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EXISTENCE OF WEAK SOLUTIONS TO STATIONARY MEAN-FIELD GAMES THROUGH VARIATIONAL INEQUALITIES*

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Abstract. Here, we consider stationary monotone mean-field games (MFGs) and study the existence of weak solutions induced by monotonicity. First, we introduce a regularized problem that preserves the monotonicity. Next, using variational inequality techniques, we prove the existence of solutions to the regularized problem. Then, using Minty's method, we establish the existence of weak solutions for the original MFG. Finally, we examine the properties of these weak solutions in several examples. Our methods provide a general framework to construct weak solutions to stationary MFGs with local, nonlocal, or congestion terms.

Key words. mean-field games, variational inequalities, stationary problems

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1. Introduction.

In recent years, mean-field games (MFGs) have become an active area of research in both the mathematics [49, 50, 51] and engineering communities [43, 44]. In spite of substantial progress, many questions remain open. Some of the more fundamental issues regard the existence and uniqueness of solutions. Various authors have attempted to answer these questions through explicit solutions and transformations [4, 5, 6, 19, 24, 25, 41, 42, 53], a priori estimates [23, 28, 29, 30, 32, 32, 33, 34, 35, 38, 54], penalization ideas [26, 27], random-variable techniques [15, 16, 39, 48, 57], weak and renormalized solutions [55, 56], and variational methods [9, 10, 11, 12, 18]. Here, we develop a new approach to investigate the existence of weak solutions induced by monotonicity to stationary MFGs using variational inequalities.

We consider MFGs given by the system of partial differential equations

(1.1)
$$\begin{cases} -u - H(x, Du, D^2u, m, h(\boldsymbol{m})) = 0, \\ m - \operatorname{div}(mD_pH(x, Du, D^2u, m, h(\boldsymbol{m}))) + \left(mD_{M_{ij}}H(x, Du, D^2u, m, h(\boldsymbol{m}))\right)_{x_ix_j} = 1. \end{cases}$$

For convenience, we take periodic boundary conditions for (1.1) and work in the d-dimensional torus, $\mathbb{T}^d$, $d \in \mathbb{N}$. Moreover, $i, j \in \{1, \dots, d\}$ and we use the Einstein convention on repeated indices. In (1.1), the Hamiltonian

$$H : \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{E} \times \mathbb{R} \to \mathbb{R}, \qquad (x, p, M, m, \theta) \mapsto H(x, p, M, m, \theta), \tag{1.2}$$

satisfies the regularity and growth assumptions discussed in section 2. The set $\mathbb{E}$ is either $\mathbb{R}^+$ or $\mathbb{R}^+_0$, and $h : \mathcal{M}_{ac}(\mathbb{T}^d) \to C(\mathbb{T}^d)$ is a nonlocal operator also examined later. Here, $\mathcal{M}_{ac}(\mathbb{T}^d)$ stands for the space of positive measures on $\mathbb{T}^d$, absolutely

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continuous with respect to the Lebesgue measure. We use a boldface font to denote elements of $\mathcal{M}_{ac}(\mathbb{T}^d)$ and represent their densities with the same nonboldface letter.

Our assumptions encompass a broad class of Hamiltonians that includes the following important examples:

$$\begin{aligned} H(x,p,M,m,\theta) &:= \bar{H}(x,p,m) - V(x) - g(m,\theta) - \sigma(x)\sum_{i=1}^{d} M_{ii}, \\ h(\boldsymbol{m}) &:= c_1\,\zeta * \boldsymbol{m} + c_2\,\zeta * \left((\zeta * \boldsymbol{m})^{\bar{\alpha}}\right), \end{aligned} \tag{1.3}$$

where

$$\bar{H}(x,p,m) := a(x)(1+|p|^2)^{\frac{\gamma}{2}} \quad \text{or} \quad \bar{H}(x,p,m) := a(x)\frac{|p|^2}{2m^\tau} \tag{1.4}$$

and

$$g(m,\theta) := m^\alpha + \theta \quad \text{or} \quad g(m,\theta) := \ln(m) + \theta \tag{1.5}$$

with V, σ, ζ, $a : \mathbb{T}^d \to \mathbb{R}$ smooth, periodic functions such that $\sigma \geqslant 0$, $\inf_{\mathbb{T}^d} \zeta \geqslant 0$, $\inf_{\mathbb{T}^d} a > 0$, c_1, $c_2 \geqslant 0$, $\gamma > 1$, $0 \leqslant \tau < 1$, and $\bar{\alpha}$, $\alpha > 0$.

MFGs model problems with large numbers of competing rational agents who seek to optimize an individual utility; see, for example, the lectures in [52]. In the time-dependent case, these games are given by a (time-dependent) Hamilton–Jacobi equation coupled with a transport or Fokker–Planck equation. The stationary case captures ergodic equilibria and corresponds to the long-time limit of time-dependent MFGs [13, 14]. For up-to-date developments on MFGs, we refer the reader to the recent monographs [7, 31], the survey paper [36], and the notes [8]. For numerical aspects, we recommend [1] and the references therein.

Uniformly elliptic MFGs and their corresponding weak solutions were introduced in [49]. The systematic study of the regularity theory for these MFGs was developed in [28, 33, 35, 54]. Those references establish the existence of classical solutions for local MFGs with logarithmic nonlinearities and powerlike nonlinearities with certain growth conditions. A particular stationary congestion model was considered in [23]. Little is known about the general stationary congestion problem. Finally, we point out that some results for time-dependent problems relying on the variational structure of MFGs [9, 10, 11] and some weak solution methods [55, 56] may be extended to the stationary case. However, to the best of our knowledge, this has not been pursued in the literature, except in [18].

In the MFGs literature, there are several gaps in the existence of solutions that we try to address here. First, to obtain smooth solutions of local MFGs, state-of-the-art methods [54] require growth conditions in the nonlinearities. These conditions seem to be of a technical nature rather than of a fundamental nature. Second, regarding weak solutions, the uniformly parabolic case with subquadratic or quadratic Hamiltonians is well understood [55, 56]. In contrast, the degenerate parabolic case and the uniformly parabolic with superquadratic Hamiltonians case are well understood only for variational problems [9, 10, 11]. Moreover, we expect analogous results to hold for degenerate elliptic problems. Unfortunately, variational MFGs are a restricted class of problems that is unstable under perturbations. For example, adding a small, nonlocal perturbation to a variational MFG should not change the theory substantially, but it destroys the variational structure. Third, apart from the preliminary results in [23] and the short-time problems in [38, 40], little is known about weak or classical solutions of congestion models.

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Here, we present a unified approach to studying these problems and construct weak solutions based on monotone operator methods. Monotonicity assumptions are central in MFG theory and are at the heart of the uniqueness proof by Lasry and Lions [49, 50], and [52]; also, the numerical methods in [3] (for stationary MFGs) and [37] (for finite state MFGs) rely on monotonicity ideas. Monotone operator methods have several advantages. First, monotonicity is stable under perturbations; see, for example, [2]. Second, there is a well-developed theory of weak solutions that, when combined with the Minty method, makes it possible to consider various limit problems. Finally, our approach to MFGs through monotonicity answers the earlier existence questions and suggests new computational approaches.

Next, we put forward the basic definitions. A weak solution induced by monotonicity (or for brevity, weak solution) to (1.1) is a pair $(m, u) \in \mathcal{D}'(\mathbb{T}^d) \times \mathcal{D}'(\mathbb{T}^d)$ with $m \geqslant 0$ that satisfies the variational inequality

$$\left\langle \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m \\ u \end{bmatrix}, A \begin{bmatrix} \eta \\ v \end{bmatrix} \right\rangle_{\mathcal{D}'(\mathbb{T}^d)\times\mathcal{D}'(\mathbb{T}^d), C^\infty(\mathbb{T}^d)\times C^\infty(\mathbb{T}^d)} \geqslant 0 \tag{1.6}$$

for all $(\eta, v) \in C^\infty(\mathbb{T}^d; \mathbb{E}) \times C^\infty(\mathbb{T}^d)$, where

$$\begin{aligned} &A \begin{bmatrix} \eta \\ v \end{bmatrix} \\ &:= \begin{bmatrix} -v - H(x, Dv, D^2v, \eta, h(\boldsymbol{\eta})) \\ \eta - \operatorname{div}(\eta D_p H(x, Dv, D^2v, \eta, h(\boldsymbol{\eta}))) + \left(\eta D_{M_{ij}} H(x, Dv, D^2v, \eta, h(\boldsymbol{\eta}))\right)_{x_i x_j} - 1 \end{bmatrix}. \end{aligned} \tag{1.7}$$

If A is a monotone operator, then any classical solution to (1.1) is a weak solution in the previous sense. Additionally, weak solutions induced by monotonicity are related to other weak solutions defined in the literature. For example, the weak solutions defined in [11] are required to do the following.

- Satisfy certain regularity and integrability conditions. For example, for Hamiltonians of the form
$$H(x, Du, D^2u, m, h(\mathbf{m})) = \frac{|Du|^2}{2} - \Delta u - m^\alpha,$$
the requirements in [11] are $u \in W^{1,2}$, $mDu \in L^1$, $m|Du|^2 \in L^1$, and $m \in L^{\alpha+1}$. Here, we only need m and u to be in $\mathcal{D}'$ to give sense to the definition. However, we prove various regularity results under mild conditions; see Theorem 1.1. In particular, for the above example, we have $(m^{\frac{\alpha+1}{2}}, u) \in W^{1,2}(\mathbb{T}^d) \times W^{1,2}(\mathbb{T}^d)$; see Theorem 6.2.
- Be a subsolution of the Hamilton–Jacobi equation in the sense of distributions—this is proven to hold for our weak solutions in section 5 under mild assumptions.
- m must solve the Fokker–Planck equation in the sense of distributions. In general, our weak solutions satisfy the Fokker–Planck equation in a relaxed sense (see Proposition 5.4), and under further assumptions (see Theorem 6.2), as a distribution and pointwise almost everywhere.
- A certain energy identity must hold. This identity roughly means that the Hamilton–Jacobi equation holds in the set $m > 0$. We recover various results along these lines (see sections 5 and 6 and in particular Corollary 6.3),

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where we prove that the Hamilton–Jacobi equation holds pointwise almost everywhere in the set $m > 0$.

Moreover, suppose that H is regular and let $(m, u) \in C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ with $m > 0$ be a weak solution. A straightforward argument shows that (m, u) solves (1.1). To see this, fix a pair $(\psi, \phi) \in C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$. Then, for all sufficiently small $\epsilon > 0$, $m + \epsilon\psi > 0$; taking $(\eta, v) = (m + \epsilon\psi, u + \epsilon\phi)$ in (1.6), dividing by ϵ, and letting $\epsilon \to 0^+$ yield

$$\left\langle \begin{bmatrix} \psi \\ \phi \end{bmatrix}, A \begin{bmatrix} m \\ u \end{bmatrix} \right\rangle_{\mathcal{D}'(\mathbb{T}^d)\times\mathcal{D}'(\mathbb{T}^d), C^\infty(\mathbb{T}^d)\times C^\infty(\mathbb{T}^d)} \geqslant 0.$$

Because ϕ and ψ are arbitrary, we conclude that (m, u) solves (1.1).

Our goal is to prove the existence of weak solutions and to study their properties. Our main result is the following.

THEOREM 1.1. *Suppose that assumptions* (h1)*,* (g1)*,* (g2)*, and* (H1)–(H3) *are satisfied. Then, there exists a weak solution,* $(m, u) \in \mathcal{D}'(\mathbb{T}^d) \times \mathcal{D}'(\mathbb{T}^d)$ *with* $m \geqslant 0$ *to* (1.1)*. Moreover,* $(\boldsymbol{m}, u) \in \mathcal{M}_{ac}(\mathbb{T}^d) \times W^{1,\gamma}(\mathbb{T}^d)$ *and* $\int_{\mathbb{T}^d} m \,\mathrm{d}x = 1$.

This foregoing theorem gives the existence of solutions for a minimal set of assumptions and is a substantial improvement on prior results. First, the theorem is valid for degenerate elliptic MFGs. These are technically challenging since various analytical techniques for first-order MFGs do not apply, and the regularizing effects due to ellipticity are mild to nonexistent. In this case, prior results only apply to problems with a variational structure. Second, the theorem holds for congestion MFGs. These were studied before for a particular problem in [23]; little is known about the existence of solutions for general stationary problems. In the time-dependent setting, only the short-time problem has been considered in the literature [38, 40].

While Theorem 1.1 gives the existence of solutions for MFGs, these solutions have low regularity. In section 5, we consider the degenerate diffusion case and investigate further properties of the weak solutions constructed in Theorem 1.1. Under appropriate conditions, we prove that such solutions are, in the sense of distributions, subsolutions of the first equation in (1.1) (Proposition 5.1), relaxed solutions of the second equation in (1.1) (Proposition 5.4), and relaxed supersolutions of the first equation in (1.1) (Proposition 5.5). The quadratic case is examined in detail in section 6, where we establish higher integrability and Sobolev estimates for m. Moreover, under appropriate assumptions, we prove that the second equation in (1.1) is satisfied pointwise in $\mathbb{T}^d$ (Theorem 6.2) and that the first equation in (1.1) is satisfied pointwise in the set where m is positive (Corollary 6.3).

This paper is structured as follows. Our main assumptions are discussed in section 2. Then, in section 3, we begin the study of (1.1). First, we introduce a regularized problem that involves two positive, small parameters. Next, we prove a priori estimates for solutions to the regularized problem. Combining these estimates with a continuation argument, we show the existence of solutions to the regularized problem. In section 4, we establish further uniform estimates with respect to the parameters that allow us to pass the regularized problem to the limit as these parameters tend to zero. Then, this limiting procedure enables us to prove Theorem 1.1 by using the Minty device. Finally, as mentioned before, sections 5 and 6 are devoted to establishing further properties of weak solutions for specific Hamiltonians. We conclude this paper in section 7 with some remarks on possible extensions and variations.

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2. Main assumptions. Here, we discuss the main assumptions of this paper. In our choice of assumptions, we have attempted to balance generality with brevity and clarity. Naturally, this requires some compromises in illustrating the core ideas, and several extensions and variations of our results are possible. Nevertheless, our assumptions encompass a broad range of examples, including variational MFGs with power and logarithmic nonlinearities, congestion problems, elliptic, degenerate elliptic, and first-order Hamiltonians with both sub- and super-quadratic growth. Furthermore, in section 7, we discuss several extensions and variations including fully nonlinear problems, existence of solutions under lower regularity requirements, and Hamiltonians that are only partially monotone.

We recall that $\mathbb{T}^d$ represents the d-dimensional torus, which we identify with the quotient space $\mathbb{R}^d/\mathbb{Z}^d$. As before, $\mathcal{M}_{ac}(\mathbb{T}^d)$ is the space of positive measures on $\mathbb{T}^d$ that are absolutely continuous with respect to the Lebesgue measure. The elements of $\mathcal{M}_{ac}(\mathbb{T}^d)$ are denoted with boldface font while their densities are represented with the same letter in a nonboldface font. For instance, if $\boldsymbol{m} \in \mathcal{M}_{ac}(\mathbb{T}^d)$, then $m \in L^1(\mathbb{T}^d)$ represents the corresponding density function. Similarly, if $m \in L^1(\mathbb{T}^d)$ is such that $m \geqslant 0$ a.e. in $\mathbb{T}^d$ and $\mathcal{L}^d$ represents the d-dimensional Lebesgue measure, then $\boldsymbol{m} := m\mathcal{L}^d{}_{\lfloor \mathbb{T}^d}$ denotes the corresponding measure in $\mathcal{M}_{ac}(\mathbb{T}^d)$. We also recall that the set $\mathbb{E}$ is either $\mathbb{R}^+$ or $\mathbb{R}^+_0$.

To avoid longer expressions, we often omit some explicit dependencies on the space variable. For example, we write $H(x, Du, D^2u, m, h(\boldsymbol{m}))$ in place of

$$H(x, Du(x), D^2u(x), m(x), h(\boldsymbol{m})(x));$$

similarly, we write $g(x, m, h(\boldsymbol{m}))$ in place of $g(x, m(x), h(\boldsymbol{m})(x))$, and so on.

In what follows,

$$\begin{aligned} h : \mathcal{M}_{ac}(\mathbb{T}^d) &\to C(\mathbb{T}^d), \\ \boldsymbol{m} &\mapsto h(\boldsymbol{m}) \end{aligned}$$

is a (possibly nonlinear) operator. In most of our statements, we suppose the following regularity assumption on h:

(h1) For all $\kappa > \frac{d}{2} + 1$,
- (a) $\{h(\boldsymbol{m})\colon m \in W^{\kappa,2}(\mathbb{T}^d)\} \subset W^{\kappa,2}(\mathbb{T}^d)$;
- (b) $m \in W^{\kappa,2}(\mathbb{T}^d) \mapsto h(\boldsymbol{m}) \in W^{\kappa,2}(\mathbb{T}^d)$ defines a Fréchet differentiable map.

If h satisfies assumption (h1), then for all $m_0 \in W^{\kappa,2}(\mathbb{T}^d)$, there exists a bounded linear operator, $\mathfrak{H}_{m_0} \in \mathcal{L}(W^{\kappa,2}(\mathbb{T}^d); W^{\kappa,2}(\mathbb{T}^d))$, such that, for all $m \in W^{\kappa,2}(\mathbb{T}^d)$,

$$\begin{aligned} &\|h(\boldsymbol{m}) - h(\boldsymbol{m}_0)\|_{W^{\kappa,2}(\mathbb{T}^d)} \\ &\quad \leqslant \|\mathfrak{H}_{m_0}\|_{\mathcal{L}(W^{\kappa,2}(\mathbb{T}^d);W^{\kappa,2}(\mathbb{T}^d))}\|m - m_0\|_{W^{\kappa,2}(\mathbb{T}^d)} + o\big(\|m - m_0\|_{W^{\kappa,2}(\mathbb{T}^d)}\big). \end{aligned} \tag{2.1}$$

Moreover, taking $m_0 = 0$ in (2.1), we get

$$\|h(\boldsymbol{m})\|_{W^{\kappa,2}(\mathbb{T}^d)} \leqslant C_0\big(1 + \|m\|_{W^{\kappa,2}(\mathbb{T}^d)}\big) \tag{2.2}$$

for some positive constant $C_0 = C_0(\kappa, \mathbb{T}^d, \|\mathfrak{H}_0\|_{\mathcal{L}(W^{\kappa,2}(\mathbb{T}^d);W^{\kappa,2}(\mathbb{T}^d))}, \|\mathfrak{H}(\mathbf{0})\|_{W^{\kappa,2}(\mathbb{T}^d)})$.

Examples of operators satisfying the previous assumption are those in (1.3). The regularity requirements in the preceding assumption are minimal as in many applications the convolution kernel ζ in (1.3) is C^∞ and consequently $h : \mathcal{M}_{ac}(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$.

