Residual time extrapolation operators for efficient wavefield construction

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SUMMARY

Solving the wave equation using finite-difference approximations suffers from dispersion and stability related issues that might limit efficient or proper extrapolation of high frequencies. Spectral-based time extrapolation methods tend to mitigate these problems, but at an additional cost to the extrapolation. I investigate the prospective of using a residual formulation of the spectral approach, along with utilizing Shanks transform based expansions, that adheres to the residual requirements, to improve accuracy and reduce the cost. Utilizing the fact that spectral methods excel (time steps are allowed to be large) in homogeneous and smooth media, the residual implementation allows for velocity discretization that optimizes the use of this feature.

INTRODUCTION

In imaging and inversion applications, and with the recent advances in our computational capacity, short changing the wave equation is no longer necessary, and thus, wavefield time extrapolation methods are now commonly used in practice. Wave equation solutions based on wavefield time extrapolation are still relatively expensive. Specifically, finite-difference methods for wave field extrapolation suffers from dispersion and stability related issues requiring fine space sampling, and consequently small time extrapolation steps. It also does not allow for simple separation of wave modes (solutions) necessary to isolate desired wavefield solutions like the P-wave mode in anisotropic acoustic media.

On the other hand, spectral, including pseudo spectral and analytical, methods for extrapolation have emerged recently to address these limitations (Reshef et al., 1988; Etgen and Brandsberg-Dahl, 2009). Spectral methods extrapolate wavefields by applying the Laplace operator as a phase shift in the Fourier (wavenumber) domain. Thus, the extrapolation is overall dispersion free and unconditionally stable for homogeneous media regardless of the length of the time step. The Fourier implementation, however, poses challenges in handling inhomogeneous media in an efficient way. These challenges are addressed using smart variable separation methods (Etgen and Brandsberg-Dahl, 2009; Zhang and Zhang, 2009; Pestana and Stoffa, 2010; Fomel et al., 2010), including simple Taylor's series expansion of the extrapolator operator (Etgen, 1989) to implement pseudo spectral methods. Despite the effectiveness of these methods they are not cheap. More importantly they rarely add to our understanding of the subsurface, especially with regard to medium parameter dependency of the extrapolator operator.

The first-order dependence of wavefields, and specifically the acoustic wave equation, on media parameters is described by the Born approximation. It is a single scattering approximation, used in seismic applications to approximate the perturbed wavefield due to small perturbations in the reference medium (Cohen and Bleistein, 1977; Panning et al., 2009). Though perturbation theory is limited to a homogeneous medium approximation in the perturbed parameter, the small time extrapolation step, which is based on a homogeneous medium approximation within the time step (Huygens's principle), allows us to readily handle any complexity within the wavelength and the grid spacing involved. Based on this concept, I apply Shank's transform (Bender and Orszag, 1978) based expansions (to improve on the Taylor's series accuracy) that allow for easy separation of the wavefield to a background part and a perturbation part. Though residual operators may be inherently addressed in many implementations, I highlight this aspect here to focus on the value of the velocity perturbation in enhancing performance and accuracy.

THE RESIDUAL SPECTRAL APPROACH

The wave equation in the Fourier space-wavenumber domain reduces to a first-order ordinary differential operator in one-way time. An approximate solution equips us with the wave extrapolation formula, valid for small time step, Δt, as follows

$$P(t+\Delta t,\mathbf{x}) \approx \int P(t,\mathbf{k})\, e^{i\,\phi(\mathbf{x},\mathbf{k})\,\Delta t + i\,\mathbf{k}\cdot\mathbf{x}}\,\mathbf{dk}, \qquad (1)$$

where $\widehat{P}(t,\mathbf{k})$ is the wavefield in the wavenumber domain given by $\mathbf{k}=(k_x,k_y,k_z)$ at time t, and thus, $\mathbf{x}=(x,y,z)$ is a vector representing the space coordinates in 3D, and $i=\sqrt{-1}$. In the geometrical (high-frequency) approximation, the function $\phi(\mathbf{x},\mathbf{k})$ appearing in equation (1), for isotropic media, is given by $\phi = v|\mathbf{k}| = v\sqrt{k_x^2+k_y^2+k_z^2}$.

