

Modeling Binary Time Series Using Gaussian Processes With Application to Predicting Sleep States

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Abstract

Motivated by the problem of predicting sleep states, we develop a mixed effects model for binary time series with a stochastic component represented by a Gaussian process. The fixed component captures the effects of covariates on the binary-valued response. The Gaussian process captures the residual variations in the binary response that are not explained by covariates and past realizations. We develop a frequentist modeling framework that provides efficient inference and more accurate predictions. Results demonstrate the advantages of improved prediction rates over existing approaches such as logistic regression, generalized additive mixed model, models for ordinal data, gradient boosting, decision tree and random forest. Using our proposed model, we show that previous sleep state and heart rates are significant predictors for future sleep states. Simulation studies also show that our proposed method is promising and robust. To handle computational complexity, we utilize Laplace approximation, golden section search and successive parabolic interpolation. With this paper, we also submit an R-package (HIBITS) that implements the proposed procedure.

Keywords: Binary time series; classification; Gaussian process; latent process; sleep state.

1 Introduction

The American Academy of Sleep Medicine indicates that humans go through several cycles during sleep with each cycle comprised of different stages. It is important to study sleep in humans because the lack of sleep is associated with psychiatric diseases (e.g. depression and ADHD) and chronic diseases (e.g. diabetes, heart disease and hypertension). In particular, understanding sleep state (asleep versus awake) and uncovering its latent pattern play a critical role in people’s daily routine. For example, many young mothers wonder if their infant’s sleep state can be predicted in advance; physicians are interested in forecasting their patient’s anesthesia level/sleep state for surgery. The goal of this paper is to develop statistical inference for studying changes in the sleep state (in particular, asleep versus awake) and the potential roles of covariates such as heart rate, respiration rate and body temperature on sleep states. A plot of the sleep states and the exogenous time series of heart rate and temperature, given in Figure (1), suggest a lead-lag dependence between sleep states and the exogenous time series. In this paper, we develop a model that formally tests for these lead-lag dependence and predict future sleep states.

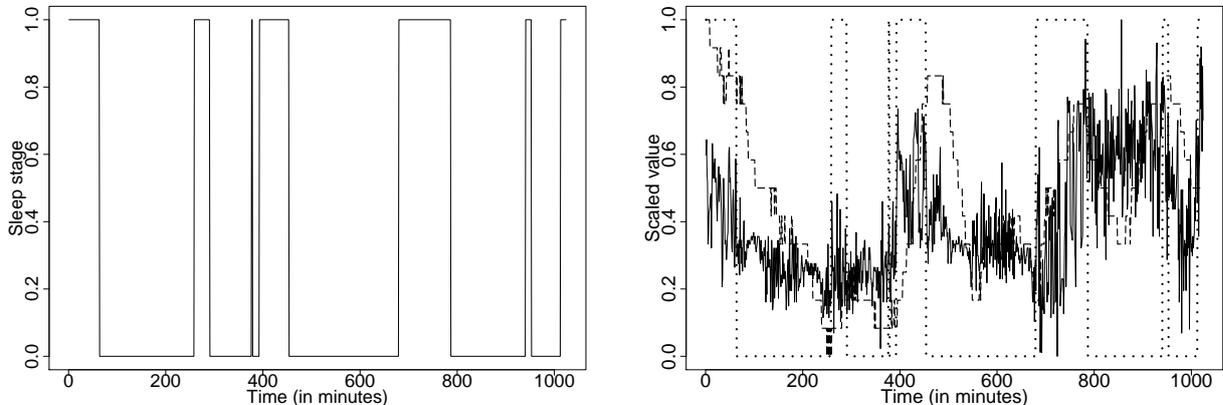


Figure 1: Left: sleep state. Right: sleep state plot (dotted line) overlaid by scaled heart rate (solid line) and body temperature (dashed line) time plots.

Various approaches have been proposed to model and predict sleep states. Since sleep states can be measured sequentially in time during an experiments or in observational studies, typical strategies for modeling categorical time series have been implemented. Caiado et al. (2006) introduced new measurements in classifying time series based on periodograms. Ma-

haraj (2002) put forward a framework of comparing time series in frequency domain. Wavelet based clustering method was also introduced by Maharaj et al. (2010). Jacobs and Lewis (1978) proposed a discrete autoregressive-moving average (DARMA) model by utilizing probabilistic mixtures. Recently, Gao et al. (2017) proposed efficient statistical inference on logistic autoregressive model, which is a widely used example of binary time series. A comprehensive modeling framework based on generalized linear models and partial likelihood inference have been developed in Fokianos and Kedem (2002) and Fokianos and Kedem (2003). Fokianos and Kedem (1998) extended the partial likelihood analysis to non-stationary categorical time series including stochastic time dependent covariates. With the Markovian structure, Meyn and Tweedie (2012), Bonney (1987) and Keenan (1982) developed inferential procedures based on the conditional likelihood. These previous studies provide inference on binary time series. Their main drawback is that they involve massive computation for high dimensional integrals, which results in poor prediction accuracy. Lindquist and McKeague (2009) introduced a logistic regression model with functional predictors and extended it to generalized linear model. Their substantial work was superior in detecting sensitive and interpretable time points that were most predictive to the response. However, when it is applied to this study, the drawbacks are: (1) the Brownian motion assumption is unlikely to be satisfied in practice because the covariates in this study hardly have the property of increment independence; (2) the influence of covariates on responses is assumed to spread across the entire trajectory and hence implies the non-existence of “sensitive time points”; (3) prediction of the time series is not developed, which could be a serious limitation for this project since we are also interested in such predictions.

From the view of the machine learning community, typical classification methodologies such as decision tree, random forest and strategies such as boosting can also be used for predicting sleep states. Although such approaches are able to achieve predictions with high accuracy, the major drawback is that they give very little guidance to sleep researchers who are measuring the impact of previous heart rate, temperature and respiratory rate on future sleep state. In this paper, we develop a statistical model that can provide us simultaneously with convincing inference and interpretation at the same time produce prediction accuracy that is higher than that achieved by typical machine learning classification approaches.

This work is inspired by Keenan (1982) which developed a binary time series using a latent

strictly stationary process. The focus here is to provide an accurate, interpretable, efficient yet computationally less demanding approach for estimation and prediction. When prior information indicates that a binary time series is determined by a process comprised with fixed and random components, we decompose the unobserved latent process into linear and stochastic effects with different covariates. On stage one, inference on the fixed effects is conducted using maximum likelihood estimation. On stage two, conditioned on the estimated fixed effect, a Gaussian process will be used to represent the random components. Predictions are obtained by combining inference on these two components. In addition, based on the results from these two stages, we use parametric bootstrap samplers from the estimated Gaussian process to obtain the final point and interval estimates of parameters.

Using the proposed procedure, we can identify the dependence of the endogenous time series (sleep state) on potential covariates (e.g., heart rate and body temperature) by providing the point and interval estimates of the coefficients from linear effects based on the results from the two-stage algorithm. Inference can be easily and directly performed by maximum likelihood using existing software. Moreover, results are easily interpretable under the framework of generalized linear model. On stage two, which is derived from Gaussian process classification strategy, we can predict the sleep state with high accuracy. Laplace approximation was implemented to reduce the computation cost. This work is also inspired by Brillinger (1983) which, to the best of our knowledge, is the first to introduce this notion of a Gaussian random effect as random intercept in a logit model. Here, we generalize this by representing the random component as a stochastic process rather than just a scalar random variable.

The main advantages of our proposed approach, which we call the hybrid inference method for binary time series (HIBITS), are the following: (1) it accounts for the linear and non-linear stochastic effects of covariates and endogenous variables on sleep states; (2) it provides efficient point and interval estimates of the coefficients from the linear effects while maintaining type I error rates; (3) it produces more accurate predictions compared to other existing approaches; (4) it is easily implemented with low computational cost; and (5) unlike other classification approaches, it gives more straightforward interpretation of the results.

The remainder of this paper is organized as follows. Section 2 is devoted to brief introduction of Gaussian process and its existing applications in regression and classification. In Section 3,

we develop our proposed methodology and discuss the motivation and the technical derivation of the proposed HIBITS method. A complete algorithm that yields prediction and inference on the coefficients of covariates and endogenous variables is also provided. Model selection strategy is also developed to address application problems. Section 4 presents the simulation results that show the benefits of the proposed method over the existing methods in terms of the significant higher prediction accuracy and narrower confidence intervals. In Section 5, we apply our proposed model and inference procedure to identify predictors of sleep states and to predict future sleep states. The results are promising in terms of prediction accuracy at low computational cost and interpretability. Moreover, the proposed method can also be modified when there are missing values.

2 Background on Gaussian Processes in Binary Time Series

2.1 Gaussian process and regression models

Gaussian process have been widely developed in spatial-temporal modeling (Williams and Rasmussen, 2006; Banerjee et al., 2008, 2014; Gelfand et al., 2005; Quick et al., 2013; Stein, 2012; Zhou et al., 2015; Vandenberg-Rodes and Shahbaba, 2015; Wang and Gelfand, 2014). It provides a framework that can capture the non-linear and stochastic components of exogenous and endogenous variables based on generalized linear models, which makes it useful for modeling binary time series and classification.

The definition of a Gaussian process is as follows.

Definition 1. A stochastic process is a Gaussian process if and only if for every finite set of indices t_1, \dots, t_k in the index set T , $\mathbf{x} = (x_{t_1}, \dots, x_{t_k})^T$ is a multivariate Gaussian random variable.