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To describe the behavior of the Hamiltonian (1.2) in the variables (x, m, θ), we introduce a continuous function, $g : \mathbb{T}^d \times \mathbb{E} \times \mathbb{R} \to \mathbb{R}$, and the following assumptions (g1) and (g2). Roughly speaking, to the best of our knowledge, these assumptions correspond to the minimal set of conditions for which our proof gives a priori estimates on the unknown m, and are based on the assertion that g may be decomposed into a sum $g = g_1 + g_2$ as follows:

(g1) (a) There exist continuous functions, g_1 and g_2, such that, for all $(x, m, \theta) \in \mathbb{T}^d \times \mathbb{E} \times \mathbb{R}$,

$$g(x, m, \theta) = g_1(x, m, \theta) + g_2(x, m, \theta)$$

and, for all $\boldsymbol{m} \in \mathcal{M}_{ac}(\mathbb{T}^d)$,

$$g_2(x, m(x), h(\boldsymbol{m})(x)) \geqslant 0.$$

(b) There exists a constant, $c > 0$, such that, for all $\boldsymbol{m} \in \mathcal{M}_{ac}(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} g_2(x, m, h(\boldsymbol{m}))\,\mathrm{d}x \leqslant c\left(1 + \int_{\mathbb{T}^d} m\,\mathrm{d}x\right).$$

(c) For all $\delta > 0$, there exists a constant, $C_\delta > 0$, such that, for all $\boldsymbol{m} \in \mathcal{M}_{ac}(\mathbb{T}^d)$,

$$\max\left\{\int_{\mathbb{T}^d} g_1(x, m, h(\boldsymbol{m}))\,\mathrm{d}x, \int_{\mathbb{T}^d} m\,\mathrm{d}x\right\} \leqslant \delta \int_{\mathbb{T}^d} m g_1(x, m, h(\boldsymbol{m}))\,\mathrm{d}x + C_\delta.$$

(d) There exists $R \geqslant 0$ such that, for all $\boldsymbol{m} \in \mathcal{M}_{ac}(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} m g_1(x, m, h(\boldsymbol{m}))\,\mathrm{d}x \geqslant -R.$$

The previous decomposition of g can be checked directly through simple computations. For example, the functions g in (1.5) with h as in (1.3) satisfy assumption (g1) for $g_1(x, m, \theta) := m^\alpha$ or $g_1(x, m, \theta) := \ln(m)$, and $g_2(x, m, \theta) := \theta$. These functions also satisfy the following compactness criterion in L^1.

(g2) If $(\boldsymbol{m}_j)_{j\in\mathbb{N}} \subset \mathcal{M}_{ac}(\mathbb{T}^d)$ is such that

$$\sup_{j\in\mathbb{N}} \int_{\mathbb{T}^d} m_j g_1(x, m_j, h(\boldsymbol{m}_j))\,\mathrm{d}x < +\infty,$$

then there exists a subsequence of $(m_j)_{j\in\mathbb{N}}$ that converges weakly in $L^1(\mathbb{T}^d)$.

The function g and the two auxiliary functions g_1 and g_2 are used next to state the main assumptions on the Hamiltonian. We begin with assumption (H1) that, together with assumption (g1), will enable a priori bounds for u, Du, and m. These a priori bounds can be obtained via a more general assumption than assumption (H1b). In fact, the way it is written, assumption (H1b) allows for linear dependence of H on second-order derivatives in divergence form because these terms integrate to zero due to periodicity. Nevertheless, a simple adaptation allows us to also address fully nonlinear Hamiltonians. For the sake of clarity and readability, we opted for the simplified version of assumption (H1b) below and, whenever appropriate, we explain how to handle the fully nonlinear case; see, for example, Remarks 3.5 and 4.3. We refer to subsection 7.2 for a brief discussion of this issue.

Recall that $M_1 : M_2 = \mathrm{tr}(M_1 M_2)$ whenever M_1 and M_2 are symmetric matrices.

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(H1) There exist constants, $\gamma > 1$, $0 \leqslant \tau < 1$, $C_1 > 0$, $C_2 > 0$, $C_3 > 0$, and $C_4 > 0$, and a continuous function, $g : \mathbb{T}^d \times \mathbb{E} \times \mathbb{R} \to \mathbb{R}$, such that

(a) for all $(x, p, M, m, \theta) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{E} \times \mathbb{R}$,

$$\begin{aligned} &- H(x, p, M, m, \theta) + D_pH(x, p, M, m, \theta) \cdot p + D_M H(x, p, M, m, \theta) : M \\ &\qquad \geqslant \frac{1}{C_1} m^{-\tau} |p|^\gamma + C_2 g(x, m, \theta) - C_1; \end{aligned} \tag{2.3}$$

(b) for all $(\boldsymbol{m}, u) \in \mathcal{M}_{ac}(\mathbb{T}^d) \times W^{1,\gamma}(\mathbb{T}^d)$ with D^2u a measurable function,

$$\begin{aligned} &- C_3 \int_{\mathbb{T}^d} g(x, m, h(\boldsymbol{m}))\, \mathrm{d}x + \frac{1}{C_1} \int_{\mathbb{T}^d} |Du|^\gamma m^{-\tau}\, \mathrm{d}x - C_1 \\ &\leqslant \int_{\mathbb{T}^d} H(x, Du, D^2u, m, h(\boldsymbol{m}))\, \mathrm{d}x \\ &\leqslant -C_4 \int_{\mathbb{T}^d} g(x, m, h(\boldsymbol{m}))\, \mathrm{d}x + C_1 \left(1 + \int_{\mathbb{T}^d} |Du|^\gamma m^{-\tau}\, \mathrm{d}x\right). \end{aligned}$$

Assumption (H1a) can be regarded as a lower bound for the Lagrangian of the control problem associated with the MFG. For example, if $H(x, p, m) = H_0(x, p) - g(m)$, we introduce the Lagrangian

$$L_0(x, v) = \sup_p \left(-v \cdot p - H_0(x, p)\right).$$

Then, for $v = -D_pH_0(x, p)$, $L_0(x, v) = D_pH_0(x, p)p - H_0(x, p)$. Thus, the left-hand side of (2.3) in assumption (H1a) is

$$L(x, v) - g(m) = D_pH_0(x, p)p - H_0(x, p) - g(m).$$

Assumption (H1b) is simply a coercivity and a growth condition on H.

We proceed with the following regularity assumption on the Hamiltonian. This assumption allows us to prove existence of smooth solutions of a regularized version of our problem through the continuation method. As before, for the sake of clarity and readability, we opted to assume that H is smooth. However, a broad class of less regular Hamiltonians can be also addressed. We refer to subsection 7.2, where we detail this discussion.

(H2) The map $(x, p, M, m, \theta) \mapsto H(x, p, M, m, \theta)$ from $\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{E} \times \mathbb{R}$ to $\mathbb{R}$ is of class C^∞.

We end with the following monotonicity assumption on the Hamiltonian, which is at the core of our definition of weak solutions. Again, for the sake of clarity and readability, we opted to assume this monotonicity with respect to all the arguments of H. Nevertheless, it is possible to also consider the case in which the monotonicity with respect to the nonlocal term is not satisfied. This extension is addressed in subsection 7.3.

(H3) The operator, A, defined in (1.7) is monotone with respect to the $L^2 \times L^2$-inner product. More precisely, for all (η, v), $(\bar\eta, \bar v) \in C^\infty(\mathbb{T}^d; \mathbb{E}) \times C^\infty(\mathbb{T}^d)$, we have

$$\left(A\begin{bmatrix}\eta \\ v\end{bmatrix} - A\begin{bmatrix}\bar\eta \\ \bar v\end{bmatrix}, \begin{bmatrix}\eta \\ v\end{bmatrix} - \begin{bmatrix}\bar\eta \\ \bar v\end{bmatrix} \right)_{L^2(\mathbb{T}^d)\times L^2(\mathbb{T}^d)} \geqslant 0.$$

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Examples of Hamiltonians satisfying (H1)–(H3) are given by (1.3)–(1.5). The great majority of nonlinear Hamiltonians in the MFGs literature satisfy these assumptions, and Theorem 1.1 holds under them.

As mentioned in the introduction, we also investigate further properties of the weak solutions constructed in Theorem 1.1. This study requires additional assumptions on the Hamiltonian. The precise statements of the additional assumptions are given in section 5 (see assumption (H4)) and in section 6 (see assumption (H5)).

3. Regularized MFG. To construct solutions to (1.1), we introduce the regularized problem

$$\begin{bmatrix} -u-H \\ m-\operatorname{div}(mD_pH)+\left(mD_{M_{ij}}H\right)_{x_ix_j}-1 \end{bmatrix} + \begin{bmatrix} \varepsilon_1(m+\Delta^{2p}m)+\varepsilon_2(m-\Delta m)+\beta_{\varepsilon_1}(m) \\ \varepsilon_1(u+\Delta^{2p}u)+\varepsilon_2(u-\Delta u) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{3.1}$$

where ε_1, $\varepsilon_2 > 0$, $p \in \mathbb{N}$ satisfies $2p-4 > \frac{d}{2}+1$, and $\beta_{\varepsilon_1} : (0,+\infty) \to (-\infty, 0]$ is a nondecreasing, C^∞ function satisfying $\beta_{\varepsilon_1}(s) = 0$ if $s \geqslant \varepsilon_1$ and $\beta_{\varepsilon_1}(s) = -\frac{1}{s^q}$ if $0 < s \leqslant \frac{\varepsilon_1}{2}$, where $q > d$. Moreover, the function H (and, analogously, D_pH and $D_{M_{ij}}H$) is identified with the map

$$x \mapsto H(x, Du(x), D^2u(x), m(x), h(\boldsymbol{m})(x)).$$

As shown in Lemma 4.1, the regularization in (3.1) preserves the monotonicity of the operator in the left-hand side. Moreover, for $\varepsilon_1 > 0$, the terms $m + \Delta^{2p}m$ and $u+\Delta^{2p}u$ ensure the solvability and regularity of solutions and the penalization β_{ε_1}, the strict positivity of m. A remarkable feature in this regularization is that the roles of the Hamilton–Jacobi equation and the Fokker–Planck equation are reversed. Usually, we regard the Hamilton–Jacobi equation as an equation for the value function, u. Here, due to the regularization term it is treated as an equation for m. Similarly, the second equation becomes an equation for u. The parameter ε_1 controls the main regularizing effects; the parameter ε_2 is a technical device that is used later to improve the regularity of the weak solutions.

Here, we prove the following result.

PROPOSITION 3.1. *Suppose that assumptions* (h1), (g1), *and* (H1)–(H3) *hold. Then, for all* ε_1, $\varepsilon_2 \in (0,1)$, (3.1) *admits a* C^∞ *solution* (m,u) *with* $\inf_{\mathbb{T}^d} m > 0$.

We postpone the proof of this proposition to the end of this section because we need to establish certain a priori bounds first. After proving these bounds, we use the continuation method to show the existence of a solution.

3.1. The perturbed problem. Fix ε_1, $\varepsilon_2 \in (0,1)$, $\mu \in [0,1]$, $q > d$, and $p \in \mathbb{N}$ such that $2p-4 > \frac{d}{2}+1$. Assume that H satisfies assumption (H1), and define

$$H_\mu(x,p,M,m,\theta) := (1-\mu)H(x,p,M,m,\theta) + \mu\widetilde{H}(p,m),$$

where

$$\widetilde{H}(p,m) := \frac{(1+|p|^2)^{\frac{\gamma}{2}}}{\gamma m^\tau} - m.$$

Our choice of $\widetilde{H}$ ensures the three following properties. First, for $\mu = 1$, we can prove the existence of a solution to the corresponding MFG. Second, H_μ satifies the same

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assumptions as H with bounds that are uniform in μ. Finally, we can prove a priori estimates with bounds that are also uniform in μ. Note that these a priori estimates do not require $\widetilde{H}$ to depend on second derivatives.

Set $\varepsilon := (\varepsilon_1, \varepsilon_2)$, and define the operator $A^\varepsilon_\mu : D(A^\varepsilon_\mu) \subset L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$, where

$$D(A^\varepsilon_\mu) := \left\{ (m,u) \in W^{4p,2}(\mathbb{T}^d) \times W^{4p,2}(\mathbb{T}^d) \colon \inf_{\mathbb{T}^d} m > 0 \right\}, \tag{3.2}$$

by

$$\begin{aligned} A^\varepsilon_\mu \begin{bmatrix} m \\ u \end{bmatrix} \\ := \begin{bmatrix} -u - H_\mu + \varepsilon_1(m + \Delta^{2p} m) + \varepsilon_2(m - \Delta m) + \beta_{\varepsilon_1}(m) \\ m - \operatorname{div}(m D_p H_\mu) + \big(m D_{M_{ij}} H_\mu\big)_{x_i x_j} - 1 + \varepsilon_1(u + \Delta^{2p} u) + \varepsilon_2(u - \Delta u) \end{bmatrix} \end{aligned} \tag{3.3}$$

for $(m,u) \in D(A^\varepsilon_\mu)$. As before, the function H_μ (and, analogously, $D_p H_\mu$ and $D_{M_{ij}} H_\mu$) is identified with the map

$$x \mapsto H_\mu(x, Du(x), D^2u(x), m(x), h(\boldsymbol{m})(x)).$$

Remark 3.2. Observe that $\widetilde{H}$ satisfies assumptions (H2) and (H3). Moreover, if $\tilde{g}_1(x,m,\theta) := m$ and $\tilde{g}_2(x,m,\theta) := 0$, then $\tilde{g} := \tilde{g}_1 + \tilde{g}_2$ satisfies assumptions (g1) and (g2). Furthermore, assumption (H1) is also satisfied, where C_1 is replaced by some positive constant that depends on γ, $\widetilde{C}_1 = \widetilde{C}_1(\gamma)$, and with $C_2 = C_3 = C_4 = 1$. Without loss of generality, we may assume that $\widetilde{C}_1 = C_1$; otherwise, we simply replace both C_1 and $\widetilde{C}_1$ by $\max\{C_1, \widetilde{C}_1\}$ and relabel the constants conveniently.

Recall that $W^{k,2}(\mathbb{T}^d)$ is an algebra for $k > \frac{d}{2}$. Moreover, since $2p - 4 > \frac{d}{2} + 1$, Morrey's theorem yields that the embedding $W^{2p-4,2}(\mathbb{T}^d) \hookrightarrow C^\lambda(\mathbb{T}^d)$ is continuous for some $\lambda \in (0,1)$. In particular, there exists a positive constant, C, depending only on p, d, and λ, such that, for all $w \in W^{2p-4,2}(\mathbb{T}^d)$,

$$\|w\|_{L^\infty(\mathbb{T}^d)} \leqslant C \|w\|_{W^{2p-4,2}(\mathbb{T}^d)}. \tag{3.4}$$

Finally, consider the problem of finding $(m^\varepsilon_\mu, u^\varepsilon_\mu) \in D(A^\varepsilon_\mu)$ satisfying the perturbed MFG

$$A^\varepsilon_\mu \begin{bmatrix} m^\varepsilon_\mu \\ u^\varepsilon_\mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.5}$$

Note that (3.5) with $\mu = 0$ reduces to (3.1).

We set

$$\Lambda^\varepsilon := \{\mu \in [0,1] \colon \exists\, (m^\varepsilon_\mu, u^\varepsilon_\mu) \in D(A^\varepsilon_\mu) \cap C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d) \text{ solving } (3.5)\}. \tag{3.6}$$

Our goal is to establish the equality

$$\Lambda^\varepsilon = [0,1]. \tag{3.7}$$

To this end, next, we begin by proving a priori estimates for classical solutions of (3.5). These are essential to the application of the continuation method. In subsection 3.3, we use this method to show that Λ^ε is a nonempty set, relatively closed, and open in $[0,1]$. Consequently, (3.7) holds.

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3.2. A priori estimates for classical solutions to the perturbed MFG. We start by establishing preliminary a priori estimates for classical solutions of (3.5). These estimates involve constants whose main feature is their independence of any particular choice of solutions to problem (3.5). To simplify the notation, we introduce some nomenclature to specify these constants. We say that *a constant depends only on the problem data* if it is a function of the dimension, d, of p, and of the constants in assumptions (h1), (g1), and (H1). We say that *a constant depends only on the problem data and on* ε if it is a function of d, of p, of the constants in assumptions (h1), (g1), and (H1), of λ, of ε_1, of ε_2, and of the L^∞-estimates of $\beta_{\varepsilon_1}(\cdot)$ and its derivatives on a compact subset of $(0,\infty)$ that depends only on ε_1, on ε_2, and on a constant that depends only on the problem data. We stress that these constants do not depend on the choice of solutions to (3.1) nor on μ.

LEMMA 3.3. *Suppose that assumptions* (g1) *and* (H1) *hold. Let* $\varepsilon \in (0,1)^2$ *and* $\mu \in [0,1]$. *Assume that* $(m_\mu^\varepsilon, u_\mu^\varepsilon) \in C^\infty(\mathbb{T}^d)\times C^\infty(\mathbb{T}^d)$ *solves* (3.5) *in* $\mathbb{T}^d$. *Suppose that* $m_\mu^\varepsilon > 0$ *in* $\mathbb{T}^d$. *Then, there exists a positive constant,* C, *that depends only on the problem data such that*

$$\begin{aligned}&\int_{\mathbb{T}^d} |Du_\mu^\varepsilon|^\gamma (m_\mu^\varepsilon)^{1-\tau}\mathrm{d}x + \int_{\mathbb{T}^d} |Du_\mu^\varepsilon|^\gamma (m_\mu^\varepsilon)^{-\tau}\,\mathrm{d}x + \int_{\mathbb{T}^d} -\beta_{\varepsilon_1}(m_\mu^\varepsilon)\,\mathrm{d}x\\ &\quad + \int_{\mathbb{T}^d} \beta_{\varepsilon_1}(m_\mu^\varepsilon)(m_\mu^\varepsilon - \varepsilon_1)\,\mathrm{d}x + (1-\mu)\int_{\mathbb{T}^d} m_\mu^\varepsilon g_2(x, m_\mu^\varepsilon, h(\boldsymbol{m}_\mu^\varepsilon))\,\mathrm{d}x\\ &\quad + \varepsilon_1 \int_{\mathbb{T}^d} \left[(u_\mu^\varepsilon)^2 + (\Delta^p u_\mu^\varepsilon)^2 + (m_\mu^\varepsilon)^2 + (\Delta^p m_\mu^\varepsilon)^2\right]\mathrm{d}x\\ &\quad + \varepsilon_2 \int_{\mathbb{T}^d} \left((u_\mu^\varepsilon)^2 + |Du_\mu^\varepsilon|^2 + (m_\mu^\varepsilon)^2 + |Dm_\mu^\varepsilon|^2\right)\mathrm{d}x \leqslant C.\end{aligned} \tag{3.8}$$

Proof. In this proof, to simplify the notation, we drop the dependence on ε and μ of $\boldsymbol{m}_\mu^\varepsilon$, m_μ^ε, and u_μ^ε. Accordingly, we simply write $\boldsymbol{m}$, m, and u in place of $\boldsymbol{m}_\mu^\varepsilon$, m_μ^ε, and u_μ^ε, respectively.

By the statement of the lemma, the two following equalities hold in $\mathbb{T}^d$:

$$-u - H_\mu + \varepsilon_1(m + \Delta^{2p}m) + \varepsilon_2(m - \Delta m) + \beta_{\varepsilon_1}(m) = 0, \tag{3.9}$$

$$m - \operatorname{div}(mD_pH_\mu) + (mD_{M_{ij}}H_\mu)_{x_ix_j} - 1 + \varepsilon_1(u + \Delta^{2p}u) + \varepsilon_2(u - \Delta u) = 0, \tag{3.10}$$

where H_μ, D_pH_μ, and $D_{M_{ij}}H_\mu$ are computed at $(x, Du(x), D^2u(x), m(x), h(\boldsymbol{m})(x))$.

