi_p(x)
\]  

(3.52)

The general solution of Equation (3.51) is given by

\[
\varphi(x) = c_1 \cos(s_1 x) + c_2 \sin(s_1 x) + c_3 \cosh(s_2 x) + c_4 \sinh(s_2 x) + c_5 \cos(2\pi x)
\]  

(3.53)

with the associated boundary condition:

\[
\varphi(0) = \varphi(1) = 0 \quad \text{and} \quad \left. \frac{d\varphi}{dx} \right|_{x=0} = \left. \frac{d\varphi}{dx} \right|_{x=1} = 0
\]  

(3.54)

where

\[
s_{1,2} = \sqrt{\frac{1}{2} \zeta + \frac{1}{4} \omega^2 + \omega_s^2}. 
\]

The \( c_i \) are constants determined by substituting Equation (3.53) into the boundary conditions Equation (3.54), and Equation (3.52) into Equation (3.51) using the fact that \( \varphi_p(x) = c_5 \cos(2\pi x) \). This yields five algebraic equations for the coefficients \( c_i \) and the natural frequency \( \omega \), which are solved for a fixed electrothermal voltage.

### 3.2.2 Experimental setup

The experimental investigation was conducted on in-plane MEMS shallow arches. These are designed with initial curvature and printed on a mask, and then the pattern is transferred to an SOI wafer with highly conductive 25 \( \mu \)m Si device layer using the photolithography process developed by MEMSCAP [138]. A stroboscopic microscope from Polytec [139] is used to determine the in-plane deflection as well as the resonance frequencies of the arches.
Here, we consider two case studies of initially curved beams of length 600 μm (Arch 1) and 500 μm (Arch 2), 25 μm width, and nominal thickness of 2 μm and 3 μm, and nominal initial rise of 2 μm and 3 μm, respectively. Note here that the final initial rise and thickness of the arches can vary due to the fabrication imperfections. The initial profile of the arches is described according to Equation (3.21). The arches are actuated electrothermally by passing a DC current through them, Figure 3.10. Also, they can be actuated electrostatically to excite the structure into vibration as resonators.

Figure 3.10: An SEM image of the arch beam with a schematic drawing clarifying the electrothermal and electrostatic actuation.

The DC thermal current heats up the arch and then controls its internally induced axial stress. The topographical characterization, through optical interferometry, is used to determine the static deflection of the midpoint of the arch as varying the electrothermal voltage. The measured static deflection of both arches as tuning the electrothermal voltage is depicted in Figures 3.11a and 3.11b.
Figure 3.11: The measured midpoint deflection of the shallow arches under electrothermal actuation. (a) Arch 1, (b) Arch 2.

The resonance frequencies of the arches are measured, as varying the DC electrothermal voltage, using ring down measurements with a DC electrostatic voltage load; after which a fast Fourier transform (FFT) is conducted. The FFT for a zero electrothermal voltage of both arch beams under consideration is depicted in Figures 3.12a and 3.12b.

Figure 3.12: FFT of the in-plane clamped-clamped arches at zero electrothermal voltage. (a) Arch 1, (b) Arch 2.

For measuring the third resonance frequency (second symmetric) of both arch beams as varying the electrothermal voltage, we conducted frequency sweeps around the third resonance frequency while maintaining a small electrostatic load. This is because the ring down measurements yielded small peaks at near the higher order modes resonances that sometimes are buried in noise, as clear in Figure 3.12. The measured resonance frequencies of both arches while changing $V_{Th}$ are shown in Figures 3.13a and 3.13b. The
figures demonstrate experimentally that the first resonance frequency of both arches is tuned for high values that reach as high as twice the initial value (at zero thermal voltage). The third resonance frequency decreases, for both arches, as increasing the electrothermal voltage. At a critical electrothermal voltage, the third natural frequency starts to increase and the first resonance frequency starts to saturate and be less sensitive to the thermal stress. Such behavior is clearer for Arch 1. This phenomenon can be explained through the veering (avoided-crossing) phenomenon, which occurs when the natural frequencies of two modes get close to each other [80].

Figure 3.13: First and third resonance frequencies of the shallow arch under electrothermal actuation. (a) Arch 1, (b) Arch 2.

One can note that a high tunability is shown in both case studies in Figure 3.13. At $V_{Th}=6.5V$, we show 168% and 101% increase of the first resonance frequency with respect to the fundamental resonance frequency at zero electrothermal voltage for Arch beam 1 and 2, respectively. Such tunability is higher compared to the one achieved in Section 3.1 based on an electrothermally buckled straight beam (66% increase). Commonly in the
literature, the resonance frequencies of electrothermally bridge resonators are shown to decrease [63, 70], for instance by 42% [70].

### 3.2.3 Finite element model

A 3D multi-physics finite-element simulation is conducted for the arch under consideration, using the commercial finite element software COMSOL [147], in order to verify that the various assumptions made in the analytical model are valid. The exact shape of the fabricated arch beam, which slightly differs from Equation (3.21), part of a circle curve, is implemented in the model. The Solid Mechanics, Electric Currents, and Heat Transfer interfaces are considered to account for the various physical domains of the problem. The anchors of the arch beam are assigned a fixed constraint boundary condition with ambient temperature at their bottom. The rest of the faces of the structure are set to a convective heat boundary condition, where the heat flux option is used for an external natural convection with air as an external fluid. We neglect the effect of radiation and the heat conduction to the substrate through the air gap conduction. For the Electric Currents module, an electrical potential and a ground were defined on the top of the anchors to allow passing an electrical current through a conductor and simulate the Joule heating. Figure 3.14 shows the temperature variation with the electrothermal voltage actuation between the two anchors. The analytical results based on Equation (3.25) are also compared, which shows a good agreement among the results.
3.2.4 Results and discussion

In order to extract the unknown parameters (residual stress and initial rise), we follow the parameters extraction procedure in [148], which relies on matching the measured linear natural frequencies of the arch to the theory. Also, we measure the effective thickness, initial rise, and length that generally different from the designed values (presented in Section 3.2.2) due to the fabrication process (mainly etching process). Then all the extracted and measured parameters of the arch beam under consideration are given by Table 3.2.

Table 3-2: Measured and extracted thermal and mechanical parameters of Arch 2.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured rise due to the residual stress</td>
<td>2.2 µm</td>
</tr>
<tr>
<td>Length ( l )</td>
<td>500 µm</td>
</tr>
<tr>
<td>Property</td>
<td>Value</td>
</tr>
<tr>
<td>----------------------------------</td>
<td>-------------</td>
</tr>
<tr>
<td>Thickness ( h )</td>
<td>2.8 µm</td>
</tr>
<tr>
<td>Width ( b )</td>
<td>25 µm</td>
</tr>
<tr>
<td>First natural frequency</td>
<td>139.576 kHz</td>
</tr>
<tr>
<td>Third natural frequency</td>
<td>546 kHz</td>
</tr>
<tr>
<td>Dimensional residual stress ( N_0 )</td>
<td>9.5 MPa</td>
</tr>
<tr>
<td>Initial rise ( \tilde{b}_0 )</td>
<td>2.8 µm</td>
</tr>
<tr>
<td>Young’s modulus ( E )</td>
<td>160 GPa</td>
</tr>
<tr>
<td>Density ( \rho )</td>
<td>2332 kg/m³</td>
</tr>
<tr>
<td>Resistivity temperature coefficient ( \eta )</td>
<td>2.13 10⁻³ 1/K</td>
</tr>
<tr>
<td>Electrical resistivity ( \rho_0 )</td>
<td>1.19 10⁻⁴ Ωm</td>
</tr>
</tbody>
</table>

Solving the eigenvalue problem analytically for different initial rises and using the same parameters in Table 3.2, we show in Figure 3.15 that for low curvatures the resonance frequency decreases before reaching a critical compressive load after which it starts to increase. To completely eliminate the resonance frequency dip, we need to consider relatively high curvature as shown in Figure 3.15.

Figure 3.15: The variation of the first resonance frequency as varying the electrothermal voltage for various initial rises.
Figure 3.16 shows experimentally, analytically, and numerically (FEM) the midpoint deflection of the microbeam due to the electrothermal actuation. As seen from the figure, the arch starts to deflect continuously in the same direction of its initial rise with the increase of the electrothermal load. The advantage of the thermal actuation is that it leads to large deflection for low applied voltages. A good agreement is shown among the results.

![Graph showing deflection versus electrothermal voltage](image)

Figure 3.16: The midpoint deflection of the shallow arch under electrothermal actuation.

The variation of the first and third resonance frequencies of the arch as varying $V_{Th}$ is shown in Figure 3.17. For the analytical results, the eigenvalue problem is solved. As shown in Figure 3.17, the first natural frequency increases as we increase the compressive load induced by the electrothermal voltage. The third natural frequency decreases as increasing the electrothermal load and then starts to increase when the first natural frequency seems to start to flatten. Hence, a significant tunability for the resonator under consideration was shown for the first resonance frequency that can reach more than 100% for low values of electrothermal voltages. A good agreement is shown among the
analytical, finite element, and experimental results. The deviation between the analytical and experimental and finite element results can be attributed due to approximating the actual shape of the arch as in Equation (3.21). As was reported in [149], the shape has significant effect on the dynamics.

Figure 3.17: The variation of the first and third resonance frequencies, respectively, as varying the electrothermal voltage.

To further investigate the saturation of the first resonance frequency and the increase of the third resonance frequency after a critical value, analytical results showing the variation of these resonance frequencies as varying the electrothermal load are shown in Figure 3.18a. This saturation of the first resonance frequency and the increase of the third could be explained by the veering phenomenon [80]. A wide tunability is shown for the first and the third resonance frequencies as varying the electrothermal voltage. Figure 3.18b shows the variation of the ratio between the third and the first resonance frequencies as varying the electrothermal voltage. It shows that the resonator can be operated for a wide range of ratios between the two first symmetric resonances. For some of these ratios, internal resonance among the vibration modes can be triggered.
Figure 3.18: (a) The simulated first and third resonance frequencies under electrothermal voltage. (b) The ratio between the third and the first resonance frequencies as varying the electrothermal voltage.
Chapter 4  Linear mode coupling: Veering

In this chapter, we study the linear coupling between the first two symmetric modes of an arch beam electrothermally tuned and electrostatically actuated as coming close to each other. Such phenomenon is known as the veering (avoid-crossing) phenomenon.

4.1 Problem formulation

The device under consideration is made of silicon and consists of an initially a curved clamped-clamped beam, Figure 4.1, of an initial shape \( \dot{w}_0(\hat{x}) \) governed by Equation (3.21) for a cosine wave (buckled) shape and Equation (4.1) for an arc shape:

\[
\dot{w}_0(\hat{x}) = \left( \hat{b}_0 - R + \sqrt{R^2 - \left( \hat{x} - \frac{l}{2} \right)^2} \right)
\]  

(4.1)

where \( \hat{x} \) is the position along the microbeam and \( \hat{b}_0 \) represents the rise at the midpoint of the arch. \( R \) represents the radius of the arc.