We will write the Gaussian process $f(\mathbf{x})$ as $f(\mathbf{x}) \sim \mathcal{GP}(m(\mathbf{x}), K(\mathbf{x}, \mathbf{x}'))$, where $m(\mathbf{x}) = \mathbb{E}[f(\mathbf{x})]$ and $K(\mathbf{x}, \mathbf{x}') = \mathbb{E}[(f(\mathbf{x}) - m(\mathbf{x}))(f(\mathbf{x}') - m(\mathbf{x}'))]$. Let us now denote the observed data to be $\{(\mathbf{x}_i, y_i), i = 1, \dots, n + n_*\}$, where $\mathbf{x}_i \in \mathbf{R}^p$ and y_i is the response data. We split the dataset into n training points and n_* testing points. Let $(\mathbf{X}_*, \mathbf{y}_*)$ represent the testing datasets and (\mathbf{X}, \mathbf{y}) represent the training datasets respectively. Define $\mu = K(\mathbf{X}_*, \mathbf{X})K(\mathbf{X}, \mathbf{X})^{-1}\mathbf{y}$, $\Sigma =$

$K(\mathbf{X}_*, \mathbf{X}_*) - K(\mathbf{X}_*, \mathbf{X})K(\mathbf{X}, \mathbf{X})^{-1}K(\mathbf{X}, \mathbf{X}_*)$. It follows that

$$\mathbf{y}_* | \mathbf{X}, \mathbf{X}_*, \mathbf{y} \sim N(\mu, \Sigma). \quad (1)$$

The distribution of the response \mathbf{y}_* can be determined by Equation (1). Point estimates, interval estimates and sampling distribution of \mathbf{y}_* can be derived accordingly.

Remark 1. On stage two of the proposed method (discussed in Section 3.2), results in Equation (1) will be utilized to achieve the distribution of the stochastic component which captures the variation in the binary time series beyond which are explained by the covariates.

2.2 Gaussian process in modeling binary time series

2.2.1 Model formulation

Denote the observed training data as $\{(\mathbf{x}_i, y_i), i = 1, \dots, n\}$, where $y_i \in \{1, 0\}$ and $\mathbf{x}_i \in \mathbf{R}^p$. For our data in this paper, y_i denotes the sleep state at time point i and \mathbf{x}_i can be heart rate or body temperature at time point i . We define a latent Gaussian process indexed by \mathbf{x} as $f(\mathbf{x})$. The relationship between \mathbf{x}_i and y_i is characterized by $P(y_i = 1 | \mathbf{x}_i) = t(f(\mathbf{x}_i))$, where t is a link function that determines the relation between \mathbf{x} and the probability of the sleep state. To name a few, t can be a logit, probit or complementary log-log link functions (McCullagh, 1984).

2.2.2 Classification method

For a given link, the inferential procedure will be divided into two steps. First, we compute the distribution of the latent process on the test data

$$p(f_* | \mathbf{X}, \mathbf{y}, \mathbf{X}_*) = \int p(f_* | \mathbf{X}, \mathbf{f}, \mathbf{X}_*) p(\mathbf{f} | \mathbf{X}, \mathbf{y}) d\mathbf{f}, \quad (2)$$

where $p(\mathbf{f} | \mathbf{X}, \mathbf{y}) = p(\mathbf{y} | \mathbf{f}) p(\mathbf{f} | \mathbf{X}) / p(\mathbf{y} | \mathbf{X})$. Then, we estimate the conditional probability of $y_* = 1$ by

$$p(y_* = 1 | \mathbf{X}, \mathbf{y}, \mathbf{X}_*) = \int t(f_*) p(f_* | \mathbf{X}, \mathbf{y}, \mathbf{X}_*) df_*, \quad (3)$$

which is approximately a weighted average of the probability of $y_* = 1$ over all possible realizations of predicted stochastic components that is a Gaussian process.

It should be pointed out that both of the two integrands in Equations (2) and (3) do not have closed forms. For Equation (3), following the argument in Williams and Rasmussen (2006), numerical tools such as Monte Carlo method can be used to obtain the approximate value of the integral given $p(f_*|\mathbf{X}, \mathbf{y}, \mathbf{X}_*)$. To obtain Equation (2), Williams and Barber (1998) introduced Laplace approximation for this problem. Minka (2001) proposed an alternative expectation propagation (EP). Besides these methods, a number of MCMC algorithms have also been considered. In the following section, we will follow the direct Laplace approximation.

From Equation (2), we can write the approximate distribution of $p(\mathbf{f}|\mathbf{X}, \mathbf{y})$ as $N(\hat{\mathbf{f}}, \hat{I}^{-1})$, where $\hat{\mathbf{f}}$ is the MLE of the distribution and \hat{I} is the observed Fisher information matrix. To find the value of $\hat{\mathbf{f}}$, Newton's method can be implemented, where in each iteration $\mathbf{f}^{\text{new}} = \mathbf{f}^{\text{old}} - \nabla^2 \log p(\mathbf{f}^{\text{old}}|\mathbf{X}, \mathbf{y})^{-1} \nabla \log p(\mathbf{f}^{\text{old}}|\mathbf{X}, \mathbf{y}) = (K^{-1}(\mathbf{X}, \mathbf{X}) + W)^{-1} (W\mathbf{f}^{\text{old}} + \nabla \log p(\mathbf{y}|\mathbf{f}^{\text{old}}))$, where $W = -\nabla^2 \log p(\mathbf{y}|\mathbf{f}^{\text{old}})$ and $K(\mathbf{X}, \mathbf{X})$ is the covariance matrix of $f(\mathbf{X})$. Thus, the distribution $p(\mathbf{f}|\mathbf{X}, \mathbf{y})$ can be approximated by $N(\hat{\mathbf{f}}, (K^{-1}(\mathbf{X}, \mathbf{X}) + W)^{-1})$.

Opper and Winther (1999) suggested the conditional expectation of f_* could be obtained by $\mathbb{E}(f_*|\mathbf{X}, \mathbf{y}, \mathbf{X}_*) = K(\mathbf{X}_*, \mathbf{X})^T K^{-1}(\mathbf{X}, \mathbf{X}) \hat{\mathbf{f}} = K(\mathbf{X}_*, \mathbf{X})^T \nabla \log p(\mathbf{y}|\hat{\mathbf{f}})$. Following similar arguments, the conditional variance of f_* can be obtained by $\mathbb{V}(f_*|\mathbf{X}, \mathbf{y}, \mathbf{X}_*) = K(\mathbf{X}_*, \mathbf{X}_*) - K(\mathbf{X}_*, \mathbf{X})^T (K^{-1}(\mathbf{X}, \mathbf{X}) + W)^{-1} K(\mathbf{X}, \mathbf{X}_*)$. Given the mean and variance, at the last step, the probability of $y_* = 1$ can be approximated by $\int t(f_*) \hat{p}(f_*|\mathbf{X}, \mathbf{y}, \mathbf{X}_*) df_*$. It should be pointed out that the Gaussian process essentially captures information beyond those provided by past value of both endogenous and exogenous time series.

Remark 2. $\frac{\partial^2}{\partial \mathbf{f}_i^2} \log p(y_i|\mathbf{f}_i)$ takes the following forms for the logit and probit links, respectively,

$$\begin{aligned} \frac{\partial^2}{\partial \mathbf{f}_i^2} \log p(y_i|\mathbf{f}_i) &= -p(y_i = 1|\mathbf{f}_i)p(y_i = 0|\mathbf{f}_i) \\ \frac{\partial^2}{\partial \mathbf{f}_i^2} \log p(y_i|\mathbf{f}_i) &= -\frac{\varphi(\mathbf{f}_i)^2}{\Phi((2y_i - 1)\mathbf{f}_i)^2} - \frac{(2y_i - 1)\mathbf{f}_i\varphi(\mathbf{f}_i)}{\Phi((2y_i - 1)\mathbf{f}_i)} \end{aligned}$$

Here $\varphi(\cdot)$ and $\Phi(\cdot)$ are the normal probability density function and the cumulative distribution function, respectively.

3 HIBITS: The hybrid estimation method for modeling and predicting binary time series

Building on the established theoretical foundations of Gaussian processes, we now develop a novel two-stage inference and classification method. This section is organized as follows: in Section 3.1, we discuss the motivation of using the hybrid strategy in modeling sleep stage; followed by details of the two-stage hybrid method in Section 3.2; in Section 3.3, we discuss our model selection strategy; and in Section 3.4, we provide a method in providing point and interval estimates of the coefficients of the covariates and endogenous variables.

3.1 Motivation

The common approach is to use a Gaussian distribution with zero mean value as a random effect if the latent process yields, equally likely, positive and negative fluctuations around 0 (Kuss, 2006). Yet, when it comes to real data, this set up overlooks the linear structure between covariates and the actual response of interest. For instance, to model the binary sleep state, scientists believe that body temperature and heart rate should be involved as potential predictors. In Fokianos and Kedem (2002), a regression-based approach for modeling covariates is proposed. However, if we naively utilize the existing Gaussian distribution with zero mean function to model the data, the latent process equally produces positive and negative value fluctuating around 0 which can produce misleading results because it will render the effects of covariates (body temperature and heart rate) to be insignificant. In addition, incorporating those covariates in the covariance function is a reasonable approach to modeling the association. However, the interpretation is complicated. Much work has been done to overcome the aforementioned limitations. To name a few, Snelson et al. (2004) proposed an approach to transform data in agreement with the Gaussian process model. Their work generalized the Gaussian process by warping the observational space. Although the transformed data can be fitted by Gaussian process, it leads to difficulty in the interpretation of the transform. Another drawback is that the effects of particular covariates could be lost (or difficult to interpret). Cornford (1998) suggested a Gaussian process regression model with mean function $m(x) = \beta^T \mathbf{x}$. Their work incorporates the effect of particular covariates. The main drawback is the computational burden that results from the choice of hyperparameter and MCMC sampler when it applies to

classification problem. Building on the prior work, we develop a two-stage method that takes advantage of the strengths of the existing methods. It is able to model the linear association with particular covariates while maintaining computational efficiency.

3.2 The proposed HIBITS method

Consider the data $\{(\mathbf{x}_{1,i}, \mathbf{x}_{2,i}, y_i)\}$ where $y_i \in \{1, 0\}$, $\mathbf{x}_{1,i} \in \mathbf{R}^p$, $\mathbf{x}_{2,i} \in \mathbf{R}^q$. Here, $\mathbf{x}_{1,i}$ are the covariates in the fixed effects part and $\mathbf{x}_{2,i}$ are covariates in the stochastic part. Then, $P(y_i = 1 | \mathbf{x}_{1,i}, \mathbf{x}_{2,i}) = t(\eta(\mathbf{x}_{1,i}, \mathbf{x}_{2,i}))$. We now propose the systematic component of the generalized linear model to take the form

$$\eta(\mathbf{x}_{1,i}, \mathbf{x}_{2,i}) = \beta^T \mathbf{x}_{1,i} + \mathbf{f}(\mathbf{x}_{2,i})$$

where $\beta^T \in \mathbf{R}^p$ and $\mathbf{f}(\mathbf{x}_{2,i}) \sim \mathcal{GP}(\mathbf{0}, K(\mathbf{x}_{2,i}))$. The systematic component with fixed and random effects follow a linear mixed effect model with the first part capturing the fixed effect and the second part describing the randomness that is not covered by the first part. Note that $\eta(\mathbf{x}_{1,i}, \mathbf{x}_{2,i})$ does not include an intercept term on this stage. Following the same notation as previous sections, we denote $\mathbf{X}_d = (\mathbf{x}_{d,1}, \dots, \mathbf{x}_{d,n})$, $d = 1, 2$ as the training dataset and $\mathbf{X}_{d*} = (\mathbf{x}_{d,n+1}, \dots, \mathbf{x}_{d,n+n_*})$, $d = 1, 2$ as the testing subsets. The proposed inference method proceeds as follows.