The linear nature of the wave equation allows us to propagate the full wavefield or part of it to allow for the following representation

$$\begin{aligned} P(t+\Delta t,\mathbf{x}) &= P_0(t+\Delta t,\mathbf{x}) + \Delta P(t+\Delta t,\mathbf{x}) \\ &\approx \int P_0(t,\mathbf{k})\, e^{i\,\phi(\mathbf{x},\mathbf{k})\,\Delta t + i\,\mathbf{k}\cdot\mathbf{x}}\,\mathbf{dk} \\ &+ \int \Delta P(t,\mathbf{k})\, e^{i\,\phi(\mathbf{x},\mathbf{k})\,\Delta t + i\,\mathbf{k}\cdot\mathbf{x}}\,\mathbf{dk}, \qquad (2) \end{aligned}$$

where $\Delta P(t,\mathbf{k})$ is part of the wavefield that satisfies the wave equation for the same velocity model (Born approximation). I refer to this part as the residual wavefield. A perturbation in the velocity model results in a perturbation in the extrapolation operator and more

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precisely in the phase, $\Delta\phi$. Thus,

$$P(t+\Delta t,\mathbf{x}) \approx \int P(t,\mathbf{k})\, e^{i\,\phi_0\,\Delta t}\, e^{i\,\Delta\phi\,\Delta t}\, e^{i\,\mathbf{k}\cdot\mathbf{x}}\, \mathbf{dk}, \quad (3)$$

where ϕ_0 is the phase corresponding to the background medium (velocity, v_0). Taking $\Delta\phi$ to be small, we can expand the exponential in equation 3 that includes $\Delta\phi$ using any one of the many series or rational approximations, some will be discussed in the next section, but most admit the following form:

$$e^{i\,\Delta\phi\,\Delta t} = 1 + F(\Delta\phi\Delta t), \quad (4)$$

where F includes the expansion terms as a function $\Delta\phi\Delta t$, using any approach including Taylor's series (Etgen, 1989). Inserting equation 4 into equation 3 yields,

$$\begin{aligned} P(t+\Delta t,\mathbf{x}) &\approx \int P(t,\mathbf{k})\, e^{i\,\phi_0\,\Delta t + i\mathbf{k}\cdot\mathbf{x}}\, \mathbf{dk} \\ &+ \int P(t,\mathbf{k})\, F(\Delta\phi\Delta t)\, e^{i\phi_0\Delta t + i\mathbf{k}\cdot\mathbf{x}}\, \mathbf{dk}. \end{aligned} \quad (5)$$

Combining formulas 2 and 5, I obtain a standalone residual wavefield extrapolation:

$$\begin{aligned} \Delta P(t+\Delta t,\mathbf{x}) &\approx \int \Delta P(t,\mathbf{k})\, e^{i\,\phi_0\,\Delta t + i\mathbf{k}\cdot\mathbf{x}}\, \mathbf{dk} \\ &+ \int P(t,\mathbf{k})\, F(\Delta\phi)\, e^{i\phi_0\Delta t + i\mathbf{k}\cdot\mathbf{x}}\, \mathbf{dk}, \end{aligned} \quad (6)$$

where $P = P_0 + \Delta P$. Note that the second term on the right hand side need to be computed only when $\Delta\phi \neq 0$. On the other hand, the first term propagates the residual wavefield using the background, possibly homogenous, model represented by ϕ_0. If the background wavefield is obtained for a homogeneous medium, which can be done analytically, the residual wavefield will approximately address the inhomogeneity. Up to this point there is no advantage to extrapolating the residual operator instead of the full one, other than its potential usage for velocity estimation.

THE EXPANSION

To derive the residual exponential operator using a form that allows us to separate the background contribution from the residual one, we utilize Taylor's series expansion (to the second order) of the residual exponential extrapolation operator (Etgen, 1989), as follows

$$F = e^{i\,\Delta t\Delta\phi} - 1 = i\Delta t\Delta\phi - \frac{\Delta t^2\Delta\phi^2}{2} + O\left(\Delta\phi^3\right). \quad (7)$$