We multiply (3.9) by $(m - \varepsilon_1 - 1)$ and (3.10) by u. Next, adding the resulting expressions, integrating over $\mathbb{T}^d$, and integrating by parts yield

$$\begin{aligned}&\int_{\mathbb{T}^d} m\big(-H_\mu + D_pH_\mu\cdot Du + D_MH_\mu : D^2u\big)\,\mathrm{d}x + (\varepsilon_1+1)\int_{\mathbb{T}^d} H_\mu\,\mathrm{d}x + \int_{\mathbb{T}^d} -\beta_{\varepsilon_1}(m)\,\mathrm{d}x\\ &\quad + \int_{\mathbb{T}^d} \beta_{\varepsilon_1}(m)(m-\varepsilon_1)\,\mathrm{d}x + \varepsilon_1\int_{\mathbb{T}^d} \left[u^2 + (\Delta^p u)^2 + m^2 + (\Delta^p m)^2\right]\mathrm{d}x\\ &\quad + \varepsilon_2 \int_{\mathbb{T}^d} \left(u^2 + |Du|^2 + m^2 + |Dm|^2\right)\mathrm{d}x\\ &= -\varepsilon_1\int_{\mathbb{T}^d} u\,\mathrm{d}x + (\varepsilon_1+\varepsilon_2)(\varepsilon_1+1)\int_{\mathbb{T}^d} m\,\mathrm{d}x \leqslant \frac{\varepsilon_1}{2}\int_{\mathbb{T}^d} u^2\,\mathrm{d}x + \frac{\varepsilon_1+\varepsilon_2}{2}\int_{\mathbb{T}^d} m^2\,\mathrm{d}x + \frac{5}{2}.\end{aligned} \tag{3.11}$$

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Let c, C_1, C_2, and C_3 be as in assumptions (g1b) and (H1) and as in Remark 3.2; let $\mathfrak{c} := \max\{C_1, C_3\}$, and set

$$\begin{aligned} F_\mu^\varepsilon :=& \frac{1}{\mathfrak{c}}\int_{\mathbb{T}^d} |Du|^\gamma m^{1-\tau} \mathrm{d}x + \frac{1}{\mathfrak{c}}\int_{\mathbb{T}^d} |Du|^\gamma m^{-\tau}\,\mathrm{d}x \\ &+ (1-\mu)C_2\int_{\mathbb{T}^d} mg_2(x,m,h(\boldsymbol{m}))\,\mathrm{d}x + \int_{\mathbb{T}^d} -\beta_{\varepsilon_1}(m)\,\mathrm{d}x \\ &+ \int_{\mathbb{T}^d} \beta_{\varepsilon_1}(m)(m-\varepsilon_1)\,\mathrm{d}x + \frac{\varepsilon_1}{2}\int_{\mathbb{T}^d}\left[u^2 + (\Delta^p u)^2 + m^2 + (\Delta^p m)^2\right]\mathrm{d}x \\ &+ \frac{\varepsilon_2}{2}\int_{\mathbb{T}^d}\left(u^2 + |Du|^2 + m^2 + |Dm|^2\right)\mathrm{d}x. \end{aligned}$$

Due to (g1b) and the definition of $\beta_{\varepsilon_1}(\cdot)$, the integral terms defining F_μ^ε are nonnegative.

Using in (3.11) the assumptions (H1a) and (H1b) for H and $\widetilde{H}$, it follows that

$$\begin{aligned} &F_\mu^\varepsilon + (1-\mu)C_2\int_{\mathbb{T}^d} mg_1(x,m,h(\boldsymbol{m}))\,\mathrm{d}x + \mu\int_{\mathbb{T}^d} m^2\,\mathrm{d}x \\ &\quad\leqslant \frac{5}{2} + 2\mathfrak{c} + (\mathfrak{c}+2)\int_{\mathbb{T}^d} m\,\mathrm{d}x + 2\mathfrak{c}\int_{\mathbb{T}^d} g_2(x,m,h(\boldsymbol{m}))\,\mathrm{d}x \\ &\qquad + C_3(\varepsilon_1+1)(1-\mu)\int_{\mathbb{T}^d} g_1(x,m,h(\boldsymbol{m}))\,\mathrm{d}x \\ &\quad\leqslant \mathfrak{c}_1 + \mathfrak{c}_2\mu\int_{\mathbb{T}^d} m\,\mathrm{d}x + \mathfrak{c}_2(1-\mu)\int_{\mathbb{T}^d} m\,\mathrm{d}x + \mathfrak{c}_3(1-\mu)\int_{\mathbb{T}^d} g_1(x,m,h(\boldsymbol{m}))\,\mathrm{d}x, \end{aligned} \tag{3.12}$$

where $\mathfrak{c}_1 := \frac{5}{2} + 2\mathfrak{c} + 2c\mathfrak{c}$, $\mathfrak{c}_2 := \mathfrak{c} + 2 + 2c\mathfrak{c}$, and $\mathfrak{c}_3 \in \{C_3, 2C_3\}$, and where we also used (g1a) and (g1b). Using now (g1c) with $\delta := \frac{C_2}{2(\mathfrak{c}_2+\mathfrak{c}_3)}$, we conclude that

$$F_\mu^\varepsilon + \frac{(1-\mu)C_2}{2}\int_{\mathbb{T}^d} mg_1(x,m,h(\boldsymbol{m}))\,\mathrm{d}x + \frac{\mu}{2}\int_{\mathbb{T}^d} m^2\,\mathrm{d}x \leqslant \mathfrak{c}_1 + \frac{\mathfrak{c}_2^2}{2} + (\mathfrak{c}_2+\mathfrak{c}_3)C_\delta. \tag{3.13}$$

Finally, by (g1d), we obtain

$$0 \leqslant F_\mu^\varepsilon \leqslant \mathfrak{c}_1 + \frac{\mathfrak{c}_2^2}{2} + (\mathfrak{c}_2+\mathfrak{c}_3)C_\delta + \frac{C_2}{2}R, \tag{3.14}$$

and this completes the proof of (3.8). □

Remark 3.4. If $\mu = 0$, then (3.13) and the condition $F_0^\varepsilon \geqslant 0$ yield

$$\int_{\mathbb{T}^d} m_0^\varepsilon g_1(x, m_0^\varepsilon, h(\boldsymbol{m}_0^\varepsilon))\,\mathrm{d}x \leqslant C,$$

where C is a positive constant that depends only on the problem data. Moreover, using (g1c) with $\delta = \frac{C_2}{2\mathfrak{c}_2}$ in (3.12) first, and then (g1d) together with the condition $F_0^\varepsilon \geqslant 0$, it follows that

$$\int_{\mathbb{T}^d} -g_1(x, m_0^\varepsilon, h(\boldsymbol{m}_0^\varepsilon))\,\mathrm{d}x \leqslant C,$$

where C is a positive constant that depends only on the problem data. These two estimates are uniform in ε and play a major role in the study of the limit of (3.1) as $\varepsilon_1 \to 0^+$ and $\varepsilon_2 \to 0^+$.

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Remark 3.5 (fully nonlinear Hamiltonians). Suppose that H is as in Lemma 3.3, let $H^0 : \mathbb{R}^{d\times d} \to \mathbb{R}$ be a C^1 convex function, and define $\hat{H} : \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{E} \times \mathbb{R} \to \mathbb{R}$ by $\hat{H}(x,p,M,m,\theta) := H(x,p,M,m,\theta) + H^0(M)$. We claim that if we replace H by $\hat{H}$ in (3.1), the bounds in Lemma 3.3 and Remark 3.4 remain unchanged. Moreover,

$$\left| \int_{\mathbb{T}^d} H^0(D^2 u_0^\varepsilon)\,\mathrm{d}x \right| \leqslant C, \tag{3.15}$$

where C is a positive constant that depends only on the problem data.

In fact, for the Hamiltonian $\hat{H}$, (3.11) remains valid if we add to its left-hand side the term

$$\begin{aligned}(1-\mu)\int_{\mathbb{T}^d} &\Big[-(m-\varepsilon_1-1)H^0(D^2u) + u(mD_{M_{ij}}H^0(D^2u))_{x_ix_j}\Big]\mathrm{d}x \\ &= (1-\mu)\int_{\mathbb{T}^d}\Big[-m\big(H^0(D^2u) - D_{M_{ij}}H^0(D^2u)D^2_{ij}u\big) + (\varepsilon_1+1)H^0(D^2u)\Big]\mathrm{d}x.\end{aligned}$$

By convexity of H^0, we have the bound

$$\begin{aligned}(1-\mu)\int_{\mathbb{T}^d} &\Big[-m\big(H^0(D^2u) - D_{M_{ij}}H^0(D^2u)D^2_{ij}u\big) + (\varepsilon_1+1)H^0(D^2u)\Big]\mathrm{d}x \\ &\geqslant -(1-\mu)H^0(0)\int_{\mathbb{T}^d} m\,\mathrm{d}x + (1-\mu)(\varepsilon_1+1)\int_{\mathbb{T}^d} H^0(D^2u)\,\mathrm{d}x.\end{aligned} \tag{3.16}$$

Thus, setting

$$\hat{F}^\varepsilon_\mu := F^\varepsilon_\mu + (1-\mu)(\varepsilon_1+1)\int_{\mathbb{T}^d} H^0(D^2u)\,\mathrm{d}x,$$

(3.12), (3.13), and the second estimate in (3.14), with F^ε_μ replaced by $\hat{F}^\varepsilon_\mu$, remain valid (up to a convenient change of the value of the constants $\mathfrak{c}$ and $\mathfrak{c}_1$). To conclude, it suffices to observe that the convexity of H^0 and Jensen's inequality yield

$$\int_{\mathbb{T}^d} H^0(D^2u)\,\mathrm{d}x \geqslant H^0(0). \tag{3.17}$$

In fact, this estimate implies, up to a convenient change of the value of the constants $\mathfrak{c}$, $\mathfrak{c}_1$, and $\mathfrak{c}_2$, that (3.12), (3.13), and (3.14) hold. Consequently, so do the statements in Lemma 3.3 and Remark 3.4. Finally, the estimate (3.15) follows from (3.14), from the second estimate in (3.14) with F^ε_μ replaced by $\hat{F}^\varepsilon_\mu$, and from (3.17).

In what follows, if $\varepsilon \in (0,1)^2$ and $\mu \in \Lambda^\varepsilon$, where Λ^ε is the set defined in (3.6), then $(m^\varepsilon_\mu, u^\varepsilon_\mu)$ represents an arbitrary solution to (3.5), which belongs to $D(A^\varepsilon_\mu)\cap C^\infty(\mathbb{T}^d)\times C^\infty(\mathbb{T}^d)$. The next lemma provides a uniform bound with respect to $\mu \in [0,1]$ on the infimum of such solutions.

Lemma 3.6. *Assume that assumptions* (g1) *and* (H1) *hold. Let* $\varepsilon \in (0,1)^2$ *be such that* $\Lambda^\varepsilon \neq \emptyset$*, where* Λ^ε *is the set defined in* (3.6)*. Then,*

$$\inf_{\mu\in\Lambda^\varepsilon} \inf_{\mathbb{T}^d} m^\varepsilon_\mu > 0. \tag{3.18}$$

Proof. We begin by proving that

$$L^{\varepsilon_1} := \sup_{\mu\in\Lambda^\varepsilon} \|m^\varepsilon_\mu\|_{W^{1,\infty}(\mathbb{T}^d)} < +\infty. \tag{3.19}$$

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Because the integral terms in (3.8) are nonnegative, we deduce that

$$\sup_{\mu\in\Lambda^\varepsilon}\left[\varepsilon_1\int_{\mathbb{T}^d}\left((u_\mu^\varepsilon)^2+(\Delta^p u_\mu^\varepsilon)^2+(m_\mu^\varepsilon)^2+(\Delta^p m_\mu^\varepsilon)^2\right)\mathrm{d}x\right]\leqslant C$$

for some positive constant, C, that depends only on the problem data. From this estimate and using integration by parts, we conclude that

$$\sup_{\mu\in\Lambda^\varepsilon}\left[\|u_\mu^\varepsilon\|_{W^{2p,2}(\mathbb{T}^d)}+\|m_\mu^\varepsilon\|_{W^{2p,2}(\mathbb{T}^d)}\right]\leqslant C \tag{3.20}$$

for some positive constant, C, that depends only on the problem data and on ε. Hence, using (3.4), u_μ^ε, $m_\mu^\varepsilon\in C^{4,\lambda}(\mathbb{T}^d)$ with

$$\sup_{\mu\in\Lambda^\varepsilon}\left[\|u_\mu^\varepsilon\|_{W^{4,\infty}(\mathbb{T}^d)}+\|m_\mu^\varepsilon\|_{W^{4,\infty}(\mathbb{T}^d)}\right]\leqslant C \tag{3.21}$$

for some positive constant, C, that depends only on the problem data and on ε. In particular, (3.19) holds.

We now establish (3.18). By contradiction, assume that $\inf_{\mu\in\Lambda^\varepsilon}\inf_{\mathbb{T}^d} m_\mu^\varepsilon=0$. Set $\bar m_\mu^\varepsilon:=\inf_{\mathbb{T}^d} m_\mu^\varepsilon$ and let $(\mu_n)_{n\in\mathbb{N}}\subset\Lambda^\varepsilon$ be such that

$$\lim_{n\to\infty}\bar m_{\mu_n}^\varepsilon=0.$$

Note that $\bar m_{\mu_n}^\varepsilon>0$ by the definition of $D(A_{\mu_n}^\varepsilon)$ (see (3.2)). Because $m_{\mu_n}^\varepsilon$ is continuous in $\mathbb{T}^d$, there exists $\bar x_{\mu_n}^\varepsilon\in\mathbb{T}^d$ satisfying $m_{\mu_n}^\varepsilon(\bar x_{\mu_n}^\varepsilon)=\bar m_{\mu_n}^\varepsilon$. Let $Q(\bar x_{\mu_n}^\varepsilon,\bar m_{\mu_n}^\varepsilon)$ be the cube centered at $\bar x_{\mu_n}^\varepsilon$ and with side length $m_{\mu_n}^\varepsilon$. For all sufficiently large $n\in\mathbb{N}$, $0<\bar m_{\mu_n}^\varepsilon\leqslant\min\{\frac{\varepsilon_1}{4},\frac{\varepsilon_1}{2L^{\varepsilon_1}}\}$. Therefore, for all such $n\in\mathbb{N}$ and for a.e. $x\in Q(\bar x_{\mu_n}^\varepsilon,\bar m_{\mu_n}^\varepsilon)$, we have

$$0<m_{\mu_n}^\varepsilon(x)\leqslant\frac{\bar m_{\mu_n}^\varepsilon L^{\varepsilon_1}}{2}+\bar m_{\mu_n}^\varepsilon\leqslant\frac{\varepsilon_1}{2}.$$

Consequently, using (3.8), the nonnegativeness of $-\beta_{\varepsilon_1}(m_{\mu_n}^\varepsilon)$, and the identity $\beta_{\varepsilon_1}(s)=-s^{-q}$ if $0<s\leqslant\frac{\varepsilon_1}{2}$, we obtain

$$\begin{aligned}C&\geqslant\int_{\mathbb{T}^d}-\beta_{\varepsilon_1}(m_{\mu_n}^\varepsilon)\,\mathrm{d}x\geqslant\int_{Q(\bar x_{\mu_n}^\varepsilon,\bar m_{\mu_n}^\varepsilon)}\frac{1}{(m_{\mu_n}^\varepsilon)^q}\,\mathrm{d}x\\&\geqslant\frac{(\bar m_{\mu_n}^\varepsilon)^d}{\left(\frac{\bar m_{\mu_n}^\varepsilon L^{\varepsilon_1}}{2}+\bar m_{\mu_n}^\varepsilon\right)^q}=\left(\frac{2}{L^{\varepsilon_1}+2}\right)^q\frac{1}{(\bar m_{\mu_n}^\varepsilon)^{q-d}},\end{aligned}$$

where C is a positive constant that depends only on the problem data. Because $q>d$ and $\lim_{n\to\infty}\bar m_{\mu_n}^\varepsilon=0$, we have a contradiction when we let $n\to\infty$ in the estimate above. ∎

Remark 3.7. Under the conditions of the previous lemma, the estimates in (3.18) and (3.21) imply that

$$\underline{m}^\varepsilon:=\inf_{\mu\in\Lambda^\varepsilon}\inf_{\mathbb{T}^d} m_\mu^\varepsilon\quad\text{and}\quad\overline{m}^\varepsilon:=\sup_{\mu\in\Lambda^\varepsilon}\sup_{\mathbb{T}^d} m_\mu^\varepsilon$$

satisfy

$$0<\underline{m}^\varepsilon\leqslant\overline{m}^\varepsilon\leqslant C, \tag{3.22}$$

where C is a positive constant that depends only on the problem data and on ε.

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Next, we prove uniform bounds in $\mu \in [0,1]$ for classical solutions to (3.5) with respect to the norm $\|\cdot\|_{W^{k,\infty}(\mathbb{T}^d)}$. These bounds rely on the previous two lemmas and play an important role in subsection 3.3, where we establish the existence of classical solutions to (3.5).

LEMMA 3.8. *Suppose that assumptions* (h1), (g1), (H1), *and* (H2) *are satisfied. Let* $\varepsilon \in (0,1)^2$ *be such that* $\Lambda^\varepsilon \neq \emptyset$, *where* Λ^ε *is the set defined in* (3.6). *Then, for all* $k \in \mathbb{N}_0$, *there exists a positive constant,* C, *that depends only on the problem data and on* ε, *such that*

$$\sup_{\mu\in\Lambda^\varepsilon} \big(\|m^\varepsilon_\mu\|_{W^{k,\infty}(\mathbb{T}^d)} + \|u^\varepsilon_\mu\|_{W^{k,\infty}(\mathbb{T}^d)}\big) \leqslant C.$$

Proof. To simplify the notation, we drop the dependence on ε and μ of $\boldsymbol{m}^\varepsilon_\mu$, m^ε_μ, and u^ε_μ. Accordingly, we write $\boldsymbol{m}$, m, and u in place of $\boldsymbol{m}^\varepsilon_\mu$, m^ε_μ, and u^ε_μ, respectively. Moreover, C is a positive constant that depends only on the problem data and on ε and whose value may change from one expression to another.