The curved beam has Young’s modulus \( E \), material density \( \rho \), length \( l \), width \( b \), and thickness \( h \). The cross-section area of both configurations is assumed to be rectangular \( A = bh \) with a moment of inertia given by \( I = bh^3/12 \). The curved beam is separated from a stationary electrode with a gap width \( d \). The curved beam is actuated electrostatically by a DC polarization voltage \( V_{DC} \) and an AC harmonic voltage of amplitude \( V_{AC} \) and frequency \( \hat{\Omega} \) and is subjected to viscous damping of coefficient \( \dot{c} \). The electrothermal voltage \( V_{Th} \) is
applied between the anchors of the curved beam inducing a current $I_{th}$ passing through the beam that heats up it and controls its internally induced axial stress.

![Diagram](image)

Figure 4.1: Schematic of an electrothermally actuated clamped-clamped shallow arch (a) cosine wave (b) arc shape. (c) The profile of both cosine wave and arc configurations highlighting their difference.

The governing equation of motion of the curved beam under consideration, Figure 4.1, describing its transverse deflection $\dot{w}(\xi, \hat{t})$ in space and time $\hat{t}$ is written as [36, 137]

$$
\rho bh \frac{\partial^2 \dot{w}}{\partial \hat{t}^2} + E I \frac{\partial^4 \dot{w}}{\partial \xi^4} + c \frac{\partial \dot{w}}{\partial \hat{t}} = \left( \frac{\partial^2 \dot{w}}{\partial \xi^2} + \frac{d^2 \dot{w}_0}{d\hat{x}^2} \right) \left[ \frac{\hat{N}}{2l} \left( \frac{\partial \dot{w}}{\partial \hat{x}} \right)^2 + \frac{\partial \dot{w}}{\partial \hat{x}} \frac{d\dot{w}_0}{d\hat{x}} \right] d\hat{x} \\
+ \frac{1}{2} \varepsilon b \frac{(V_{DC} + V_{AC} \cos(\hat{\Omega} t))^2}{(d - \dot{w} - \dot{w}_0)^2}
$$

(4.2)

The microbeam is subjected to the following fixed-fixed boundary conditions:
\[ \hat{w}(0, \hat{t}) = \hat{w}(1, \hat{t}) = 0 \text{ and } \frac{\partial \hat{w}}{\partial \hat{x}} \bigg|_{(0, \hat{t})} = \frac{\partial \hat{w}}{\partial \hat{x}} \bigg|_{(1, \hat{t})} = 0 \]  

(4.3)

The term \( \hat{N} = \hat{N}_0 - \hat{S}_{th} \) represents the tensile axial load, where \( \hat{N}_0 \) is arising from the fabrication process and \( \hat{S}_{th} \) denotes the thermal compressive stress given by Equation (3.24). \( \varepsilon \) represents the dielectric constant of the medium.

For convenience, we introduce the nondimensional variables

\[
w = \frac{\hat{w}}{d}; x = \frac{\hat{x}}{l}; t = \frac{\hat{t}}{T_s}; w_0 = \frac{\hat{w}_0}{d} \text{ and } b_0 = \frac{\hat{b}_0}{d} \]

(4.4)

where \( T_s = \sqrt{\rho \beta h l^4 / EI} \) is a time scale. Substituting Equation (4.4) into Equation (4.2) and Equation (4.3), we obtain the nondimensional equation of motion of the beam

\[
\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + c \frac{\partial w}{\partial t} = \left( \frac{\partial^2 w}{\partial x^2} + \frac{d^2 w_0}{dx^2} \right) \left[ N + \alpha_1 \left( \int_0^1 \left( \frac{\partial w}{\partial x} \right)^2 + 2 \frac{\partial w}{\partial x} \frac{d w_0}{dx} \right) dx \right] \]

\[
+ \alpha_2 \frac{(V_{DC} + V_{AC} \cos(\Omega t))^2}{(1 - w - w_0)^2}
\]

subjected to the nondimensional boundary conditions

\[
w(0, t) = w(1, t) = 0 \text{ and } \left. \frac{\partial w}{\partial x} \right|_{(0, \hat{t})} = \left. \frac{\partial w}{\partial x} \right|_{(1, \hat{t})} = 0 \]

(4.6)

The parameters appearing in Equation (4.5) are defined as

\[
\alpha_1 = 6 \left( \frac{d}{h} \right)^2; N = N_0 + S_{th}; N_0 = \frac{l^2}{EI} \hat{N}_0; S_{th} = \frac{l^2}{EI} \hat{S}_{th};
\]

\[
\alpha_2 = \frac{6 \varepsilon l^4}{E h^3 d^4}; c = \frac{l^4}{EI} \hat{c} \text{ and } \Omega = T_s \hat{\Omega}
\]

(4.7)
4.2 Experimental setup

The experimental validation was conducted on intentionally fabricated arc beams with specific initial shapes. The arc beams were fabricated by MEMCAP [138], from SOI wafers with highly conductive Si device layer. To determine the resonance frequencies as well as the frequency response of the structures, we use stroboscopic video microscopy from Polytec [139].

The arc beams are actuated electrothermally by passing a DC current, \( I_{Th} \), through them, Figure 4.2. Also, they are actuated electrostatically to excite the structure into vibration. Three case studies presented in Table 4.1 will be investigated.

![Figure 4.2: An SEM image of the arch beam showing in schematic the electrothermal and electrostatic actuation circuits.](image)

Table 4-1: Geometrical properties of the silicon microbeam.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Arc Beam 1</th>
<th>Arc Beam 2</th>
<th>Arc Beam 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (( \mu \m )</td>
<td>600</td>
<td>600</td>
<td>800</td>
</tr>
<tr>
<td>Thickness (( \mu \m )</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Width (( \mu \m )</td>
<td>25</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>Initial rise at the midpoint (( \mu \m )</td>
<td>2</td>
<td>3</td>
<td>2.6</td>
</tr>
<tr>
<td>Gap (( \mu \m )</td>
<td>7</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>
4.3 Eigenvalue problem

In this section, we examine the variation of the resonance frequencies of both shape configurations, under consideration, as a function of the compressive load induced by the thermal voltage. Hence, the electrostatic force term is dropped from the equations. The deflection of the arch under the electrothermal force is split into a static deflection $w_s(x)$ due to $V_{Th}$ and a small dynamic deflection $w_d(x,t)$ [36, 137].

The static equation is governed by

$$\frac{d^4w_s}{dx^4} = \left(\frac{d^2w_s}{dx^2} + \frac{d^2w_0}{dx^2}\right) \left[N + \alpha I_\Omega \left(\frac{dw_s}{dx}\right)^2 + 2 \frac{dw_s}{dx} \frac{dw_0}{dx} \right] \right)$$ (4.8)

with the associated boundary conditions

$$w_s(0) = w_s(1) = 0 \quad \text{and} \quad \frac{dw_s}{dx} \bigg|_{x=0} = \frac{dw_s}{dx} \bigg|_{x=1} = 0$$ (4.9)

The linearized equation of motion describing the small dynamic behavior of the curved beam around the new static configuration induced by $V_{Th}$ and governed by Equation (4.10) is derived by substituting $w(x,t) = w_s(x) + w_d(x,t)$ into Equation (4.2) and dropping the terms representing the equilibrium position, the electrostatic force, and the nonlinear terms. The outcome equation becomes

$$\frac{\partial^2 w_d}{\partial t^2} + \frac{\partial^4 w_d}{\partial x^4} = \left[N + \alpha I_\Omega \left(\frac{dw_s}{dx}\right)^2 + 2 \frac{dw_s}{dx} \frac{dw_0}{dx} \right] \frac{\partial^2 w_d}{\partial x^2} + 2\alpha I_\Omega \left(\frac{dw_s}{dx} \frac{\partial w_d}{\partial x} + \frac{dw_0}{dx} \frac{\partial^2 w_d}{\partial x^2}\right) dx \left(\frac{d^2 w_s}{dx^2} + \frac{d^2 w_0}{dx^2}\right)$$ (4.10)
with the associated boundary conditions

\[ w_d(0, t) = w_d(1, t) = 0 \text{ and } \frac{\partial w_d}{\partial x} \bigg|_{x=0, t} = \frac{\partial w_d}{\partial x} \bigg|_{x=1, t} = 0 \] (4.11)

We resort to the Galerkin discretization to represent the dynamic deflection \( w_d(x, t) \) and to solve the eigenvalue problem of the curved beam under the DC thermal voltage [137]. Toward this, we let

\[ w_d(x, t) = \sum_{i=0}^{n} u_i(t) \phi_i(x) \] (4.12)

where \( u_i(t) (i=0,1,2..n) \) denotes the nondimensional modal coordinates and \( \phi_i(x) (i=0,1,2..n) \) denotes the mode shape of the unactuated straight clamped-clamped beam [137].

Then, we substitute Equation (4.12) into Equation (4.10), multiplying the outcome by the mode shape \( \phi_j \) and integrating over the beam domain (from 0 to 1), which yields the below equation [137]

\[
\ddot{u}_j = -\int_0^1 \phi_j \left( \sum_{i=0}^{n} u_i \phi_i^{(iv)} \right) dx + \left[ N + \alpha \int_0^1 \left( \frac{d^2 w_d}{dx^2} + 2 \frac{d w_d}{dx} \frac{d w_0}{dx} \right) dx \right] \int_0^1 \phi_j \left( \sum_{i=0}^{n} u_i \phi_i^{n} \right) dx \\
+ 2 \alpha \int_0^1 \left( \frac{d w_d}{dx} + \frac{d w_0}{dx} \right) \sum_{i=0}^{n} u_i \phi_i^{(iv)} \right) dx \int_0^1 \phi_j \left( \frac{d^2 w_d}{dx^2} + \frac{d^2 w_0}{dx^2} \right) dx 
\] (4.13)

Using five symmetric modes, we compute the Jacobian of the system of the five obtained equations, for each \( V_{Th} \), and find the corresponding eigenvalues and mode shapes. Then, we compute the resonance frequencies of the resonators, at constant \( V_{Th} \), by taking the square root of these eigenvalues.
Figure 4.3 shows the variation of the first two symmetric resonance frequencies of the curved beams while tuning $V_{Th}$, experimentally and analytically for the two configurations. One can note that for all the case studies, the arc configuration shows a good agreement with the experimental results compared to the classically assumed cosine-wave (buckled) configuration. As shown in Figures 4.3a, 4.3b, and 4.3c, for the buckled configuration, the first resonance frequency increases while increasing $V_{Th}$ and then slows down, almost flattens, as it gets close and passes the third resonance frequency. The third resonance frequency decreases slightly with increasing $V_{Th}$ then starts to increase when getting close to the first resonance frequency. As noted, the first and third resonance frequencies never get too close to each other except for arc beam 3, Figure 4.3c.

On the other hand, and as proven experimentally, for the arc configuration, higher tunability is achieved for both resonance frequencies. The first resonance frequency increases as high as twice the fundamental frequency at zero electrothermal voltage. The third resonance frequency decreases and gets much closer to the first resonance frequency, where both modes deviate from each other. Figure 4.3c shows that for arc beam 3 the first and third resonance frequencies get very close at a critical electrothermal voltage. Then, they alter their directions, and each frequency continues along the path that the other frequency would have taken if they crossed, i.e., the first resonance frequency decreases while the third resonance frequency increases. This demonstrates the veering phenomenon (avoided-crossing), which is a mechanical way to linearly couple the two involved modes.
Figure 4.3: The variation of the first two symmetric resonance frequencies while varying the electrothermal voltage of the curved beams for (a) Arc beam 1, (b) Arc beam 2, and (c) Arc beam 3.