Stage 1. Inference on the fixed effect.

The joint likelihood function $L(\beta | \mathbf{X}_1, \mathbf{X}_2, \mathbf{y}, \mathbf{f}(\mathbf{X}_2))$ can be written as

$$L(\beta | \mathbf{X}_1, \mathbf{X}_2, \mathbf{y}, \mathbf{f}(\mathbf{X}_2)) = \prod_{i=1}^n t(\eta(\mathbf{x}_{1,i}, \mathbf{x}_{2,i}))^{y_i} (1 - t(\eta(\mathbf{x}_{1,i}, \mathbf{x}_{2,i})))^{1-y_i}. \quad (4)$$

On the first stage, we consider the latent Gaussian process $\mathbf{f}(\mathbf{X}_2)$ fixed across time i . Numerical algorithms such as Newton-Raphson method can be used to obtain $\hat{\beta}$, the MLE of the joint likelihood function. In fact, in this stage, we regard the latent Gaussian process $\mathbf{f}(\mathbf{X}_2)$ as the time-invariant intercept of the logistic regression, which is considered fixed but unknown.

Stage 2. Inference on the stochastic components.

On the second stage, we make use of the result of inference on the fixed effect from Stage 1

and adjust the estimates by introducing the latent Gaussian process $\mathbf{f}(\mathbf{X}_2)$. Conditional on $\hat{\beta}$, we define $\tilde{\eta}(\mathbf{x}_{1,i}, \mathbf{x}_{2,i}|\hat{\beta}) = \hat{\beta}^T \mathbf{x}_{1,i}$, then it follows that

$$P(y_i = 1|\mathbf{x}_{1,i}, \mathbf{x}_{2,i}, \hat{\beta}) = t(\tilde{\eta}(\mathbf{x}_{1,i}, \mathbf{x}_{2,i}|\hat{\beta}) + \mathbf{f}(\mathbf{x}_{2,i})).$$

Here, we model the stochastic component $\mathbf{f}(\mathbf{X}_2)$ as a Gaussian process with covariance function

$$Cov(\mathbf{f}(\mathbf{x}_{2,i}), \mathbf{f}(\mathbf{x}_{2,j})) = \lambda \exp(-\rho \|\mathbf{x}_{2,i} - \mathbf{x}_{2,j}\|^2) + \sigma^2 \delta_{ij} \quad (5)$$

and δ_{ij} takes value 1 when $i = j$ and 0 otherwise. The parameters ρ, σ and λ are estimated by the strategy proposed by Section 3.3 and we will not specify any prior on those parameters. Since $\tilde{\eta}(\mathbf{x}_{1,i}, \mathbf{x}_{2,i}|\hat{\beta})$ is known, we can implement the strategy in Section 2.2 in dealing with the predictive probability from Equation (3). The complete hybrid method can be summarized in the following Algorithm 1.

Algorithm 1 The proposed binary hybrid method

Stage 1.

Input: \mathbf{y} , $K(\mathbf{X}_2, \mathbf{X}_2)$ (covariance matrix), $p(\mathbf{y}|\mathbf{X}_1, \mathbf{f})$ (the likelihood function)

Compute the MLE $\hat{\beta}$ of $L(\beta|\mathbf{X}_1, \mathbf{X}_2, \mathbf{y}, \mathbf{f}(\mathbf{X}_2))$ using Newton-Raphson method (see Equation (4)).

$\mathbf{f} := \mathbf{0}$ initialization

While (iter < Max-iter)

Repeat

$W := -\nabla^2 \log p(\mathbf{y}|\hat{\beta}, \mathbf{f})$

$C := W * \mathbf{f} + \nabla \log p(\mathbf{y}|\hat{\beta}, \mathbf{f})$

$\mathbf{f} = (K^{-1}(\mathbf{X}_2, \mathbf{X}_2) + W)^{-1} * C$

If the difference of successive value of \mathbf{f} is small enough, **break**

else continue this procedure.

Return: $\hat{\mathbf{f}} := \mathbf{f}$

Stage 2.

Input: \mathbf{y} , $\hat{\beta}$ (the estimates of coefficients of the fixed effect), $\hat{\mathbf{f}}$ (the mean of the Laplace approximation), $K(\mathbf{X}_2, \mathbf{X}_2)$, $K(\mathbf{X}_{2*}, \mathbf{X}_2)$, $K(\mathbf{X}_{2*}, \mathbf{X}_{2*})$ (covariance matrix), $p(\mathbf{y}|\mathbf{X}_1, \mathbf{f})$ (the likelihood function), $\mathbf{X}_{1*}, \mathbf{X}_{2*}$ (test input)

$W := -\nabla^2 \log p(\mathbf{y}|\hat{\beta}, \hat{\mathbf{f}})$

$\bar{\mathbf{f}}_* = K(\mathbf{X}_{2*}, \mathbf{X}_2)^T \nabla \log p(\mathbf{y}|\hat{\beta}, \hat{\mathbf{f}})$

$\mathbf{v}_* = K(\mathbf{X}_{2*}, \mathbf{X}_{2*}) - K(\mathbf{X}_{2*}, \mathbf{X}_2)^T W^{\frac{1}{2}} (I + W^{\frac{1}{2}} K(\mathbf{X}_2, \mathbf{X}_2) W^{\frac{1}{2}})^{-1} W^{\frac{1}{2}} K(\mathbf{X}_2, \mathbf{X}_{2*})$

$\bar{\pi}_* = \int t(\hat{\beta}^T \mathbf{X}_{1*} + z) N(z|\bar{\mathbf{f}}_*, \mathbf{v}_*) dz$

Return: $\bar{\pi}_*$ (the predictive probability of test input $\mathbf{X}_{1*}, \mathbf{X}_{2*}$)

In the implementation of this method, we conducted a model selection strategy on the covariance matrix K based on maximum likelihood in Equation (6).

Remark 3. The Hessian matrix W is a diagonal matrix with the following elements for the logit and probit link respectively,

$$\begin{aligned} W_{ii} &= -p(y_i = 1|\hat{\beta}, \mathbf{f}_i)p(y_i = 0|\hat{\beta}, \mathbf{f}_i), \\ W_{ii} &= -\frac{\varphi^2((2y_i - 1)(\hat{\beta}^T \mathbf{x}_{1,i} + \mathbf{f}_i))(\hat{\beta}^T \mathbf{x}_{1,i} + \mathbf{f}_i)}{\Phi^2((2y_i - 1)(\hat{\beta}^T \mathbf{x}_{1,i} + \mathbf{f}_i))} - \frac{(2y_i - 1)(\hat{\beta}^T \mathbf{x}_{1,i} + \mathbf{f}_i)\varphi(y_i(\hat{\beta}^T \mathbf{x}_{1,i} + \mathbf{f}_i))}{\Phi((2y_i - 1)(\hat{\beta}^T \mathbf{x}_{1,i} + \mathbf{f}_i))}. \end{aligned}$$

3.3 Model selection

Strategies on model selection are also presented in two steps.

Step 1. In this study, we will use exploratory analysis to choose variables. Alternatively, we could use AIC or BIC focusing on the fixed effects. Using automatic variable selection strategies based on AIC or BIC, we can choose a model with a subset of predictors. AIC value is defined as $AIC = 2k - 2 \log L$ and BIC is defined as $BIC = k \log n - 2 \log L$, where k is the number of parameters, n is the number of observations and L is the maximum value of likelihood.

Step 2. We select the parameters for the covariance matrix by maximum likelihood estimation. The strategy is inspired by the work of Williams and Rasmussen (2006). Our work is similar in terms of maximizing the marginal likelihood but differs in the way that the both fixed and random effects are involved.

We denote θ as the parameters in the covariance structure $Cov(\mathbf{y})$. The approximate log marginal likelihood is

$$\log q(\mathbf{y}|\mathbf{X}_1, \mathbf{X}_2, \theta) = -\frac{1}{2}\hat{\mathbf{f}}^T K^{-1}(\mathbf{X}_1, \mathbf{X}_1)\hat{\mathbf{f}} + \log p(\mathbf{y}|\mathbf{X}_1, \hat{\mathbf{f}}) - \frac{1}{2} \log |B|, \quad (6)$$

where $B = I + W^{\frac{1}{2}}K(\mathbf{X}_1, \mathbf{X}_1)W^{\frac{1}{2}}$ and $\hat{\mathbf{f}}$ is defined in Section 2.2.2. The strategy is to choose the value of θ that maximizes Equation (6). Note that the covariance matrix K ($K(\mathbf{X}_1, \mathbf{X}_1)$) and $\hat{\mathbf{f}}$ involve parameters θ , the partial derivative of $\frac{\partial \log q(\mathbf{y}|\mathbf{X}_1, \mathbf{X}_2, \theta)}{\partial \theta_j}$ is therefore

$$\frac{\partial \log q(\mathbf{y}|\mathbf{X}_1, \mathbf{X}_2, \theta)}{\partial \theta_j} = A + B,$$

where A and B are defined as follows

$$A = \frac{1}{2} \hat{\mathbf{f}}^T K^{-1} \frac{\partial K}{\partial \theta_j} K^{-1} \hat{\mathbf{f}} - \frac{1}{2} \text{tr}((W^{-1} + K)^{-1} \frac{\partial K}{\partial \theta_j}),$$

$$B = \sum_{i=1}^n -\frac{1}{2} [(K^{-1} + W)^{-1}]_{ii} \frac{\partial^3}{\partial f_i^3} \log p(\mathbf{y} | \mathbf{X}_1, \hat{\mathbf{f}}) [(I + KW)^{-1} \frac{\partial K}{\partial \theta_j} \nabla \log p(\mathbf{y} | \mathbf{X}_1, \hat{\mathbf{f}})]_i.$$

Newton-Raphson method or coordinate descent will be applied to optimize the log marginal likelihood in Equation (6).

