Asymptotic Analysis of Regularized Zero-Forcing in Double Scattering Channels

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***Abstract*—This paper studies the sum-rate performance of regularized zero-forcing (RZF) precoding in a multi-user multiple-input single-output (MISO) system, where the channel between the base station (BS) and each user is modeled by the double scattering channel model. This non-Gaussian channel accounts for both the spatial correlation in the antenna arrays and the structure of scattering in the propagation environment. The user population is divided into G groups, where the users in the same group experience similar propagation conditions and are characterized by common correlation matrices. Under this setting, we derive deterministic approximations of the signal-to-interference-plus-noise ratio (SINR) and the sum-rate with RZF precoding, which are almost surely tight in the large system limit. Simulation results confirm the close match provided by the asymptotic analysis for moderate system dimensions.**

I. Introduction

Pioneer works on multiple-input multiple-output (MIMO) systems have shown their capacity to scale linearly with the minimum number of transmit (Tx) and receive (Rx) antennas, provided that the fades between the Tx-Rx antenna pairs are independent and identically Rayleigh [1], [2]. Although the use of full rank Rayleigh channel matrices facilitates the derivation of closed-form capacity expressions, these models do not capture the characteristics of realistic propagation channels, which are likely to be non-Gaussian distributed [3], [4].

Many works have considered correlated Rayleigh fading channels [5]–[7] in the last decade to study the impact of antenna correlation on the performance of MIMO systems. However, low rank channels have been observed in systems with low antenna correlation [8], leading to the realization that MIMO capacity is governed by both the spatial correlation at the transmission ends and the structure of scattering in the propagation environment. Motivated by this, Gesbert *et al* devised a “double-scattering channel model” in [3], which utilizes the geometry of the propagation environment to model spatial correlation, limited scattering and rank deficiency. A special case of this is the keyhole channel [4], [9], which exhibits null correlation between the entries of the channel matrix but only a single degree of freedom.

Several theoretical works have investigated the double scattering model since its introduction in 2002. The authors in [10] analyzed the ergodic MIMO capacity taking into account the presence of spatial correlation, double scattering, and keyhole effects in the propagation environment and showed that the use of multiple antennas in keyhole channels only offers diversity gains, but no spatial multiplexing gain. The authors in [11] studied the diversity order of the double scattering model and showed that a MIMO system with t Tx and r Rx antennas and s scatterers achieves the diversity of order $trs/\max(t,r,s)$. The MIMO multiple access channel (MAC) with double-scattering fading was studied in [12], where the authors obtained closed-form upper-bounds on the sum-capacity and proved that signals sent along the eigenvectors of the Tx correlation matrix maximize capacity.

The non-Gaussian nature of the double scattering channel makes the development of tractable closed-form expressions for different performance metrics very hard. To overcome this challenge, the authors in [13] studied the performance of MIMO MAC with double-scattering in the asymptotic regime and derived almost surely tight deterministic approximations of the mutual information and the signal-to-interference-plus-noise ratio (SINR) of the minimum-mean-square error (MMSE) detector. To the best of the authors’ knowledge, the performance of double scattering channel model in a multi-user multiple-input single-output (MISO) setting utilizing a linear precoding scheme has not been studied so far.

This work considers the downlink of a single-cell large-scale MISO system in which the base station (BS) makes use of regularized zero-forcing (RZF) precoding to communicate with the users, who are divided into G groups in a way that the users in the same group experience similar scattering conditions and are therefore characterized by a common correlation matrix between the BS antennas and the Tx scatterers and a common correlation matrix between the Tx and Rx scatterers. Under the assumption that the number of BS antennas, scatterers and users grow infinitely large while their ratio remains bounded, we derive almost surely tight deterministic approximations of the SINR of each user and the sum-rate under RZF precoding. Compared to the Rayleigh fading channel, the double-scattering model makes the asymptotic analysis much more involved and we resort to recent results from random matrix theory (RMT) to overcome this challenge [13]. The derived approximations are expressed in a closed-form for the special case of Rayleigh product channel. Simulation results show that our analysis is quite accurate for moderate system dimensions as well.

The rest of the paper is organized as follows. Section II

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presents the system and channel model. In section III, the deterministic approximations of the SINR and sum rate under RZF precoding are derived. Simulation results are provided in Section IV and section V concludes the paper.

II. System Model

This section outlines a multi-user MISO system with flat-fading double-scattering channels between the BS and users.

A. Transmission Model

Consider a multi-user MISO system composed of a central BS equipped with N antennas serving K non-cooperative single-antenna users. The K users are divided into G groups of K_g, $g = 1, \dots, G$, co-located users such that the users in the same group experience similar propagation conditions. For a narrow-band transmission model, the signal $y_{k,g}$ received by user k in group g is given as,

$$y_{k,g} = \mathbf{h}_{k,g}^H \mathbf{x} + n_{k,g}, \quad k = 1, \dots, K_g, \; g = 1, \dots, G, \quad (1)$$

where $\mathbf{h}_{k,g} \in \mathbb{C}^{N \times 1}$ is the channel vector from the BS to user k in group g, $\mathbf{x} \in \mathbb{C}^{N \times 1}$ is the Tx vector and $n_{k,g} \sim \mathcal{CN}(0, \sigma^2)$ is the noise term with variance σ^2.

The Tx vector $\mathbf{x}$ is given by,

$$\mathbf{x} = \sum_{g=1}^{G} \sum_{k=1}^{K_g} \sqrt{p_{k,g}} \mathbf{g}_{k,g} s_{k,g}, \quad (2)$$

where $\mathbf{g}_{k,g} \in \mathbb{C}^{N \times 1}$ is the precoding vector for user k in group g, and $p_{k,g} \geq 0$ and $s_{k,g}$ are the signal power and the data symbol for user k in group g respectively. The precoding vectors satisfy the average total power constraint as,

$$\mathbb{E}[||\mathbf{x}||^2] = \text{tr } (\mathbf{P}\mathbf{G}^H \mathbf{G}) \leq \bar{P}, \quad (3)$$

where $\bar{P}$ is the total Tx power, $\mathbf{P} = \text{diag}(p_{1,1}, p_{2,1}, \dots, p_{K_1,1}, p_{1,2}, \dots, p_{K_G-1,G}, p_{K_G,G}) \in \mathbb{R}^{K \times K}$ and $\mathbf{G} = [\mathbf{G}_1, \mathbf{G}_2, \dots, \mathbf{G}_G] \in \mathbb{C}^{N \times K}$ is the precoding matrix, where $\mathbf{G}_g = [\mathbf{g}_{1,g}, \dots, \mathbf{g}_{K_g,g}] \in \mathbb{C}^{N \times K_g}$.