To further verify the avoided-crossing behavior for arc beam 3, we analytically study the variation of the resonance frequency as varying the compressive load for the same beam but with different thicknesses, Figure 4.4. Figure 4.4a indicates that veering cannot be activated for cosine-wave configuration. Figure 4.4b shows that for a specific thickness the third resonance frequency decreases until getting very close to the first resonance frequency. The figure verifies the avoided-crossing behavior. This presents a way to choose the geometric parameters of such curved beams carefully to activate or avoid the veering phenomenon depending on the targeted application.
Figure 4.4: The variation of the first two symmetric resonance frequencies of Arc beam 3 for different thicknesses and while changing the nondimensional compressive stress. (a) Cosine wave shape. (b) Arc shape.

For arc beam 3, we investigate the variation of the different mode shapes around veering by solving the mode shapes problem associated with Equation (4.13). Table 4.2 shows the variation of the first two symmetric modes as varying the electrothermal voltage. One can note the interchange of mode shapes among the first and third vibrational modes at veering. This explains the increase in sensitivity, after veering, of the third mode compared to the first mode.

Table 4-2: The first two symmetric mode shapes for various electrothermal voltages corresponding to Figure 4.3c.
To prove the validity of the different assumptions made by the analytical model, we conduct a 3D multi-physics finite-element FE simulation using the finite element software COMSOL [147]. The arc shape (arc beam 3) is implemented in the model since it better describes the behaviors of the fabricated curved beams. For the different physical domains in the developed model, the Solid Mechanics, Electric Currents, and Heat Transfer interfaces are implemented. For the Electric Currents module, an electrical potential and a
ground were defined on top of the anchors to allow passing an electrical current through a conductor and to simulate the Joule’s heating. The detailed FE procedure is presented in Chapter 3. A good agreement is shown in Figure 4.5a among the experimental, analytical, and finite element results. The first and third mode shapes for different electrothermal voltages are depicted in Figures. 4.5b, 4.5c, 4.5d, and 4.5e. A good agreement is shown among the mode shapes and those in Table 4.2.

(a)
115

Figure 4.5: (a) The variation of the first two symmetric resonance frequencies while varying the electrothermal voltage of arc beam 3 experimentally, analytically, and using a finite element model. (b)-(e): The first three mode shapes of the arc beam for (b) $V_{Th}=0$ V, (c) $V_{Th}=4.625$ V, (d) $V_{Th}=4.75$ V, and (e) $V_{Th}=9$ V.

4.4 Dynamic analysis

Next, we further analyze the forced vibration response near the veering phenomenon of arc beam 3. While exciting the beam electrostatically, we study the dynamic response before, at, and after veering; i.e., for different values of electrothermal voltage.

Figure 4.6 shows different frequency responses, obtained experimentally, of the arc beam for various electrothermal voltages and different excitation electrostatic voltages. Figure 4.6a shows the dynamic response before veering at $V_{Th}=3.5$ V. The amplitude of vibration of the first mode is higher than the third mode, as expected since it is more sensitive than the third mode to the electrostatic forcing. As increasing the electrostatic voltage, the first mode exhibits softening (dominated by the quadratic nonlinearity coming from the beam curvature and the electrostatic force) and the third mode exhibits hardening (dominated by the cubic nonlinearity coming from mid-plane stretching; where the quadratic nonlinearity seems weaker here).

Getting much closer to the veering regime, $V_{Th}=4$ V, the amplitude of vibration of the third mode starts to increase compared to the one of the first mode as increasing the electrostatic force, as shown in Figure 4.6b. This suggests that the third mode starts to take energy from the first mode. At veering, $V_{Th}=4.5$ V, both modes, first and third, start to
exchange energy. Figure 4.6c displays equal amplitude of vibration for both modes for different electrostatic voltages (i.e., same sensitivity to the electrostatic force for both modes).

After the veering zone, the third mode starts to be more sensitive than the first mode as shown in Figure 4.6d ($V_{Th} = 5$ V) and Figure 4.6e ($V_{Th} = 6$ V). Figures 4.6d and 4.6e demonstrate that after veering the third mode takes the nonlinear properties of the first mode before veering. Indeed, it starts to exhibit softening behavior for high electrostatic forcing instead of hardening behavior. That means that the cubic nonlinearity starts to dominate the quadratic nonlinearity after veering. One can note that, for the same applied electrostatic force, the maximum amplitude of vibration at the third resonance frequency at $V_{Th} = 5$ V is higher than at $V_{Th} = 4.5$ V, contrary to what is expected since the stiffness and the rise at midpoint increase more by increasing the electrothermal voltage. Figures 4.6d and 4.6e demonstrate that the response of the first mode has weakened after veering. The first mode shows also signs of hardening behavior in this regime.
Figure 4.6: Frequency response of arc beam (3) under different electrostatic loads for different constant electrothermal voltages. (a) $V_{Th} = 3.5$ V (before veering), (b) $V_{Th} = 4$ V (close to veering), (c) $V_{Th} = 4.5$ V (on veering), (d) $V_{Th} = 5$ V (after veering) and (e) $V_{Th} = 6$ V (after veering).

To simulate the dynamic response of the arc beam under electrostatic forcing and at a constant electrothermal voltage, we discretize Equation (4.5) using the Galerkin procedure, which yields a reduced order model (ROM). To do so, the transverse deflection of the arc beam is written as [137]
where \( q_i(t) \) \((i=0...n)\) are the nondimensional modal coordinates and \( \varphi_i(x) \) \((i=0...n)\) are the mode shapes obtained either by solving the eigenvector problem associated with Equation (4.13) at a constant electrothermal voltage or those of the unactuated straight beam.

Following [36, 137], we first multiply Equation (4.5) by \((1-w-w_0)^2\) in order to reduce the computational costs (integration of a numerator term is much less expensive than a dominator term). Then, by substituting Equation (4.14) into Equation (4.5), multiplying by \( \varphi_j(x) \) and integrating along the arc beam, this yields \( n \) algebraic equation in terms of \( q_i(t) \)

\[
\sum_{i=0}^{n} M_{ij}(q) \ddot{q}_i(t) + \sum_{i=0}^{n} c_{ij}(q) \dot{q}_i(t) + \sum_{i=0}^{n} K_{ij}(q) q_i(t) = Fm_j(q) + Fe_j(t) \quad \forall(j = 0...n) \quad (4.15)
\]

where

\[
M_{ij}(q) = \int_{0}^{1} \left[ \varphi_i(x) \varphi_j(x) \left( 1 - \sum_{i=0}^{n} q_i(t) \varphi_i(x) - w_0(x) \right)^2 \right] dx
\]

\[
c_{ij}(q) = c \int_{0}^{1} \left[ \varphi_i(x) \varphi_j(x) \left( 1 - \sum_{i=0}^{n} q_i(t) \varphi_i(x) - w_0(x) \right)^2 \right] dx
\]

\[
K_{ij}(q) = \int_{0}^{1} \left[ \varphi_i^{(iv)}(x) \varphi_j(x) \left( 1 - \sum_{i=0}^{n} q_i(t) \varphi_i(x) - w_0(x) \right)^2 \right] dx
\]

\[(4.16)\]
\[ F_{e_j}(t) = \alpha_2 (V_{DC} + V_{AC} \cos(\Omega t))^2 \int_0^1 \phi_j(x)dx \]

\[ F_{m_j}(q) = \alpha_1 \Gamma \left[ \phi_j(x) \left( \sum_{i=0}^n q_i(t)\varphi_i''(x) + w_0''(x) \right) \right] \left( 1 - \sum_{i=0}^n q_i(t)\varphi_i(x) - w_0(x) \right)^2 dx \]

\[ \Gamma(q) = N + \int_0^1 \left( \sum_{i=0}^n q_i(t)\varphi_i'(x) \right)^2 dx + 2 \int_0^1 \left( \sum_{i=0}^n q_i(t)\varphi_i'(x) \right) w_0'(x)dx \]

Starting by computing the integrals of Equation (4.16) and then by using the Runge-Kutta technique, the dynamic response of the arc beam is obtained by time-integrating Equation (4.15). Using four symmetric mode shapes (by solving the eigenvector problem in Eq. (16)), good agreement is reported in Figures 4.7a and 4.7b between the simulations and the experimental results for \( V_{Th} = 3.5 \text{ V} \) \( (V_{DC} = 15 \text{ V}, V_{AC} = 15 \text{ V}) \) and \( V_{Th} = 4 \text{ V} \) \( (V_{DC} = 10 \text{ V}, V_{AC} = 10 \text{ V}) \), respectively. In Figures 4.7a and 4.7c, we show that using either four exact mode shapes of the arc beam under electrothermal voltage or using five symmetric mode shapes of unactuated straight beam yields the same result. For \( V_{Th} = 4 \text{ V} \), Figure 4.7c, increasing more the electrostatic force, we start to have a mismatch between the experimental and the analytical results around the first mode. On the other hand, the ROM was able to detect the response of the system around the third mode accurately, Figures 4.7c and 4.7d. Increasing more the electrothermal voltage \( V_{Th} = 5 \text{ V} \) (after veering regime), Figure 4.7d, the ROM was able to detect the change of the nonlinear behavior of the third mode from hardening to softening behavior while failing to accurately predict the response around the first mode.
Figure 4.7: Analytical and experimental frequency response curves of arc beam (3) under electrostatic force and different constant electrothermal voltages. (a) \( V_{Th} = 3.5 \) V, \( V_{DC} = 15 \) V and \( V_{AC} = 15 \) V, (b) \( V_{Th} = 4 \) V, \( V_{DC} = 10 \) V and \( V_{AC} = 10 \) V, (c) \( V_{Th} = 4 \) V, \( V_{DC} = 15 \) V and \( V_{AC} = 15 \) V, (d) \( V_{Th} = 5 \) V, \( V_{DC} = 20 \) V and \( V_{AC} = 20 \) V. The assumed exact mode shapes are obtained by solving the eigenvector problem associated with Equation (4.13) at a constant electrothermal voltage and SB (straight beam) mode shapes refers to those of a straight unactuated beam.

The assumed exact mode shapes are obtained by solving the eigenvector problem associated with Equation (4.13) at a constant electrothermal voltage. SB (straight beam) mode shapes refer to those of a straight unactuated beam.
One can note that the ROM could not accurately capture the dynamic behavior near and after veering. One reason might be that the ROM does not take into consideration the contribution of the out-of-plane and rotational modes in the response, which experimentally are observed to affect the response. We have noticed high sensitivity of such modes as increasing the electrothermal voltage as seen during the experiment using the high speed camera. In addition to the fabrication imperfections, another potential source of error is the clamping condition that is assumed in the model to be with zero slopes. Experimental images suggest that this might not be accurate. Other techniques can be employed to demonstrate theoretically the veering phenomenon, mainly the multiple scale methods [80], which will be considered in future work.
Chapter 5  Nonlinear mode coupling: Internal resonance

In this chapter, we study experimentally multiple types of internal resonances among the first two symmetric modes of an arch beam electrothermally tuned and electrostatically actuated. We will focus in the second section on investigating theoretically and experimentally the two-to-one internal resonance as a case study.