Truncating F to only the first term amounts to a split-step type implementation. Using this polynomial based expansion allows us to readily separate the space dependence part (residual velocity) from the wavenumber part (the Laplace operator), in a form commonly referred to as the pseudo-spectral implementation (Reshef et al., 1988). The accuracy of the residual extrapolation relies on the order of the expansion, the amount of velocity residual and the time step of the extrapolation. To improve the accuracy of the Taylor's series expansion I use Shanks transform to predict the higher-order behavior of the series by introducing a denominator, thus requiring fewer terms. The first-order Shanks transform (Bender and Orszag, 1978) admits the following form

$$F = e^{i\,\Delta t\Delta\phi} - 1 = \frac{i\Delta\phi}{1 - \frac{i\,\Delta t\Delta\phi}{2}}. \quad (8)$$

This form, unlike Taylor's series, does not allow us to factor out the velocity requiring a full spectral implementation. We will later address this issue through additional approximations. Pade approximations are also possible (Zhang et al., 2004) for spectral implementations, but they do not admit a series form required by the residual formulation given by equation 4.

ACCURACY TESTS

Considering an isotropic homogeneous model with a velocity of 3 km/s, I analyze the accuracy of computing the phase of the extrapolation operator using the various approximations of the residual operator considering a background velocity of $v = 2km/s$ and as a result a velocity perturbation of $v = 1km/s$. Despite the relatively large perturbation, I consider a time step of 0.01 s (of course this time step will have to be reduced considerably for extrapolation in inhomogeneous media). Figure 1 demonstrates the accuracy of the approximations given in the previous section, and specifically the accuracy of the Shanks transform representation (dashed black curves) given by equation 8 as a function of wavenumber (a) and a function of velocity perturbation (b). Errors increase with an increase in the wavenumber or an increase in the velocity perturbation. Nevertheless, the increase in error with velocity is exponential, which suggests that a residual implementation with small steps in velocity perturbation is worth exploring. Clearly all approximations are extremely ac-

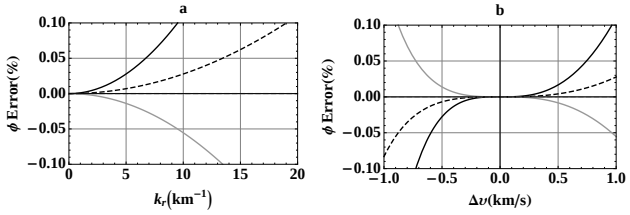

Figure 1: Percent errors in the phase of the extrapolator operator given by the first-order (solid black curve) and second-order (solid grey curve) Taylor's series expansions, and the first-order Shanks transform equation 8 (dashed black curve), as a function of wavenumber for a perturbation $\Delta v = 1km/s$ (a) and a function of perturbation velocity for $k_r = 10km^{-1}$ (b). The background model has a velocity of 2 km/s and a time step of 0.01 s.

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curate for small velocity perturbations or equivalently small time steps, with errors increasing exponentially with an increase in these two variables. Since a larger time step speeds up the extrapolation to the limit that velocity variations allow us, we focus on the velocity perturbation. Shanks transform form provided high accuracy, higher than the second order with fewer terms. However, it does not allow for straightforward separation of the velocity for a pseudo-spectral implementation. A simple remedy is provided by replacing $\Delta\phi$ in the denominator of equation 8 by a constant value given by the average velocity perturbation. Considering we are dealing with controllable perturbations, the deviation introduced by this approximation can be small, and the accuracy through tests not shown here is much higher than the second order. A constant average velocity for the denominator reduces the number of required inverse Fourier transforms for the perturbation part to one instead of 2 for the second-order expansion.

SMALLER CONSTANT-VELOCITY STEPS

The residual implementation can be used to apply as many residuals as we want. ? investigated formulations for a single velocity perturbation in a full implementation (not a residual one). To demonstrate the advantages of considering small successive velocity perturbations, we divide the perturbation in velocity to n smaller perturbations each corresponding to $\Delta v_i = \frac{\Delta v}{n}$. Considering the first term of a Taylor's series expansion

$$\begin{aligned} e^{i\Delta t\phi} &= e^{i\Delta t\phi_0}\prod_{i=1}^{n}(1+i\Delta t\Delta\phi_i) \\ &= e^{i\Delta t\phi_0}\left(1+i\Delta t\sum_{i=1}^{n}\Delta\phi_i+\Delta t^2\sum_{(i\neq j)}\Delta\phi_i\Delta\phi_j\right. \\ &- \left. i\Delta t^3\sum_{(i\neq j\neq k)}\Delta\phi_i\Delta\phi_j\Delta\phi_k+R\left(\Delta t,n\right)\right), \quad (9) \end{aligned}$$