Under the assumption of Gaussian signaling, i.e., $s_k \sim \mathcal{CN}(0, 1)$ and perfect channel state information at the BS and the users, the SINR $\gamma_{k,g}$ of user k in group g is defined as,

$$\gamma_{k,g} = \frac{p_{k,g} |\mathbf{h}_{k,g}^H \mathbf{g}_{k,g}|^2}{\mathbf{h}_{k,g}^H \mathbf{G}_{[k,g]} \mathbf{P}_{[k,g]} \mathbf{G}_{[k,g]}^H \mathbf{h}_{k,g} + \sigma^2} \quad (4)$$

where $\mathbf{G}_{[k,g]} = [\mathbf{G}_1, \dots, \mathbf{G}_{g-1}, \mathbf{g}_{1,g}, \dots, \mathbf{g}_{k-1,g}, \mathbf{g}_{k+1,g}, \dots, \mathbf{g}_{K_g,g}, \mathbf{G}_{g+1}, \dots, \mathbf{G}_G] \in \mathbb{C}^{N \times K-1}$ and $\mathbf{P}_{[k,g]} = \text{diag}(p_{1,1}, p_{2,1}, \dots, p_{K_{g-1},g-1}, p_{1,g}, \dots, p_{k-1,g}, p_{k+1,g}, \dots, p_{K_g,g}, \dots, p_{K_G,G}) \in \mathbb{C}^{K-1,K-1}$.

B. Double-Scattering Channel Model

The double-scattering channel model considered in this work provides non-Gaussian channels between the BS and the users with ranks that are determined by both the spatial correlation between the antennas at the BS and the structure of scattering in the propagation environment, and thus is a way to address the limitations of the uncorrelated Rayleigh fading as well as the correlated Rayleigh fading models. The expression

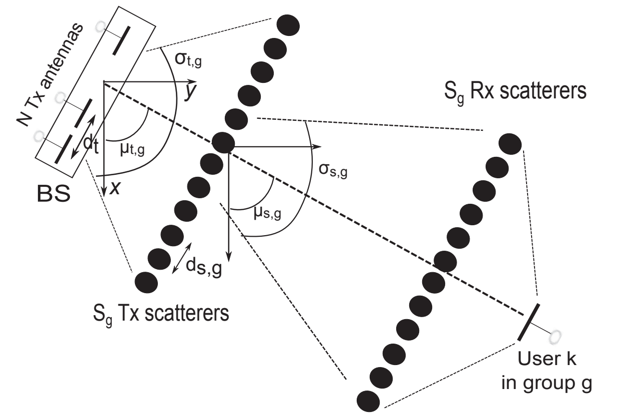

Fig. 1. Geometric model of the double scattering channel between the BS and the user k in group g.

of the channel vector $\mathbf{h}_{k,g}$ described by the double scattering model is given as [3],

$$\mathbf{h}_{k,g} = \sqrt{S_g} \left(\frac{1}{\sqrt{S_g}} \mathbf{R}_{BS_g}^{1/2} \mathbf{W}_g \bar{\mathbf{S}}_g^{1/2} \right) \tilde{\mathbf{w}}_{k,g}, \quad (5)$$

where S_g is the number of scatterers at the Tx and the Rx sides in group g, $\mathbf{R}_{BS_g} \in \mathbb{C}^{N \times N}$ is the correlation matrix between the BS antennas and the S_g Tx scatterers in group g, $\bar{\mathbf{S}}_g \in \mathbb{C}^{S_g \times S_g}$ is the correlation matrix between the Tx and Rx scatterers in group g, $\mathbf{W}_g \sim$ i.i.d. $\mathcal{CN}(0, 1) \in \mathbb{C}^{N \times S_g}$ describes the small-scale fading between the BS and the scattering cluster at the Tx side in group g and $\tilde{\mathbf{w}}_{k,g} \sim$ i.i.d. $\mathcal{CN}(0, \frac{1}{S_g}) \in \mathbb{C}^{S_g \times 1}$ describes the small-scale fading between the user k in group g and the scattering cluster at the Rx side. Since the distributions of $\mathbf{W}_g$ and $\tilde{\mathbf{w}}_{k,g}$ are unitarily invariant, we can assume $\bar{\mathbf{S}}_g$ to be diagonal, i.e. $\bar{\mathbf{S}}_g = \text{diag}(\bar{s}_{1,g}, \bar{s}_{2,g}, \dots, \bar{s}_{S_g,g})$ without any loss of generality for the statistics of the received signal.

Schematic of the double scatting channel from the BS to user k in group g has been illustrated in Fig. 1, where $\sigma_{t,g}$ and $\sigma_{s,g}$ represent the angular spread of the radiated signal from the BS array and the Tx scatterers respectively in group g, $\mu_{t,g}$ and $\mu_{s,g}$ determine the mean angle of departure (AoD) of the radiated signal from the BS array and the Tx scatterers respectively in group g, where $\mu_{t,g} = \mu_{s,g}$, and d_t and $d_{s,g}$ determine the spacing between the adjacent antennas at the BS and between the adjacent scatterers in group g respectively.

III. Main Results

This section considers RZF precoding and presents the deterministic approximations of the SINR and the sum-rate for the double-scattering channel model.

A. Regularized Zero-Forcing Precoding

RZF is a state-of-the-art heuristic precoding scheme with a simple closed-form expression given as [5], [14],

$$\mathbf{G} = \zeta (\mathbf{H}^H \mathbf{H} + K\alpha \mathbf{I}_N)^{-1} \mathbf{H}^H, \quad (6)$$

where $\mathbf{H}$ is the compound channel defined as $[\mathbf{H}_1^H, \mathbf{H}_2^H, \ldots, \mathbf{H}_G^H]^H \in \mathbb{C}^{K \times N}$, where $\mathbf{H}_g = [\mathbf{h}_{1,g}, \mathbf{h}_{2,g}, \ldots, \mathbf{h}_{K_g,g}]^H \in \mathbb{C}^{K_g \times N}$, α is the regularization parameter and ζ is a normalization factor to ensure that the power constraint in (3) is satisfied and is obtained as,

$$\zeta^2 = \frac{\bar{P}}{\text{tr}\mathbf{P}\mathbf{H}(\mathbf{H}^H\mathbf{H} + K\alpha\mathbf{I}_N)^{-2}\mathbf{H}^H} = \frac{\bar{P}}{\Theta}, \tag{7}$$

where $\Theta = \text{tr}\mathbf{P}\mathbf{H}\mathbf{V}^2\mathbf{H}^H$ and $\mathbf{V} = (\mathbf{H}^H\mathbf{H} + K\alpha\mathbf{I}_N)^{-1}$. The SINR for the user k in group g in (4) is now defined as,