Arguing as in the proof of Lemma 3.6 and using the hypothesis, we conclude that m, $u \in W^{4,\infty}(\mathbb{T}^d) \cap W^{2p,2}(\mathbb{T}^d)$, that (3.20) and (3.21) hold, and that

$$m + \Delta^{2p} m = f_1(m,u), \tag{3.23}$$

$$u + \Delta^{2p} u = f_2(m,u), \tag{3.24}$$

where

$$f_1(m,u) := \frac{1}{\varepsilon_1}\Big[u + H_\mu(x, Du, D^2u, m, h(\boldsymbol{m})) - \beta_{\varepsilon_1}(m) - \varepsilon_2(m - \Delta m)\Big], \tag{3.25}$$

$$\begin{aligned} f_2(m,u) := \frac{1}{\varepsilon_1}\Big[& - m + \operatorname{div}(m D_p H_\mu(x, Du, D^2u, m, h(\boldsymbol{m}))) \\ & - (m D_{M_{ij}} H_\mu(x, Du, D^2u, m, h(\boldsymbol{m})))_{x_i x_j} + 1 - \varepsilon_2(u - \Delta u)\Big]. \end{aligned} \tag{3.26}$$

By (2.2), (3.4), and (3.20), we also have that

$$\|h(\boldsymbol{m})\|_{L^\infty(\mathbb{T}^d)} \leqslant C\|h(\boldsymbol{m})\|_{W^{2p-4,2}(\mathbb{T}^d)} \leqslant C(1 + \|m\|_{W^{2p-4,2}(\mathbb{T}^d)}) \leqslant C. \tag{3.27}$$

Hence, when

$$K := \{(x, Du(x), D^2u(x), m(x), h(\boldsymbol{m})(x)) \colon x \in \mathbb{T}^d\}$$

is thus defined, the estimates (3.21), (3.22), and (3.27) yield that $\overline{K}$ is a compact subset of $\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{E} \times \mathbb{R}$ and there exists a ball, $B(0,C)$, centered at the origin and of radius C such that $\overline{K} \subset B(0,C)$. Using in addition assumption (H2), it follows that, for all $\kappa \in \mathbb{N}_0$,

$$\|H_\mu\|_{W^{\kappa,\infty}(\overline{K})} \leqslant \|H\|_{W^{\kappa,\infty}(\overline{K})} + \|\widetilde{H}\|_{W^{\kappa,\infty}(\overline{K})} \leqslant C. \tag{3.28}$$

Next, we recall that if $q > \frac{d}{2}$, then $W^{q,2}(\mathbb{T}^d)$ is an algebra. Thus, because p is such that $2p - 4 > \frac{d}{2} + 1$, from Remark 3.7, (3.20), (2.2), and (3.28), it follows that

$$\begin{aligned} & f_1(m,u) \in W^{2,2}(\mathbb{T}^d), && \|f_1(m,u)\|_{W^{2,2}(\mathbb{T}^d)} \leqslant c = c(C, \|\beta_{\varepsilon_1}\|_{W^{2,2}([\underline{m}^\varepsilon, \overline{m}^\varepsilon])}), \\ & f_2(m,u) \in L^2(\mathbb{T}^d), && \|f_2(m,u)\|_{L^2(\mathbb{T}^d)} \leqslant \bar{c} = \bar{c}(C). \end{aligned}$$

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Therefore, (3.23), (3.24), and the elliptic regularity theory yield

$$m \in W^{4p+2,2}(\mathbb{T}^d) \text{ with } \|m\|_{W^{4p+2,2}(\mathbb{T}^d)} \leqslant c_1 = c_1(c),$$
$$u \in W^{4p,2}(\mathbb{T}^d) \text{ with } \|u\|_{W^{4p,2}(\mathbb{T}^d)} \leqslant \bar{c}_1 = \bar{c}_1(\bar{c}).$$

Then, going back to (3.25) and (3.26), and arguing as above, we have

$$f_1(m,u) \in W^{4p-2,2}(\mathbb{T}^d),\ \|f_1(m,u)\|_{W^{4p-2,2}(\mathbb{T}^d)} \leqslant c_2 = c_2(c_1, \bar{c}_1, \|\beta_{\varepsilon_1}\|_{W^{4p-2,2}([\underline{m}^\varepsilon, \overline{m}^\varepsilon])}),$$
$$f_2(m,u) \in W^{4p-4,2}(\mathbb{T}^d),\ \|f_2(m,u)\|_{W^{4p-4,2}(\mathbb{T}^d)} \leqslant \bar{c}_2 = \bar{c}_2(c_1, \bar{c}_1).$$

Arguing as before, we conclude that

$$m \in W^{8p-2,2}(\mathbb{T}^d) \text{ with } \|m\|_{W^{8p-2,2}(\mathbb{T}^d)} \leqslant c_3 = c_3(c_2, \bar{c}_2),$$
$$u \in W^{8p-4,2}(\mathbb{T}^d) \text{ with } \|u\|_{W^{8p-4,2}(\mathbb{T}^d)} \leqslant \bar{c}_3 = \bar{c}_3(c_2, \bar{c}_2).$$

The conclusion follows by iterating this "bootstrap" argument and using the Sobolev embedding theorem. ∎

3.3. Existence of classical solutions to the perturbed MFG.

Here, we prove that $\Lambda^\varepsilon = [0,1]$ for all $\varepsilon \in (0,1)^2$. Consequently, by the definition of Λ^ε in (3.6), we have that (3.1) has a solution, $(u^\varepsilon, m^\varepsilon)$, with $m^\varepsilon > 0$. To prove that $\Lambda^\varepsilon = [0,1]$, we show that Λ^ε is a nonempty set, relatively closed, and open in $[0,1]$. These properties are established in the next three lemmas.

LEMMA 3.9. *Let Λ^ε be the set defined in* (3.6). *Then, $\Lambda^\varepsilon \neq \emptyset$.*

Proof. To prove the lemma, we show that $1 \in \Lambda^\varepsilon$. We set

$$f(c) := -c - \frac{1}{\gamma(1 - c(\varepsilon_1+\varepsilon_2))^\tau} + (\varepsilon_1+\varepsilon_2)(1 - c(\varepsilon_1+\varepsilon_2)) + \beta_{\varepsilon_1}(1 - c(\varepsilon_1+\varepsilon_2)).$$

Recall that $\gamma \geqslant 1 > 0$, $\tau \geqslant 0$, $\beta_{\varepsilon_1}(s) = 0$ if $s \geqslant \varepsilon_1$, and $\lim_{s\to 0^+} \beta_{\varepsilon_1}(s) = -\infty$. Consequently, we have that $\lim_{c\to -\infty} f(c) = +\infty$ and $\lim_{c \to (\frac{1}{\varepsilon_1+\varepsilon_2})^-} f(c) = -\infty$. Therefore, there exists $\bar{c} \in (-\infty, \frac{1}{\varepsilon_1+\varepsilon_2})$ such that $f(\bar{c}) = 0$.

Set $m_1^\varepsilon := 1 - \bar{c}(\varepsilon_1+\varepsilon_2)$ and $u_1^\varepsilon := \bar{c}$. Then, $(m_1^\varepsilon, u_1^\varepsilon) \in D(A_1^\varepsilon) \cap C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ and

$$A_1^\varepsilon \begin{bmatrix} m_1^\varepsilon \\ u_1^\varepsilon \end{bmatrix}$$
$$= \begin{bmatrix} -\bar{c} - \frac{1}{\gamma(1-\bar{c}(\varepsilon_1+\varepsilon_2))^\tau} + (\varepsilon_1+\varepsilon_2)(1-\bar{c}(\varepsilon_1+\varepsilon_2)) + \beta_{\varepsilon_1}(1-\bar{c}(\varepsilon_1+\varepsilon_2)) \\ 1 - \bar{c}(\varepsilon_1+\varepsilon_2) + \varepsilon_1\bar{c} + \varepsilon_2\bar{c} - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This completes the proof. ∎

LEMMA 3.10. *Suppose that assumptions* (h1), (g1), (H1), *and* (H2) *hold. Let Λ^ε be the set defined in* (3.6). *Then, Λ^ε is a closed subset of* $[0,1]$.

Proof. Let $\{\mu_n\}_{n\in\mathbb{N}} \subset \Lambda^\varepsilon$ be a sequence converging to some $\mu_0 \in [0,1]$. We claim that $\mu_0 \in \Lambda^\varepsilon$. To see this, let $\{(m_{\mu_n}^\varepsilon, u_{\mu_n}^\varepsilon)\}_{n\in\mathbb{N}} \subset D(A_{\mu_n}^\varepsilon) \cap C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ be such that

$$A_{\mu_n}^\varepsilon \begin{bmatrix} m_{\mu_n}^\varepsilon \\ u_{\mu_n}^\varepsilon \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.29}$$

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By Lemma 3.8,

$$\sup_{n\in\mathbb{N}} \left(\|m^\varepsilon_{\mu_n}\|_{W^{4p+1,2}(\mathbb{T}^d)} + \|u^\varepsilon_{\mu_n}\|_{W^{4p+1,2}(\mathbb{T}^d)}\right) \leqslant C$$

for some positive constant, C, that depends only on the problem data and on ε. Hence, by the Rellich–Kondrachov theorem, we can find $(m^\varepsilon, u^\varepsilon) \in W^{4p+1,2}(\mathbb{T}^d) \times W^{4p+1,2}(\mathbb{T}^d)$ such that, up to a (not relabeled) subsequence, $(m^\varepsilon_{\mu_n}, u^\varepsilon_{\mu_n})_{n\in\mathbb{N}}$ converges to $(m^\varepsilon, u^\varepsilon)$ weakly in $W^{4p+1,2}(\mathbb{T}^d)$ and strongly in $W^{4p,2}(\mathbb{T}^d)$ and

$$D^\alpha m^\varepsilon_{\mu_n}(x) \to D^\alpha m^\varepsilon(x) \text{ and } D^\alpha u^\varepsilon_{\mu_n}(x) \to D^\alpha u^\varepsilon(x)$$

as $n \to \infty$ for a.e. $x \in \mathbb{T}^d$ and for all $\alpha \in \mathbb{N}^d_0$ with $|\alpha| \leqslant 4p$. Using (2.1) with $\boldsymbol{m}$ and $\boldsymbol{m}_0$ replaced by $\boldsymbol{m}^\varepsilon_{\mu_n}$ and $\boldsymbol{m}^\varepsilon$, respectively, up to a further (not relabeled) subsequence, we also have that for a.e. $x \in \mathbb{T}^d$ and for all $\alpha \in \mathbb{N}^d_0$ with $|\alpha| \leqslant 4p$,

$$D^\alpha h(\boldsymbol{m}^\varepsilon_{\mu_n})(x) \to D^\alpha h(\boldsymbol{m}^\varepsilon)(x)$$

as $n \to \infty$. Moreover, using Lemma 3.8 once more and in view of Lemma 3.6, $(m^\varepsilon, u^\varepsilon) \in W^{4p+1,2}(\mathbb{T}^d) \times W^{4p+1,2}(\mathbb{T}^d) \cap C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ with $\inf_{\mathbb{T}^d} m^\varepsilon > 0$. By assumption (H2) and because $\beta_{\varepsilon_1}(\cdot)$ is smooth, passing (3.29) to the limit as $n \to \infty$, we conclude that

$$A^\varepsilon_{\mu_0} \begin{bmatrix} m^\varepsilon \\ u^\varepsilon \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence, $\mu_0 \in \Lambda^\varepsilon$. ∎

LEMMA 3.11. *Under assumptions* (h1)*,* (g1)*, and* (H1)–(H3)*, the set* Λ^ε *defined in* (3.6) *is an open subset of* $[0,1]$*.*

Proof. Fix $\mu_0 \in \Lambda^\varepsilon$. We want to show that there exists a neighborhood of μ_0 in $[0,1]$ contained in Λ^ε. That will be a consequence of the implicit function theorem in Banach spaces (see, for example, [17]).

Fix a solution, $(m_0, u_0) \in D(A^\varepsilon_\mu) \cap C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$, to (3.5) with $\mu = \mu_0$.

In this proof, to simplify the notation, we omit the dependence on ε and μ_0.

To use the implicit function theorem, we have to prove that the Fréchet derivative, $\mathcal{L} : W^{4p,2}(\mathbb{T}^d) \times W^{4p,2}(\mathbb{T}^d) \to L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$, of

$$\begin{bmatrix} m \\ u \end{bmatrix} \mapsto A^\varepsilon_{\mu_0} \begin{bmatrix} m \\ u \end{bmatrix}$$

at (m_0, u_0) is an isomorphism from $W^{4p,2}(\mathbb{T}^d) \times W^{4p,2}(\mathbb{T}^d)$ to $L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$.

Given $(\eta, v) \in W^{4p,2}(\mathbb{T}^d) \times W^{4p,2}(\mathbb{T}^d)$, we have that

$$\mathcal{L}\begin{bmatrix} \eta \\ v \end{bmatrix} = \begin{bmatrix} -v - \ell_1(\eta, v) + \ell_2(\eta) \\ \eta - \operatorname{div}(\eta D_p H_{\mu_0} + m_0 \ell_3(\eta, v)) + (\eta D_{M_{ij}} H_{\mu_0} + m_0 \ell_4(\eta, v))_{x_i x_j} + \ell_5(v) \end{bmatrix},$$

where, recalling that $\mathfrak{H}_{m_0}$ is the Fréchet derivative of $m \in W^{4p,2}(\mathbb{T}^d) \mapsto h(\boldsymbol{m}) \in W^{4p,2}(\mathbb{T}^d)$ at m_0 (see (2.1)),

$$\begin{aligned}
\ell_1(\eta, v) &:= D_p H_{\mu_0} \cdot Dv + D_M H_{\mu_0} : D^2 v + D_m H_{\mu_0} \eta + D_\theta H_{\mu_0} \mathfrak{H}_{m_0}(\eta), \\
\ell_2(\eta) &:= \varepsilon_1(\eta + \Delta^{2p}\eta) + \varepsilon_2(\eta - \Delta\eta) + \beta'_\varepsilon(m_0)\eta, \\
\ell_3(\eta, v) &:= D^2_{pp} H_{\mu_0} Dv + D^2_{M_{ij}p} H_{\mu_0} D^2_{ij} v + D^2_{mp} H_{\mu_0} \eta + D^2_{\theta p} H_{\mu_0} \mathfrak{H}_{m_0}(\eta), \\
\ell_4(\eta, v) &:= D^2_{pM_{ij}} H_{\mu_0} \cdot Dv + D^2_{MM_{ij}} H_{\mu_0} : D^2 v + D^2_{mM_{ij}} H_{\mu_0} \eta + D_{\theta M_{ij}} H_{\mu_0} \mathfrak{H}_{m_0}(\eta), \\
\ell_5(v) &:= \varepsilon_1(v + \Delta^{2p} v) + \varepsilon_2(v - \Delta v),
\end{aligned}$$

and the argument of H_{μ_0} and all its partial derivatives is $(x, Du_0, D^2u_0, m_0, h(\boldsymbol{m}_0))$.

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Let

$$K_0 := \{(x, Du_0(x), D^2u_0(x), m_0(x), h(\boldsymbol{m}_0)(x)) \colon x \in \mathbb{T}^d\}.$$

Arguing as in Lemma 3.8, for all $\kappa \in \mathbb{N}_0$, we have

$$\|H_\mu\|_{W^{\kappa,\infty}(\overline{K}_0)} \leqslant \|H\|_{W^{\kappa,\infty}(\overline{K}_0)} + \|\widetilde{H}\|_{W^{\kappa,\infty}(\overline{K}_0)} \leqslant C \tag{3.30}$$

for some positive constant, C, that depends only on the problem data and on ε.

Note also that $\mathcal{L}(\eta, v) \in L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ because $W^{4p-4,2}(\mathbb{T}^d)$ is an algebra, $\inf_{\mathbb{T}^d} m_0 > 0$, and in view of (2.2) and (3.30). We will use the Lax–Milgram theorem to prove that $\mathcal{L}$ defines an isomorphism.

Fix $\bar{z} = (\bar{z}_1, \bar{z}_2) \in L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$. Next, we show that there exists a unique $\bar{w} = (\bar{\eta}, \bar{v}) \in W^{4p,2}(\mathbb{T}^d) \times W^{4p,2}(\mathbb{T}^d)$ such that $\mathcal{L}\bar{w} = \bar{z}$.

Define a map $L : W^{2p,2}(\mathbb{T}^d) \times W^{2p,2}(\mathbb{T}^d) \to \mathbb{R}$ by $L[w] := (\bar{z}, w)_{L^2(\mathbb{T}^d)\times L^2(\mathbb{T}^d)}$ for $w = (\eta, v) \in W^{2p,2}(\mathbb{T}^d) \times W^{2p,2}(\mathbb{T}^d)$. Clearly, L is linear and bounded in $W^{2p,2}(\mathbb{T}^d) \times W^{2p,2}(\mathbb{T}^d)$.

Now, let $B : W^{2p,2}(\mathbb{T}^d) \times W^{2p,2}(\mathbb{T}^d) \to \mathbb{R}$ be the bilinear form defined for $w_1 = (\eta_1, v_1)$, $w_2 = (\eta_2, v_2) \in W^{2p,2}(\mathbb{T}^d) \times W^{2p,2}(\mathbb{T}^d)$ by

$$\begin{aligned} B[w_1, w_2] := \int_{\mathbb{T}^d} \big[&- v_1\eta_2 + \eta_1 v_2 - \ell_1(\eta_1, v_1)\eta_2 + m_0 D_p H_{\mu_0} \cdot Dv_2 + \eta_1 \ell_3(\eta_1, v_1) \cdot Dv_2 \\ &+ \eta_1 D_M H_{\mu_0} : D^2 v_2 + m_0 \ell_4(\eta_1, v_1) D^2_{ij} v_2 \\ &+ \varepsilon_1(\eta_1\eta_2 + \Delta^p \eta_1 \Delta^p \eta_2 + v_1 v_2 + \Delta^p v_1 \Delta^p v_2) + \beta'_{\varepsilon_1}(m_0)\eta_1\eta_2 \\ &+ \varepsilon_2(\eta_1\eta_2 + \nabla\eta_2\nabla\eta_2 + v_1 v_2 + \nabla v_1 \nabla v_2)\big]\, \mathrm{d}x. \end{aligned}$$

We start by observing that if $w_1 = (\eta_1, v_1) \in W^{4p,2}(\mathbb{T}^d) \times W^{4p,2}(\mathbb{T}^d)$ and $w_2 = (\eta_2, v_2) \in W^{2p,2}(\mathbb{T}^d) \times W^{2p,2}(\mathbb{T}^d)$, then

$$B[w_1, w_2] = (\mathcal{L}w_1, w_2)_{L^2(\mathbb{T}^d)\times L^2(\mathbb{T}^d)}. \tag{3.31}$$

Moreover, for $w = (\eta, v) \in W^{2p,2}(\mathbb{T}^d) \times W^{2p,2}(\mathbb{T}^d)$,

$$\begin{aligned} B[w, w] = B_1[w, w] &+ \int_{\mathbb{T}^d} \varepsilon_1(\eta^2 + (\Delta^p\eta)^2 + v^2 + (\Delta^p v)^2) + \beta'_{\varepsilon_1}(m_0)\eta^2\, \mathrm{d}x \\ &+ \int_{\mathbb{T}^d} \varepsilon_1(\eta^2 + |\nabla\eta|^2 + v^2 + |\nabla v|^2)\, \mathrm{d}x, \end{aligned}$$

where

$$\begin{aligned} B_1[w, w] := \int_{\mathbb{T}^d} \big[&- \ell_1(\eta, v)\eta + m_0 D_p H_{\mu_0} \cdot Dv + \eta\ell_3(\eta, v) \cdot Dv + \eta D_M H_{\mu_0} : D^2 v \\ &+ m_0 \ell_4(\eta, v) D^2_{ij} v\big]\, \mathrm{d}x. \end{aligned}$$

We claim that

$$B_1[w, w] \geqslant 0. \tag{3.32}$$

Assume that (3.32) holds. Then, using integration by parts, Gagliardo–Nirenberg's interpolation inequalities, and the fact that $m_0 > 0$ and $\beta'_{\varepsilon_1}(m_0) \geqslant 0$ in $\mathbb{T}^d$, it follows that there exists a positive constant, $C = C(\varepsilon, p, d, \mathbb{T}^d)$, such that

$$B[w, w] \geqslant C\|w\|^2_{W^{2p,2}(\mathbb{T}^d)\times W^{2p,2}(\mathbb{T}^d)}.$$

Hence, $B[\cdot, \cdot]$ is coercive in $W^{2p,2}(\mathbb{T}^d) \times W^{2p,2}(\mathbb{T}^d)$.