In this study, the parameters θ from Equations (5) are ρ, σ and λ . Through our simulation studies, we specify the parameters σ and ρ and apply the aforementioned strategy on estimating λ for the following reasons: (1) it might lead to identifiability problem if we do not fix some of the parameters in this frequentist setting; (2) results do not show much difference if parameters σ and ρ are not fixed; (3) computation will be demanding if no parameter is fixed.

3.4 Inference on the effects of covariates

We propose to use bootstrap sampler to provide point and confidence intervals of the linear coefficients of the covariates \mathbf{X}_1 . Our approach is based on resampling the stochastic component and maximum likelihood. The algorithm is summarized in Algorithm 2.

Algorithm 2 Inference on the linear effects

Input: \mathbf{y} , \hat{K} (the estimated covariance matrix derived from Section 3.3), $\hat{\beta}$ (the estimates of coefficients of the fixed effect derived from Stage 1 in Section 3.2)

Procedure:

$\tilde{\eta}(\mathbf{X}_1) := \hat{\beta}^T \mathbf{X}_1$

While (Iter < Max-iter)

Repeat

Generate $\mathbf{f}^{(\text{iter})}(\mathbf{X}_2)$ from \mathcal{GP} with covariance function \hat{K}

$\eta^{(\text{iter})}(\mathbf{X}_1, \mathbf{X}_2) := \tilde{\eta}(\mathbf{X}_1) + \mathbf{f}^{(\text{iter})}(\mathbf{X}_2)$

Compute the MLE $\hat{\beta}^{(\text{iter})}$ of $L(\beta | \mathbf{X}_1, \mathbf{X}_2, \mathbf{y}, \mathbf{f}(\mathbf{X}_2))$ using Newton-Raphson method, where $L(\beta | \mathbf{X}_1, \mathbf{X}_2, \mathbf{y}, \mathbf{f}(\mathbf{X}_2)) = \prod_{i=1}^n t(\eta^{(\text{iter})}(\mathbf{x}_{1,i}, \mathbf{x}_{2,i}))^{y_i} (1 - t(\eta^{(\text{iter})}(\mathbf{x}_{1,i}, \mathbf{x}_{2,i})))^{1-y_i}$

End of while

Compute $\hat{\beta}_* = \frac{1}{\text{Max-iter}} \sum_{i=1}^{\text{Max-iter}} \hat{\beta}^{(i)}$

$\hat{\beta}_{0.025} = 2.5\text{-th percentile of } \{\hat{\beta}^{(i)}\}_{i=1}^{\text{Max-iter}}$

$\hat{\beta}_{0.975} = 97.5\text{-th percentile of } \{\hat{\beta}^{(i)}\}_{i=1}^{\text{Max-iter}}$

Return: $\hat{\beta}_*$ (The point estimates of the parameters from linear effects); $(\hat{\beta}_{0.025}, \hat{\beta}_{0.975})$ (The 95% bootstrap confidence interval of the parameters from linear effects).

3.5 Summary

In summary, the proposed method on inference, prediction and model selection maintain the following strengths: (1.) it uses linear and non-linear stochastic components to model the effect of the covariates on the response; (2.) it provides point and interval estimates of the linear effects that are more efficient than the existing methods as demonstrated in Section 4; (3.) it is able to make accurate predictions as shown in Section 4; (4.) the computational cost is not demanding; (5.) similarly to generalized linear models, it provides results that are straightforward to interpret.

4 Simulations

In this section, simulations are implemented to test the performance of the proposed method. In Section 4.1, binary time series \mathbf{y}_i are generated by the logit model. We compared the classification error rates derived from the proposed method with 6 other competing statistical and machine learning approaches, namely, the ordinal model, logistic regression, generalized additive mixed model, random forest, decision tree and gradient boosting. We also compute the point and confidence intervals of the coefficients of the covariates and endogenous variables in comparison with other existing methods. To test the robustness of our method, in Section 4.2, we generate time series \mathbf{y}_i from the probit model but use the logit model to fit the data. In Section 4.3, we utilize mixture kernels to generate the Gaussian process and then apply the proposed HIBITS method. Classification error rates, point estimates and confidence intervals are also utilized as measures for comparison.

4.1 Prediction and inference performance on logit model

To evaluate the prediction power and robustness of the proposed method, binary time series y_i are generated under two scenarios:

- **Scenario 1 (with a stochastic process).**

$$P(y_i = 1) = \text{logit}^{-1}(\beta_0 x_{1i} + \beta_1 y_{i-1} + \mathbf{f}(x_{2i}));$$

- **Scenario 2 (without a stochastic process).**

$$P(y_i = 1) = \text{logit}^{-1}(\beta_0 x_{1i} + \beta_1 y_{i-1}).$$

Here, $\mathbf{f}(\mathbf{x}_2)$ follows Gaussian process with

$$\text{Cov}(\mathbf{f}(x_{2i}), \mathbf{f}(x_{2j})) = \lambda \exp(-\rho(x_{2i} - x_{2j})^2) + \sigma^2 \delta_{ij}$$

and δ_{ij} takes value 1 when $i = j$ and 0 otherwise. The parameter β_1 controls the strength of dependence on previous realizations y_{i-1} and it denotes the log odds ratios of $y_{i-1} = 1$ versus $y_{i-1} = 0$. β_0 is the linear coefficients with respect to covariates at current time point. λ is the parameter that determines the strength of dependence across adjacent time points. In this simulation, parameters $\beta = (\beta_0, \beta_1)$ and λ vary in different scenarios. 1000 simulations are conducted in each scenario. Figure 2 shows plots of the simulated data. In this scenario, $\beta = (0.5, 4)$, $\lambda = 1$, $\rho = 1$, $\sigma = 0.1$. 500 sleep stages were generated.

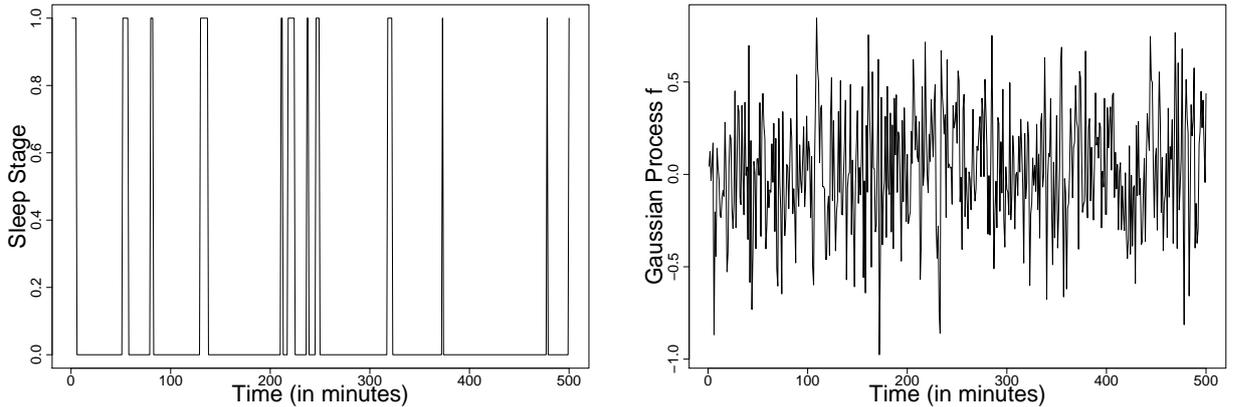


Figure 2: Plots of the generated sleep stage (left) and the simulated Gaussian process(right).

Alternative Methods. To evaluate the prediction power of the proposed method, we compare the classification error rates with other six competing approaches. In general, those approaches include regression and tree based classification strategies. Generalized linear model with logit link is fitted as the first competing method. Further, to respect the correlated structure of the binary time series, we consider the generalized additive mixed models (GAMMs) as the second regression based competing approach. In the work of Lin and Zhang (1999), linear structures of covariates are extended to be smooth functions. Following the notation in Section 3, the

GAMM model is defined as

$$\eta(\mathbf{x}_{\mathbf{1},i}) = \beta_0 + f_1(x_{1,i}^1) + \cdots + f_p(x_{1,i}^p) + \mathbf{z}_i^T \mathbf{b}_i,$$

where $x_{1,i}^j$ denotes the j^{th} component of vector $\mathbf{x}_{\mathbf{1},i}$, $f_j(\cdot)$ is a centered twice-differentiable smooth function, the random effects \mathbf{b} are assumed to be distributed as $N(0, D(\theta))$ and θ is the variance components. Lin and Zhang (1999) estimated nonparametric functions and parameters by using smoothing splines and marginal quasi-likelihood. In this simulation, R package ‘`gamm4`’ was implemented to test the performance of GAMMs. We also considered the regression models for nominal and ordinal time series introduced by Fokianos and Kedem (2002). As is discussed in their concrete work, we implemented ordinal time series model in the simulation. It should be pointed out that due to the binary response, ordinal time series model is degenerated into logistic regression. Simulation results also suggest the equivalence of these two approaches. In addition, we compared our method to tree-based classification approaches. In general, we split the feature space (heart rate and previous sleep states in this study) into “subspaces” and fit simple models within each region. Following the derivation in Friedman et al. (2001), for a node m denoting a region R_m with N_m observations, we let

$$\hat{p}_{mk} = \frac{1}{N_m} \sum_{\mathbf{x}_{\mathbf{1},i} \in R_m} \mathbf{1}(y_i = k),$$

where class k is either 0 or 1 and $\mathbf{1}$ is the indicator function. We assign the observations in node m to class $k(m) = \arg \max_k \hat{p}_{mk}$. Measures of node impurity, denoted as $Q_m(T)$, can be chosen as the misclassification error, Gini index and cross-entropy or deviance.