$$\gamma_{k,g\text{RZF}} = \frac{p_{k,g}|\mathbf{h}_{k,g}^H\mathbf{V}\mathbf{h}_{k,g}|^2}{\mathbf{h}_{k,g}^H\mathbf{V}\mathbf{H}_{[k,g]}^H\mathbf{P}_{[k,g]}\mathbf{H}_{[k,g]}\mathbf{V}\mathbf{h}_{k,g} + \frac{\Theta}{\rho}}, \tag{8}$$

where $\rho = \frac{\bar{P}}{\sigma^2}$, $\mathbf{h}_{k,g}$ is given by (5) and $\mathbf{H}_{[k,g]} = [\mathbf{H}_1^H, \ldots, \mathbf{H}_{g-1}^H, \ \mathbf{h}_{1,g}, \ldots, \mathbf{h}_{k-1,g}, \mathbf{h}_{k+1,g}, \ldots, \mathbf{h}_{K_g,g}, \mathbf{H}_{g+1}^H, \ldots, \mathbf{H}_G^H]^H \in \mathbb{C}^{K-1 \times N}$.

B. Deterministic Equivalent

In order to derive a deterministic equivalent of the SINR under RZF precoding and the double scattering channel in (5), we make the following two assumptions.

A-1. For all g, N, S_g, K_g and K tend to infinity such that,

$$0 < \lim\inf \frac{S_g}{N} \leq \lim\sup \frac{S_g}{N} < \infty, \tag{9}$$

$$0 < \lim\inf \frac{K_g}{N} \leq \lim\sup \frac{K_g}{N} < \infty, \tag{10}$$

$$0 < \lim\inf \frac{K_g}{K} \leq \lim\sup \frac{K_g}{K} < \infty. \tag{11}$$

In the sequel, the notation $N \to \infty$ denotes Assumption **A-1**.

A-2. For all g, $\lim\sup_N ||\mathbf{R}_{BS_g}|| < \infty$ and $\lim\sup_{S_g} ||\bar{\mathbf{S}}_g|| < \infty$, where $||.||$ is the spectral norm.

The derivation of the deterministic equivalent in this paper interprets the double-scattering channel model in (5) as,

$$\mathbf{h}_{k,g} = \sqrt{S_g}\mathbf{Z}_g\tilde{\mathbf{w}}_{k,g}, \tag{12}$$

where

$$\mathbf{Z}_g = \frac{1}{\sqrt{S_g}}\mathbf{R}_{BS_g}^{1/2}\mathbf{W}_g\bar{\mathbf{S}}_g^{1/2}. \tag{13}$$

This is essentially a correlated Rayleigh fading channel model [14] but with a random Tx correlation matrix $\mathbf{Z}_g$, which can be represented as a Kronecker model. The idea is to first assume $\mathbf{Z}_g$ to be deterministic and obtain the deterministic equivalent of the SINR for the channel in (12) in terms of certain fixed point equations that depend on the matrices $\mathbf{Z}_g$, using a RMT result from [15]. Next, we obtain the deterministic equivalent of these fixed point equations under the actual random $\mathbf{Z}_g$'s.

The resulting asymptotically tight deterministic approximation of the SINR under double-scattering fading and RZF precoding is provided in following theorem.

Theorem 1: Under the setting of assumptions **A-1** and **A-2** and for $\alpha > 0$, the SINR of user k in group g defined in (8) converges almost surely as,

$$\gamma_{k,g\text{RZF}} - \gamma_{k,g\text{RZF}}^o \xrightarrow[N\to\infty]{a.s.} 0, \tag{14}$$

where,

$$\gamma_{k,g\text{RZF}}^o = \frac{p_{k,g}m_g^2}{\Upsilon_{k,g}^o + \frac{\Theta^o(1+m_g)^2}{\rho}}, \tag{15}$$

where,

$$\Upsilon_{k,g}^o = \frac{1}{K_g}\sum_{\substack{g'=1 \\ g'\neq g}}^{G}\sum_{k'=1}^{K_{g'}}\frac{p_{k',g'}M'_{g',g}}{(1+m_{g'})^2} + \frac{1}{K_g}\sum_{\substack{j=1 \\ j\neq k}}^{K_g}\frac{p_{j,g}M'_{g,g}}{(1+m_g)^2}, \tag{16}$$

$$\Theta^o = \frac{1}{K}\sum_{g=1}^{G}\sum_{k=1}^{K_g}\frac{-p_{k,g}m'_g}{(1+m_g)^2}, \tag{17}$$

and $(m_g, \bar{m}_g, \delta_g)$ form the unique positive solution of,

$$m_g = \frac{1}{K_g}\sum_{j=1}^{S_g}\frac{\frac{K_g}{K}\bar{s}_{g,j}\delta_g}{1 + \frac{K_g}{K}\bar{s}_{g,j}\bar{m}_g\delta_g},$$
$$\bar{m}_g = \frac{1}{1+m_g}, \tag{18}$$
$$\delta_g = \frac{1}{S_g}\text{tr }\mathbf{R}_{BS_g}\left(\sum_{i=1}^{G}\frac{\bar{m}_i m_i K_i \mathbf{R}_{BS_i}}{S_i\delta_i} + \alpha\mathbf{I}_N\right)^{-1},$$

for $g = 1, \ldots, G$.

Also $\mathbf{m}' = [m'_1, m'_2, \ldots, m'_G]^T$ and $\mathbf{M}' = \begin{bmatrix} M'_{1,1} & M'_{1,2} & \ldots & M'_{1,G} \\ \vdots & \vdots & \ddots & \vdots \\ M'_{G,1} & M'_{G,2} & \ldots & M'_{G,G} \end{bmatrix}$ are given by,

$$\mathbf{m}' = (\mathbf{I}_G - \mathbf{A}\mathbf{L})^{-1}\mathbf{A}\mathbf{v}, \tag{19}$$

$$\mathbf{M}' = (\mathbf{I}_G - \mathbf{A}\mathbf{L})^{-1}(\mathbf{A}\mathbf{V} + (\mathbf{I}_G - \bar{\mathbf{J}})^{-1}\bar{\mathbf{V}}), \tag{20}$$

where,

$$\mathbf{A} = (\mathbf{I}_G - \bar{\mathbf{J}})^{-1}\bar{\mathbf{L}}(\mathbf{I}_G - \mathbf{J})^{-1}, \tag{21}$$

$$[\mathbf{J}]_{g,i} = \frac{1}{S_g}tr(\mathbf{R}_{BS_g}\mathbf{T}\mathbf{R}_{BS_i}\mathbf{T})\frac{K_iS_im_i\bar{m}_i}{(S_i\delta_i)^2}, \tag{22}$$