5.1  Multiple internal resonances

Based on electrical characterization techniques, this chapter aims to investigate different types of internal resonances of an initially curved beam and to characterize in-depth the dynamics during these internal resonances.

5.1.1  Experimental setup

The experiments were conducted on electrothermally tuned and electrostatically driven arch MEMS resonators. These arch beams are fabricated on a highly conductive 30μm Si device layer of a silicon-on-insulator (SOI) wafer by a two-mask process using standard photo-lithography, metal sputtering, deep reactive ion etch (DRIE), and vapor hydrofluoric acid dry (vpour-HF) etch [150]. Here the arch beam has a length of 700 μm, a depth of 30 μm, a width of 2 μm, and an initial rise 2.6 μm. The arch beam is biased with a DC voltage source. The drive electrode provides an AC harmonic voltage. A low-noise amplifier (LNA) is used to amplify the output current induced at the sense electrode due to the microbeam motion. Then, the output current is fed-back, in parallel, to a network analyzer (Agilent 5071C) for transmission signal measurements, and to a spectrum
analyzer to concurrently identify the output power spectrum in the vicinity of the contributed frequency on the overall response, Figure 5.1. All the experiments are conducted at room temperature and in vacuum (pressure 6 mTor).

![Diagram](image)

**Figure 5.1**: A schematic of the experimental setup.

To control the stiffness of the arch beam, and hence its resonance frequency, a separate DC voltage source, $V_{Th}$, is connected to the arch beam anchors to induce a constant current flowing through the beam, and hence heating it up by Joule's heating. Figure 5.2 shows the measured first two symmetric resonance frequencies of the arch beam. Upon changing the electrothermal voltage, the first resonance frequency increases up to twice its fundamental value (71 kHz); due to the continuous increase in the arch curvature (stiffness) [8]; and the third resonance frequency decreases, Figure 5.2. Hence the ratios of two-to-one and three-to-one between the two modes can be achieved as varying $V_{Th}$, as shown in the inset of Figure 5.2, which presents a necessary condition to activate the corresponding internal resonances.
Figure 5.2: The variation of the first two symmetric frequencies of the arch beam; 700 μm length, 30 μm depth, and 2 μm width; with the electrothermal voltage $V_{Th}$. The arch beam has a 2.6 μm initial rise (distance between its midpoint to its straight beam level). The inset shows the variation of the ratio between the 3$^{rd}$ and 1$^{st}$ mode frequencies with $V_{Th}$.

5.1.2 Results and discussions

a Two-to-one internal resonance

As fixing $V_{Th}$ at a ratio of two ($V_{Th} = 1.509$ V), we sweep the excitation frequency around the 1$^{st}$ mode. Meanwhile, the output signal spectrum is examined, using the spectrum analyzer, in the vicinity of the 3$^{rd}$ mode to identify its contribution on the overall response shown by the network analyzer, Figure 5.3. The figures indicate a frequency component around the 3$^{rd}$ mode. Figure 5.3a shows that even for small AC excitation voltage the response of the beam is dominated by the cubic nonlinearities showing a hardening response. This illustrates the strong interaction between both modes since the natural response of the arch beam around the 1$^{st}$ resonance frequency is dominated by the quadratic nonlinearity (softening effect). Increasing the AC bias voltage gradually results in the splitting of the overall vibration amplitude into two peaks where the arch beam is
experiencing internal resonance. In Figures 5.3b-5.3d, the presence of two peaks suggests a nonlinear modal interaction among the 1st and 3rd resonance frequencies. This proves the transfer of energy from the first mode to the third mode and vice versa. This energy transfer can be shown by the detected current around the 3rd resonance frequency while exciting the 1st resonance frequency.

![Figure 5.3: (a), (b), (c) and (d) The frequency response of the arch beam around the 1st resonance frequency for various AC harmonic excitations showing the two-to-one internal resonance, at $V_{Th} = 1.509$ V. The inset in (b) shows the contribution of the 1st mode on the overall response.](image)

Then, as applying larger driving voltages while sweeping around the 1st mode, the arch beam passes from stable to unstable branches showing a possible quasi-periodic
motion (the green regions) which could be a consequence of a reverse Hopf bifurcation, Figures 5.3b-5.3d. The quasi-periodic motion presents a common characteristic of the internal resonance suggesting the immersion of a finite number of incommensurate frequencies on the dynamic response of the arch beam. Driving the system more strongly enhances the coexistence of multi-stable states for larger bandwidth and hence might lead to more complicated dynamic phenomena such as chaotic responses.

For the same electrothermal voltage, we sweep the excitation frequency around the 3rd mode while examining the contribution of the 1st mode to the total response, Figure 5.4. The figure illustrates the energy transfer from the 3rd mode to the 1st mode as experiencing two-to-one internal resonance. Figure 5.4 shows that while sweeping around the 3rd mode, the arch beam does not experience any unstable motion. However, Figure 5.4a shows that even when if no oscillation is detected around the 3rd mode (dashed line section), a current is sensed around the 1st mode that acts as a reservoir of the energy of the higher mode at the steady state.
Figure 5.4: (a), (b), and (c) The frequency response of the arch beam around the 3rd frequency for various AC harmonic excitations showing the two-to-one internal resonance, at \( V_{th} = 1.509 \text{ V} \).

b) Three-to-one and four-to-one internal resonances

Next, we tune \( V_{th} \) to a ratio three \( (V_{th} = 0.845 \text{ V}) \). As sweeping the frequency around the first mode, the 1st and 3rd frequencies start to interact leading to sharp dips in frequency that could be explained by the falling to a lower stable branch due to the transfer of the energy to the 3rd mode, Figures 5.5a-5.5c. The figures show that the quadratic nonlinearity dominates the response despite the mode interaction. The amplitude of the 3rd mode contribution appears to have a hardening behavior (due to the cubic nonlinearity), which is normal nonlinear response of an arch beam around the 3rd mode.
Figure 5.5: (a), (b), and (c) The frequency response of the arch beam around the 1<sup>st</sup> frequency for various AC harmonic excitations showing the three-to-one internal resonance, at $V_{Th} = 0.845$ V. (d), (e), and (f) The frequency response of the arch beam around the 1<sup>st</sup> frequency for various AC harmonic excitations showing the four-to-one internal resonance, at $V_{Th} = 0.76$ V.

For a similar arch beam with length 650 μm and a thickness of 3 μm, a ratio four-to-one between the 1<sup>st</sup> and 3<sup>rd</sup> modes was achieved. As known in the literature that the 4<sup>th</sup> order coupling between the 1<sup>st</sup> and 3<sup>rd</sup> resonance frequencies is weak compared to other couplings, hence, can be thought as negligible. However, in this work, we show that the four-to-one internal resonance could be activated by driving the arch beam highly at
vacuum, Figures 5.5d-5.5f. This interaction is attributed to a quartic nonlinearity, which can be from the higher order nonlinearity geometric nonlinearity or that from the electrostatic force. Also, one should mention here that the qualitative response of the four-to-one internal resonance appears similar to the three-to-one internal resonance.

**c Theory**

To model the nonlinear behavior of the arch resonator, we refer to the Euler Bernoulli beam theory. The transverse vibration of the arch beam of an initial shape $w_0(x)$ under double-side electrostatic actuation is governed by [137, 140]:

$$
\rho bh \frac{\ddot{w}}{\partial t^2} + c \frac{\dot{w}}{\partial t} + EI \frac{\dddot{w}}{\partial x^4} = \left( \frac{\dddot{w}}{\partial x^4} + \frac{d^2w_0}{dx^2} \left[ N + \frac{EA}{2I_0} \left( \frac{\dot{w}}{\partial x} + 2 \frac{\dot{w} \ddot{w}_0}{\partial x} \right) dx \right] \right) \left( \text{Curvature (quadratic)} + \text{Midplane Stretching (cubic)} \right) + \frac{1}{\varepsilon b} V_{DC}^2 - V_{AC}^2 \cos(\Omega t) - \frac{1}{\varepsilon b} \frac{V_{DC}^2}{2} \left( \frac{d-w+w_0}{d+w+w_0} \right)^2 \left( \text{Electrostatic (quadratic, cubic, ...)} \right) \tag{5.1}
$$

The arch beam is subjected to the clamped-clamped boundary conditions as

$$
w(0,t) = w(l,t) = 0 \text{ and } \left. \frac{\dot{w}}{\partial x} \right|_{0,l} = \left. \frac{\ddot{w}}{\partial x} \right|_{1,l} = 0 \tag{5.2}
$$

In Equation (5.1), $x$ is the position along the arch beam, $t$ is time, $l$, $b$, and $h$, are, respectively, the length, width, and thickness of the arch beam, $\rho$ is the material density, $I$ is the moment of inertia of the rectangular cross-sectional area $A=bh$, and $E$ is Young’s modulus. The term $N$ represents the total axial load on the structure accounting for the residual load arising from the fabrication process, and the thermal compressive load induced by the applied $V_{Th}$. The last two terms in Equation (5.1) represent the electrostatic force when the two-port measurement is used, where $V_{DC}$ is the DC polarization voltage and $V_{AC}$ is the AC harmonic voltage of frequency $\Omega$. The arch resonator is separated from the
driving/sensing electrodes with gap widths $d$ and $d_1$, respectively. The beam is subjected to viscous damping of coefficient $c$.

According to Equation (5.1), the response of the arch is governed by two types of nonlinearity: quadratic and cubic nonlinearities. The quadratic nonlinearity is originated from the arch curvature and the electrostatic force. The cubic nonlinearity is a geometric nonlinearity originated from the midplane stretching. Next, using a three-mode Galerkin reduced-order model and referring to Chapter 4, the frequency response of the arch beam is simulated for different AC excitations for a case of two-to-one, Figure 5.6, and three-to-one Figure 5.7 internal resonances.

As the arch starts to experience the two-to-one internal resonance, the frequency response is split into two peaks, Figures 5.6a-5.6b. Increasing the forcing further, Figures 5.6c-5.6d, the arch response passes through a quasi-periodic motion, as evident in the Poincaré section in the insets of the figures, which can be a consequence of a Hopf bifurcation that is common in such cases [98]. Evidence of complex (potentially chaotic) behavior can be seen in Figure 5.6d.

![Graphs showing frequency response](image-url)
Figure 5.6: Frequency response curves of the arch beam around the 1st resonance frequency for various AC harmonic excitations near the two-to-one internal resonance. $V_{DC}$ is fixed at 65 V. The assumed damping ratio is 0.00013. The red scattered data show the quasi-periodic motion of the arch. Insets show Poincaré sections of selected points.

For the three-to-one internal resonance, a higher excitation AC is needed to activate the nonlinear interaction between the contributing modes. Figure 5.7 shows the simulated results for the cases of backward and forward frequency sweeps. As experiencing internal resonance, Figure 5.7a shows a simple softening behavior, which is then interrupted by jumps as AC is further increased, Figure 5.7b, Figure 5.7c. Figure 5.7c also shows signs of complex behavior.
Figure 5.7: The frequency response of the arch beam around the 1st resonance frequency showing the three-to-one internal resonance for different AC excitation. $V_{DC}$ is fixed at 65 V. The assumed damping ratio is 0.00018. Insets in (c) show Poincaré sections of selected points.