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Next, we prove that (3.32) is a consequence of the monotonicity of H and $\widetilde{H}$ encoded in assumption (H3). In fact, let $\widetilde{A}$ and A_{μ_0} be the operators associated with $\widetilde{H}$ and H_{μ_0}, respectively (see (1.7)). Then, $A_{\mu_0} = (1-\mu_0)A + \mu_0\widetilde{A}$. Since both A and $\widetilde{A}$ satisfy assumption (H3), the same holds for A_{μ_0}. Consequently,

$$0 \leqslant \lim_{\delta\to 0^+} \frac{1}{\delta^2}\left(A^\varepsilon_{\mu_0}\begin{bmatrix} m_0+\delta\eta \\ u_0+\delta v\end{bmatrix} - A^\varepsilon_{\mu_0}\begin{bmatrix} m_0 \\ u_0\end{bmatrix}, \begin{bmatrix}\delta\eta \\ \delta v\end{bmatrix}\right)_{L^2(\mathbb{T}^d)\times L^2(\mathbb{T}^d)} = B_1[w,w],$$

which proves (3.32).

We also have that $B[\cdot,\cdot]$ is bounded in $W^{2p,2}(\mathbb{T}^d)\times W^{2p,2}(\mathbb{T}^d)$; more precisely,

$$|B[w_1,w_2]| \leqslant C\|w_1\|_{W^{2p,2}(\mathbb{T}^d)\times W^{2p,2}(\mathbb{T}^d)}\|w_2\|_{W^{2p,2}(\mathbb{T}^d)\times W^{2p,2}(\mathbb{T}^d)}$$

for some positive constant, C, that depends only on the problem data and on ε in view of Remark 3.7, Lemma 3.8, (2.2), and (3.30).

By the Lax–Milgram theorem, there exists a unique $\bar{w} = (\bar{\eta},\bar{v}) \in W^{2p,2}(\mathbb{T}^d)\times W^{2p,2}(\mathbb{T}^d)$ such that

$$B[\bar{w},w] = L[w] = (\bar{z},w)_{L^2(\mathbb{T}^d)\times L^2(\mathbb{T}^d)} \tag{3.33}$$

for all $w \in W^{2p,2}(\mathbb{T}^d)\times W^{2p,2}(\mathbb{T}^d)$. From (3.31), the proof is complete once we show that $\bar{w} \in W^{4p,2}(\mathbb{T}^d)\times W^{4p,2}(\mathbb{T}^d)$.

Equality (3.33) yields that $\bar{\theta} \in W^{2p,2}(\mathbb{T}^d)$ and $\bar{v} \in W^{2p,2}(\mathbb{T}^d)$ are weak solutions of

$$\varepsilon(\bar{\eta} + \Delta^{2p}\bar{\eta}) = \bar{v} + \ell_1(\bar{\eta},\bar{v}) - \ell_2(\bar{\eta}) + \bar{z}_1 =: g_1[\bar{\eta},\bar{v}]$$

and

$$\begin{aligned}\varepsilon(\bar{v} + \Delta^{2p}\bar{v}) =& -\bar{\eta} + \operatorname{div}(m_0 D_p H_{\mu_0} + \bar{\eta}\ell_3(\bar{\eta},\bar{v})) \\ &- (\bar{\eta} D_{M_{ij}} H_{\mu_0} + m_0\ell_4(\bar{\eta},\bar{v}))_{x_i x_j} - \ell_5(\bar{v}) + \bar{z}_2 =: g_2[\bar{\eta},\bar{v}],\end{aligned}$$

respectively, where the argument of H_{μ_0} and all its partial derivatives is $(x, Du_0, D^2u_0, m_0, h(\boldsymbol{m}_0))$. Arguing as above, we have that $g_1[\bar{\eta},\bar{v}]$, $g_2[\bar{\eta},\bar{v}] \in L^2(\mathbb{T}^d)$; hence, by the elliptic regularity theory, it follows that $\bar{\eta} \in W^{4p,2}(\mathbb{T}^d)$ and $\bar{v} \in W^{4p,2}(\mathbb{T}^d)$. ∎

As a result of the previous lemmas, we obtain (3.7).

Corollary 3.12. *Assume that assumptions* (h1)*,* (g1)*, and* (H1)–(H3) *are satisfied and let Λ^ε be the set defined in* (3.6)*. Then,* $\Lambda^\varepsilon = [0,1]$.

The proof of Proposition 3.1 is now a simple matter.

Proof of Proposition 3.1. By Corollary 3.12, $0 \in \Lambda^\varepsilon$. Therefore, the claim in the statement holds. ∎

4. Existence of weak solutions.

Our next goal is to establish the existence of weak solutions to (1.1). Because the parameter μ does not play any role in this section, we set $\mu = 0$ and $A^\varepsilon = A_0^\varepsilon$, and we recall that (3.1) corresponds to (3.5) with $\mu = 0$. We proceed in three steps. First, we prove that the operator A^ε is monotone. Next, we prove estimates for solutions of (3.1) that are uniform in ε. Finally, combining the monotonicity and these estimates, we use Minty's method to obtain a weak solution to (1.1).

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LEMMA 4.1. *Suppose that assumption* (H3) *holds. Then, the operator* A^ε *given by* (3.2)–(3.3) *(with* $\mu = 0$*) is monotone in* $L^2(\mathbb{T}^d)\times L^2(\mathbb{T}^d)$*. More precisely, for all* (m_1,u_1), $(m_2,u_2)\in D(A^\varepsilon)\cap C^\infty(\mathbb{T}^d)\times C^\infty(\mathbb{T}^d)$*, we have that*

$$\left(A^\varepsilon\begin{bmatrix}m_1\\u_1\end{bmatrix}-A^\varepsilon\begin{bmatrix}m_2\\u_2\end{bmatrix},\begin{bmatrix}m_1\\u_1\end{bmatrix}-\begin{bmatrix}m_2\\u_2\end{bmatrix}\right)_{L^2(\mathbb{T}^d)\times L^2(\mathbb{T}^d)}\geqslant 0.$$

Proof. Let (m_1,u_1), $(m_2,u_2)\in D(A^\varepsilon)\cap C^\infty(\mathbb{T}^d)\times C^\infty(\mathbb{T}^d)$. Using the definition of A^ε and A (see (3.3) and (1.7)), we obtain

$$\begin{aligned}&\left(A^\varepsilon\begin{bmatrix}m_1\\u_1\end{bmatrix}-A^\varepsilon\begin{bmatrix}m_2\\u_2\end{bmatrix},\begin{bmatrix}m_1\\u_1\end{bmatrix}-\begin{bmatrix}m_2\\u_2\end{bmatrix}\right)_{L^2(\mathbb{T}^d)\times L^2(\mathbb{T}^d)}\\
&\quad=\left(A\begin{bmatrix}m_1\\u_1\end{bmatrix}-A\begin{bmatrix}m_2\\u_2\end{bmatrix},\begin{bmatrix}m_1\\u_1\end{bmatrix}-\begin{bmatrix}m_2\\u_2\end{bmatrix}\right)_{L^2(\mathbb{T}^d)\times L^2(\mathbb{T}^d)}\\
&\qquad+\int_{\mathbb{T}^d}(\beta_{\varepsilon_1}(m_1)-\beta_{\varepsilon_1}(m_2))(m_1-m_2)\,\mathrm{d}x\\
&\qquad+\varepsilon_1\int_{\mathbb{T}^d}\left[(m_1-m_2)^2+(\Delta^p m_1-\Delta^p m_2)^2+(u_1-u_2)^2+(\Delta^p u_1-\Delta^p u_2)^2\right]\mathrm{d}x\\
&\qquad+\varepsilon_2\int_{\mathbb{T}^d}\left[(m_1-m_2)^2+(Dm_1-Dm_2)^2+(u_1-u_2)^2+(Du_1-Du_2)^2\right]\mathrm{d}x.\end{aligned}$$

The conclusion follows because $\beta_{\varepsilon_1}(\cdot)$ is a nondecreasing function and A satisfies (H3). ∎

Next, we give bounds for solutions of (3.1).

PROPOSITION 4.2. *Suppose that assumptions* (g1) *and* (H1) *hold. Then, there exists a positive constant,* C*, such that, for any* $\varepsilon\in(0,1)^2$ *and any solution* $(u^\varepsilon,m^\varepsilon)\in D(A^\varepsilon)$ *to* (3.1)*, we have*

$$\int_{\mathbb{T}^d}|Du^\varepsilon|^\gamma\,\mathrm{d}x\leqslant C, \tag{4.1}$$

$$\left|\int_{\mathbb{T}^d}u^\varepsilon\,\mathrm{d}x\right|\leqslant C, \tag{4.2}$$

$$\int_{\mathbb{T}^d}|\beta_{\varepsilon_1}(m^\varepsilon)|\,\mathrm{d}x\leqslant C, \tag{4.3}$$

$$\int_{\mathbb{T}^d}m^\varepsilon g_1(x,m^\varepsilon,h(\boldsymbol{m}^\varepsilon))\,\mathrm{d}x\leqslant C, \tag{4.4}$$

and

$$\int_{\mathbb{T}^d}-g_1(x,m^\varepsilon,h(\boldsymbol{m}^\varepsilon))\,\mathrm{d}x\leqslant C. \tag{4.5}$$

Proof. To simplify the notation, C represents a positive constant depending only on the problem data and whose value may change from one expression to another.

Estimates (4.4) and (4.5) follow from Remark 3.4.

Next, we prove (4.1) and (4.3). Because all integral terms in (3.8) are nonnegative, $0\leqslant\tau<1$, $m^\varepsilon>0$, and $\beta_{\varepsilon_1}(\cdot)\leqslant 0$, it follows from Lemma 3.3 with $\mu=0$ that

$$C\geqslant\int_{\mathbb{T}^d}-\beta_{\varepsilon_1}(m^\varepsilon)\,\mathrm{d}x=\int_{\mathbb{T}^d}|\beta_{\varepsilon_1}(m^\varepsilon)|\,\mathrm{d}x$$

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and

$$
\begin{aligned}
C &\geqslant \int_{\mathbb{T}^d} |Du^\varepsilon|^\gamma (m^\varepsilon)^{1-\tau} \mathrm{d}x + \int_{\mathbb{T}^d} |Du^\varepsilon|^\gamma (m^\varepsilon)^{-\tau} \, \mathrm{d}x \\
&\geqslant \int_{\{x\in\mathbb{T}^d\colon m^\varepsilon(x)\geqslant 1\}} |Du^\varepsilon|^\gamma \mathrm{d}x + \int_{\{x\in\mathbb{T}^d\colon m^\varepsilon(x)<1\}} |Du^\varepsilon|^\gamma \, \mathrm{d}x \\
&= \int_{\mathbb{T}^d} |Du^\varepsilon|^\gamma \mathrm{d}x.
\end{aligned}
$$

Therefore, (4.1) and (4.3) hold.

Finally, we show (4.2). Integrating the first equation in (3.1) over $\mathbb{T}^d$, we obtain

$$
\begin{aligned}
\int_{\mathbb{T}^d} u^\varepsilon \, \mathrm{d}x = &- \int_{\mathbb{T}^d} H(x, Du^\varepsilon, D^2u^\varepsilon, m^\varepsilon, h(\boldsymbol{m}^\varepsilon)) \, \mathrm{d}x + \int_{\mathbb{T}^d} \beta_{\varepsilon_1}(m^\varepsilon) \, \mathrm{d}x \\
&+ (\varepsilon_1 + \varepsilon_2) \int_{\mathbb{T}^d} m^\varepsilon \, \mathrm{d}x.
\end{aligned} \tag{4.6}
$$

By (3.8) with $\mu = 0$ and (4.3), we have that

$$
\begin{aligned}
&\int_{\mathbb{T}^d} \left|\beta_{\varepsilon_1}(m^\varepsilon)\right| \mathrm{d}x + (\varepsilon_1 + \varepsilon_2) \int_{\mathbb{T}^d} \left|m^\varepsilon\right| \mathrm{d}x \\
&\qquad \leqslant C + 2 + (\varepsilon_1 + \varepsilon_2) \int_{\mathbb{T}^d} (m^\varepsilon)^2 \, \mathrm{d}x \leqslant C.
\end{aligned} \tag{4.7}
$$

Next, we estimate the upper and lower bounds of $\int_{\mathbb{T}^d} H \, \mathrm{d}x$ in (H1b). According to (g1a), (4.5), and (3.8) with $\mu = 0$, we have that

$$
- C_4 \int_{\mathbb{T}^d} g(x, m^\varepsilon, h(\boldsymbol{m}^\varepsilon)) \, \mathrm{d}x + C_1 \left(1 + \int_{\mathbb{T}^d} |Du^\varepsilon|^\gamma (m^\varepsilon)^{-\tau} \, \mathrm{d}x \right) \leqslant C. \tag{4.8}
$$

Similarly, from (g1a), (g1b), (g1c) with $\delta = 1$, and (4.4), we get

$$
\begin{aligned}
&C_3 \int_{\mathbb{T}^d} g(x, m^\varepsilon, h(\boldsymbol{m}^\varepsilon)) \, \mathrm{d}x - \frac{1}{C_1} \int_{\mathbb{T}^d} |Du^\varepsilon|^\gamma (m^\varepsilon)^{-\tau} \, \mathrm{d}x + C_1 \\
&\quad \leqslant C_3(1+c) \max \left\{ \int_{\mathbb{T}^d} g_1(x, m, h(\boldsymbol{m})) \, \mathrm{d}x, \int_{\mathbb{T}^d} m \, \mathrm{d}x \right\} + C_3 c + C_1 \leqslant C.
\end{aligned} \tag{4.9}
$$

Hence, in view of (H1b), (4.8) and (4.9) yield

$$
\left| - \int_{\mathbb{T}^d} H(x, Du^\varepsilon, D^2u^\varepsilon, m^\varepsilon, h(\boldsymbol{m}^\varepsilon)) \, \mathrm{d}x \right| \leqslant C. \tag{4.10}
$$

Consequently, owing to (4.6), (4.7), and (4.10), we conclude that (4.2) holds. ◻

Remark 4.3 (fully nonlinear Hamiltonians). Suppose that H is as in Proposition 4.2, let $H^0 : \mathbb{R}^{d\times d} \to \mathbb{R}$ be a C^1 convex function, and define $\hat{H} : \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{E} \times \mathbb{R} \to \mathbb{R}$ by $\hat{H}(x,p,M,m,\theta) := H(x,p,M,m,\theta) + H^0(M)$. We claim that if we replace H by $\hat{H}$ in (3.1), the bounds in Proposition 4.2 remain unchanged.

In fact, in view of Remark 3.5, the proofs of (4.1) and (4.3)–(4.5) are as in the preceding proof. The proof of (4.2) is also similar to the preceding proof observing that (4.6) remains valid if we add to its right-hand side the term $\int_{\mathbb{T}^d} H^0(D^2u^\varepsilon) \, \mathrm{d}x$ and using the bound (3.15) established in Remark 3.5.

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Next, we present the proof of Theorem 1.1.

Proof of Theorem 1.1. As mentioned at the beginning of this section, we set $A^\varepsilon = A_0^\varepsilon$. In view of Corollary 3.12 with $\mu = 0$, there exists $(m^\varepsilon, u^\varepsilon) \in D(A^\varepsilon) \cap C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ such that

$$A^\varepsilon \begin{bmatrix} m^\varepsilon \\ u^\varepsilon \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.11}$$

pointwise in $\mathbb{T}^d$.

As usual, we assume that ε_1 and ε_2 take values on fixed sequences of positive numbers converging to zero. To simplify the notation, we write $\varepsilon \to 0^+$ instead of $\varepsilon \to (0^+, 0^+)$.

We now proceed in two steps.

Step 1. In this step, we prove that there exists $(\boldsymbol{m}, u) \in \mathcal{M}_{ac}(\mathbb{T}^d) \times W^{1,\gamma}(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} m \, \mathrm{d}x = 1$ such that, up to a not relabeled subsequence, $(m^\varepsilon, u^\varepsilon) \rightharpoonup (m, u)$ weakly in $L^1(\mathbb{T}^d) \times W^{1,\gamma}(\mathbb{T}^d)$ as $\varepsilon = (\varepsilon_1, \varepsilon_2) \to 0^+$.

From (4.4) and assumption (g2), it follows that there exists $m \in L^1(\mathbb{T}^d)$ such that, up to a not relabeled subsequence, $m^\varepsilon \rightharpoonup m$ weakly in $L^1(\mathbb{T}^d)$ as $\varepsilon \to 0^+$. Because m^ε is nonnegative in $\mathbb{T}^d$, we conclude that $\boldsymbol{m} \in \mathcal{M}_{ac}(\mathbb{T}^d)$.

Next, we observe that $(u^\varepsilon)_\varepsilon$ is a bounded sequence in $W^{1,\gamma}(\mathbb{T}^d)$ by the Poincaré–Wirtinger inequality, (4.1), and (4.2). Therefore, there is $u \in W^{1,\gamma}(\mathbb{T}^d)$ satisfying, up to a not relabeled further subsequence, $u^\varepsilon \rightharpoonup u$ weakly in $W^{1,\gamma}(\mathbb{T}^d)$ as $\varepsilon \to 0^+$.

Finally, integrating the second equation in (4.11) over $\mathbb{T}^d$ and letting $\varepsilon \to 0^+$, we conclude that $\int_{\mathbb{T}^d} m \, \mathrm{d}x = 1$. This completes Step 1.

Step 2. In this step, we show that (m, u) satisfies (1.6).