where $\Delta\phi_i = \Delta v_i|\mathbf{k}|$, n is the number of velocity discretization terms ($n > 2$), and $\prod$ stands for the product symbol. The function R contains the other terms of the expansion. For $n = 3$, $R = 0$, and for $n > 3$ R contains terms up to the order Δt^n. For $n = 2$, only the first two terms of the expansion remain. In this case, $\Delta v_1 = \Delta v_2 = \frac{1}{2}\Delta v$, and thus

$$F \approx i\Delta t\Delta\phi - \Delta t^2\Delta\phi^2. \quad (10)$$

Compared to the Taylor's series expansion 8, we are missing the division over two in the second term. A similar result is obtainable by dividing the time step to artificially smaller ones, but the velocity implementation allows for more options as velocity may vary with position.

We now test the accuracy of this new implementation. To reduce the number of inverse Fourier transforms required I will investigate the cases where n=2 and n=4. For $n = 4$, we also truncate the product expansion back to two terms stopping at Δt^2, and thus only consider the first three terms in brackets in equation 9, which is equivalent to a second order approximation. Figure 2a shows the percent error in the phase operator as a function of wavenumber. Among the second-order equivalent (in cost) implementations, the truncated four-step ($n = 4$) implementation (dashed grey curve) has the highest accuracy. This is appreciated even more in Figure 2b where the errors in the operators are given as a function of the velocity perturbation.

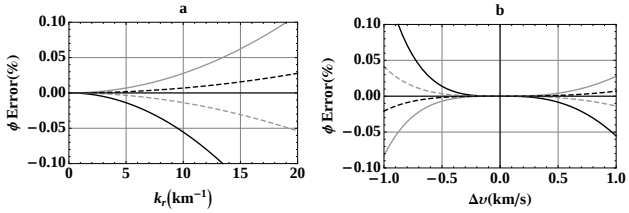

Figure 2: Percent errors in the phase of the extrapolator operator given by the second-order expansion (solid black curve) and two-step velocity perturbation, which is equivalent to second order (solid grey curve), four-step velocity perturbation, which is equivalent to fourth order (dashed black curve), and the four step truncated to second order (dashed grey curve), as a function of wavenumber for a perturbation $\Delta v = 1km/s$ (a) and as a function of the perturbation velocity for $k_r = 10km^{-1}$ (b). The background model has a velocity of 2 km/s and an extrapolation time step of 0.01 s.

WAVEFIELD EXTRAPOLATION

Using the low rank approach of Fomel et al. (2010) to deal with the mixed space-wavenumber domain nature of the phase operator function in equation 1, allows us to extrapolate waves in inhomogeneous media. The rank of the separable operators (space and wavenumber dependent) control the number of inverse Fourier transforms required in each extrapolation step, and thus, the cost. This is typically dependent on the complexity of the velocity model and the size of time step used in the extrapolation.

We apply the residual extrapolator on the BP model (Billette and Brandsberg-Dahl, 2004) using a Ricker-wavelet at a point source with a maximum frequency of 50 Hz. The horizontal grid size Δx is 37.5 m, the vertical grid size Δz is 12.5 m and the time step is 1 ms. Figure 3 shows a part of the model with a salt body. A snap shot of the wavefield at time 3.2 s extracted from wavefield extrapolation using the low rank method applied to the conventional operator given by equation 1 is shown in Figure 4(a). We compare this reference wavefield with that obtained using the residual extrapolator based on a linear approximation of the Taylor's series expansion

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in equation 8 (using only the first term on the right hand side); a snap shot at 3.2 s shown in Figure 4(b). The difference plotted at the same scale is shown in Figure 4(c). The difference between the conventional and using Shanks transform instead (which requires the same number of pseudo-spectral inverse Fourier transform as the linear) plotted again at the same scale is shown in Figure 4(d). Clearly, the difference associated with the Shanks transform implementation, considering same cost, is lower.