**d Discussion**

One should note that the arch response is highly dependent on the surrounding pressure. Pressure variation leads to a different operating $V_{th}$ to activate different internal resonances. Also, the pressure variation affects the interaction region and bandwidths. Figure 5.8a shows that as increasing the pressure, the bandwidth of the unstable motion shrinks down with conserving the same interaction behavior, at the same excitation voltage.
However, as increasing pressure, the dip of the frequency response induced by the three-to-one internal resonance shown previously decreases until disappearing at a relatively higher pressure, Figure 5.8b. Hence, at a relatively higher pressure, the response is dominated by the quadratic nonlinearity and then passes by the internal resonance without falling to the other lower branch (frequency dip) [100].

Figure 5.8: (a) The frequency response as experiencing the two-to-one internal resonance for 0.2236 V (RMS) excitation voltage for different surrounding pressure. At 450 mTorr pressure, the applied $V_{Th}$ is equal to 1.521 V. At 700 mTorr pressure, the applied $V_{Th}$ is equal to 1.53 V. (b) The frequency response as experiencing the three-to-one internal resonance at 450 mTorr and for different AC harmonic excitations, at $V_{Th} = 0.86$ V. (c) The
frequency response as experiencing the three-to-one internal resonance at 700 mTorr and for 0.2236 V (RMS) excitation voltage, at $V_{Th} = 0.875$ V.

Next, we investigate and characterize the effect of different internal resonances (two-to-one and three-to-one) on the frequency stabilization of the studied system. Hence, a frequency noise analysis can be obtained by analyzing the Allan deviation for different cases: outside and inside the internal resonance region. The effect of the noise on the resonator oscillation is given by the average of the fractional frequency fluctuation over an interval $\tau$, as a function of that averaging time $\tau$. Then, the Allan deviation [151, 152] $\sigma_y(\tau)$ can be expressed as:

$$
\sigma_y(\tau) = \sqrt{\frac{1}{2(N-1)}} \left( \sum_{i=1}^{N-1} \left( \bar{f}_{i+1}^{\tau} - \bar{f}_i^{\tau} \right)^2 \right)
$$

(5.3)

where $\bar{f}_i^{\tau}$ denotes the relative frequency fluctuation averaged over the $i^{th}$ discrete time interval of $\tau$ defined as

$$
\bar{f}_i^{\tau}(\tau) = \frac{f(i) - f_0}{f_0}
$$

(5.4)

where $f_0$ is the driving frequency.

Figure 5.9 shows the Allan deviation of the arch resonator, in an open-loop configuration, at a constant AC voltage (0.22 V RMS) outside and within the internal resonance regime. The measurements are conducted by monitoring the amplitude fluctuation for a specified period of time, at the constant $f_0$. The Allan deviation curves seem to have a standard curve behavior as expected in a MEMS oscillator. At low
integration time, the noise is dominated by white noise while at higher integration time the fluctuation is governed by the thermal drift.

Figure 5.9: Allan deviation of the arch resonator showing the improvement of frequency stabilization inside the internal resonance regime. The driving power is fixed at 0.22 V (RMS).

Figure 5.9 demonstrates one-order-of-magnitude enhanced stability of the resonator response inside the internal resonance regime. Also, the frequency fluctuation around the driving frequency decreases from ±1 Hz to ±0.45 Hz as operating in the internal resonance regions. Both internal resonance types (two-to-one and three-to-one) show almost similar improvement on the stability of the resonator response. At low pressure, operating the arch beam in the three-to-one internal resonance interaction may lead to a safe operation since it gives the same frequency stability and it prevents the oscillation from the unstable branches due to the Hopf bifurcations. One should mention here that these results can be improved by using a closed-loop configuration. The activated internal
resonances are independent of the arch size. Hence, the proposed mechanism can be used to enhance the stability of nanoscale resonator.

5.2 Two-to-one internal resonance

5.2.1 Experimental setup

The experiments were conducted on initially curved beams fabricated by MEMCAP [138], from SOI wafers with highly conductive silicon device layer using two-mask process. The stroboscopic video microscopy from Polytec [139] was used to study and investigate the free vibration and the dynamic response of the resonator.

Figure 5.10 shows the arch beam under consideration, which is similar to the curved beam studied in Chapter 4 (Arch beam 3). The dimensions of the arch beam are given in Table 5.2. Similar principle of operation and testing procedure to Chapter 4 are used in this section. By applying a DC voltage $V_{Th}$ between the anchors of the arch beam, a DC current $I_{Th}$ passes through it heating it up, which controls the induced axial load (stiffness). The arch beam is separated from a stationary electrode with a transduction gap of width $d$. The curved beam is actuated electrostatically by a DC bias voltage $V_{DC}$ and an AC harmonic voltage of amplitude $V_{AC}$ and frequency $\hat{\Omega}$.

Figure 5.10: A schematic of the clamped-clamped arch beam electrothermally tuned and electrostatically actuated.
Table 5-1: Geometrical properties of the arch beam made of doped silicon.

<table>
<thead>
<tr>
<th>Length, ( l ) (( \mu )m)</th>
<th>Thickness, ( h ) (( \mu )m)</th>
<th>Width, ( b ) (( \mu )m)</th>
<th>Gap, ( d ) (( \mu )m)</th>
<th>Initial rise, ( \dot{b}_0 ) (( \mu )m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>800</td>
<td>1.75</td>
<td>30</td>
<td>8.4</td>
<td>2.6</td>
</tr>
</tbody>
</table>

5.2.2 Problem formulation

The initial shape \( \dot{w}_0(x, \bar{x}) \) of the clamped-clamped beam (arc shape) is governed by Equation (4.1). Similar numerical procedure to Chapter 4 is used to solve the eigenvalue problem and dynamic response of the arch beam under different electrothermal and electrostatic voltages.

In the present work, large AC voltages are used to excite the beam dynamically due to the high damping (under atmospheric pressure). Hence, the forcing amplitude term \( \frac{1}{2} V_{ac}^2 \) can reach an order of magnitude similar to that of the \( 2V_{ac}V_{dc} \) term in the expansion of the electrostatic force as indicated below.

\[
(V_{dc} + V_{ac} \cos(\Omega))^2 = V_{dc}^2 + \frac{1}{2} V_{ac}^2 + \left\{ 2V_{ac}V_{dc} \cos(\Omega) + \frac{1}{2} V_{ac}^2 \cos(2\Omega) \right\} \quad (5.5)
\]

Here the influence of the \( 2\Omega \) term cannot be neglected when exciting the first primary resonance. Also, it indicates that the level of nonlinear interaction is influenced by the simultaneous excitation of the two modes, 1\(^{st}\) and 3\(^{rd}\) modes, as well as the internal resonance. One should mention here that as the AC load increases, the ratio between the 1\(^{st}\) and 3\(^{rd}\) frequencies varies due to the static effect of the term \( \frac{1}{2} V_{ac}^2 \) on the frequencies of the arch beam.
5.2.3 Results and discussion

a Linear frequency results

As increasing the electrothermal voltage, the arch beam heats up, and hence its stiffness accordingly increases. Figure 5.11a shows the variation of the first two symmetric resonance frequencies of the arch beam while varying the electrothermal voltage, experimentally and analytically (by solving the corresponding eigenvalue problem presented in Chapter 4). Initially, increasing the compressive load raises the first resonance frequency while the third resonance frequency decreases until both approach each other; after which they veer away. Then the first resonance frequency flattens while the third resonance frequency continuously increases. Next, computing the ratio between both frequencies, Figure 5.11b, reveals that different internal resonances can be activated (three-to-one and two-to-one). In this study, we focus on the nonlinear aspects of the two-to-one internal resonance.

Figure 5.11: (a) The variation of the first and third resonance frequency with the electrothermal voltage $V_{Th}$. (b) The variation of the ratio between both frequencies with $V_{Th}$. 
b  Nonlinear vibrations results

To investigate the dynamics of the two-to-one internal resonance, we experimentally sweep the frequency for different electrostatic forces around both the first and third resonance frequencies and at $V_{Th}=2.45$ V, Figure 5.12. The experiments were conducted in atmospheric pressure (high damping) requiring high AC harmonic excitation amplitudes to activate the internal resonance. At lower AC excitations, the system yields a linear response, Figure 5.12a. However, increasing the electrostatic force gradually results in the splitting of the vibration amplitude into two peaks where the arch beam starts to experience internal resonance, Figures 5.12a and 5.12b. The presence of two peaks suggests a nonlinear modal interaction among the first and third modes. This indicates the transfer of energy from the first vibrational mode to the third mode. Also, the high AC voltages indicate that the nonlinear coupling between the two modes is influenced by the simultaneous direct excitation of the two modes.

On the other hand, we excite the arch beam for the same $V_{Th}$, around its third resonance frequency for different electrostatic forces, Figures 5.12c and 5.12d. As the forcing increases, the arch beam starts to experience internal resonance, which is demonstrated by the splitting of the frequency response curve. The separation between the two existing peaks increases with the excitation. For larger forcing, a potential coexistence of states is shown in Figure 5.12d (the red stars).
Figure 5.12: (a), (b) Experimentally measured frequency responses around the first resonance frequency at $V_{Th}=2.45$ V, $V_{DC}=15$ V, and for different AC bias voltages. (c), (d) Experimentally measured frequency responses around the third resonance frequency at $V_{Th}=2.45$ V.

Due to the limitations of experiments to provide thorough insights, we use the numerical simulation of the Galerkin model to analyze the involved nonlinear interactions around the first mode. As exciting the beam with relatively low AC excitation, a good qualitative and quantitative agreement is shown among the theoretical (Figure 5.13) and experimental results, Figure 5.12a.
Figure 5.13: The analytical simulations of the dynamic response of the arch beam at $V_{Th}=2.45$ V, $V_{DC}=15$ V and for different $V_{AC}$ loads.

To better understand the rich dynamics involved for different excitation conditions, we compute for several frequencies the Poincare section, time history, and power spectrum density (PSD). We start by simulating the response at $V_{AC}=30$ V, Figure 5.14a. Figure 5.14b shows the modal contribution of the $1^{st}$ and $3^{rd}$ modes in the total response, Figure 5.14a. By looking at the different lines on the frequency response curve Figures 5.15c-5.14e, the Poincare sections show that the arch beam has a periodic motion. However, the PSDs at different frequency lines show that despite the periodicity of the motion, two frequencies ($1^{st}$ and $3^{rd}$) are contributing to the response. Also, the PSD at the first line, Figure 5.14c, shows that even far from the internal resonance interaction two contributing frequencies exist: the first and the third, which are produced due to the simultaneous excitation of both frequencies at the high AC voltage.
Figure 5.14: (a) Simulations of the dynamic response of the arch beam at $V_{TH} = 2.45$ V, $V_{DC} = 15$ V, and $V_{AC} = 30$ V. (b) the associated modal coordinates of the 1st and 3rd modes.
(c), (d), (e) The time history, Poincare section, and PSD at different lines A, B and C as shown in (a).