To further extend the decision tree approach, we also consider random forest and gradient boosting algorithms in the simulation. The essence of random forest is to average many noisy but asymptotically unbiased classifiers and hence reduce the variation. It requires bootstrapping samples and selection features from the training dataset. Since there exist only a few features in this model, the benefit from using random forest approach is mainly derived from the bootstrapping strategy. For each bootstrap training sample set, we grow a random forest tree $T_b, b = 1, \dots, B$. The final output is the ensemble of trees and then predictions are made by majority vote. In addition to random forest, gradient boosting is another extension of decision

tree based method. Similar to the general boosting methods, gradient boosting searches for a strategy to combine multiple weak classifiers in an iterative manner. As discussed in Friedman (2001) and Friedman et al. (2001), the method generically starts from a model with constant value. At iteration $m(1 \leq m \leq M)$, suppose the classifier is denoted as $F_{m-1}(\mathbf{x}_1, \mathbf{x}_2)$, we calculate pseudo-residuals by

$$r_{im} = - \left[\frac{\partial L(y_i, F(\mathbf{x}_{1,i}, \mathbf{x}_{2,i}))}{\partial F(\mathbf{x}_{1,i}, \mathbf{x}_{2,i})} \right]_{F(\mathbf{x}_{1,i}, \mathbf{x}_{2,i})=F_{m-1}(\mathbf{x}_{1,i}, \mathbf{x}_{2,i})},$$

where $L(y, F(x))$ is a loss function. Then, we fit a classifier $h_m(x)$ to the pseudo-residuals and implement a line search algorithm in solving the optimization problem

$$\gamma_m = \arg \min_{\gamma} \sum_{i=1}^n L(y_i, F_{m-1}(\mathbf{x}_{1,i}, \mathbf{x}_{2,i}) + \gamma h_m(\mathbf{x}_{1,i}, \mathbf{x}_{2,i})).$$

At the end of this iteration, we update the model by

$$F_m(\mathbf{x}_{1,i}, \mathbf{x}_{2,i}) = F_{m-1}(\mathbf{x}_{1,i}, \mathbf{x}_{2,i}) + \gamma_m h_m(\mathbf{x}_{1,i}, \mathbf{x}_{2,i}).$$

We keep repeating the full sweep until convergence. The final classifier is denoted as $F_M(\mathbf{x}_{1,i}, \mathbf{x}_{2,i})$.

Model Evaluation. To formally evaluate the performance of all the aforementioned approaches, we calculate the classification error rates under both scenarios. In particular, we fit the results in linear mixed effect model to account for the correlation among classification errors across different methods that result from the same simulated dataset. We consider the model

$$E_{ij} = \mu_i + z_j + \epsilon_{ij},$$

where E_{ij} denotes the classification error rate of approach i on dataset j ; μ_i is the mean error rate of method i , which is well-defined by the law of large numbers. $z_j \stackrel{iid}{\sim} N(0, \sigma^2)$, $\epsilon_{ij} \stackrel{iid}{\sim} N(0, \tau^2)$, $i = 1, \dots, 6$ and $j = 1, \dots, 1000$. We calculate the simultaneous 95% Bonferroni confidence intervals of $(\mu_1 - \mu_i), i = 2, \dots, 6$ to detect the difference in the mean error rates between the proposed method with all the other approaches. In particular, μ_1, \dots, μ_6 denote the mean

error rates of HIBITS, Ordinal model (logistic regression), GAMMs, Random forest, Gradient boosting and Decision tree respectively.

Table 1 provides a summary of the simulation studies for various parameters. It can be seen that for datasets with Gaussian process, there is statistically significant difference in comparison with the competing methods. In particular, the proposed HIBITS method produces significantly lower prediction error rates compared to existing methods. The advantage of the proposed approach is even more obvious when compared with gradient boosting and decision tree approaches. The results show that the proposed HIBITS method captures effective information from covariates x_{1i} , \mathbf{y}_{i-1} and also the stochastic process. The covariate \mathbf{y}_{i-1} serves as a significant predictor as we increase the ratio of β_1 over β_0 .

For the datasets generated *without* the Gaussian process (Scenario 2) shown in Table 2, the accuracy prediction from the two-stage approach is significantly higher than some of the existing approaches such as decision tree and gradient boosting. Among all the other competitors, the proposed method behaves equally competitive. This shows the robustness of the proposed approach when data have no Gaussian process components. This is partly due to the strategy on choosing hyperparameters. By controlling their values, the effects of Gaussian process will be adjusted to the data.

Table 1: Summary of simulation results. μ_1, \dots, μ_6 denote the mean error rates of HIBITS, Ordinal model (logistic regression), GAMMs, Random forest, Gradient boosting and Decision tree respectively. 1000 simulated datasets were generated under the scenario: $P(y_i = 1) = \text{logit}^{-1}(\beta_0 x_{1i} + \beta_1 y_{i-1} + \mathbf{f}(x_{2i}))$ (“Scenario 1”). We calculated the 95% Bonferroni-corrected confidence intervals of the prediction error difference from the testing dataset, $\mu_1 - \mu_i, i = 2, \dots, 6$ that the classification error rate for the proposed method is lower than that for each of competing methods.

Parameters(β, λ)	Competing Method	Scenario 1	
		95% confidence interval of $\mu_1 - \mu_i, i = 2, \dots, 7$	
$\beta = (0.5, 3), \lambda = 10$	Ordinal model*	(-0.052	, -0.032)
	GAMMs	(-0.052	, -0.032)
	Random forest	(-0.029	, -0.009)
	Gradient boosting	(-0.075	, -0.055)
	Decision tree	(-0.070	, -0.050)
$\beta = (0.5, 3), \lambda = 5$	Ordinal model	(-0.015	, -0.001)
	GAMMs	(-0.017	, -0.001)
	Random forest	(-0.017	, -0.002)
	Gradient boosting	(-0.038	, -0.022)
	Decision tree	(-0.048	, -0.032)
$\beta = (0.5, 3.5), \lambda = 10$	Ordinal model	(-0.036	, -0.013)
	GAMMs	(-0.030	, -0.011)
	Random forest	(-0.021	, -0.001)
	Gradient boosting	(-0.046	, -0.028)
	Decision tree	(-0.055	, -0.037)
$\beta = (0.5, 3.5), \lambda = 5$	Ordinal model	(-0.010	, -0.001)
	GAMMs	(-0.011	, -0.001)
	Random forest	(-0.015	, -0.001)
	Gradient boosting	(-0.020	, -0.006)
	Decision tree	(-0.035	, -0.021)

* For binary time series, ordinal model is equivalent to logistic regression.

Table 2: Summary of simulation results. μ_1, \dots, μ_6 denote the mean error rates of HIBITS, Ordinal model (logistic regression), GAMMs, Random forest, Gradient boosting and Decision tree respectively. 1000 simulated datasets were generated under the scenario: $P(y_i = 1) = \text{logit}^{-1}(\beta_0 x_{1i} + \beta_1 y_{i-1})$ (“Scenario 2”). We calculated the 95% Bonferroni-corrected confidence intervals of the prediction error difference from the testing dataset, $\mu_1 - \mu_i, i = 2, \dots, 6$ that the classification error rate for the proposed method is lower than that for each of competing methods.

Parameters(β)	Competing Method	Scenario 2	
		95% confidence interval of $\mu_1 - \mu_i, i = 2, \dots, 7$	
$\beta = (0.5, 3)$	Ordinal model	(-0.006	, +0.010)
	GAMMs	(-0.005	, +0.009)
	Random forest	(-0.004	, +0.012)
	Gradient boosting	(-0.022	, -0.001)
	Decision tree	(-0.023	, -0.002)
$\beta = (0.5, 3.5)$	Ordinal model	(-0.003	, +0.010)
	GAMMs	(-0.002	, +0.010)
	Random forest	(-0.015	, -0.001)
	Gradient boosting	(-0.020	, -0.006)
	Decision tree	(-0.018	, -0.001)

We also evaluate the performance of modeling the linear effects of covariates x_{1i} , \mathbf{y}_{i-1} by comparing the 95% confidence intervals of β_0 and β_1 with the corresponding interval estimates from the other existing methods. Table 3 summarizes the results under the same scenarios in Table 1. It shows that compared with ordinal model, the proposed HIBITS method produces narrower confidence intervals of parameters β_0 while maintaining high capture rates of the true values. The length difference is obvious and it can gain almost 60% shorter confidence intervals in some scenario. It should be noted that using ordinal model, the capture rate of β_1 is extremely low while HIBITS method provides promising performance. The same pattern can also be found in Table 4. Under Scenario 2, the benefits of using HIBITS is even more obvious in terms of shorter confidence interval length and high capture rate.

Table 3: Summary of simulation results. 1000 simulations were generated under the scenario: $P(y_i = 1) = \text{logit}^{-1}(\beta_0 x_{1i} + \beta_1 y_{i-1} + \mathbf{f}(x_{2i}))$ (“Scenario 1”). We present the 95% confidence intervals β_0 and β_1 from the training dataset.

Parameters(β, λ)	Method	Scenario 1	
		95% confidence interval of	
		β_0	β_1
$\beta = (0.5, 3), \lambda = 10$	HIBITS method	(0.113, 0.547)	(1.385, 3.424)
	Ordinal model	(-0.292, 0.586)	(0.695, 2.473)
$\beta = (0.5, 3), \lambda = 5$	HIBITS method	(0.163, 0.572)	(1.570, 3.700)
	Ordinal model	(-0.267, 0.637)	(0.850, 2.676)
$\beta = (0.5, 3.5), \lambda = 10$	HIBITS method	(0.092, 0.535)	(1.628, 4.082)
	Ordinal model	(-0.358, 0.625)	(0.806, 2.593)
$\beta = (0.5, 3.5), \lambda = 5$	HIBITS method	(0.182, 0.582)	(1.820, 3.985)
	Ordinal model	(-0.286, 0.694)	(0.991, 2.841)

Table 4: Summary of simulation results. 1000 simulations were generated under the scenario: $P(y_i = 1) = \text{logit}^{-1}(\beta_0 x_{1i} + \beta_1 y_{i-1})$ (“Scenario 2”). We present the 95% confidence intervals β_0 and β_1 from the training dataset.