$$[\mathbf{L}]_{g,i} = \frac{1}{S_g}tr(\mathbf{R}_{BS_g}\mathbf{T}\mathbf{R}_{BS_i}\mathbf{T})\left(\frac{K_im_i}{(1+m_i)^2S_i\delta_i} - \frac{K_i\bar{m}_i}{S_i\delta_i}\right), \tag{23}$$

$$[\bar{\mathbf{J}}]_{g,g} = \frac{K_g\delta_g^2}{K^2(1+m_g)^2}tr(\bar{\mathbf{S}}_g\mathbf{B}_g^2\bar{\mathbf{S}}_g), \tag{24}$$

$$[\bar{\mathbf{L}}]_{g,g} = \frac{1}{K}tr(\bar{\mathbf{S}}_g\mathbf{B}_g) - \frac{K_g}{K^2}\delta_g\bar{m}_gtr(\bar{\mathbf{S}}_g\mathbf{B}_g^2\bar{\mathbf{S}}_g), \tag{25}$$

$$\mathbf{v} = -\left[\frac{1}{S_1}tr(\mathbf{R}_{BS_1}\mathbf{T}^2), \ldots, \frac{1}{S_G}tr(\mathbf{R}_{BS_G}\mathbf{T}^2)\right]^T, \tag{26}$$

$$[\mathbf{V}]_{g',g} = \frac{1}{S'_g}tr(\mathbf{R}_{BS_{g'}}\mathbf{T}\mathbf{R}_{BS_g}\mathbf{T})\frac{K_gm_g}{S_g\delta_g}, \tag{27}$$

$$[\bar{\mathbf{V}}]_{g,g} = \frac{K_g}{K^2}\delta_g^2tr(\bar{\mathbf{S}}_g\mathbf{B}_g^2\bar{\mathbf{S}}_g), \tag{28}$$

$$\mathbf{T} = \left(\sum_{i=1}^{G}\frac{K_i\bar{m}_im_i\mathbf{R}_{BS_i}}{S_i\delta_i} + \alpha\mathbf{I}_N\right)^{-1}, \tag{29}$$

$$\mathbf{B}_i = \left(\mathbf{I}_{S_i} + \delta_i\bar{m}_i\frac{K_i}{K}\bar{\mathbf{S}}_i\right)^{-1}, \text{ for } g, g', i = 1, \ldots, G. \tag{30}$$

Proof: Sketch of proof of **Theorem 1** is provided in the appendix.

Remark: One can prove that $m_g, \bar{m}_g$ and δ_g can be computed using a classical fixed-point algorithm which recursively computes (18), starting from an arbitrary initialization $m_g, \bar{m}_g, \delta_g > 0$. The algorithm converges in a few iterations and does not pose any computational challenge. The proof of uniqueness of such solutions relies on arguments of the standard interference functions [see Definition 2 and Theorem 8 in [16]].

Corollary 1: Assume that **A-1** and **A-2** hold true. Then the individual rates $R_{k,g}$ of users converge almost surely as,

$$R_{k,g} - R_{k,g}^o \xrightarrow[N\to\infty]{a.s.} 0, \tag{31}$$

where,

$$R_{k,g}^o = \log(1+\gamma_{k,g\text{RZF}}^o), \tag{32}$$

and $\gamma_{k,g\text{RZF}}^o$ is given by (15).

Proof: The proof follows from the application of the continuous mapping theorem [17] to the logarithm function and the almost sure convergence of $\gamma_{k,g\text{RZF}}$ in (14).

An approximation of the average system sum rate can be obtained by replacing the instantaneous SINR $\gamma_{k,g\text{RZF}}$ with its asymptotic approximation as,

$$\bar{R}_{sum} = \sum_{g=1}^{G}\sum_{k=1}^{K_g} \log(1+\gamma_{k,g\text{RZF}}^o). \tag{33}$$

It follows that,

$$\frac{1}{K}(R_{sum} - \bar{R}_{sum}) \xrightarrow[N\to\infty]{a.s.} 0. \tag{34}$$

A special case of the double-scattering channel is the Rayleigh product channel which does not exhibit any form of correlation between the Tx antennas or the scatterers [18]. For this model, **Theorem 1** can be given in a closed-form as shown in the next corollary.

Corollary 2: For $G=1$, let $S_1 = S$, $K_1 = K = N$ and assume $\bar{\mathbf{S}}_1 = \mathbf{I}_S$ and $\mathbf{R}_{BS_1} = \mathbf{I}_N$. Then $\gamma_{k\text{RZF}}^o$ defined in **Theorem 1** can be given in a closed-form as,

$$\gamma_{k\text{RZF}}^o = \frac{\frac{p_k}{P/K}(1-\bar{m})^2}{(1-\frac{p_k}{P})\bar{m}^4 M' - \frac{\bar{m}^2 m'}{\rho}}, \tag{35}$$

where,

$$m' = \frac{S(\bar{m}-1)}{N\bar{m}^2(3\bar{m}^2 - 2(1-\frac{S}{N})\bar{m} + \frac{S\alpha}{N})}, \tag{36}$$

$$M' = \frac{(1+\frac{S}{N})\bar{m}^2 + (\frac{2S\alpha}{N} + \frac{S}{N} - 1)\bar{m} - \frac{2S\alpha}{N}}{\bar{m}^2\left((1-\frac{S}{N})\bar{m}^2 - \frac{2S\alpha}{N}\bar{m} + \frac{3S\alpha}{N}\right)}, \tag{37}$$

and $\bar{m}$ is given as the unique root to,

$$\bar{m}^3 - \left(1-\frac{S}{N}\right)\bar{m}^2 + \frac{S\alpha}{N}\bar{m} - \frac{S\alpha}{N} = 0, \tag{38}$$

such that $\bar{m} \in \left(1-\frac{S}{N}, 1\right)$.

Proof: Sketch of proof of *Corollary 2* is provided in the appendix.

Corollary 3: Under the setting of Corollary 2, let $\frac{S}{N} \to \infty$. Then $\gamma_{k\text{RZF}}^o$ defined in **Theorem 1** can be given in a closed-form as,

$$\gamma_{k\text{RZF}}^o = \frac{\frac{p_k}{P/K}(\frac{1}{\bar{m}} - 1)(1+\frac{\alpha}{\bar{m}^2})}{(1-\frac{p_k}{P}) + \frac{1}{\rho\bar{m}^2}}, \tag{39}$$

where,

$$\bar{m} = \frac{-\alpha + \sqrt{4\alpha + \alpha^2}}{2}. \tag{40}$$

Note that as $\frac{S}{N} \to \infty$, $\mathbf{h}$ behaves as a Rayleigh fading channel, whose SINR is given as (39). A similar result has also been obtained in Corollary 2 of [14].