Then, increasing the AC load to 35V, Figures 5.15a-5.15b, the arch beam starts to show a quasi-periodic motion (the green scattered points). This presents a common characteristic of the internal resonance suggesting that the response of the system experiences bifurcation as the AC load is increased. As sweeping the frequency around the first natural frequency, the system jumps from periodic motion to quasi-periodic motion. This is confirmed by the Poincare section, Figure 5.15e, showing a closed curve that presents the main characteristic of a quasi-periodic motion. Also, the corresponding PSD, Figure 5.15e, shows the creation of various periods originated from the incommensurate ratio between the excitation and the contributing frequencies. Also, the time history during quasi-periodic motion shows a fluctuation of the amplitude at steady-state. Figures 5.15d-5.15f present the transition frequency from periodic to quasi-periodic and vice versa, respectively.
Figure 5.15: (a) Simulations of the dynamic response of the arch beam at $V_{Th}=2.45$ V, $V_{DC} = 15$ V, and $V_{AC} = 35$ V. (b) The associated modal coordinates of the 1$^{st}$ and 3$^{rd}$ modes. (c), (d), (e), (f), (g) The time history, Poincare section, and PSD at different lines A, B, C, D, and E as shown in (a). The green scattered data denote the quasi-periodic motion.
We further increase the AC bias voltage, which leads to more complicated dynamical phenomena. Figure 5.16a shows different jumps to some unstable points (discrete red) in the frequency response curve obtained experimentally. The vibration videos at these frequencies show erratic motions that are aperiodic. To investigate the nature of the dynamic response, we simulate the response of the arch beam at higher AC bias voltage, Figure 5.16b. Then, as sweeping the frequency, the response jumps from the periodic to aperiodic solutions, presented by red points. Poincare sections at these frequencies show the presence of a chaotic motion that might describe the strange motion observed experimentally. At section point B, Figure 5.16d, the system passes from stable to chaotic motion, which might present a boundary crises bifurcation. Then, before returning to periodic motion, the system passes throughout 4-periods and 2-periods motion, Figures 5.16e-5.16f. In between the chaotic motions presented in red, the response emerges from aperiodic to periodic and then aperiodic solution, which consists of the contribution of both frequencies. Next, the system passes through a quasi-periodic motion (green scatters), from 44.75 kHz to 45.4 kHz, which might be a consequence of a reversed Hopf bifurcation. More investigation is needed to explore the dynamic response and identify whether this aperiodic motion is originated only from the mode coupling of both modes via internal resonance or from the combination of internal resonance and direct excitation of the third mode due to the high AC bias voltage.
(f) Line D

(g) Line E

(h) Line F

(i) Line G

(j) Line H
Figure 5.16: (a) and (b) Experimental and numerical results of the dynamic response of the arch beam at $V_{Th}=2.45$ V, $V_{DC}=15$ V, and $V_{AC}=45$ V, respectively. (c), (d), (e), (f), (g), (h), (i), (j), (k) The time history, Poincare section, and PSD at different sections A, B, C, D, E, F, G, H, and I as shown in (b). The red stars denote the aperiodic (potentially chaotic) motion. The green scattered data denote the quasi-periodic motion.
Chapter 6  Applications: Bandpass filter & pressure sensor

In this chapter, we present two potential applications that can be exploited based on the theoretical and experimental analysis presented in the previous chapters. We will present the exploitation of the veering phenomenon of the arch beam to build a bandpass filter of much improved rolls from pass to stop bands. The second section will focus on exploiting the tunability of buckled beams and arches for a pressure sensing application.

6.1  Bandpass filter

6.1.1  Principle of operation

The arch beam under electrostatic force is subjected to a cubic nonlinearity, from mid-plane stretching, and a quadratic geometric nonlinearity, from curvature and electrostatic force, which can yield hardening and softening behavior of the various modes. The nonlinear frequency response of the arch around the first and third (second symmetric) resonance frequencies is known to be dominated by the quadratic (softening behavior) and cubic (hardening behavior) nonlinearities, respectively. By changing a control parameter, the voltage in this case, it is possible to bring both frequencies close to each other, veering, thus forming a band of frequency of high amplitude (the passband). This band will be determined by the two jumping frequencies, due to the softening and hardening behavior of the two modes. These jumps lead to sharp transition from the stopbands to the passband; thereby yielding near ideal bandpass filter. The concept is illustrated in Figure 6.1.
Figure 6.1: Schematic illustrating the proposed bandpass filter. (a) Shows the nonlinear response of the first (softening) and third (hardening) modes. (b) Shows the responses upon increasing the DC electrothermal voltage, thus bringing the two modes closer to each other. (c) Shows the two modes brought very close to each other (veering) and indicates the realization of the bandpass filter with jumps on the sides of the passband.

### 6.1.2 Experimental setup

The resonator is fabricated on a highly conductive Si device layer of a silicon-on-insulator wafer by a two-mask process using standard photo-lithography, electron beam evaporation for metal layer deposition for actuating pad, deep reactive ion etching for silicon device layer etching and vapor hydrofluoric acid etch to remove the oxide layer underneath the resonating structure[150]. The clamped-clamped arch beam, under consideration, is sandwiched between two adjacent electrodes, Figure 6.2, to induce the vibration by exciting it electrostatically. The fabricated arch beam is of length 800 μm, width 30 μm, thickness 2 μm, and 2.6 μm initial rise.
The arch is actuated electrothermally by a DC voltage $V_{Th}$ and electrostatically by a DC polarization voltage $V_{DC}$ and an AC harmonic voltage of amplitude $V_{AC}$, Figure 6.3. The electrothermal voltage $V_{Th}$ is applied between the anchors of the arch inducing a current $I_{Th}$ flowing through the microbeam that generates heat, which causes thermal expansion and controls its internally induced axial stress (compressive stress). This compressive force causes an increase in the microbeam curvature, and hence increases its stiffness.

Stroboscopic video microscopy from Polytec is used to determine the resonance frequencies as well as the frequency response of the arch beam. We measured the resonance...
frequencies of the arch beam using the ring down measurement and the fast Fourier transform (FFT) while varying the DC electrothermal voltage. To conduct frequency sweeps, we generate an amplified periodical sine signal to excite the in-plane arch. The stroboscopic video microscopy generates the frequency response curves in two different scales decibel and linear. It provides as well the frequency at -3 db. All the experiments are conducted in air and at room temperature.

6.1.3 Results and discussion

Figure 6.4a shows the FFT of the arch beam under consideration for a zero electrothermal voltage. The first and third resonance frequencies of the unactuated arch beam are found to be around 38 kHz and 104 kHz, respectively.

Figure 6.4: (a) FFT of the in-plane clamped-clamped arches at zero electrothermal voltage. (b) The variation of the first and third resonance frequencies of the shallow arch under electrothermal actuation.
One should note that the time associated with the electrothermal cooling and heating is much longer than the time associated with the vibration of the studied arch[68]. The associated thermal time coefficient could be calculated using the equation

$$\tau = \left[ \frac{\pi^2 K_{Si}}{c_l h^2} + \frac{F_s K_{air}}{gbc \rho} \right]^{-1}.$$  

In the above equation, $l$, $b$ and $g$ present the length, width of the arch beam and the gap between the arch beam and the substrate, respectively. $\rho$, $c$, $K_{Si}$, and $K_{air}$ are the silicon density, the silicon heat capacitance and the thermal conductivity of the silicone and the air, respectively. $F_s$, the beam shape factor, is the correction term calculated based on the geometry of the arch beam using the formula[153] given by

$$F_s = \frac{b}{h} \left( \frac{2g}{b} + 1 \right) + 1,$$

where $h$ is the thickness of the arch beam. In the studied case, $F_s$ is calculated to be equal to 17.995. Then, the thermal time constant of the arch beam under consideration is 162.833 µs.

The variation in the measured resonance frequencies, while changing the electrothermal voltage $V_{Th}$, is depicted in Figure 6.4b. It can be observed that the first resonance frequency increases with the increase in the electrothermal voltage and reaches the value as high as twice of the initial value at zero electrothermal voltage. On the other hand, the third resonance frequency decreases while increasing the electrothermal voltage until getting very close to the first resonance frequency. At this critical electrothermal voltage, the third resonance frequency starts to increase while the first resonance frequency starts to become flat. Indeed, each frequency continues along the path that the other frequency would have taken. This phenomenon can be explained through the veering (avoided-crossing) phenomenon, which occurs when the resonance frequencies of two
modes get close to each other [77, 80]. Veering can be viewed as a way to mechanically couple the two involved modes.

Next, we excite the arch beam electrostatically by applying $V_{DC}=20$ V and $V_{AC}=20$ V for a range of electrothermal voltages where the first and third resonance frequencies are close to each other. For $V_{Th}=1.6$ V, 1.65 V and 1.7 V, Figure 6.5a, the first and third modes show softening and hardening nonlinear behavior, respectively, as expected of an arch microbeam[36]. This nonlinear behavior breaks the symmetry of the frequency response curve, which becomes no longer Lorentzian. However, for $V_{Th}=1.65$ V, and 1.7 V, the third resonance frequency shows an amplitude of vibration more than the amplitude of vibration of the first resonance frequency and even more than the amplitude of vibration for both resonances for $V_{Th}=1.6$ V. These results suggest that both modes start to interact with each other, where the third mode takes energy from the first mode, and hence eventually becomes of large amplitude.

For $V_{Th}=1.7$ V, Figure 6.5a, after a small softening jump, characterizing the first resonance frequency, a flat band of frequency starts to appear around the third resonance frequency. Upon increasing the electrothermal voltage to 1.8 V or 1.9 V, we are able to obtain a flat wide frequency band due to full interaction between the first and the third modes of vibration, Figure 6.5b.
Figure 6.5: (a),(b) Frequency response curves for $V_{DC}=20\, V$ and $V_{AC}=20\, V$ for different electrothermal voltages. (c) Frequency and phase response for $V_{DC}=20\, V$, $V_{AC}=20\, V$ and $V_{Th}=1.8\, V$. (d) Frequency response for $V_{DC}=20\, V$, $V_{AC}=20\, V$ at various electrothermal voltages. (e) Enlarged view of the bandpass in decibel scale.
The flat band is a result of the combination of two resonance modes, the first and the third. We plot the experimentally obtained phase response for $V_{Th} = 1.8$ V, Figure 6.5c. The total phase shift is shown to be around $300^\circ$, which suggests that there are two vibrational modes, the first and the third, getting close to each other, hence creating a flat passband around their corresponding resonance frequencies. This kind of phase response is typical for a bandpass filter [118, 119, 154]. Therefore, it is inferred that a flat bandpass filter can be realized based on electrothermally tuned single arch resonator by coupling the first and third mode of vibration.

Next, we characterize the tunability feature of this bandpass filter by varying the electrothermal voltage centered around $1.8$ V, while keeping the electrostatic excitation force unchanged, Figures 6.5d and 6.5e. Figure 6.5e is displayed in decibel scale to show the frequency at -3db and then to extract the center frequency and the bandwidth accurately. It is shown that both the center frequency and the bandwidth can be moderately tuned by varying the electrothermal voltage.