Parameters(β)	Method	Scenario 2	
		95% confidence interval of	
		β_0	β_1
$\beta = (0.5, 3)$	HIBITS method	(0.467, 0.600)	(2.838, 3.173)
	Ordinal model	(-0.177, 1.420)	(1.677, 4.515)
$\beta = (0.5, 3.5)$	HIBITS method	(0.422, 0.556)	(3.468, 3.771)
	Ordinal model	(-0.081, 1.202)	(2.275, 5.096)

Overall, the proposed method outperforms competing approaches when comparing the results from data both with and without Gaussian process. Through the model selection strategy discussed in Section 3.3, the proposed approach can adjust the covariance matrix to the data, which in return produces lower prediction error rate and more efficient inference on covariates

than existing methods.

4.2 Investigating robustness of the estimation method

Our goal is to investigate robustness of the proposed model by applying the logistic-based model on data that are generated using a probit model. We generate binary time series \mathbf{y}_i following the scenarios:

- **Scenario 3 (with a stochastic process).**

$$P(y_i = 1) = \Phi(\beta_0 x_{1i} + \beta_1 y_{i-1} + \mathbf{f}(x_{2i}));$$

- **Scenario 4 (without a stochastic process).**

$P(y_i = 1) = \Phi(\beta_0 x_{1i} + \beta_1 y_{i-1})$. $\Phi(\cdot)$ is the cumulative distribution function of standard normal distributions and $\mathbf{f}(\mathbf{x}_2)$ is defined in the same manner as in Section 4.1.

Parameters $\beta = (\beta_0, \beta_1)$ and λ vary in different scenarios. 1000 simulations are conducted in each scenario. We fit the same linear mixed effect model discussed in Section 4.1. Tables 5 and 6 show the summary of confidence intervals $\mu_1 - \mu_i, i = 2, \dots, 6$. Similar to the results in Section 4.1, for dataset with Gaussian process, most of the confidence intervals do not cover 0. The negative values of the classification error rates imply remarkable benefits of using the proposed method over the other competing methods. Note that when comparing with the gradient boosting and decision tree approaches, the proposed method behaves significantly better in terms of extraordinary higher prediction accuracy. In Scenario 4, we tested the proposed method on the dataset without Gaussian process. It is shown that although there is no significant difference in comparison with other competing methods, the proposed approach produces the same prediction power as other competing methods, which implies the robustness with regard to various link functions. In addition, Table 7 shows the 95% confidence intervals of the coefficients β_0, β_1 derived from the proposed method and the ordinal model. Similar to the results in Section 4.1, the proposed method yields much narrower confidence intervals while maintaining good properties of capturing true values.

Table 5: Summary of simulation results. μ_1, \dots, μ_6 denote the mean error rates of HIBITS, Ordinal model (logistic regression), GAMMs, Random forest, Gradient boosting and Decision tree respectively. 1000 simulated datasets were generated under the scenario: $P(y_i = 1) = \Phi(\beta_0 x_{1i} + \beta_1 y_{i-1} + \mathbf{f}(x_{2i}))$ (“Scenario 3”). We calculated the 95% Bonferroni-corrected confidence intervals of the prediction error difference from the testing dataset, $\mu_1 - \mu_i, i = 2, \dots, 6$ that the classification error rate for the proposed method is lower than that for each of competing methods.

Parameters(β, λ)	Competing Method	Scenario 3	
		95% confidence interval of $\mu_1 - \mu_i, i = 2, \dots, 7$	
$\beta = (0.5, 3), \lambda = 10$	Ordinal model	(-0.042 , -0.023)	
	GAMMs	(-0.041 , -0.022)	
	Random forest	(-0.025 , -0.006)	
	Gradient boosting	(-0.064 , -0.045)	
	Decision tree	(-0.060 , -0.041)	
$\beta = (0.5, 3), \lambda = 5$	Ordinal model	(-0.015 , -0.002)	
	GAMMs	(-0.016 , -0.002)	
	Random forest	(-0.021 , -0.007)	
	Gradient boosting	(-0.024 , -0.009)	
	Decision tree	(-0.040 , -0.026)	
$\beta = (0.5, 3.5), \lambda = 10$	Ordinal model	(-0.030 , -0.001)	
	GAMMs	(-0.029 , -0.007)	
	Random forest	(-0.006 , +0.026)	
	Gradient boosting	(-0.057 , -0.035)	
	Decision tree	(-0.024 , +0.002)	
$\beta = (0.5, 3.5), \lambda = 5$	Ordinal model	(-0.014 , +0.001)	
	GAMMs	(-0.013 , +0.001)	
	Random forest	(-0.019 , -0.002)	
	Gradient boosting	(-0.120 , -0.008)	
	Decision tree	(-0.031 , -0.014)	

Table 6: Summary of simulation results. μ_1, \dots, μ_6 denote the mean error rates of HIBITS, Ordinal model (logistic regression), GAMMs, Random forest, Gradient boosting and Decision tree respectively. 1000 simulated datasets were generated under the scenario: $P(y_i = 1) = \Phi(\beta_0 x_{1i} + \beta_1 y_{i-1})$ (“Scenario 4”). We calculated the 95% Bonferroni-corrected confidence intervals of the prediction error difference from the testing dataset, $\mu_1 - \mu_i, i = 2, \dots, 6$ that the classification error rate for the proposed method is lower than that for each of competing methods.

Parameters(β)	Competing Method	Scenario 4	
		95% confidence interval of $\mu_1 - \mu_i, i = 2, \dots, 7$	
$\beta = (0.5, 3)$	Ordinal model	(-0.003 , +0.015)	
	GAMMs	(-0.006 , +0.005)	
	Random forest	(-0.002 , +0.015)	
	Gradient boosting	(-0.012 , +0.009)	
	Decision tree	(-0.016 , +0.008)	
$\beta = (0.5, 3.5)$	Ordinal model	(-0.003 , +0.007)	
	GAMMs	(-0.002 , +0.008)	
	Random forest	(-0.005 , +0.011)	
	Gradient boosting	(-0.010 , +0.016)	
	Decision tree	(-0.011 , +0.002)	

Table 7: Summary of simulation results. 1000 simulations were generated under the scenario: $P(y_i = 1) = \Phi(\beta_0 x_{1i} + \beta_1 y_{i-1} + \mathbf{f}(x_{2i}))$ (“Scenario 3”). We present the 95% confidence intervals β_0 and β_1 from the training dataset.

Parameters(β, λ)	Method	Scenario 3	
		95% confidence interval of	
		β_0	β_1
$\beta = (0.5, 3), \lambda = 10$	HIBITS method	(0.129, 0.564)	(1.529, 3.574)
	Ordinal model	(−0.247, 0.564)	(0.756, 2.549)
$\beta = (0.5, 3), \lambda = 5$	HIBITS method	(0.191, 0.502)	(1.784, 3.956)
	Ordinal model	(−0.273, 0.668)	(0.966, 2.831)
$\beta = (0.5, 3.5), \lambda = 10$	HIBITS method	(0.129, 0.579)	(1.766, 3.871)
	Ordinal model	(−0.406, 0.734)	(0.875, 2.678)
$\beta = (0.5, 3.5), \lambda = 5$	HIBITS method	(0.200, 0.509)	(2.111, 4.310)
	Ordinal model	(−0.239, 0.666)	(1.156, 3.046)

4.3 Investigating the misspecification of the covariance function

The objective of this section is to study the effects of misspecification on the covariance function. We will assume the true covariance function follows mixtures of different kernels and apply the proposed HIBITS method to the generated dataset. In particular, we generate binary time series y_i under the following scenario:

- **Scenario 5 (with a mixture covariance function).**

$$P(y_i = 1) = \text{logit}^{-1}(\beta_0 x_{1i} + \beta_1 y_{i-1} + \mathbf{f}(x_{2i})).$$

Here, $\mathbf{f}(x_{2i})$ follows Gaussian process with

$$\text{Cov}(\mathbf{f}(x_{2i}), \mathbf{f}(x_{2j})) = \eta [\lambda \exp(-\rho(x_{2i} - x_{2j})^2) + \sigma^2 \delta_{ij}] + (1 - \eta) \left[\frac{1}{1 + \tau(x_{2i} - x_{2j})^2} \right].$$

Note that we assume the covariance function is a mixture of exponential and Cauchy kernels. This setting serves as an approach of modeling the long-term and short-term correlation on \mathbf{x}_2 . By increasing the value of trade-off parameter η , the mixture kernel will weight more on the exponential kernel, which captures the short-term dependence. Table 8 summarizes the results of mean error rates under Scenario 5. It is shown that the proposed HIBITS is able to maintain significant lower error rates compared to the other competing methods when the trade-off parameter η is 0.2. As we increase the value to be 0.8, HIBITS performs almost as good as all the other methods and significantly better than decision tree. Table 9 presents the confidence intervals in Scenario 5. Similar to the previous results, HIBITS is capable of yielding narrower intervals and high capture rates even when the trade-off parameter η is large.

In summary, through this section, simulation results show that the proposed HIBITS method is robust to the misspecification of covariance function. This is partly due to the fact that we are able to dynamically “learn” the hyperparameter through model selection. The fine-tuned covariance function could capture the long-term and short-term correlation from the generated dataset.

Table 8: Summary of simulation results. μ_1, \dots, μ_6 denote the mean error rates of HIBITS, Ordinal model (logistic regression), GAMMs, Random forest, Gradient boosting and Decision tree respectively. 1000 simulated datasets were generated under the scenario: $P(y_i = 1) = \text{logit}^{-1}(\beta_0 x_{1i} + \beta_1 y_{i-1} + \mathbf{f}(x_{2i}))$ (“Scenario 5”). We calculated the 95% Bonferroni-corrected confidence intervals of the prediction error difference from the testing dataset, $\mu_1 - \mu_i, i = 2, \dots, 6$ that the classification error rate for the proposed method is lower than that for each of competing methods.

Parameters(β, η)	Competing Method	Scenario 5	
		95% confidence interval of $\mu_1 - \mu_i, i = 2, \dots, 7$	
$\beta = (0.5, 3), \eta = 0.2$	Ordinal model	(-0.024	, -0.011)
	GAMMs	(-0.023	, -0.010)
	Random forest	(-0.022	, -0.002)
	Gradient boosting	(-0.038	, -0.023)
	Decision tree	(-0.061	, -0.037)
$\beta = (0.5, 3), \eta = 0.8$	Ordinal model	(-0.008	, +0.005)
	GAMMs	(-0.008	, +0.007)
	Random forest	(-0.005	, +0.001)
	Gradient boosting	(-0.004	, +0.001)
	Decision tree	(-0.016	, -0.002)

Table 9: Summary of simulation results. 1000 simulations were generated under the scenario: $P(y_i = 1) = \text{logit}^{-1}(\beta_0 x_{1i} + \beta_1 y_{i-1} + \mathbf{f}(x_{2i}))$ (“Scenario 5”). We present the 95% confidence intervals β_0 and β_1 from the training dataset.