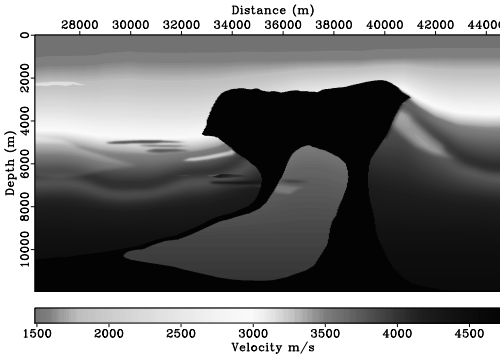

Figure 3: Portion of BP 2004 synthetic velocity model.

COST

Though the emphasis here is on accuracy, a residual implementation has potential cost advantages. Considering a background time extrapolation operator, ϕ_0, for homogeneous or smoothly inhomogeneous media, the extrapolation of such operators tend to be cheap, as approximations in the spectral implementation requires less inverse Fourier transforms per time step (or, in other words, a lower rank representation, especially for a relatively large time step). Thus, using the residual implementation given by equation 6, we need to evaluate the second term, with the extrapolation operator for the more complex media, only when residuals in the medium exists ($\Delta v \neq 0$). A single combined extrapolation operator will not recognize such simplicity in the model in certain areas (like within a salt body). This opens the door for many smart velocity seperation based operators that can help in optimally improving the speed of the extrapolation.

CONCLUSIONS

Isolating the residual part of the wavefield in its time extrapolation allows us to utilize accuracy enhancements obtainable from expansions based on small velocity perturbations. Further accuracy is accessible through Shanks transform, which admits a residual implementation friendly form. The accuracy of the residual implementation supports its usefulness in wavefield extrapolation speed up.

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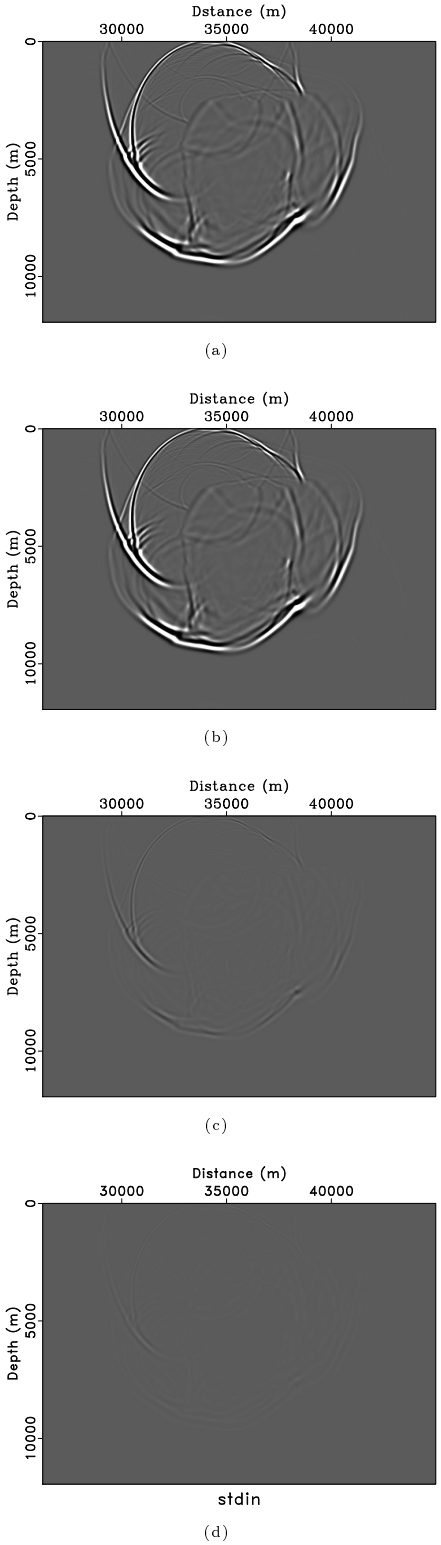

Figure 4: Snap shots of the wavefield at 3.2 s using the conventional extrapolation operator, equation 1 (a), using the linear approximation (b), the difference (c), and the difference between the conventional operator and the Shanks transform approximation (d) for the BP model shown in Figure 3, with a source near the surface.

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EDITED REFERENCES

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