We apply a variation of Minty's device (see, for instance, [20, 47]). Let $(\eta, v) \in C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ with $\inf_{\mathbb{T}^d} \eta > 0$. By Lemma 4.1 and (4.11), we have

$$\begin{aligned}
0 &\leqslant \left(A^\varepsilon \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m^\varepsilon \\ u^\varepsilon \end{bmatrix} \right)_{L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d)} \\
&= \int_{\mathbb{T}^d} \Big(-v - H(x, Dv, D^2 v, \eta, h(\boldsymbol{\eta})) \Big) (\eta - m^\varepsilon) \, \mathrm{d}x \\
&\quad + \int_{\mathbb{T}^d} \Big(\varepsilon_1 (\eta + \Delta^{2p} \eta) + \varepsilon_2 (\eta - \Delta \eta) + \beta_{\varepsilon_1}(\eta) \Big) (\eta - m^\varepsilon) \, \mathrm{d}x \\
&\quad + \int_{\mathbb{T}^d} \Big(\eta - \operatorname{div}(\eta D_p H(x, Dv, D^2 v, \eta, h(\boldsymbol{\eta}))) \\
&\qquad\qquad + \big(\eta D_{M_{ij}} H\big(x, Dv, D^2 v, \eta, h(\boldsymbol{\eta})\big)\big)_{x_i x_j} - 1 \Big) (v - u^\varepsilon) \, \mathrm{d}x \\
&\quad + \int_{\mathbb{T}^d} \Big(\varepsilon_1 (v + \Delta^{2p} v) + \varepsilon_2 (v - \Delta v) \Big) (v - u^\varepsilon) \, \mathrm{d}x.
\end{aligned}$$

Letting $\varepsilon \to 0^+$ and using (H2), (h1), the convergence results from Step 1, and the convergence $\varepsilon_1(\eta + \Delta^{2p}\eta) + \varepsilon_2(\eta - \Delta\eta) + \beta_{\varepsilon_1}(\eta) + \varepsilon_1(v + \Delta^{2p} v) + \varepsilon_2(v - \Delta v) \to 0$ uniformly in $\mathbb{T}^d$ as $\varepsilon \to 0^+$, which holds since $\inf_{\mathbb{T}^d} \eta > 0$, we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^d} \Big[-v - H(x, Dv, D^2 v, \eta, h(\boldsymbol{\eta})) \Big] (\eta - m) \, \mathrm{d}x \\
&\quad + \int_{\mathbb{T}^d} \big[\eta - \operatorname{div}(\eta D_p H(x, Dv, D^2 v, \eta, h(\boldsymbol{\eta}))) \\
&\qquad\qquad + \big(\eta D_{M_{ij}} H\big(x, Dv, D^2 v, \eta, h(\boldsymbol{\eta})\big)\big)_{x_i x_j} - 1 \big] (v - u) \, \mathrm{d}x \geqslant 0.
\end{aligned} \tag{4.12}$$

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This concludes the proof in the $\mathbb{E} = \mathbb{R}^+$ case because $(m, u) \in L^1(\mathbb{T}^d) \times L^1(\mathbb{T}^d)$; thus, the left-hand side of (4.12) coincides with

$$\left\langle \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m \\ u \end{bmatrix}, A \begin{bmatrix} \eta \\ v \end{bmatrix} \right\rangle_{\mathcal{D}'(\mathbb{T}^d)\times\mathcal{D}'(\mathbb{T}^d), C^\infty(\mathbb{T}^d)\times C^\infty(\mathbb{T}^d)}.$$

In the $\mathbb{E} = \mathbb{R}_0^+$ case, we proceed as follows. Let $\tilde{\eta} \in C^\infty(\mathbb{T}^d)$ with $\tilde{\eta} \geqslant 0$ and for $\delta > 0$, set $\eta_\delta := \tilde{\eta} + \delta$; using (4.12) with η replaced by η_δ and letting $\delta \to 0^+$, by (H2), (2.1), and (3.4), we conclude that (4.12) holds for all $(\eta, v) \in C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ with $\eta \geqslant 0$. This concludes Step 2, as well as the proof of Theorem 1.1. ∎

We end this section with a corollary to the previous results.

COROLLARY 4.4. *Suppose that assumptions* (h1), (g1), (g2), *and* (H1)–(H3) *are satisfied. Then, there exist* $(\boldsymbol{m}, u) \in \mathcal{M}_{ac}(\mathbb{T}^d) \times W^{1,\gamma}(\mathbb{T}^d)$ *with* $\int_{\mathbb{T}^d} m \,\mathrm{d}x = 1$ *and a sequence of positive numbers,* $(\varepsilon_j)_{j\in\mathbb{N}}$*, convergent to zero such that*

(i) (m, u) *is a weak solution to* (1.1)*;*

(ii) (m, u) *is the weak limit in* $L^1(\mathbb{T}^d) \times W^{1,\gamma}(\mathbb{T}^d)$ *of a sequence* $(m_{\varepsilon_j}, u_{\varepsilon_j})_{j\in\mathbb{N}} \subset D(A^{\varepsilon_j})$*, where each* $(m_{\varepsilon_j}, u_{\varepsilon_j})$ *satisfies* (3.1) *with* $\varepsilon = \varepsilon_j$*.*

Proof. We established (ii) in the first step of the proof of Theorem 1.1 and (i) in the second step of the same proof. ∎

5. Properties of weak solutions.

Here, we examine some properties of weak solutions. To illustrate our methods, we consider the degenerate diffusion case, which corresponds to Hamiltonians of the form (5.1) below. We show that the weak solutions given by Theorem 1.1 are subsolutions in the sense of distributions of the first equation in (1.1), the Hamilton–Jacobi equation. Next, we recover an analog to the second equation in (1.1), the Fokker–Planck equation. Finally, we show that in the set where m is positive, the Hamilton–Jacobi equation in (1.1) holds in a relaxed sense.

The main assumption of this section is the following.

(H4) The Hamiltonian, H, can be written as

$$H(x, p, M, m, \theta) = H_0(x, p, m, \theta) - \sum_{i,j=1}^{d} a_{ij}(x) M_{ij}, \tag{5.1}$$

where H_0 is a real-valued C^∞ function and $(a_{ij})_{1\leqslant i,j\leqslant d}$ is a C^∞ matrix-valued function.

(a) Let γ be as in assumption (H1). For all $\varphi \in C^\infty(\mathbb{T}^d)$ with $\varphi \geqslant 0$, the functional

$$(m, u) \in L^1(\mathbb{T}^d) \times W^{1,\gamma}(\mathbb{T}^d) \mapsto \int_{\mathbb{T}^d} \varphi \, H_0(x, Du, m, h(\boldsymbol{m})) \,\mathrm{d}x$$

is sequentially weakly lower semicontinuous in $L^1(\mathbb{T}^d) \times W^{1,\gamma}(\mathbb{T}^d)$.

(b) Let $\gamma > 1$ be as in assumption (H1) with $\tau = 0$ and g satisfying (g1a). There exist constants, $C_5 > 0$ and $\alpha > 0$, such that, for all $(x, p, m, \theta) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{E} \times \mathbb{R}$,

i. $|D_p H_0(x, p, m, \theta)| \leqslant C_5(1 + |p|^{\gamma-1})$;

ii. $g_1(x, m, \theta) \geqslant \dfrac{1}{C_5} m^\alpha - C_5$.

(c) Let $\gamma > 1$ and $\alpha > 0$ be as in assumption (H4b). Let L_0 be the Lagrangian associated with H_0, defined for $(x, v, m, \theta) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{E} \times \mathbb{R}$

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by

$$L_0(x,v,m,\theta)=\sup_{p\in\mathbb{R}^d}\left\{-p\cdot v-H_0(x,p,m,\theta)\right\}.$$

For all $(x,m,\theta)\in\mathbb{T}^d\times\mathbb{E}\times\mathbb{R}$, the map $p\mapsto H_0(x,p,m,\theta)$ is convex; moreover, the functional

$$(m,J)\in L^{\alpha+1}(\mathbb{T}^d)\times L^{\frac{(\alpha+1)\gamma}{(\alpha+1)\gamma-\alpha}}(\mathbb{T}^d;\mathbb{R}^d)\mapsto\int_{\mathbb{T}^d} mL_0\left(x,\frac{J}{m},m,h(\boldsymbol{m})\right)\mathrm{d}x$$

is sequentially weakly lower semicontinuous in

$$L^{\alpha+1}(\mathbb{T}^d)\times L^{\frac{(\alpha+1)\gamma}{(\alpha+1)\gamma-\alpha}}(\mathbb{T}^d;\mathbb{R}^d).$$

We observe that the preceding assumption holds in the case given by (1.3) with a quadratic Hamiltonian for $\tau=0$, and with $g(m,\theta)=m^\alpha+\theta$, where $\alpha>0$. In general, assumptions (H4a) and (H4c) depend on convexity properties of the maps $(p,m,\theta)\in\mathbb{R}^d\times\mathbb{E}\times\mathbb{R}\mapsto H_0(x,p,m,\theta)$ and $(v,m,\theta)\in\mathbb{R}^d\times\mathbb{E}\times\mathbb{R}\mapsto mL_0(x,\frac{v}{m},m,\theta)$, respectively. Note that we make no assumption on (a_{ij}) being positive definite.

5.1. Subsolution property. We begin by examining the Hamilton–Jacobi equation in (1.1). As previously stated, we suppose that H satisfies (5.1). We say that $(\boldsymbol{m},u)\in\mathcal{M}_{ac}(\mathbb{T}^d)\times W^{1,\gamma}(\mathbb{T}^d)$ is a subsolution in the sense of distributions (i.e., in $\mathcal{D}'(\mathbb{T}^d)$) of the Hamilton–Jacobi equation in (1.1) if for all $\varphi\in C^\infty(\mathbb{T}^d)$ with $\varphi\geqslant 0$, we have

$$\int_{\mathbb{T}^d}\left[\varphi(x)\left(u+H_0(x,Du,m,h(\boldsymbol{m}))\right)-\left(\varphi(x)a_{ij}(x)\right)_{x_ix_j}u\right]\mathrm{d}x\leqslant 0. \tag{5.2}$$

The preceding implies that at a point x where there exists $r>0$ such that $u\in C^2(B_r(x))$ and $m\in C(B_r(x))$, we have the pointwise inequality

$$u+H_0(x,Du,m,h(\boldsymbol{m}))-a_{ij}u_{x_ix_j}\leqslant 0.$$

This matter is more delicate if m or u has less regularity. For example, if m is continuous and u Lipschitz, we can select in (5.2) $\varphi=\eta_\epsilon(x-y)$, with η_ϵ a standard mollifier. Then, we conclude that for $u^\epsilon=u*\eta_\epsilon$, we have

$$u^\epsilon+H_0(x,Du^\epsilon,m,h(\boldsymbol{m}))-a_{ij}u^\epsilon_{x_ix_j}\leqslant o(1).$$

Then, by the stability of viscosity subsolutions, we gather that u is a viscosity subsolution.

Next, we show that the weak solutions given by Theorem 1.1 are subsolutions in $\mathcal{D}'(\mathbb{T}^d)$.

PROPOSITION 5.1. *Suppose that assumptions* (h1), (g1), (g2), (H1)–(H3), *and* (H4a) *hold. Let* $(m^\varepsilon,u^\varepsilon)\in D(A^\varepsilon)$ *be a solution of* (3.1). *Let* $(m,u)\in L^1(\mathbb{T}^d)\times W^{1,\gamma}(\mathbb{T}^d)$ *with* $m\geqslant 0$ *and* $\int_{\mathbb{T}^d}m\,\mathrm{d}x=1$ *be a weak solution of* (1.1) *obtained as a weak sublimit of* $\{(m^\varepsilon,u^\varepsilon)\}_\varepsilon$ *in* $L^1(\mathbb{T}^d)\times W^{1,\gamma}(\mathbb{T}^d)$. *Then,* (m,u) *is a subsolution in the sense of distributions of the Hamilton–Jacobi equation in* (1.1).

Remark 5.2. We observe that there exist $(m^\varepsilon,u^\varepsilon)$ and (m,u) as in the statement of Proposition 5.1 in view of Corollary 3.12 with $\mu=0$ and Corollary 4.4.

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Proof. To simplify the notation, we do not distinguish $\{(m^\varepsilon, u^\varepsilon)\}_\varepsilon$ from its subsequence that converges to (m, u).

Take $\varphi \in C^\infty(\mathbb{T}^d)$, $\varphi \geqslant 0$. Multiplying the first equation in (3.1) by φ and integrating over $\mathbb{T}^d$ gives

$$\int_{\mathbb{T}^d} \Big[\varphi\left(u^\varepsilon + H_0(x, Du^\varepsilon, m^\varepsilon, h(\boldsymbol{m}^\varepsilon))\right) - (\varphi a_{ij})_{x_i x_j} u^\varepsilon\Big]\,\mathrm{d}x - \varepsilon_1 \int_{\mathbb{T}^d} m^\varepsilon(\varphi + \Delta^{2p}\varphi)\,\mathrm{d}x \\ - \varepsilon_2 \int_{\mathbb{T}^d} m^\varepsilon(\varphi - \Delta\varphi)\,\mathrm{d}x = \int_{\mathbb{T}^d} \beta_{\varepsilon_1}(m^\varepsilon)\varphi\,\mathrm{d}x \leqslant 0$$

since $\beta_{\varepsilon_1}(\cdot) \leqslant 0$.

Because $m^\varepsilon \rightharpoonup m$ in $L^1(\mathbb{T}^d)$, we have

$$\lim_{\varepsilon\to 0^+} \left[\varepsilon_1 \int_{\mathbb{T}^d} m^\varepsilon(\varphi + \Delta^{2p}\varphi)\,\mathrm{d}x + \varepsilon_2 \int_{\mathbb{T}^d} m^\varepsilon(\varphi - \Delta\varphi)\,\mathrm{d}x\right] = 0.$$

Next, since $u^\varepsilon \to u$ in $L^\gamma(\mathbb{T}^d)$, we have

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{T}^d} \left[\varphi u^\varepsilon - (\varphi a_{ij})_{x_i x_j} u^\varepsilon\right]\mathrm{d}x = \int_{\mathbb{T}^d} \left[\varphi u - (\varphi a_{ij})_{x_i x_j} u\right]\mathrm{d}x.$$

Finally, assumption (H4a), together with the weak convergences $m^\varepsilon \rightharpoonup m$ in $L^1(\mathbb{T}^d)$ and $u^\varepsilon \rightharpoonup u$ in $W^{1,\gamma}(\mathbb{T}^d)$, gives

$$\liminf_{\varepsilon\to 0^+} \int_{\mathbb{T}^d} \varphi\, H_0(x, Du^\varepsilon, m^\varepsilon, h(\boldsymbol{m}^\varepsilon))\,\mathrm{d}x \geqslant \int_{\mathbb{T}^d} \varphi\, H_0(x, Du, m, h(\boldsymbol{m}))\,\mathrm{d}x.$$

Consequently, we have (5.2). ∎

5.2. Fokker–Planck equation. To pursue our analysis further, we consider the case without congestion, $\tau = 0$, a natural growth condition on $D_p H_0$, and a powerlike growth in g_1. This case corresponds to assumption (H4b). We begin by examining the integrability of the drift in the Fokker–Planck equation.

PROPOSITION 5.3. *Suppose that assumptions* (g1), (H1), *and* (H4b) *hold. Let* $q = \frac{(\alpha+1)\gamma}{(\alpha+1)\gamma-\alpha} > 1$. *Then, there exists a positive constant, C, such that, for any* $\varepsilon \in (0,1)^2$ *and any solution* $(u^\varepsilon, m^\varepsilon) \in D(A^\varepsilon)$ *to* (3.1), *we have*

$$\|m^\varepsilon\|_{L^{\alpha+1}(\mathbb{T}^d)} + \|m^\varepsilon D_p H(\cdot, Du^\varepsilon, m^\varepsilon, h(\boldsymbol{m}^\varepsilon))\|_{L^q(\mathbb{T}^d;\mathbb{R}^d)} \leqslant C.$$

Proof. By (4.4), assumption (H4bii), and assumption (g1c) with $\delta = 1$, m^ε is bounded in $L^{\alpha+1}(\mathbb{T}^d)$ independently of ε. In particular, because $q < \alpha + 1$, $\{m^\varepsilon\}_\varepsilon$ is bounded in $L^q(\mathbb{T}^d)$. Then, in view of assumptions (5.1) and (H4bi), it suffices to prove that $m^\varepsilon|Du^\varepsilon|^{\gamma-1}$ is bounded in $L^q(\mathbb{T}^d)$ independently of ε.

Set $s = \gamma(\alpha+1)$ and $r = \frac{\gamma}{\gamma-1}$. Observe that

$$m^\varepsilon|Du^\varepsilon|^{\gamma-1} = \left[(m^\varepsilon)^{\alpha+1}\right]^{\frac{1}{s}} \left(|Du^\varepsilon|^\gamma m^\varepsilon\right)^{\frac{1}{r}}$$

and $\frac{1}{q} = \frac{1}{r} + \frac{1}{s}$. As we proved before, $(m^\varepsilon)^{\alpha+1}$ is bounded in $L^1(\mathbb{T}^d)$ independently of ε. Moreover, from Lemma 3.3 with $\tau = 0$, we have that $|Du^\varepsilon|^\gamma m^\varepsilon$ is bounded in $L^1(\mathbb{T}^d)$ independently of ε. The result then follows by Hölder's inequality. ∎

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PROPOSITION 5.4. *Suppose that assumptions* (h1), (g1), (g2), (H1)–(H3), *and* (H4b) *hold. Let* $(u^\varepsilon, m^\varepsilon) \in D(A^\varepsilon)$ *be a solution of* (3.1). *Let* $(m,u) \in L^1(\mathbb{T}^d) \times W^{1,\gamma}(\mathbb{T}^d)$ *with* $m \geqslant 0$ *and* $\int_{\mathbb{T}^d} m \,\mathrm{d}x = 1$ *be a weak solution to* (1.1) *obtained as a weak sublimit of* $\{(m^\varepsilon, u^\varepsilon)\}_\varepsilon$ *in* $L^1(\mathbb{T}^d) \times W^{1,\gamma}(\mathbb{T}^d)$. *Let* $q = \frac{(\alpha+1)\gamma}{(\alpha+1)\gamma-\alpha}$ *and set* $J^\varepsilon = m^\varepsilon D_pH(x, Du^\varepsilon, m^\varepsilon, h(\boldsymbol{m}^\varepsilon))$. *Then, there exists* $J \in L^q(\mathbb{T}^d;\mathbb{R}^d)$ *such that, up to a not relabeled subsequence,* $J^\varepsilon \rightharpoonup J$ *in* $L^q(\mathbb{T}^d;\mathbb{R}^d)$. *Moreover,*

$$m - \operatorname{div}(J) - (a_{ij}(x)m)_{x_ix_j} = 1 \tag{5.3}$$

in $\mathcal{D}'(\mathbb{T}^d)$.

Proof. By Proposition 5.3, up to a not relabeled subsequence, we have $J^\varepsilon \rightharpoonup J$ in $L^q(\mathbb{T}^d;\mathbb{R}^d)$ for some $J \in L^q(\mathbb{T}^d;\mathbb{R}^d)$. The result follows by convergence in $\mathcal{D}'(\mathbb{T}^d)$ together with the second equation in (3.1). ∎

5.3. Supersolution property. Here, we revisit the Hamilton–Jacobi equation in (1.1). From the results in subsection 5.1, we know that weak solutions are subsolutions. Next, we prove the converse result; that is, weak solutions solve, in a relaxed sense, the Hamilton–Jacobi equation in the set where m is positive.

PROPOSITION 5.5. *Suppose that assumptions* (h1), (g1), (g2), (H1)–(H3), (H4b), *and* (H4c) *hold. Let* $q = \frac{(\alpha+1)\gamma}{(\alpha+1)\gamma-\alpha}$. *Let* $(u^\varepsilon, m^\varepsilon)$, (u,m), *and* J *be as in Proposition* 5.4. *Then, for all* $\varphi \in C^\infty(\mathbb{T}^d)$, *we have*

$$\int_{\mathbb{T}^d} [JD\varphi - H_0(x, D\varphi, m, h(\boldsymbol{m}))m - u] \,\mathrm{d}x \leqslant 0. \tag{5.4}$$

Remark 5.6. We regard (5.4) as a weak version of

$$\int_{\mathbb{T}^d} m\left[-u - H_0(x, Du, m, h(\boldsymbol{m})) + a_{ij}(x)u_{x_ix_j}\right] \mathrm{d}x \leqslant 0. \tag{5.5}$$

To see this, assume, for example, that u is C^2 and m is continuous. Then, (5.4) for $\varphi = u$ yields

$$\int_{\mathbb{T}^d} [JDu - H_0(x, Du, m, h(\boldsymbol{m}))m - u] \,\mathrm{d}x \leqslant 0. \tag{5.6}$$

Then, due to (5.3), we get

$$\int_{\mathbb{T}^d} (JDu - u) \,\mathrm{d}x = \int_{\mathbb{T}^d} m(-u + a_{ij}(x)u_{x_ix_j}) \,\mathrm{d}x.$$

Combining the previous identity with (5.6) gives (5.5). Moreover, we have from the subsolution property that

$$-u - H_0(x, Du, m, h(\boldsymbol{m})) + a_{ij}(x)u_{x_ix_j} \geqslant 0$$

pointwise. Thus,

$$-u - H_0(x, Du, m, h(\boldsymbol{m})) + a_{ij}(x)u_{x_ix_j} = 0$$

on the set $m > 0$. In general, solutions have low regularity and the Hamilton–Jacobi equation may not hold pointwise. In the following section, we explore an example where further pointwise properties can be proven.