Parameters(β, η)	Method	Scenario 5	
		95% confidence interval of	
		β_0	β_1
$\beta = (0.5, 3), \eta = 0.2$	HIBITS method	(0.056, 0.505)	(1.796, 3.306)
	Ordinal model	(-0.232, 0.671)	(0.829, 2.769)
$\beta = (0.5, 3), \eta = 0.8$	HIBITS method	(0.160, 0.702)	(2.803, 3.309)
	Ordinal model	(-0.333, 1.142)	(1.186, 6.428)

5 Analysis of the sleep state data

In this section, we apply our method to sleep state data. People spend one third of their lifetime on sleep. Studying and predicting sleep patterns is significant because our body requires sleep in much the same way as the need of eating and breathing. Moreover, disruptions in sleep are known to be associated with both psychiatric and chronic diseases. In what follows, we will

analyze the sleep data obtained from an observational study with the goal predicting sleep states and identifying associations between sleep states and potential regulators such as temperature and heart rate.

5.1 Exploratory analysis

The data were recorded from a four month old infant who was placed to bed at night. Heart rate (H_i , beats per minute at time i), temperature (T_i , in Celsius, at time i) and sleep stage (S_i at time i) of length ($N = 1024$) were sampled every 30 seconds. Heart rate was recorded automatically using a standard ECG (electrocardiogram) monitor. The infant’s EEG (electroencephalogram) and EOG (electrooculogram) were also measured with a period of 30 seconds. The EEG captured brain waves including alpha (8 – 15 Hz), beta (16 – 31 Hz) and mu (8 – 12 Hz) rhythms; EOG recorded the eye movement. Sleep stage for each time point i was determined by the sleep lab expert visually interpreting the EEG and EOG record (Nevsimalova and Sonka, 1997). It was classified as 4 categories: (1) quiet sleep, (2) indeterminate sleep, (3) active sleep and (4) awake (Benbadis, 2006). The sleep stage S_i was measured as integers ranging from 1 to 4. In this section, following the work of Fokianos and Kedem (2002), sleep state is defined as a binary time series Y_i :

$$Y_i = \begin{cases} 1 & : \text{awake at time } i, \\ 0 & : \text{not awake at time } i. \end{cases}$$

where “not awake” stands for quiet sleep, indeterminate sleep or active sleep.

Time series plots of heart rate, temperature and sleep state are shown in Figure 1 and Figure 3. By comparing the heart rate, temperature with sleep state, we note that higher heart rate are likely to correspond to sleep state 1 (awake). While this pattern is clear for heart rate, no such pattern between temperature and sleep state can be detected by visual inspection. In addition, it can be seen that the current sleep state is highly related to previous states.

To further study the dependence of sleep state on the covariates temperature and heart rate, we conducted additional preliminary analysis. Particularly, we categorize heart rate and temperature (after taking the logarithm) into several levels and calculate the empirical log odds of awake over not-awake for each level. Figure 4 show the relationship between the empirical log odds and different levels of the underlined heart rate and temperature. We are able to identify

a positive association between heart rate with current sleep states. The effect of the lower heart rates are associated with higher probability of being asleep. Regarding temperature, one can hardly identify any definitive relationship using the log odds. Moreover, in Table 10, we report the empirical transition probability of sleep state. It shows that the current sleep state is highly dependent on the previous state. More specifically, there is a strong tendency for sleep to remain in its current state.

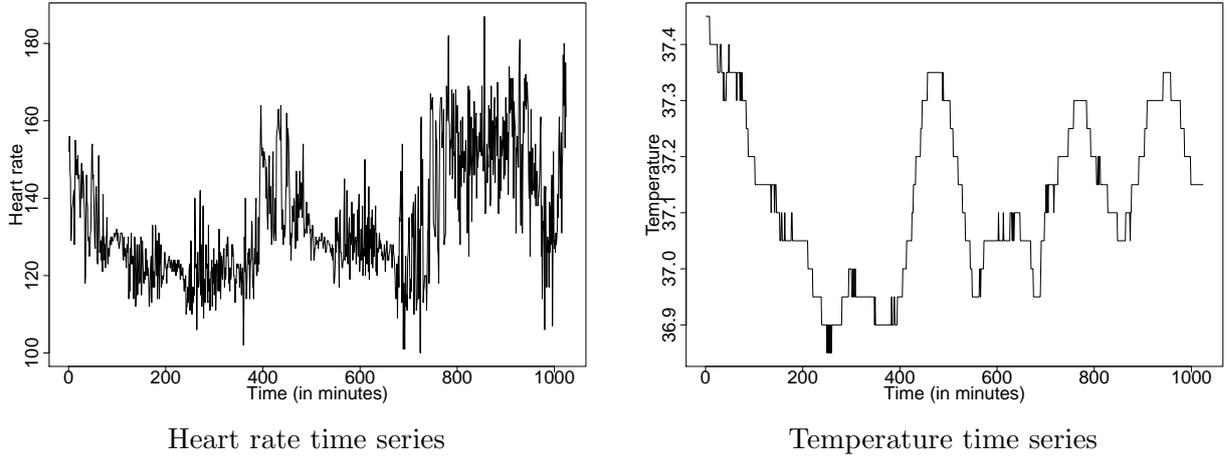


Figure 3: Left: heart rate (in beats per minute). Right: temperature (in Celsius).

Table 10: Empirical transition table of sleep state: when the current state is not awake, the sample probability of staying not wake in the next time point is $729/735$ while the sample probability of being in the awake state at the next time point is $6/735$. When the current state is awake, the sample probability of staying awake at the next time point is $282/288$ while the sample probability of changing to a non-awake state at the next time point is $6/288$.

	$Y_{i-1} = 0$	$Y_{i-1} = 1$
$Y_i = 0$	$729/1023$	$6/1023$
$Y_i = 1$	$6/1023$	$282/1023$

5.2 Modeling and results

Following the exploratory analysis, $\log H_i$, Y_{i-1} and time (in minutes) are suggested in the proposed model. Since there is strong effect of $\log H_i$, Y_{i-1} on current sleep state, we include those two covariates as fixed effect components. Gaussian process on time domain is introduced to capture the nonlinear term.

We also applied our proposed binary hybrid approach to make the inference and prediction. Summaries of point and interval estimates are shown in Table 11. It is seen that compared

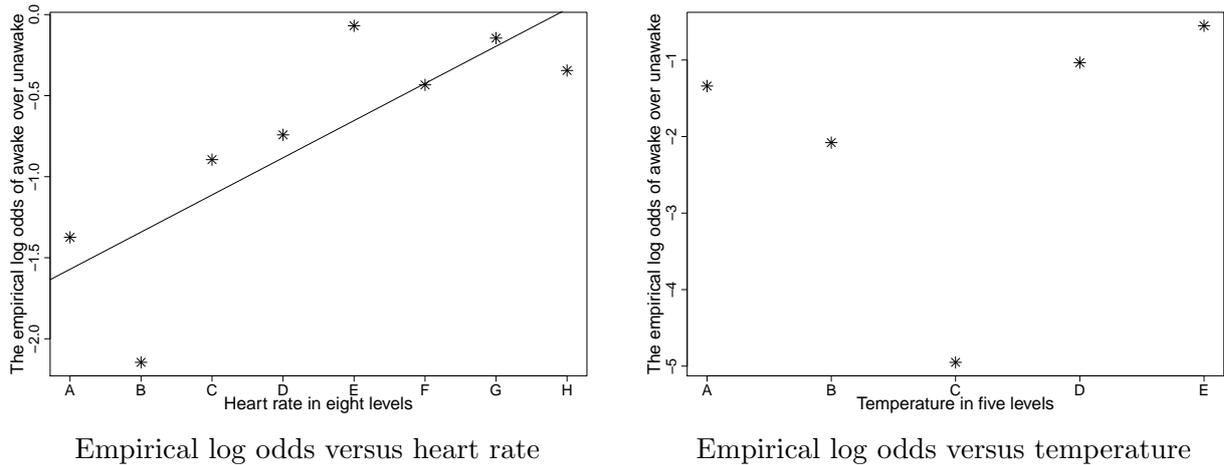


Figure 4: Scatterplots of empirical log odds versus heart rate and temperature. The left panel shows the empirical log odds over eight levels of heart rate. The right panel displays the same value versus temperature.

with the ordinal model (logistic regression), the point estimates are similar. However, there is significantly large difference among the interval estimates. Using the proposed HIBITS method, we gain substantially narrower confidence intervals than ordinal model. The benefits are up to almost 90% shorter in length. From the proposed results, we find that one unit increment in heart rate at current time point will lead to 211.4% accretion of odds. Current odd of sleep state when previous sleep state is awake is estimated to be dramatically higher than that when previous state is not awake. To test the prediction power of this model, the proposed method was implemented with various training and testing data size. Numerical results are summarized in Table 12. It can be seen that the model produces around 99% prediction accuracy while ordinal model yields about 96%. As we decrease the ratio of training over testing data size, the prediction accuracy remains stable. Time series plots of the real and predicted sleep state are presented in Figure 5. It can be shown that the proposed method produces high prediction accuracy and recover the same sleep state pattern as the real dataset. To check for the sensitivity of the proposed method to the estimated value of parameter λ , we compared the results from the data-adaptive estimate (0.730) against the following values (1.730, 2.730). The data-adaptive estimate gave roughly the same prediction error but the confidence intervals were narrower.

Table 11: Summary of the sleep state analysis. The point and interval estimates from HIBITS method are obtained by Section 3.4. It can be seen that the widths of the confidence intervals from the HIBITS method are narrower than those of the classical ordinal model.

Parameters(β_0, β_1)	Method	Point estimate	95% confidence intervals
β_0	HIBITS method	1.136	(1.000, 1.271)
	Ordinal model	1.105	(0.101, 2.109)
β_1	HIBITS method	8.275	(8.124, 8.427)
	Ordinal model	8.241	(6.669, 9.813)

Table 12: Prediction accuracy with different training and testing data size.