Table 6.1 summarizes the bandpass filter features for different electrothermal voltages presented in Figures 6.5d and 6.5e. We show a tunable center frequency $f_0$ as well as tunable bandwidth $\Delta f$. The bandwidth, $\Delta f$, is wide, flat, and can be varied by 22% from $9$ kHz at $V_{Th} = 1.76$ V to $11$ kHz at $V_{Th} = 1.848$ V. The center frequency, $f_0$, can be tuned by 3%. 

Table 6-1: Tunability of the center frequency and bandwidth by the electrothermal voltage.

<table>
<thead>
<tr>
<th>$V_{Th}$ (V)</th>
<th>1.769</th>
<th>1.8322</th>
<th>1.848</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0$ (kHz)</td>
<td>86.760</td>
<td>88.125</td>
<td>88.25</td>
</tr>
<tr>
<td>$\Delta f$ (kHz)</td>
<td>9</td>
<td>10.25</td>
<td>11</td>
</tr>
</tbody>
</table>

Note that these results can be improved for operation in the megahertz and gigahertz regimes by shrinking the dimensions of the arch beam and keeping the same concept of actuation. To demonstrate this, we conduct next theoretical simulations, referring to Chapter 4, for a case study of smaller dimensions for an arch of length 20μm, thickness 100 nm, width 850nm, and initial rise 250 nm, Figure 6.6. The figure shows the variation of the first two symmetric resonance frequencies as varying the electrothermal voltage; using the same thermal and mechanical parameters of the device under consideration in this work. The figure indicates that as tuning the electrothermal voltage, that is equivalent to a compressive load, both frequencies get very close to each other with a separation near 80 kHz. This demonstrates that the same concept can be applied to smaller devices. One should note here that to enable the veering phenomenon several parameters need to be carefully examined and chosen including the material properties, the nonlinearity from stress/strain, curvature, and the electrostatic force.
Figure 6.6: Variation of the first two symmetric resonance frequencies as tuning the electrothermal voltage for smaller arch of length 20 μm.

By increasing the electrothermal voltage for more than 2 V, Fig. 7, both resonance frequencies start to be further separated from each other. Figure 7 shows that the bandpass is destructed by the increase in the electrothermal voltage. However, the linear behavior is well-kept for the same excitation force.

Figure 6.7: Frequency responses (a) and phase responses (b) for \( V_{DC}=20 \) V, \( V_{AC}=20 \) V, and various electrothermal voltages.
6.2 Pressure sensor

6.2.1 Principle of operation

In a recent study [135] a new pressure-sensing technique was presented based on monitoring the variation of the resonance frequency of a locally heated beam resonator by an external laser source. Although a better sensitivity was demonstrated compared to conventional sensors, the performance of such devices depends on the resonator geometry and the external laser wavelength. Moreover, the resonant structure can only be miniaturized up to a certain level depending on the spot-size of the laser source used for the local heating.

Based on the cooling effect of the air surrounding an electrothermally heated bridge resonator, we propose an alternative pressure sensor that offers the flexibility of being scalable to small or large sizes. The operation principle of the sensor relies on tracking the change in the resonance frequency of the resonant structure with pressure while actuated electrothermally.

6.2.2 Experimental setup

In this work, we explore two different kinds of bridge resonators, straight and intentionally curved beams. These micro-resonators are fabricated on a highly conductive 30μm Si device layer of a silicon-on-insulator (SOI) wafer by two-mask process using standard photo-lithography, electron beam evaporation, deep reactive ion etch (DRIE), and vapor hydrofluoric acid (vpour-HF) etch [150]. The straight and arch beams have length of 700 μm, width of 30 μm, and thickness of 3 μm. The arch beam has a 2.6 μm initial rise. A drive electrode is provided with an AC actuation signal, and the resonant beam is biased
with a DC voltage source. The output current induced at the sense electrode due to the microbeam motion is amplified by a low-noise amplifier (LNA), which is then fed back to a network analyzer (Agilent 5071C), Figure 6.8, for transmission signal measurements.

![Diagram of experimental setup](image)

Figure 6.8: Schematic of the experimental setup for the proposed pressure sensor.

A separate DC voltage source $V_{Th}$ is connected to the beam anchors to induce a constant current flowing through the beam heating it up by Joule's heating. This causes thermal expansion that controls the internal axial stress on the beam, which also changes its stiffness and resonance frequency.

### 6.2.3 Results and discussion

Figure 6.9a shows the measured resonance frequency of a straight microbeam as it continuously decreases while increasing the electrothermal load due to the decrease in its stiffness until reaching the buckling limit, as studied in Chapter 2. After buckling, a sudden jump in the resonance frequency occurs due to the increase of the buckled beam stiffness. For an arch MEMS resonator, on the other hand, Figure 6.9b, the resonance frequency increases monotonically with the electrothermal load, reaching more than twice its initial
value, due to the continuous increase in the arch curvature, and hence stiffness, as studied in Chapter 2.

Figure 6.9: (a) Variation of the resonance frequency of the straight beam (with dimensions: 700 μm length, width =30 μm, thickness =3 μm, and transduction gap =8 μm) with the electrothermal voltage. The inset is a schematic of the clamped-clamped straight beam. (b) Variation of the resonance frequency of the arch beam (with dimensions: 700 μm length, width =30 μm, thickness =3 μm, initial rise at the midpoint =2.6 μm, and transduction gap =12 μm) with the electrothermal voltage. The inset is a schematic of the arch beam.

The device was placed inside a controllable pressure chamber equipped with a pressure gauge. Upon changing pressure while maintaining constant $V_{Th}$, the cooling effect of the surrounding air changes, thereby changing the overall stress of the structure, and thus its resonance frequency. This presents a way to measure the surrounding pressure by monitoring the changes in the resonance frequency of the microbeam. Thus, unlike previous studies [135], the geometrical size of the resonant structure can be scaled without compromising the performance of the sensor.
Since for doped silicon the dependence of the resistance on temperature is assumed to be linear, the variation of the resistance with pressure can be related to the variation of temperature along the beam. Figure 6.10a displays the change of resistance of the 700 μm straight beam, measured using an LCR meter, and shows as expected the decrease of the beam resistance with pressure while maintaining $V_{Th}$ constant [155]. By increasing pressure, the volume of air surrounding the beam provides sufficient cooling reducing temperature along the beam, which linearly varies the beam resistance. This validates experimentally the cooling down effect of the surrounding air on the beam while varying pressure.

To investigate the effect of damping on the resonance frequency while changing pressure, we monitored the change in resonance frequency at $V_{Th}$=0 V for both straight and arch beams, plotted in Figures 6.10b and 6.10c, respectively. As seen, the damping effect on the resonance frequency is negligible. Since the behavior of the straight beam is different before and after buckling, Figure 6.9a, two case studies are presented to study the sensitivity of the resonator in Figure 6.10b. Starting by increasing $V_{Th}$, while maintaining the beam before buckling at low pressure, the resonance frequency increases with pressure at constant $V_{Th}$. For $V_{Th}$=1.102 V, the resonance frequency increases by almost 100% from its initial value at low vacuum for pressure varying from low vacuum to 10 Torr. However, while choosing a $V_{Th}$ leading to buckling the beam at vacuum, the resonance frequency decreases at first with pressure until reaching the buckling point, and then it starts to increase since the beam is no longer buckled. Note that tracking the buckling point can be an independent mechanism to sense pressure as well.
To avoid the dip in frequency near the buckling instability, we conduct the same experiments using arch beams. Upon changing pressure while maintaining $V_{Th}$ constant, air cools down the beam due to the heat dissipation, then decrease its stiffness, and thus decreases its resonance frequency. Figure 6.10c shows that the normalized shift of the frequency with pressure for the arch beam resonator is highly dependent on and sensitive to the applied $V_{Th}$. By choosing a $V_{Th}$ in the range of high slope as shown in Figure 6.9c, the sensitivity of the arch resonator is found to be much higher.

Figure 6.10: (a) The normalized percentage of resistance change of the straight beam with pressure for different $V_{Th}$, with respect to the reference resistance for each $V_{Th}$ at 6 mTorr. (b) The normalized percentage shift of resonance frequency with pressure for different $V_{Th}$. 
The inset shows the enlarged view of the normalized percentage shift of resonance frequency for $V_{Th} = 1.102$ V and 1.3 V, between 1- to 10 Torr. (c) The normalized percentage shift of resonance frequency with pressure for different $V_{Th}$ for the arch beam. The inset shows the enlarged view of the normalized percentage shift of resonance frequency for $V_{Th} = 2.35$ V, between 1- to 10 Torr. (d) The normalized percentage shift of resonance frequency with pressure for different $V_{Th}$ for a thinner and longer arch beam of length $= 800$ μm, width $= 30$ μm, thickness $= 2$ μm, and initial rise $= 2$ μm. For (b) and (c), the difference in resonance frequency was calculated by taking the difference between the frequency at 6 mTorr (80 mTorr for figure (d)) and the values of frequency for different pressure values.

The normalized percentage shifts of resonance frequency with pressure for different $V_{Th}$ for straight and arch beams are shown in Figures 6.10b and 6.10c. The difference in resonance frequency was calculated by taking the difference between the frequency at 6 mTorr and the values of frequency for different pressure. The sensitivity to pressure of the arch resonator, in the range from 1 to 10 Torr, is 7862 ppm/mbar at $V_{Th} = 2.3$ V, when normalized with the frequency at 1 Torr, Figure 6.10c. Nevertheless, for a thinner and longer arch resonator, of length $= 800$ μm, width $= 30$ μm, thickness $= 2$ μm, and initial rise $= 2$ μm, this value increases from 12655 ppm/mbar to 24634 ppm/mbar while increasing $V_{Th}$ from 0.3 to 0.6 V, as shown in Figure 6.10d. The sensitivity of the proposed sensor for the 800 μm arch resonator at $V_{Th} = 0.6$ V is 8 times higher than the normalized sensitivity of the microdiaphragm-based pressure sensor, which was equal to 3256 ppm/mbar [156].
However, the frequency response to pressure for the 700 μm straight beam, in the range of pressure from 1 to 10 Torr, is greater compared to the arch beam, when normalized with the frequency at 1 Torr, Figure 6.10b. For $V_{th} = 1.102$ V, when the beam is not yet buckled, the sensitivity is equal to 51390 ppm/mbar. Moreover, these values are 17 times higher than the normalized sensitivity[156]. It is clear that the straight beam around buckling is more sensitive than the arch beam. This is originated from the high sensitivity of the beam to a small variation of its stiffness in that region. Note from Figures 6.10d-6.10d that the percentage resonance changes are much larger than the percentage change of resistance, Figure 6.10a; indicating that monitoring resonances leads to higher sensitivity. Note here that using either method (frequency or resistance monitoring); will yield the same range since both resistance and resonance frequency variations are due to the same effect; cooling of the heated beam. It is imperative to note also that no hysteresis was shown experimentally as varying pressure from low to high values and from high to low values, Figure 6.11.
Figure 6.11: Comparison of the variation of the resonance frequency of the straight beam as varying pressure from low to high and from high to low values showing no hysteresis behavior at $V_{Tn}=1.102$ V.