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Proof of Proposition 5.5. To simplify the notation, we do not relabel the weakly convergent subsequence of $\{(m^\varepsilon, u^\varepsilon, J^\varepsilon)\}_\varepsilon$ to (m, u, J) in $L^1(\mathbb{T}^d) \times W^{1,\gamma}(\mathbb{T}^d) \times L^q(\mathbb{T}^d;\mathbb{R}^d)$. Note that in view of Proposition 5.3, we also have $m^\varepsilon \rightharpoonup m$ weakly in $L^{\alpha+1}(\mathbb{T}^d)$.

Next, we observe that

$$\begin{aligned}
&\int_{\mathbb{T}^d} m^\varepsilon \left[-u^\varepsilon - H_0(x, Du^\varepsilon, m^\varepsilon, h(\boldsymbol{m}^\varepsilon)) + a_{ij}(x)u^\varepsilon_{x_ix_j}\right] \mathrm{d}x\\
&\quad + \int_{\mathbb{T}^d} m^\varepsilon \left[\varepsilon_1(m^\varepsilon + \Delta^{2p}m^\varepsilon) + \varepsilon_2(m^\varepsilon - \Delta m^\varepsilon) + \beta_{\varepsilon_1}(m^\varepsilon)\right] \mathrm{d}x\\
&\quad + \int_{\mathbb{T}^d} u^\varepsilon \Big[m^\varepsilon - \operatorname{div}(m^\varepsilon D_pH_0) - \left(m^\varepsilon a_{ij}\right)_{x_ix_j}\\
&\qquad\qquad - 1 + \varepsilon_1(u^\varepsilon + \Delta^{2p}u^\varepsilon) + \varepsilon_2(u^\varepsilon - \Delta u^\varepsilon)\Big] \mathrm{d}x\\
&= 0.
\end{aligned}$$

Integrating the terms multiplied by ε_1 and ε_2 by parts, and taking into account their sign, we conclude that

$$\begin{aligned}
&\int_{\mathbb{T}^d} m^\varepsilon \left[-u^\varepsilon - H_0(x, Du^\varepsilon, m^\varepsilon, h(\boldsymbol{m}^\varepsilon)) + a_{ij}(x)u^\varepsilon_{x_ix_j} + \beta_{\varepsilon_1}(m^\varepsilon)\right] \mathrm{d}x\\
&\quad + \int_{\mathbb{T}^d} u^\varepsilon \left[m^\varepsilon - \operatorname{div}(m^\varepsilon D_pH_0) - \left(m^\varepsilon a_{ij}\right)_{x_ix_j} - 1\right] \mathrm{d}x \leqslant 0.
\end{aligned}$$

By Proposition 4.2, we have that $\|\beta_{\varepsilon_1}(m^\varepsilon)\|_{L^1(\mathbb{T}^d)}$ is bounded independently of ε. Accordingly, we find

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{T}^d} m^\varepsilon \beta_{\varepsilon_1}(m^\varepsilon)\,\mathrm{d}x = 0$$

because $|m^\varepsilon \beta_{\varepsilon_1}(m^\varepsilon)| \leqslant \varepsilon_1 |\beta_{\varepsilon_1}(m^\varepsilon)|$ by definition of $\beta_{\varepsilon_1}(\cdot)$. Hence,

$$\liminf_{\varepsilon\to 0^+} \left(\int_{\mathbb{T}^d} m^\varepsilon \left(-H_0(x, Du^\varepsilon, m^\varepsilon, h(\boldsymbol{m}^\varepsilon))\right) \mathrm{d}x + \int_{\mathbb{T}^d} u^\varepsilon \left[-\operatorname{div}(m^\varepsilon D_pH_0) - 1\right] \mathrm{d}x \right) \leqslant 0.$$

Recalling that $J^\varepsilon = m^\varepsilon D_pH_0(x, Du^\varepsilon, m^\varepsilon, h(\boldsymbol{m}^\varepsilon))$, the definition of the Legendre transform and assumption (H4c) give

$$\liminf_{\varepsilon\to 0^+} \int_{\mathbb{T}^d} m^\varepsilon L_0\left(x, \frac{-J^\varepsilon}{m^\varepsilon}, m^\varepsilon, h(\boldsymbol{m}^\varepsilon)\right) \mathrm{d}x \leqslant \lim_{\varepsilon\to 0^+} \int_{\mathbb{T}^d} u^\varepsilon\,\mathrm{d}x = \int_{\mathbb{T}^d} u\,\mathrm{d}x,$$

where we used the convergence $u^\varepsilon \to u$ in $L^\gamma(\mathbb{T}^d)$. Then, because $-J^\varepsilon \rightharpoonup -J$ in $L^q(\mathbb{T}^d;\mathbb{R}^d)$ and $m^\varepsilon \rightharpoonup m$ in $L^{\alpha+1}(\mathbb{T}^d)$, assumption (H4c) implies that

$$\int_{\mathbb{T}^d} mL_0\left(x, \frac{-J}{m}, m, h(\boldsymbol{m})\right) \mathrm{d}x \leqslant \int_{\mathbb{T}^d} u\,\mathrm{d}x. \tag{5.7}$$

On the other hand, using the definition of the Legendre transform again, we have

$$mL_0\left(x, \frac{-J}{m}, m, h(\boldsymbol{m})\right) \geqslant JD\varphi - H_0(x, D\varphi, m, h(\boldsymbol{m}))m \tag{5.8}$$

for any $\varphi \in C^\infty(\mathbb{T}^d)$. Consequently, (5.7) and (5.8) yield (5.4). ∎

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6. Improved regularity. In this section, we consider the quadratic case for either first-order problems or elliptic problems with constant coefficients, encoded in assumption (H5) below. We get improved regularity for m and u, including higher integrability and Sobolev estimates for m. Further, we show that the Fokker–Plank equation holds pointwise, not just in the sense of distributions. Finally, we prove that the Hamilton–Jacobi equation holds pointwise in the set where m is positive.

The main assumption of this section is the following.

(H5) The Hamiltonian, H, is of the form

$$H(x,p,M,m,\theta) = \frac{|p|^2}{2} - V(x,m,\theta) - \sigma^2 \sum_{i=1}^{d} M_{ii}, \tag{6.1}$$

where $\sigma \in \mathbb{R}$ and $V : \mathbb{T}^d \times \mathbb{E} \times \mathbb{R} \to \mathbb{R}$ is a C^∞ function.

(a) There exist constants, $c_1 > 0$, $c_2 > 0$, and $c_3 > 0$, and a continuous function, $g : \mathbb{T}^d \times \mathbb{E} \times \mathbb{R} \to \mathbb{R}$, such that we have the following.

i. For all $(x,m,\theta) \in \mathbb{T}^d \times \mathbb{E} \times \mathbb{R}$,

$$V(x,m,\theta) \geqslant c_1 g(x,m,\theta) - \frac{1}{c_2}.$$

ii. For all $\boldsymbol{m} \in \mathcal{M}_{ac}(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} V(x,m,h(\boldsymbol{m}))\,\mathrm{d}x \leqslant c_2 + c_3 \int_{\mathbb{T}^d} g(x,m,h(\boldsymbol{m}))\,\mathrm{d}x.$$

iii. For all η, $\bar\eta \in C^\infty(\mathbb{T}^d;\mathbb{E})$,

$$\int_{\mathbb{T}^d} \left[V(x,\eta,h(\boldsymbol{\eta})) - V(x,\bar\eta,h(\bar{\boldsymbol{\eta}}))\right](\eta-\bar\eta)\,\mathrm{d}x \geqslant 0.$$

(b) There exist constants, $\alpha > 0$ and $\kappa_1 > 0$, such that we have the following.

i. For all $(x,m,\theta) \in \mathbb{T}^d \times \mathbb{E} \times \mathbb{R}$, the function g in (H5a) satisfies (g1a) with

$$g_1(x,m,\theta) \geqslant \frac{1}{\kappa_1} m^\alpha - \kappa_1.$$

ii. For all $\eta \in C^\infty(\mathbb{T}^d;\mathbb{E})$,

$$\int_{\mathbb{T}^d} \left(V(x,\eta,h(\boldsymbol{\eta}))\right)_{x_i} \eta_{x_i}\,\mathrm{d}x \geqslant \frac{1}{\kappa_1}\int_{\mathbb{T}^d} \eta^{\alpha-1}|D\eta|^2\,\mathrm{d}x - \kappa_1.$$

(c) Let $\alpha > 0$ and $\kappa_1 > 0$ be as in (H5b). For all $(x,m,\theta) \in \mathbb{T}^d \times \mathbb{E} \times \mathbb{R}$, we have

$$V(x,m,\theta) \leqslant \kappa_1(1 + m^\alpha + |\theta|).$$

We observe that a Hamiltonian satisfying (H5a) is a particular case of a Hamiltonian for which (H1)–(H3) hold (with $\gamma = 2$ and $\tau = 0$).

In some cases, we will also require the operator h to satisfy the following condition:

(h2) There exists a positive constant, C, such that, for all $\boldsymbol{m}$, $\bar{\boldsymbol{m}} \in \mathcal{M}_{ac}(\mathbb{T}^d)$, we have

$$\|h(\boldsymbol{m}) - h(\bar{\boldsymbol{m}})\|_{L^1(\mathbb{T}^d)} \leqslant C\|m - \bar m\|_{L^1(\mathbb{T}^d)}.$$

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Examples of operators satisfying assumptions (h1) and (h2) are those in (1.3).

For Hamiltonians as in (6.1), (1.1) takes the form

$$\begin{cases} -u - \dfrac{|Du|^2}{2} + V(x,m,h(\boldsymbol{m})) + \sigma^2\Delta u = 0, \\ m - \operatorname{div}(mDu) - \sigma^2\Delta m - 1 = 0. \end{cases} \tag{6.2}$$

First, we consider the regularized problem (3.1) or, equivalently, (3.5) with $\mu = 0$ and prove additional estimates that are uniform in $\varepsilon \in (0,1)^2$.

LEMMA 6.1. *Suppose that assumptions* (g1)*,* (H5a)*, and* (H5b) *hold. Fix $\varepsilon \in (0,1)^2$. Let $(m^\varepsilon, u^\varepsilon) \in C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ with $\inf_{\mathbb{T}^d} m^\varepsilon > 0$ in $\mathbb{T}^d$ be a solution of* (3.1)*. Then, there exists a positive constant, C, independent of ε such that*

$$\begin{aligned} &\|u^\varepsilon\|_{W^{1,2}(\mathbb{T}^d)} + \|(m^\varepsilon)^{\frac{\alpha+1}{2}}\|_{W^{1,2}(\mathbb{T}^d)} + \|\beta_{\varepsilon_1}(m^\varepsilon)\|_{L^1(\mathbb{T}^d)} + \|(m^\varepsilon)^{\frac{1}{2}} Du^\varepsilon\|_{L^2(\mathbb{T}^d;\mathbb{R}^d)} \\ &\quad + \|(m^\varepsilon)^{\frac{1}{2}} D^2u^\varepsilon\|_{L^2(\mathbb{T}^d;\mathbb{M}_{d\times d})} + \|(m^\varepsilon)^{\frac{\alpha+1}{2}} Du^\varepsilon\|_{W^{1,1}(\mathbb{T}^d;\mathbb{R}^d)} + \|\sqrt{\varepsilon_1} u^\varepsilon\|_{W^{2p+1,2}(\mathbb{T}^d)} \\ &\quad + \|\sqrt{\varepsilon_1} m^\varepsilon\|_{W^{2p+1,2}(\mathbb{T}^d)} + \|\sqrt{\varepsilon_2} u^\varepsilon\|_{W^{2,2}(\mathbb{T}^d)} + \|\sqrt{\varepsilon_2} m^\varepsilon\|_{W^{2,2}(\mathbb{T}^d)} \leqslant C. \end{aligned}$$

Proof. Here, to simplify the notation, we write $\boldsymbol{m}$, m, and u in place of $\boldsymbol{m}^\varepsilon$, m^ε, and u^ε, respectively. Moreover, C represents a positive constant independent of ε and whose value may change from one line to another. Also, we do not use the Einstein convention on repeated indices, and we specify all the sums.

Since (m,u) is a solution to (3.1) and H is given by (6.1), we have

$$\begin{aligned} -u - \frac{|Du|^2}{2} + V(x,m,h(\boldsymbol{m})) + \sigma^2\Delta u + \varepsilon_1(m + \Delta^{2p} m) + \beta_{\varepsilon_1}(m) & \\ + \varepsilon_2(m - \Delta m) = 0, & \end{aligned} \tag{6.3}$$

$$m - \operatorname{div}(mDu) - \sigma^2\Delta m - 1 + \varepsilon_1(u + \Delta^{2p} u) + \varepsilon_2(u - \Delta u) = 0, \tag{6.4}$$

pointwise in $\mathbb{T}^d$.

Next, we recall that under assumption (H5a), the Hamiltonian H in (6.1) satisfies assumption (H1) with $\gamma = 2$, $\tau = 0$, and g given by (H5a). Because g also satisfies (g1), Proposition 4.2 and Lemma 3.3 with $\gamma = 2$, $\tau = 0$, and $\mu = 0$ give the bounds

$$\|Du\|_{L^2(\mathbb{T}^d;\mathbb{R}^d)} + \left| \int_{\mathbb{T}^d} u \,\mathrm{d}x \right| \leqslant C, \tag{6.5}$$

$$\int_{\mathbb{T}^d} m g_1(x,m,h(\boldsymbol{m}))\,\mathrm{d}x \leqslant C, \tag{6.6}$$

and

$$\|\beta_{\varepsilon_1}(m)\|_{L^1(\mathbb{T}^d)} + \|m^{\frac{1}{2}} Du\|_{L^2(\mathbb{T}^d;\mathbb{R}^d)} \leqslant C.$$

The estimate (6.5) together with the Poincaré–Wirtinger inequality yields

$$\|u\|_{W^{1,2}(\mathbb{T}^d)} \leqslant C. \tag{6.7}$$

Combining (6.6), (H5bi), and (g1c) with $\delta = 1$, we get

$$\|m^{\frac{\alpha+1}{2}}\|_{L^2(\mathbb{T}^d)} \leqslant C. \tag{6.8}$$

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After differentiating (6.3) twice with respect to x_j, $j \in \{1, \dots, d\}$, multiplying by m, and integrating over $\mathbb{T}^d$, the resulting equality becomes

$$\begin{aligned}
&-\int_{\mathbb{T}^d} u_{x_j x_j} m \,\mathrm{d}x - \int_{\mathbb{T}^d} \sum_{i=1}^d |u_{x_i x_j}|^2 m \,\mathrm{d}x + \int_{\mathbb{T}^d} \sum_{i=1}^d (m u_{x_i})_{x_i} u_{x_j x_j} \,\mathrm{d}x \\
&\quad - \int_{\mathbb{T}^d} (V(x, m, h(\boldsymbol{m})))_{x_j} m_{x_j} \,\mathrm{d}x \\
&\quad + \sigma^2 \int_{\mathbb{T}^d} \Delta u \, m_{x_j x_j} \,\mathrm{d}x - \varepsilon_1 \int_{\mathbb{T}^d} |m_{x_j}|^2 \,\mathrm{d}x - \varepsilon_1 \int_{\mathbb{T}^d} |\Delta^p m_{x_j}|^2 \,\mathrm{d}x \\
&\quad - \int_{\mathbb{T}^d} \beta'_{\varepsilon_1}(m) |m_{x_j}|^2 \,\mathrm{d}x - \varepsilon_2 \int_{\mathbb{T}^d} |m_{x_j}|^2 \,\mathrm{d}x - \varepsilon_2 \int_{\mathbb{T}^d} |D m_{x_j}|^2 \,\mathrm{d}x = 0
\end{aligned}$$

using integration by parts. In view of (6.4), we have that

$$\sum_{i=1}^d (m u_{x_i})_{x_i} = m - \sigma^2 \Delta m - 1 + \varepsilon_1 u + \varepsilon_1 \Delta^{2p} u + \varepsilon_2 u - \varepsilon_2 \Delta u.$$

Using this identity in the preceding equality, summing over $j \in \{1, \dots, d\}$, and using (H5bii), we conclude that

$$\begin{aligned}
&\int_{\mathbb{T}^d} |D^2 u|^2 m \,\mathrm{d}x + \frac{1}{\kappa_1} \int_{\mathbb{T}^d} m^{\alpha-1} |Dm|^2 \,\mathrm{d}x + \varepsilon_1 \int_{\mathbb{T}^d} |Du|^2 \,\mathrm{d}x + \varepsilon_1 \int_{\mathbb{T}^d} |\Delta^p Du|^2 \,\mathrm{d}x \\
&\quad + \varepsilon_1 \int_{\mathbb{T}^d} |Dm|^2 \,\mathrm{d}x + \varepsilon_1 \int_{\mathbb{T}^d} |\Delta^p Dm|^2 \,\mathrm{d}x + \varepsilon_2 \int_{\mathbb{T}^d} |Du|^2 \,\mathrm{d}x + \varepsilon_2 \int_{\mathbb{T}^d} |D^2 u|^2 \,\mathrm{d}x \\
&\quad + \varepsilon_2 \int_{\mathbb{T}^d} |Dm|^2 \,\mathrm{d}x + \varepsilon_2 \int_{\mathbb{T}^d} |D^2 m|^2 \,\mathrm{d}x + \int_{\mathbb{T}^d} \beta'_{\varepsilon_1}(m) |Dm|^2 \,\mathrm{d}x \leqslant \kappa_1.
\end{aligned} \tag{6.9}$$

Because $|D(m^{\frac{\alpha+1}{2}})|^2 = \frac{(\alpha+1)^2}{4} m^{\alpha-1} |Dm|^2$ and because all terms on the left-hand side of the inequality in (6.9) are nonnegative, it follows that

$$\|m^{\frac{1}{2}} D^2 u\|_{L^2(\mathbb{T}^d;\mathbb{M}_{d\times d})} + \|m^{\frac{\alpha+1}{2}}\|_{W^{1,2}(\mathbb{T}^d)} \leqslant C, \tag{6.10}$$

where we also take into account (6.8). Moreover, invoking the Poincaré–Wirtinger inequality once more and using integration by parts, we get

$$\|\sqrt{\varepsilon_1} u\|_{W^{2p+1,2}(\mathbb{T}^d)} + \|\sqrt{\varepsilon_1} m\|_{W^{2p+1,2}(\mathbb{T}^d)} + \|\sqrt{\varepsilon_2} u\|_{W^{2,2}(\mathbb{T}^d)} + \|\sqrt{\varepsilon_2} m\|_{W^{2,2}(\mathbb{T}^d)} \leqslant C.$$

Finally, we observe that

$$\|m^{\frac{\alpha+1}{2}} Du\|_{W^{1,1}(\mathbb{T}^d;\mathbb{R}^d)} \leqslant C$$

is a consequence of (6.7) and (6.10). □

The next result refines Theorem 1.1 and improves the result in Proposition 5.4.