Training/Testing data size	Method	Prediction Accuracy
600/400	HIBITS method	99.0%
	Ordinal model	96.0%
500/500	HIBITS method	99.2%
	Ordinal model	96.1%
400/600	HIBITS method	99.1%
	Ordinal model	96.4%

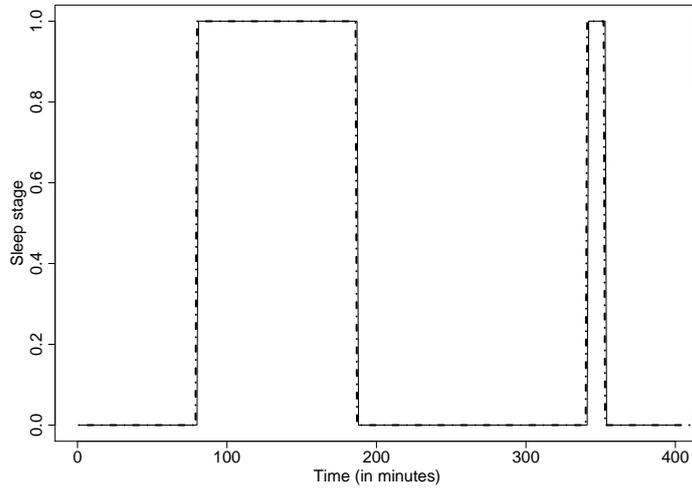


Figure 5: Predicted sleep state (solid line) overlaid with real data (dotted line) (training/testing data size 600/400).

5.3 Discussion on missing data

One advantage of the proposed prediction model is that it captures the information from its own past. Derived from the results, the odds when previous sleep state is awake is 4000-fold higher than that when the preceding state is not awake. However, if there are missing data or the observations are not collected successively, such information will be lost. This motivates us to adjust the model to fit such cases. In the adjusted model, we choose $\log H_i$ as fixed effects and still use Gaussian process on time domain. To test the prediction power, instead of fixing the training and testing dataset, we randomly pick those two pieces of data with fixed size.

The proposed HIBITS method was implemented. Summaries of the test results can be found in Table 13. The tests were conducted 10 times with training and testing data of different sizes. From the results, it is clear that as the training data size becomes larger, the prediction accuracy increases at a reasonable rate. As the training data size reaches 600, the accuracy is promising.

Table 13: Prediction accuracy with different training and testing data size, *stands for the test number.

Training/Testing data size	1*	2	3	4	5	6	7	8	9	10	Average
400/100	0.76	0.92	0.95	0.93	0.80	0.64	0.84	0.89	0.90	0.93	0.86
500/100	0.99	0.89	0.90	0.98	0.91	0.88	0.89	0.93	0.95	0.91	0.92
600/100	0.95	0.97	0.99	0.91	0.93	0.96	0.97	0.90	0.92	0.93	0.94

To further study the performance of the proposed HIBITS method, we make the ratio of training over testing data size smaller. Particularly, we change the training and testing data size to be 700 and 300 respectively. The prediction accuracy is around 0.873. If we move further to change the training data size to be 800 and testing data size to be 200, the prediction accuracy is about 90%. All the results demonstrate that the proposed binary hybrid method produces promising prediction power when the dataset are not collected successively or partly missing. Moreover, it should be pointed out that the computation is not very demanding. The tests are conducted in R programming and the operation time is approximately 90 seconds for each test.

6 Concluding remarks

The proposed hybrid inference method for binary time series (HIBITS) produces efficient inference and promising predictions with a relatively low computational cost. Compared to existing methods, our proposed approach has the following advantages: on one hand, by involving known covariates as fixed effect components, we make use of the information indicating the association between the response and covariates. On the second stage, a Gaussian process will capture the information beyond what provided by those covariates of both endogenous and exogenous time series. On the other hand, as indicated in the simulation, the proposed method is robust compared to existing methods. The proposed model selection strategy allows the model to fit the data even though not enough information is captured by the fixed effect components. The strategies in providing point and interval estimates, in addition, allows researchers to gain more

informative conclusions in the association between response and covariates. These advantages make our model easy to interpret. In summary, the proposed HIBITS method, serving as an approach with high prediction power, efficient inference capability and direct interpretability, provides a promising methodology in modeling and predicting sleep states and other binary time series.

Although the HIBITS method produced outstanding results, it is limited to binary outcomes. As a future direction, we could extend the proposed HIBITS to model general categorical responses (e.g. the 4 sleep stages). In the case of nominal categorical outcomes, we could follow the similar framework of multinomial logit model discussed by Fokianos and Kedem (2002). Specifically, the link function could be extended to softmax function where fixed and random effects can be imposed on the systematic component, which is a natural generalization of our proposed HIBITS. On the other hand, if the outcomes are ordinal categorical time series, one can impose a threshold mechanism on the systematic component of the model. Following the scenario of proportional odds models in Fokianos and Kedem (2003), the HIBITS method can be extended by incorporating fixed and random effects.

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A Manual of R package HIBITS

A.1 Description

Package: HIBITS

Type: Package

Title: The hybrid inference method of binary time series

Version: 0.1

Date: 2016-03-23

Description: Generate the simulated binary time series, compute the point and confidence intervals of the parameters in the model and make predictions of the future observations

License: GPL-2

LazyData: TRUE

RoxygenNote: 5.0.1

Imports: stats, base

NeedsCompilation: no

Built: R 3.2.3; ; 2016-03-24 05:54:32 UTC; unix

A.2 Functions

A.2.1 BHM

Description The implement of the hybrid method

Usage BHM(xtrain, ytrain, xtest, rho, sigma, n, train_index, test_index, optim = F, specify, ini_interval)

Argument

xtrain	The training datasets of features
ytrain	The training datasets of response
xtest	The testing datasets of features
rho	The hyperparameter of the kernel function
sigma	The hyperparameter of the kernel function
n	The size of the whole dataset
train_index	The index of the training dataset
test_index	The index of the testing dataset
optim	An option of using the model selection strategy
specify	Pre-specified value of parameters if optim is False
ini_interval	The range of lambda

Value

BHM A list of “predictions”, “the estimated parameters”, “the estimated intercept from the maximum likelihood ” and “fhat”.

Examples

```
x1 = rnorm(150);
time_series = gen_sim(rho = 1, sigma = 0.1, lambda = 5, beta_true = c(0.5, 2), x1, length(x1),
ini = 1, F)$realization
split_data_temp = Split_data(cbind(x1, time_series), 100);
temp = BHM(split_data_temp$xtrain, split_data_temp$ytrain, split_data_temp$xtest, rho = 1,
sigma = 0.1, length(x1), split_data_temp$strain_index, split_data_temp$stest_index, T, 5, c(0,10))
```

A.2.2 gen_sim

Description Generate binary time series for simulation purpose

Usage gen_sim(rho, sigma, lambda, beta_true, x, n, ini, GPs = T)

Argument

rho	The hyperparameter of the kernel function
sigma	The hyperparameter of the kernel function
lambda	The parameter of the kernel function
beta_true	The parameters of the systematic components
x	The covariates to generate time series
n	The size of the whole dataset
ini	The initial value of the time series
GPs	An option of involving a Gaussian process in generating the time series

Value

gen_sim A list of “realizations” of time series and the “Kernel” of the covariance matrix.

Examples

```
x1 = rnorm(150);
time_series = gen_sim(rho = 1, sigma = 0.1, lambda = 5, beta_true = c(0.5, 2), x1, length(x1),
ini = 1, F)$realization
```

A.2.3 interval_est

Description Generate the interval estimates derived from the hybrid method

Usage interval_est(xtrain, ytrain, rho, sigma, lambda, beta, intercept, fhat)

Argument

xtrain	The training dataset of features
ytrain	The training dataset of response
rho	The hyperparameter of the kernel function
sigma	The hyperparameter of the kernel function
lambda	The estimated parameter of the kernel function
beta	The estimated parameters of the systematic component
intercept	The estimated intercept derived from the maximum likelihood
fhat	Value obtained from the estimated Gaussian process.

Value

interval_est A list of samples of parameters from which we can get the interval estimates.

Examples

```
x1 = rnorm(150);
time_series = gen_sim(rho = 1, sigma = 0.1, lambda = 5, beta_true = c(0.5, 2), x1, length(x1),
ini = 1, F)$realization;
split_data_temp = Split_data(cbind(x1, time_series), 100);
temp = BHM(split_data_temp$xtrain, split_data_temp$ytrain, split_data_temp$xtest, rho = 1,
sigma = 0.1, length(x1), split_data_temp$train_index, split_data_temp$test_index, T, 5, c(0,
10));
para_sample = interval_est(split_data_temp$xtrain, split_data_temp$ytrain, rho = 1, sigma =
0.1, temp$parameter[3], temp$parameter[-3], temp$intercept, temp$fhat)$par
```

A.2.4 interval_glm

Description Generate the interval estimates derived from logistic regression (ordinal model)

Usage interval_glm(xtrain, ytrain)

Argument

xtrain	The training dataset of features
ytrain	The training dataset of response

Value

interval_glm A list of confidence intervals of parameters from logistic regression (ordinal model).

Examples

```
x1 = rnorm(150);
time_series = gen_sim(rho = 1, sigma = 0.1, lambda = 5, beta_true = c(0.5, 2), x1, length(x1),
```

```
ini = 1, F)$realization
split_data_temp = Split_data(cbind(x1, time_series), 100);
para_sample = interval_glm(split_data_temp$xtrain, split_data_temp$ytrain)
```

A.2.5 Split_data

Description Split the dataset into training and testing dataset

Usage Split_data(dataset, ntrain)

Argument

dataset The whole dataset.
ntrain The size of training dataset.

Value

Split_data A list of “feature training data”, “response training data”, “feature testing data”, “response testing data”, “training data index”, “testing data index”.

Examples

```
demo = replicate(5, rnorm(200));
Split_data(demo, 150);
```

A.2.6 like

Description Calculate the value of logistic function

Usage like(f)

Argument

f A number or a vector.

Value

The value of logistic function of f

Examples

```
like(0.5)
like(c(1, 3))
```