IV. Simulation Results

We employ the fixed quantization method from [19] for user grouping, wherein the user population is partitioned into groups based on the criteria of the minimum chordal distance between the users' correlation eigenspaces and the group subspaces $\mathbf{V}_g \in \mathbb{C}^{N\times r_g}$, $g = 1,\ldots,G$ which are fixed and known apriori based on the characteristics of the propagation environment. The method relies on choosing G AoDs $\mu_{t,g}$ and angular spreads $\sigma_{t,g}$, such that the resulting G intervals $[\mu_{t,g} - \sigma_{t,g}, \mu_{t,g} + \sigma_{t,g}]$ are disjoint. Then the eigenspace corresponding to the r_g dominant eigenvalues of the constructed correlation matrix for group g, denoted as $\mathbf{R}_{BS_g}$, using the chosen values of $\mu_{t,g}$ and $\sigma_{t,g}$, constitutes the group subspace, where r_g is selected such that $\sum_{g=1}^G r_g = N$. The algorithm has three steps:

- For $g = 1,\ldots,G$, compute $\mathbf{V}_g$, the eigenspace corresponding to the r_g dominant eigenvalues of $\mathbf{R}_{BS_g}$.
- For $k = 1,\ldots,K$, compute the chordal distances,
$$d_c(\mathbf{U}_k, \mathbf{V}_g) = ||\mathbf{U}_k\mathbf{U}_k^H - \mathbf{V}_g\mathbf{V}_g^H||, \tag{41}$$
where $\mathbf{U}_k \in \mathbb{C}^{N\times r_k}$ is the eigenspace corresponding to r_k dominant eigenvalues of user k's channel correlation matrix, denoted as $\mathbf{R}_{user_k}$.
- Find the minimum distance group index as $g = \underset{g'}{\arg\min}\, d_c(\mathbf{U}_k, \mathbf{V}_{g'})$ and add user k to group g.

Under this grouping, we can assume that the users in the same group have channel covariance eigenspaces spanning (approximately) a given common subspace, which characterizes the group. Therefore, common correlation matrices, $\mathbf{R}_{BS_g}$ and $\bar{\mathbf{S}}_g$, are assumed for users in group g in the simulations in this section. Under the double scattering channel model, these correlation matrices are given as $\mathbf{R}_{BS_g} = \mathbf{G}(\mu_{t,g}, \sigma_{t,g}, d_t, S_g)$, $\bar{\mathbf{S}}_g = \mathbf{G}(\mu_{s,g}, \sigma_{s,g}, d_{s,g}, S_g)$ and $\mathbf{R}_{user_k} = \mathbf{G}(\mu_{u,k}, \sigma_{u,k}, d_t, S)$, where $\mathbf{G}(\mu,\sigma,d,n)$ is defined as [13], [20],

$$[\mathbf{G}(\mu,\sigma,d,n)]_{k,l} = \frac{1}{n}\sum_{j=\frac{1-n}{2}}^{\frac{n-1}{2}} \exp\Big(-i2\pi d(k-l)\cos\Big(\frac{\pi}{2} + \frac{j\sigma}{n-1} + \mu\Big)\Big), \tag{42}$$

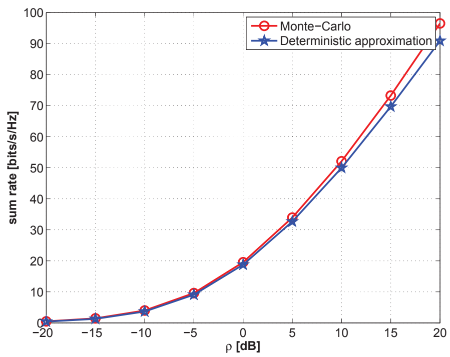

Fig. 2. Sum-rate versus SNR with $\alpha = 1/\rho$, $N = 30$, $K = 30$.

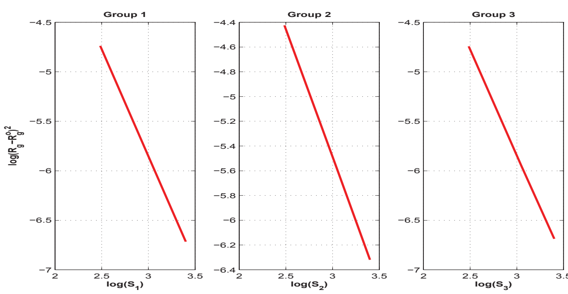

Fig. 3. Error versus S_g for fixed SNR $\rho = 1$, with $\alpha = 1/\rho$.

where $\sigma_{u,k}$ and $\mu_{u,k}$ denote the angular spread and mean AoD for user k and all other parameters are defined in Fig. 1.

We assume $G = 3$, $K = 30$, $N = 30$ and S_g to be set as $\{30, 40, 34\}$ in the simulations. The AoDs for the users are generated randomly between $-\pi/3$ and $\pi/3$ and the angular spreads are generated randomly between $\pi/12$ and $\pi/9$. The mean angles $\mu_{t,g} = \mu_{s,g}$ are set as $\{-\pi/3, 0, \pi/3\}$, $\sigma_{t,g}$ is set as $\{2\pi/7, 2\pi/6, 2\pi/7\}$ and $\sigma_{s,g}$ is set as $\{\pi/6\}$, $\forall g$. Also, d_t and $d_{s,g}$ are set as 0.5 and 2 respectively, $\forall g$. The K users are partitioned into G groups based on the user grouping method described earlier. The signal power $p_{k,g} = \frac{1}{K}$ for all users in all groups. Fig. 2 compares the system sum rate $R_{sum} = \sum_{g=1}^{G} \sum_{k=1}^{K_g} \log(1 + \gamma_{k,g_{\text{RZF}}})$ obtained using 2000 Monte-Carlo realizations of the SINR in (8) for the double scattering channel model in (5) to the deterministic approximation provided in (33), where $\gamma^o_{k,g\text{RZF}}$ is given by (15). It can be seen that the asymptotic result derived in this paper yields a tight approximation for even moderate system dimensions. Also note that the fit is very good at low SNR values but the gap starts to increase for high SNR values. This is due to the slower convergence of $\gamma_{k,g\text{RZF}}$ to its deterministic approximation at high SNR values [14]. Therefore, larger system dimensions are needed for a better approximation at high SNR values.