The total power consumption for the proposed sensor is the combination of the electrostatic drive power for the resonator (in the order of nW[150]) and the electrothermal power (in the order of mW[150]). Thus, the overall power consumption is dominated by the electrothermal voltage; increasing the resistance of the microbeam will considerably reduce it. For the 700 $\mu$m straight beam, an electrothermal voltage of $V_{Tn}=1.102$ V results in a power consumption around 3.55 mW.

One should mention here that the operation speed of the proposed sensor is limited by the thermal time constant, which is around 206.26 $\mu$s [68] for the 700 $\mu$m straight beam. Hence, the sensor will give accurate readings within milliseconds intervals. To decrease the thermal time constant for faster operation, the sensor needs to be further miniaturized, for instance to the nano and sub micro scale.

Another factor to be considered is the sensitivity of the device to temperature variations from the environment. To study this effect, we conducted experiments in which the ambient temperature surrounding the beam is varied within $\pm 10$ °C. We found that the temperature coefficient of frequency TCF [157] of the studied silicon device varies around $\pm 220$ ppm/°C (i.e., a temperature change of $\pm 10$ °C yields $\pm 115$ Hz change). Hence, the variation of the ambient temperature in this interval should not affect the operation and the calibration of the sensor. More experiments will be needed if the device is intended to be operated in an environment where the temperature variations exceed this range.
Also, one should note that the device sensitivity is dependent on the applied electrothermal voltage. A higher $V_{th}$ heats the beam more and hence more air volume (more pressure) is needed to cool down the beam. However, we should mention here that the sensed pressure range of the proposed device may not exceed 30 Torr. This limitation could be overcome by using other structure designs and geometry and using a material with higher thermal expansion. This might be perceived as a limitation of the proposed sensor, which can be used as a moderate vacuum pressure sensor. Another limitation is the quality factor that decreases with pressure, which can affect the proposed sensor resolution. Thus, for the final implementation of the proposed sensor, all these concerns need to be taken into consideration and further investigated.
Chapter 7  Summary, conclusions and future work

7.1  Summary and conclusions

We studied in Chapter 2 the combined effect of the mid-plane stretching and the electrostatic force on the variation of the natural frequencies of the electrostatically actuated straight and curved micro and nano-beams. Through careful designs of the gap-to-thickness ratios, significant tuning ranges have been presented for both straight and curved beams. For the straight beams, we found that the microbeam resonator is highly tunable for medium and high gap-to-thickness ratios at high values of DC voltages. For high gap-to-thickness ratios, the effect of mid-plane stretching due to the deformation of the beam under the DC load overpasses the effect of the electrostatic force and then the natural frequency is shifted for higher values as increasing the DC voltage, which can lead to a rise of more than 100%. For curved nanobeams, the beams have shown high tunability by electrostatic actuation due to the dominant mid-plane stretching effect over the softening effect exerted by the electrostatic force. We interpret that this mid-plane stretching effect arises due to the combined effect of the gap-to-thickness ratio and the initial curvature of the nanobeams. An increase of the resonance frequency by 108.14% is reported. It is demonstrated that the curvature is essential to reach such tunability and to increase the pull-in voltage. The high tunability of such beams, straight and curved beams, may open up the possibilities for tunable bandpass filtering in the high frequency range.
Next, we investigated in Chapter 3 experimentally and analytically the tunability of in-plane clamped-clamped microbeam, straight and initially curved, using electrothermal actuation in addition to the electrostatic actuation. For the straight microbeam case, we showed that as increasing the DC electrothermal voltage the microbeam buckles, due to the stiffness change via Joule’s heating, after a certain critical electrothermal voltage. Before buckling, the fundamental frequency decreases until the resonance frequency drops to very low values (almost zero). After buckling, the resonance frequency increases to high values, which can reach the double of the original resonance frequency. Then, we added a DC electrostatic bias in addition to the electrothermal actuation. The microbeam encounters a perturbed pitchfork bifurcation due to DC polarization voltage. We showed that the dip in the resonance frequency before bucking is reduced and the resonance frequency after buckling is increased as increasing the DC polarization voltage. For the arch microbeam case, we demonstrated that the arch beam increases its curvature as increasing the DC thermal voltage in the same direction of its initial curvature. We showed that the first natural frequency of the initially curved beams could be tuned for more than 100% as increasing the electrothermal voltage until reaching specific voltage after which it starts to saturate. We reported the possibility of achieving a monotonic and continuous increase in the resonance frequency for certain initial curvature values. The third natural frequency was shown to be decreased as increasing the electrothermal voltage and then it starts to rise when the first natural frequency begins to saturate. Hence, we demonstrated that a single resonator electrothermally and electrostatically actuated can be operated at a wide range of resonance frequency, as low as almost zero frequency to as high as twice of its unactuated resonance frequency by only controlling the electrothermal and the
electrostatic voltages. We also showed that after exceeding a certain electrothermal voltage, the third mode could be made to be as sensitive as the first mode for the voltage load, and hence, it becomes highly tunable as the first mode.

Then, we showed in Chapter 4 that by choosing carefully the geometric parameters of an initially curved arc beam, the veering phenomenon can be activated. Veering occurs when two frequencies, the first two symmetric modes in our case, approach each other as varying a control parameter, the stiffness induced by the electrothermal voltage in our case, due to a linear coupling between two modes and then deviate away from each other, and each one continues along the path that the other would have taken if they would cross. We studied theoretically, using the reduced order model, and experimentally the dynamic behavior of such resonator before, at, and after veering. In the veering regime, the first and third modes exchange energy. After veering, we demonstrated that the third mode takes the nonlinear behavior of the first mode before veering and starts to be more sensitive than the first mode. We showed that the ROM could capture most of the dynamical behavior; however suffers near the veering regime, potentially due to the activation of the rotational modes near this regime. As carefully designing the curved structure, the veering phenomenon could be exploited in various applications, such as sensing and filtering.

Next, we investigated the nonlinear mode coupling between the first two symmetric modes of an initially curved MEMS resonators via different types of internal resonances. The curved beam is electrothermally tuned and electrostatically driven. While maintaining a constant electrothermal voltage, the stiffness, and hence the frequencies, of the arch beam is monitored, and different commensurate ratios among various modes could be attained. As dynamically driving the arch beam at a fixed ratio, the storing of the energy of the
contributing mode was demonstrated for different types of internal resonances: two-to-one, three-to-one, and four-to-one. As driving the arch resonator in the interaction regime, an improvement of the frequency stabilization is demonstrated. Next, we focused on studying theoretically and experimentally the two-to-one internal resonance of an arch resonator. We investigated the two-to-one internal resonance considering the influence of the direct excitation of the third resonance frequency. Theoretically, we solved the dynamic response of the arch beam using a reduced-order model combined with longtime integration of the equation of motion. As exciting the arch beam electrostatically at a constant electrothermal voltage, when the ratio between both frequencies equals to two, we investigated the dynamic response and the activation of the internal resonance. We showed that at high AC bias voltage, a chaotic motion might appear. We also showed that due to the high AC voltages, the nonlinear interaction is influenced by the simultaneous excitation of the two modes, 1\textsuperscript{st} and 3\textsuperscript{rd} modes, as well as the internal resonance. Other methods can be used to characterize the stability of the system, such as the multiple scale technique (perturbation method) and the Floquet theory combined with continuation techniques. These studies motivate further research in this direction to exploit internal resonances of arch resonators for practical applications, such as sensors and mechanical amplifier. Otherwise, this internal resonance should be taken into consideration to avoid such phenomena once designing a system.

Finally, we demonstrated the ability to exploit the tunability and the linear coupling via different modes into different applications motivated by the results presented in Chapters 3 and 4. First, we experimentally demonstrate a bandpass filter by exploiting the nonlinear hardening, softening, and veering phenomena in MEMS arches. The arches were
electrothermally tuned and electrostatically actuated. At some critical electrothermal voltages, the first and the third resonance frequencies get close to each other and get coupled by veering, which results in a bandpass filter with flat passband and wide bandwidth. In conclusion, we demonstrated that a single arch resonator could be potentially used as a tunable bandpass filter by a simple electrothermal frequency modulation scheme. On the other hand, we investigated a moderate vacuum pressure sensor relying on convective cooling of the air surrounding heated MEMS resonators, straight and initially curved beams. The beams were electrothermally tuned and electrostatically driven. While maintaining a constant electrothermal voltage, pressure is tracked by the cooling effect. High sensitivity of the proposed sensor was demonstrated. The advantages of the proposed sensor are the simplicity of fabrication, operation, and sensing scheme, lower power consumption, and also scalability. The proposed sensor can be scaled to larger (millimeter) or smaller (nanometer) sizes, depending on the aimed application, using the standard fabrication process and using the same mechanism. Although the proposed sensor is made of silicon, known for its good mechanical and thermal properties, we expect that the sensitivity of the pressure sensor could be further improved by choosing another material of optimized thermal and electrical properties.

7.2 Future work

The research work presented in this dissertation can be extended in various directions. The main extension works are the following two points.

The first direction of this work is to investigate experimentally and theoretically the linear and nonlinear mode coupling between symmetric and antisymmetric modes as they
are tuned electrothermally. Indeed, as tuning the stiffness of an initially curved beam, the first symmetric mode increases while the first antisymmetric mode decreases until they cross. In the present work, we did not include the effect of the antisymmetric modes since a full electrode configuration is used to excite the arch beam electrostatically. Hence, we can use an antisymmetric partial electrode in order to activate both modes of vibrations. Then, we can investigate and study the linear coupling and the activation of one-to-one internal resonance as the first symmetric and antisymmetric modes cross. This might promise rich and complex dynamics.

The second direction of this work is to develop a gas sensor based on exciting and tracking the frequency shift in the first and second vibration modes (first symmetric and antisymmetric modes) of an electrothermally heated bridge resonator. As increasing the electrothermal voltage, the first symmetric and antisymmetric frequencies of a straight beam are known to show a continuous decrease as its stiffness decreases due to the induced compressive stress until reaching buckling. After buckling, a sudden increase in the first resonance frequency is expected (as shown in Chapter 6, Section 6.2) due to the increase of stiffness of the buckled microbeam. However, the second mode should remain constant since it is purely bending mode and it is not affected with the stretching. Exposing the resonator to gases with higher/lower thermal conductivity compared with Nitrogen reduces/increases the axial stress inside the microbeam, which alters the values of the resonance frequencies. Hence, operating the resonator near the buckling point might maximize the sensitivity to variation in the first mode frequency to physical phenomena that affect the axial stress such as cooling or and heating effects as shown in Chapter 6, Section 6.2. By tracking the shift in the second mode, we can identify the type of gas.
Another extension of this dissertation is by revisiting the second section of Chapter 6 and perform and theoretical analysis in order to optimize the sensor geometry to improve the sensitivity of the presented pressure sensor and to broaden the pressure range. Moreover, a read-out circuit might be developed to perform real-time tracking of the pressure alteration. On the other hand, different numerical techniques could be developed to characterize the nonlinear coupling of different modes of vibration as experiencing complex dynamic behavior such as chaotic motion. Hence, we can adopt the multiple scale technique (perturbation method) and the Floquet theory combined with continuation technique to more understand and theoretically characterize these complex dynamic features.
REFERENCES