The error between the simulated normalized sum-rate $R_g = \frac{1}{K_g} \sum_{k=1}^{K_g} \log(1+\gamma_{k,g_{\text{RZF}}})$ and the theoretical normalized sum-rate $R^o_g = \frac{1}{K_g} \sum_{k=1}^{K_g} \log(1+\gamma^o_{k,g_{\text{RZF}}})$ is plotted for each group to study the accuracy of the deterministic approximation. The results are shown in Fig. 3 for equal number of users in each group such that, $K = N = S_g$. The error decreases with a slope of 2.

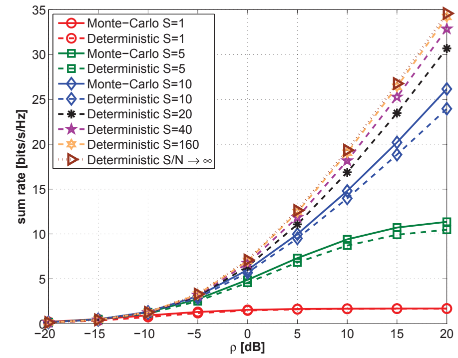

Fig. 4. Sum-rate versus SNR for a multi-keyhole channel, with $\alpha = 1/\rho$.

Fig. 4 studies the effect of the number of scatterers on the system sum-rate for a single group with multi-keyhole channel, i.e. $G = 1$, $\mathbf{R}_{BS} = \mathbf{I}_N$, $\bar{\mathbf{S}} = \mathbf{I}_S$, with $N = 10$ and $K = 10$. The sum-rate in (33) plotted using the closed-form expression of $\gamma^o_{k\text{RZF}}$ in *Corollary 2* is close to the Monte-Carlo result even for very low number of scatterers. The spatial multiplexing gains are seen to increase linearly with S. However, for $S > N$, the gains start to decrease since the degrees of freedom are limited by the number of antennas at the BS. The limiting sum-rate as $S/N \to \infty$ is also plotted using the SINR in (39) . It can be observed that as the number of scatterers increases, the performance approaches to that of a Rayleigh fading channel.

V. Conclusion

In this paper, we studied a multi-user MISO system with double-scattering fading channels. The system performance of this non-Gaussian channel is extremely difficult to study for finite dimensions but some tractable deterministic approximations for different performance metrics can be obtained in the large system limit. Under the assumptions that the users in the same group are characterized by common channel correlation matrices and that the number of antennas, scatterers and users grow infinitely large with a bounded ratio, we derived almost surely tight deterministic approximations of the SINR and the sum-rate with RZF precoding. Simulation results showed a close match between the asymptotic and the Monte-Carlo simulated sum-rate for small system dimensions and provided insights into the performance of multi-keyhole channels.

Appendix

A. Sketch of proof of Theorem 1

Following the strategy in [14], the deterministic equivalents of the energy term $|\mathbf{h}^H_{k,g}\mathbf{V}\mathbf{h}_{k,g}|^2$, the term Θ of the power normalization and the interference term are worked out separately.

1) Deterministic equivalent of $\mathbf{h}^H_{k,g}\mathbf{V}\mathbf{h}_{k,g}$: Exploiting the matrix inversion lemma, trace lemma and rank-1 perturbation lemma (*Lemma 3, 4* and *7* in [16]) with $\mathbf{h}_{k,g}$ modeled as (12),

we have,

$$\mathbf{h}_{k,g}^H \mathbf{V} \mathbf{h}_{k,g} - \frac{\frac{1}{K_g}\text{tr } \tilde{\mathbf{Z}}_g \tilde{\mathbf{Z}}_g^H \mathbf{Q}^{-1}}{1 + \frac{1}{K_g}\text{tr } \tilde{\mathbf{Z}}_g \tilde{\mathbf{Z}}_g^H \mathbf{Q}^{-1}} \xrightarrow[S_g\to\infty]{a.s.} 0, \quad (43)$$

where $\tilde{\mathbf{Z}}_g = \sqrt{\frac{K_g}{K}}\mathbf{Z}_g$ and,

$$\mathbf{Q} = \frac{1}{K}\mathbf{H}^H\mathbf{H} + \alpha\mathbf{I}_N, \quad (44)$$

$$= \sum_{g=1}^{G}\sum_{k=1}^{K_g} \frac{S_g}{K_g} \tilde{\mathbf{Z}}_g \tilde{\mathbf{w}}_{k,g} \tilde{\mathbf{w}}_{k,g}^H \tilde{\mathbf{Z}}_g^H + \alpha\mathbf{I}_N, \quad (45)$$

$$= \sum_{g=1}^{G} \tilde{\mathbf{Z}}_g \tilde{\mathbf{W}}_g \mathbf{I}_{K_g} \tilde{\mathbf{W}}_g^H \tilde{\mathbf{Z}}_g^H + \alpha\mathbf{I}_N, \quad (46)$$

where $\tilde{\mathbf{W}}_g \sim$ *i.i.d.* $\mathcal{CN}(0, \frac{1}{K_g}) \in \mathbb{C}^{S_g\times K_g}$. Now rely on the observation that $\tilde{\mathbf{Z}}_g \tilde{\mathbf{W}}_g \mathbf{I}_{K_g} \tilde{\mathbf{W}}_g^H \tilde{\mathbf{Z}}_g^H$ is a double scattering channel model with random correlation matrices $\tilde{\mathbf{Z}}_g$. For deterministic $\tilde{\mathbf{Z}}_g$, the deterministic equivalent of $\frac{1}{K_g}\text{tr } \tilde{\mathbf{Z}}_g \tilde{\mathbf{Z}}_g^H \mathbf{Q}^{-1}$ has been derived in [Corollary 1 [15]] as,

$$\frac{1}{K_g}\text{tr } \tilde{\mathbf{Z}}_g \tilde{\mathbf{Z}}_g^H \mathbf{Q}^{-1} - \frac{1}{K_g}\text{tr } \tilde{\mathbf{Z}}_g \tilde{\mathbf{Z}}_g^H (\sum_{i=1}^{G} \bar{e}_i \tilde{\mathbf{Z}}_i \tilde{\mathbf{Z}}_i^H + \alpha\mathbf{I}_N)^{-1} \xrightarrow[N\to\infty]{a.s.} 0, \quad (47)$$

where $(e_g, \bar{e}_g)$ are given as a unique solution to the following set of implicit equations,

$$\bar{e}_g = \frac{1}{1+e_g}, \quad (48)$$

$$e_g = \frac{1}{K_g}\text{tr } \tilde{\mathbf{Z}}_g \tilde{\mathbf{Z}}_g^H (\sum_{i=1}^{G} \bar{e}_i \tilde{\mathbf{Z}}_i \tilde{\mathbf{Z}}_i^H + \alpha\mathbf{I}_N)^{-1}. \quad (49)$$

Consequently, for deterministic $\tilde{\mathbf{Z}}_g$ we have,

$$\mathbf{h}_{k,g}^H \mathbf{V} \mathbf{h}_{k,g} - \frac{e_g}{1+e_g} \xrightarrow[N\to\infty]{a.s.} 0, \quad (50)$$

Now let $\mathbf{Z}_g$ to be random and modeled as $\frac{1}{\sqrt{S_g}}\mathbf{R}_{BS_g}^{1/2}\mathbf{W}_g\bar{\mathbf{S}}_g^{1/2}$. This does not affect the expression of $\bar{e}_g$, but e_g are now random quantities and we need to find a deterministic equivalent, denoted as m_g for them, such that, $\max_g|e_g - m_g| \xrightarrow[N\to\infty]{a.s.} 0$. Following the technique in [21], it can be shown that m_g is given as the unique solution to the set of implicit equations in (18).

2) Deterministic equivalent of $\Theta = \text{tr } \mathbf{P}\mathbf{H}(\mathbf{H}^H\mathbf{H} + K\alpha\mathbf{I}_N)^{-2}\mathbf{H}^H$: Again using the matrix inversion lemma, trace lemma and rank-1 perturbation lemma (*Lemma 3, 4* and *7* in [16]) with $\mathbf{h}_{k,g}$ modeled as (12), we have,

$$\Theta - \frac{1}{K}\sum_{g=1}^{G}\sum_{k=1}^{K_g} \frac{p_{k,g}\frac{1}{K_g}\text{tr } \tilde{\mathbf{Z}}_g \tilde{\mathbf{Z}}_g^H \mathbf{Q}^{-2}}{(1 + \frac{1}{K_g}\text{tr } \tilde{\mathbf{Z}}_g \tilde{\mathbf{Z}}_g^H \mathbf{Q}^{-1})^2} \xrightarrow[S_g\to\infty]{a.s.} 0, \quad (51)$$

where $\mathbf{Q}$ is defined in (44). Exploiting (47) and the fact that $\max_g|e_g - m_g| \xrightarrow[N\to\infty]{a.s.} 0$ yields the deterministic equivalent of Θ as,

$$\Theta - \frac{1}{K}\sum_{g=1}^{G}\sum_{k=1}^{K_g} \frac{p_{k,g}(-m_g')}{(1+m_g)^2} \xrightarrow[N\to\infty]{a.s.} 0, \quad (52)$$

where m_g is obtained as a solution of (18) and m_g' is the derivative of m_g with respect to α. One can show by straightforward calculations that $\mathbf{m}' = [m_1', m_2', \dots, m_G']$ can be expressed as a system of linear equations given by (19).

3) Deterministic equivalent of $Int = \mathbf{h}_{k,g}^H \mathbf{V}\mathbf{H}_{[k,g]}^H \mathbf{P}_{[k,g]} \mathbf{H}_{[k,g]} \mathbf{V} \mathbf{h}_{k,g}$: Exploiting the matrix inversion lemma, trace lemma and rank-1 perturbation lemma (*Lemma 3, 4* and *7* in [16]), we have,

$$Int - \frac{1}{K}\frac{\frac{1}{K_g}tr\mathbf{P}_{[k,g]}\mathbf{H}_{[k,g]}\mathbf{Q}^{-1}\tilde{\mathbf{Z}}_g\tilde{\mathbf{Z}}_g^H\mathbf{Q}^{-1}\mathbf{H}_{[k,g]}^H}{(1+\frac{1}{K_g}tr\tilde{\mathbf{Z}}_g\tilde{\mathbf{Z}}_g^H\mathbf{Q}^{-1})^2} \xrightarrow[S_g\to\infty]{a.s.} 0. \quad (53)$$

Define $\Upsilon_{k,g} = \frac{1}{KK_g}\text{tr } \mathbf{P}_{[k,g]}\mathbf{H}_{[k,g]}\mathbf{Q}^{-1}\tilde{\mathbf{Z}}_g\tilde{\mathbf{Z}}_g^H\mathbf{Q}^{-1}\mathbf{H}_{[k,g]}^H$. Exploiting (47) and the fact that $\max_g|e_g - m_g| \xrightarrow[N\to\infty]{a.s.} 0$, we have,

$$\mathbf{h}_{k,g}^H \mathbf{V}\mathbf{H}_{[k,g]}^H \mathbf{P}_{[k,g]} \mathbf{H}_{[k,g]} \mathbf{V} \mathbf{h}_{k,g} - \frac{\Upsilon_{k,g}}{(1+m_g)^2} \xrightarrow[N\to\infty]{a.s.} 0. \quad (54)$$

To derive the deterministic equivalent for $\Upsilon_{k,g}$, we express it as a sum of two terms as follows,

$$\Upsilon_{k,g} = \frac{1}{K}\sum_{\substack{g'=1\\g'\neq g}}^{G}\sum_{k'=1}^{K_{g'}}\frac{1}{K_g}p_{k',g'}\mathbf{h}_{k',g'}^H\mathbf{Q}^{-1}\tilde{\mathbf{Z}}_g\tilde{\mathbf{Z}}_g^H\mathbf{Q}^{-1}\mathbf{h}_{k',g'}$$

$$+ \frac{1}{K}\sum_{\substack{j=1\\j\neq k}}^{K_g}\frac{1}{K_g}p_{j,g}\mathbf{h}_{j,g}^H\mathbf{Q}^{-1}\tilde{\mathbf{Z}}_g\tilde{\mathbf{Z}}_g^H\mathbf{Q}^{-1}\mathbf{h}_{j,g}. \quad (55)$$

Apply *Lemma 3*, *Lemma 4* and *Lemma 7* in [16] to the first term, denoted as Υ_{k,g_1}, to get,

$$\Upsilon_{k,g_1} - \frac{1}{K_g}\sum_{\substack{g'=1\\g'\neq g}}^{G}\sum_{k'=1}^{K_{g'}}\frac{p_{k',g'}\frac{1}{K_{g'}}\text{tr } \tilde{\mathbf{Z}}_{g'}\tilde{\mathbf{Z}}_{g'}^H\mathbf{Q}^{-1}\tilde{\mathbf{Z}}_g\tilde{\mathbf{Z}}_g^H\mathbf{Q}^{-1}}{(1+\frac{1}{K_{g'}}\text{tr } \tilde{\mathbf{Z}}_{g'}\tilde{\mathbf{Z}}_{g'}^H\mathbf{Q}^{-1})^2} \xrightarrow[S_g\to\infty]{a.s.} 0. \quad (56)$$

The deterministic equivalent of $\frac{1}{K_{g'}}\text{tr } \tilde{\mathbf{Z}}_{g'}\tilde{\mathbf{Z}}_{g'}^H\mathbf{Q}^{-1}\tilde{\mathbf{Z}}_g\tilde{\mathbf{Z}}_g^H\mathbf{Q}^{-1}$ is computed by noting that,

$$\frac{1}{K_{g'}}\text{tr } \tilde{\mathbf{Z}}_{g'}\tilde{\mathbf{Z}}_{g'}^H\mathbf{Q}^{-1}\tilde{\mathbf{Z}}_g\tilde{\mathbf{Z}}_g^H\mathbf{Q}^{-1} = \frac{d}{dl}\frac{1}{K_{g'}}\text{tr } \tilde{\mathbf{Z}}_{g'}\tilde{\mathbf{Z}}_{g'}^H(\mathbf{Q} - l\tilde{\mathbf{Z}}_g\tilde{\mathbf{Z}}_g^H)^{-1}|_{l=0}. \quad (57)$$

Assuming $\tilde{\mathbf{Z}}_g$ to be deterministic, we can exploit the convergence results for the double scattering model in [Corollary 1 [15]] and obtain a deterministic equivalent as,

$$\frac{1}{K_{g'}}\text{tr } \tilde{\mathbf{Z}}_{g'}\tilde{\mathbf{Z}}_{g'}^H(\mathbf{Q} - l\tilde{\mathbf{Z}}_g\tilde{\mathbf{Z}}_g^H)^{-1} - \frac{1}{K_{g'}}\text{tr } \tilde{\mathbf{Z}}_{g'}\tilde{\mathbf{Z}}_{g'}^H (\sum_{i=1}^{G}\bar{E}_{i,g}\tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i^H - l\tilde{\mathbf{Z}}_g\tilde{\mathbf{Z}}_g^H + \alpha\mathbf{I}_N)^{-1} \xrightarrow[N\to\infty]{a.s.} 0, \quad (58)$$

where $(E_{g',g}, \bar{E}_{g',g})$ are given as a unique solution to the following set of implicit equations,

$$\bar{E}_{g',g} = \frac{1}{1 + E_{g',g}}, \tag{59}$$

$$E_{g',g} = \frac{1}{K_{g'}} \text{tr}\ \tilde{\mathbf{Z}}_{g'} \tilde{\mathbf{Z}}_{g'}^H (\sum_{i=1}^{G} \bar{E}_{i,g} \tilde{\mathbf{Z}}_i \tilde{\mathbf{Z}}_i^H - l \tilde{\mathbf{Z}}_g \tilde{\mathbf{Z}}_g^H + \alpha \mathbf{I}_N)^{-1}. \tag{60}$$

Now let $\mathbf{Z}_g$ to be random and modeled as $\frac{1}{\sqrt{S_g}} \mathbf{R}_{BS_g}^{1/2} \mathbf{W}_g \bar{\mathbf{S}}_g^{1/2}$. This does not affect the expression of $\bar{E}_{g',g}$. However, $E_{g',g}$ are now random quantities and we need to find a deterministic equivalent, denoted as $M_{g',g}$ for them, such that, $\max_{g',g} |E_{g',g} - M_{g',g}| \xrightarrow[N\to\infty]{a.s.} 0$. Following the technique in [21], it can be shown that $M_{g',g}$ is given as the unique solution to the following set of implicit equations,

$$M_{g',g} = \begin{cases} \frac{1}{K_{g'}} \sum_{j=1}^{S_{g'}} \frac{\frac{K_{g'}}{K} \bar{s}_{g',j} \Delta_{g',g}}{1 + \frac{K_{g'}}{K} \bar{s}_{g',j} \bar{M}_{g',g} \Delta_{g',g}}, \text{ if } g' \neq g, \\ \frac{1}{K_g} \sum_{j=1}^{S_g} \frac{\frac{K_g}{K} \bar{s}_{g,j} \Delta_{g,g}}{1 + \frac{K_g}{K} \bar{s}_{g,j} \bar{M}_{g,g} \Delta_{g,g} - l \frac{K_g}{K} \bar{s}_{g,j} \Delta_{g,g}}, \text{ if } g' = g, \end{cases}$$

$$\bar{M}_{g',g} = \frac{1}{1 + M_{g',g}}, \tag{61}$$

$$\Delta_{g',g} = \frac{1}{S_{g'}} \text{tr}\ \mathbf{R}_{BS_{g'}} \left(\sum_{i=1}^{G} \frac{\mathbf{R}_{BS_i} K_i \bar{M}_{i,g} M_{i,g}}{S_i \Delta_{i,g}} - \frac{\mathbf{R}_{BS_g} K_g l M_{g,g}}{S_g \Delta_{g,g}} + \alpha \mathbf{I}_N \right)^{-1}$$

Consequently using (57) we have the following convergence,

$$\frac{1}{K_{g'}} \text{tr}\ \tilde{\mathbf{Z}}_{g'} \tilde{\mathbf{Z}}_{g'}^H \mathbf{Q}^{-1} \tilde{\mathbf{Z}}_g \tilde{\mathbf{Z}}_g^H \mathbf{Q}^{-1} - M'_{g',g}|_{l=0} \xrightarrow[N\to\infty]{a.s.} 0, \tag{62}$$

where $M'_{g',g}$ is the derivative of $M_{g',g}$ with respect to l and can be expressed as a system of linear equations as shown in (20).

Repeating similar steps for the second term in $\Upsilon_{k,g}$ yields the deterministic equivalent of $\Upsilon_{k,g}$ as,

$$\Upsilon_{k,g}^o = \frac{1}{K_g} \left(\sum_{\substack{g'=1 \\ g' \neq g}}^{G} \sum_{k'=1}^{K_{g'}} \frac{p_{k',g'} M'_{g',g}|_{l=0}}{(1 + m_{g'})^2} + \sum_{\substack{j=1 \\ j \neq k}}^{K_g} \frac{p_{j,g} M'_{g,g}|_{l=0}}{(1 + m_g)^2} \right). \tag{63}$$

Plugging it into (54) yields the deterministic equivalent of the interference term.

This completes the proof of **Theorem 1**.

B. Sketch of proof of Corollary 2:

One can show by straightforward but tedious calculations that the fundamental equations in (18) can be reduced to a single polynomial equation (38) in $\bar{m}$, for $S_1 = S$, $K = N$, $\bar{\mathbf{S}}_1 = \mathbf{I}_{S_1}$ and $\mathbf{R}_{BS_1} = \mathbf{I}_N$. We then use implicit function theorem on the polynomial equation to obtain expressions for m' and M'. Simplifying $\gamma_{k,g\text{RZF}}^o$ in (15) for one group and representing all expressions in terms of $\bar{m}$ leads to the desired result.

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