THEOREM 6.2. *Suppose that assumptions* (h1), (h2), (g1), *and* (H5) *hold. Then, there exists a weak solution,* $(m, u) \in \mathcal{D}'(\mathbb{T}^d) \times \mathcal{D}'(\mathbb{T}^d)$ *with* $m \geqslant 0$ *to* (6.2). *Moreover,* $(m^{\frac{\alpha+1}{2}}, u) \in W^{1,2}(\mathbb{T}^d) \times W^{1,2}(\mathbb{T}^d)$, $\int_{\mathbb{T}^d} m \,\mathrm{d}x = 1$,

$$-u - \frac{|Du|^2}{2} + V(x, m, h(\boldsymbol{m})) + \sigma^2 \Delta u \geqslant 0 \tag{6.11}$$

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in the sense of distributions, and

$$m - \operatorname{div}(mDu) - \sigma^2 \Delta m - 1 = 0 \tag{6.12}$$

a.e. in $\mathbb{T}^d$.

Proof. By Corollary 3.12, for each $\varepsilon = (\varepsilon_1, \varepsilon_2) \in (0,1)^2$, there exists $(m^\varepsilon, u^\varepsilon) \in C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ with $\inf_{\mathbb{T}^d} m^\varepsilon > 0$ satisfying (6.3)–(6.4).

We assume that ε_1 and ε_2 take values on fixed sequences of positive numbers converging to zero. We proceed with the proof in four steps.

Step 1. Here, we establish preliminary convergences of $(m^\varepsilon, u^\varepsilon)$ first as $\varepsilon_1 \to 0^+$, and then as $\varepsilon_2 \to 0^+$.

We recall that the embeddings $W^{1,2}(\mathbb{T}^d) \hookrightarrow L^q(\mathbb{T}^d)$ and $W^{2,2}(\mathbb{T}^d) \hookrightarrow L^r(\mathbb{T}^d)$ are compact for all $q \in [1, 2^*)$ and $r \in [1, (2^*)^*)$ and continuous for all $q \in [1, 2^*]$ and $r \in [1, (2^*)^*]$. Hence, the uniform bounds in Lemma 6.1 with respect to ε_1 give that, for each ε_2, there exist $u^{\varepsilon_2} \in W^{2,2}(\mathbb{T}^d)$ and $m^{\varepsilon_2} \in W^{2,2}(\mathbb{T}^d)$ with $(m^{\varepsilon_2})^{\frac{\alpha+1}{2}} \in W^{1,2}(\mathbb{T}^d)$ such that, up to a not relabeled subsequence of $\{(m^\varepsilon, u^\varepsilon)\}_{\varepsilon_1}$,

$$u^\varepsilon \rightharpoonup u^{\varepsilon_2} \text{ weakly in } W^{2,2}(\mathbb{T}^d) \text{ as } \varepsilon_1 \to 0^+, \tag{6.13}$$

$$m^\varepsilon \rightharpoonup m^{\varepsilon_2} \text{ weakly in } W^{2,2}(\mathbb{T}^d) \text{ as } \varepsilon_1 \to 0^+, \tag{6.14}$$

$$(m^\varepsilon)^{\frac{\alpha+1}{2}} \rightharpoonup (m^{\varepsilon_2})^{\frac{\alpha+1}{2}} \text{ weakly in } W^{1,2}(\mathbb{T}^d) \text{ as } \varepsilon_1 \to 0^+, \tag{6.15}$$

$$\sqrt{\varepsilon_1} u^\varepsilon \to 0 \text{ in } W^{2p,2}(\mathbb{T}^d) \text{ as } \varepsilon_1 \to 0^+, \tag{6.16}$$

$$\sqrt{\varepsilon_1} m^\varepsilon \to 0 \text{ in } W^{2p,2}(\mathbb{T}^d) \text{ as } \varepsilon_1 \to 0^+, \tag{6.17}$$

$$(m^\varepsilon)^{\frac{1}{2}} Du^\varepsilon \rightharpoonup (m^{\varepsilon_2})^{\frac{1}{2}} Du^{\varepsilon_2} \text{ weakly in } L^2(\mathbb{T}^d;\mathbb{R}^d) \text{ as } \varepsilon_1 \to 0^+, \tag{6.18}$$

$$(m^\varepsilon)^{\frac{1}{2}} D^2u^\varepsilon \rightharpoonup (m^{\varepsilon_2})^{\frac{1}{2}} D^2u^{\varepsilon_2} \text{ weakly in } L^2(\mathbb{T}^d;\mathbb{M}_{d\times d}) \text{ as } \varepsilon_1 \to 0^+, \tag{6.19}$$

$$(m^\varepsilon)^{\frac{\alpha+1}{2}} Du^\varepsilon \rightharpoonup (m^{\varepsilon_2})^{\frac{\alpha+1}{2}} Du^{\varepsilon_2} \text{ weakly in } W^{1,1}(\mathbb{T}^d;\mathbb{R}^d) \text{ as } \varepsilon_1 \to 0^+, \tag{6.20}$$

$$h(\boldsymbol{m}^\varepsilon) \to h(\boldsymbol{m}^{\varepsilon_2}) \text{ in } L^1(\mathbb{T}^d) \text{ as } \varepsilon_1 \to 0^+, \tag{6.21}$$

where we also used the Rellich–Kondrachov theorem, the uniqueness of weak limits, and assumption (h2). Observe that (6.20) holds weakly in $W^{1,1}(\mathbb{T}^d;\mathbb{R}^d)$, and not only weakly-$\star$ in $BV(\mathbb{T}^d;\mathbb{R}^d)$, by (6.13), (6.15), and the Rellich–Kondrachov theorem. Moreover, in view of (6.13), (6.14), and (6.21), we can assume that the subsequence also satisfies, for a.e. $x \in \mathbb{T}^d$,

$$Du^\varepsilon(x) \to Du^{\varepsilon_2}(x), \;\; m^\varepsilon(x) \to m^{\varepsilon_2}(x), \text{ and } h(\boldsymbol{m}^\varepsilon)(x) \to h(\boldsymbol{m}^{\varepsilon_2})(x) \text{ as } \varepsilon_1 \to 0^+. \tag{6.22}$$

Note also that by (6.14), $(m^\varepsilon)^{\frac{1}{2}} \to (m^{\varepsilon_2})^{\frac{1}{2}}$ in $L^2(\mathbb{T}^d)$ as $\varepsilon_1 \to 0^+$, which, together with (6.18) and (6.19), yields

$$\begin{aligned} m^\varepsilon Du^\varepsilon \rightharpoonup m^{\varepsilon_2} Du^{\varepsilon_2} \text{ weakly in } L^1(\mathbb{T}^d;\mathbb{R}^d) \text{ as } \varepsilon_1 \to 0^+,\\ m^\varepsilon D^2u^\varepsilon \rightharpoonup m^{\varepsilon_2} D^2u^{\varepsilon_2} \text{ weakly in } L^1(\mathbb{T}^d;\mathbb{R}^d) \text{ as } \varepsilon_1 \to 0^+. \end{aligned} \tag{6.23}$$

Additionally, using the sequential lower semicontinuity of the norm in Lebesgue and Sobolev spaces with respect to the corresponding weak convergence, it follows from Lemma 6.1 and (6.13)–(6.20) that there is a positive constant, C, independent of ε_2 such that

$$\begin{aligned} &\|u^{\varepsilon_2}\|_{W^{1,2}(\mathbb{T}^d)} + \|(m^{\varepsilon_2})^{\frac{\alpha+1}{2}}\|_{W^{1,2}(\mathbb{T}^d)}\\ &\quad + \|(m^{\varepsilon_2})^{\frac{1}{2}} Du^{\varepsilon_2}\|_{L^2(\mathbb{T}^d;\mathbb{R}^d)} + \|(m^{\varepsilon_2})^{\frac{1}{2}} D^2u^{\varepsilon_2}\|_{L^2(\mathbb{T}^d;\mathbb{M}_{d\times d})}\\ &\quad + \|(m^{\varepsilon_2})^{\frac{\alpha+1}{2}} Du^{\varepsilon_2}\|_{W^{1,1}(\mathbb{T}^d;\mathbb{R}^d)} + \|\sqrt{\varepsilon_2} u^{\varepsilon_2}\|_{W^{2,2}(\mathbb{T}^d)} + \|\sqrt{\varepsilon_2} m^{\varepsilon_2}\|_{W^{2,2}(\mathbb{T}^d)} \leqslant C. \end{aligned}$$

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Thus, arguing as above, there exist $u \in W^{1,2}(\mathbb{T}^d) \cap L^{2^*}(\mathbb{T}^d)$ and $m \in L^{\frac{\alpha+1}{2}2^*}(\mathbb{T}^d)$ with $m^{\frac{\alpha+1}{2}} \in W^{1,2}(\mathbb{T}^d)$ such that, up to a not relabeled subsequence of $\{(m^{\varepsilon_2}, u^{\varepsilon_2})\}_{\varepsilon_2}$,

$$u^{\varepsilon_2} \rightharpoonup u \text{ weakly in } W^{1,2}(\mathbb{T}^d) \text{ as } \varepsilon_2 \to 0^+, \tag{6.24}$$

$$(m^{\varepsilon_2})^{\frac{\alpha+1}{2}} \rightharpoonup m^{\frac{\alpha+1}{2}} \text{ weakly in } W^{1,2}(\mathbb{T}^d) \text{ as } \varepsilon_2 \to 0^+, \tag{6.25}$$

$$(m^{\varepsilon_2})^{\frac{1}{2}} Du^{\varepsilon_2} \rightharpoonup m^{\frac{1}{2}} Du \text{ weakly in } L^2(\mathbb{T}^d;\mathbb{R}^d) \text{ as } \varepsilon_2 \to 0^+,$$

$$(m^{\varepsilon_2})^{\frac{\alpha+1}{2}} Du^{\varepsilon_2} \overset{\star}{\rightharpoonup} m^{\frac{\alpha+1}{2}} Du \text{ weakly-}\star \text{ in } BV(\mathbb{T}^d;\mathbb{R}^d) \text{ as } \varepsilon_2 \to 0^+, \tag{6.26}$$

$$m^{\varepsilon_2} Du^{\varepsilon_2} \rightharpoonup mDu \text{ weakly in } L^1(\mathbb{T}^d;\mathbb{R}^d) \text{ as } \varepsilon_2 \to 0^+, \tag{6.27}$$

$$\sqrt{\varepsilon_2} u^{\varepsilon_2} \to 0 \text{ in } W^{1,2}(\mathbb{T}^d) \text{ as } \varepsilon_2 \to 0^+, \tag{6.28}$$

$$\sqrt{\varepsilon_2} m^{\varepsilon_2} \to 0 \text{ in } W^{1,2}(\mathbb{T}^d) \text{ as } \varepsilon_2 \to 0^+, \tag{6.29}$$

$$\varepsilon_2 \Delta m^{\varepsilon_2} \to 0 \text{ in } L^1(\mathbb{T}^d) \text{ as } \varepsilon_2 \to 0^+, \tag{6.30}$$

$$h(\boldsymbol{m}^{\varepsilon_2}) \to h(\boldsymbol{m}) \text{ in } L^1(\mathbb{T}^d) \text{ as } \varepsilon_2 \to 0^+. \tag{6.31}$$

Moreover, using in addition the compact embedding $BV(\mathbb{T}^d) \hookrightarrow L^s(\mathbb{T}^d)$ for all $s \in [1, 1^*)$, it follows from (6.24), (6.25), (6.26), (6.30), and (6.31) that, up to a further not relabeled subsequence of $\{(m^{\varepsilon_2}, u^{\varepsilon_2})\}_{\varepsilon_2}$, we have for a.e. $x \in \mathbb{T}^d$,

$$\begin{aligned} &u^{\varepsilon_2}(x) \to u(x), \quad m^{\varepsilon_2}(x) \to m(x), \quad (m^{\varepsilon_2})^{\frac{\alpha+1}{2}}(x) Du^{\varepsilon_2}(x) \to m^{\frac{\alpha+1}{2}}(x) Du(x), \\ &\varepsilon_2 \Delta m^{\varepsilon_2}(x) \to 0, \quad h(\boldsymbol{m}^{\varepsilon_2})(x) \to h(\boldsymbol{m})(x) \text{ as } \varepsilon_2 \to 0^+. \end{aligned} \tag{6.32}$$

Finally, because $\inf_{\mathbb{T}^d} m^\varepsilon > 0$ and $\{m^\varepsilon\}_{\varepsilon_1}$ converges a.e. in $\mathbb{T}^d$ to m^{ε_2} as $\varepsilon_1 \to 0^+$, we conclude that $m^{\varepsilon_2} \geqslant 0$ a.e. in $\mathbb{T}^d$. In turn, the latter condition together with the second convergence in (6.32) yields $m \geqslant 0$ a.e. in $\mathbb{T}^d$. This concludes Step 1.

Step 2. We claim that the pair (m, u) with $m \geqslant 0$ a.e. in $\mathbb{T}^d$ determined in Step 1 is a weak solution to (6.2).

The argument is the same as the one in the proof of Theorem 1.1.

Step 3. We prove that the pair (m, u) with $m \geqslant 0$ a.e. in $\mathbb{T}^d$ determined in Step 1 satisfies $\int_{\mathbb{T}^d} m \, \mathrm{d}x = 1$, $\operatorname{div}(mDu) + \sigma^2 \Delta m \in L^{\frac{\alpha+1}{2}2^*}(\mathbb{T}^d)$, and (6.12).

Let $\varphi \in C^\infty(\mathbb{T}^d)$. Multiplying (6.4) by φ, integrating the resulting equality over $\mathbb{T}^d$, and using integration by parts, we obtain

$$\begin{aligned} &\int_{\mathbb{T}^d} (m^\varepsilon - 1)\varphi \, \mathrm{d}x + \varepsilon_1 \int_{\mathbb{T}^d} u^\varepsilon \varphi \, \mathrm{d}x + \varepsilon_1 \int_{\mathbb{T}^d} \Delta^p u^\varepsilon \Delta^p \varphi \, \mathrm{d}x \\ &\quad + \varepsilon_2 \int_{\mathbb{T}^d} u^\varepsilon \varphi \, \mathrm{d}x - \varepsilon_2 \int_{\mathbb{T}^d} u^\varepsilon \Delta \varphi \, \mathrm{d}x \\ &\quad = -\int_{\mathbb{T}^d} m^\varepsilon Du^\varepsilon \cdot D\varphi \, \mathrm{d}x + \sigma^2 \int_{\mathbb{T}^d} m^\varepsilon \Delta \varphi \, \mathrm{d}x. \end{aligned}$$

Letting $\varepsilon_1 \to 0^+$ first and using (6.14), (6.16), (6.13), and (6.23), and then letting $\varepsilon_2 \to 0^+$ and invoking (6.25), (6.28), and (6.27), we conclude that

$$\begin{aligned} \int_{\mathbb{T}^d} (m-1)\varphi \, \mathrm{d}x &= -\int_{\mathbb{T}^d} mDu \cdot D\varphi \, \mathrm{d}x + \sigma^2 \int_{\mathbb{T}^d} m\Delta\varphi \, \mathrm{d}x \\ &= \langle \operatorname{div}(mDu) + \sigma^2 \Delta m, \varphi \rangle_{\mathcal{D}'(\mathbb{T}^d), C^\infty(\mathbb{T}^d)}. \end{aligned}$$

Because $m - 1 \in L^{\frac{\alpha+1}{2}2^*}(\mathbb{T}^d)$ and $\varphi \in C^\infty(\mathbb{T}^d)$ is arbitrary, it follows that $\operatorname{div}(mDu) + \sigma^2 \Delta m \in L^{\frac{\alpha+1}{2}2^*}(\mathbb{T}^d)$ and (6.12) holds a.e. in $\mathbb{T}^d$.

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Finally, taking $\varphi = 1$ in the equality above yields $\int_{\mathbb{T}^d} m \, \mathrm{d}x = 1$. This completes Step 3.

Step 4. We prove that the pair (m, u) with $m \geqslant 0$ a.e. in $\mathbb{T}^d$ determined in Step 1 satisfies (6.11).

Let $\varphi \in C^\infty(\mathbb{T}^d)$ with $\varphi \geqslant 0$. Multiplying (6.3) by φ, integrating the resulting equality over $\mathbb{T}^d$, and using the integration by parts formula and the condition $\beta_{\varepsilon_1}(\cdot) \leqslant 0$, we get

$$
\begin{aligned}
&\int_{\mathbb{T}^d} -u^\varepsilon \varphi \, \mathrm{d}x + \int_{\mathbb{T}^d} V(x, m^\varepsilon, h(\boldsymbol{m}^\varepsilon))\varphi \, \mathrm{d}x + \sigma^2 \int_{\mathbb{T}^d} u^\varepsilon \Delta\varphi \, \mathrm{d}x + \varepsilon_1 \int_{\mathbb{T}^d} m^\varepsilon \varphi \, \mathrm{d}x \\
&\quad + \varepsilon_1 \int_{\mathbb{T}^d} \Delta^p m^\varepsilon \Delta^p \varphi \, \mathrm{d}x + \varepsilon_2 \int_{\mathbb{T}^d} m^\varepsilon \varphi \, \mathrm{d}x - \varepsilon_2 \int_{\mathbb{T}^d} m^\varepsilon \Delta\varphi \, \mathrm{d}x \\
&\quad = \int_{\mathbb{T}^d} \frac{|Du^\varepsilon|^2}{2} \varphi \, \mathrm{d}x - \int_{\mathbb{T}^d} \beta_{\varepsilon_1}(m^\varepsilon)\varphi \, \mathrm{d}x \geqslant \int_{\mathbb{T}^d} \frac{|Du^\varepsilon|^2}{2} \varphi \, \mathrm{d}x.
\end{aligned} \tag{6.33}
$$

First, we let $\varepsilon_1 \to 0^+$ and then $\varepsilon_2 \to 0^+$. According to (6.13), (6.14), (6.17), (6.21), (6.22), (6.24), (6.25), (6.29), (6.31), (6.32), the sequential lower semicontinuity of the convex functional

$$
w \in L^2(\mathbb{T}^d; \mathbb{R}^d) \mapsto \int_{\mathbb{T}^d} \frac{|w(x)|^2}{2} \varphi(x) \, \mathrm{d}x \in \mathbb{R}_0^+
$$

with respect to the weak convergence in $L^2(\mathbb{T}^d; \mathbb{R}^d)$, and the continuity of V together with Fatou's lemma and assumption (H5c), we deduce that

$$
\int_{\mathbb{T}^d} -u\varphi \, \mathrm{d}x + \int_{\mathbb{T}^d} V(x, m, h(\boldsymbol{m}))\varphi \, \mathrm{d}x + \sigma^2 \int_{\mathbb{T}^d} u \Delta\varphi \, \mathrm{d}x \geqslant \int_{\mathbb{T}^d} \frac{|Du|^2}{2} \varphi \, \mathrm{d}x.
$$

Equivalently,

$$
\left\langle -u - \frac{|Du|^2}{2} + V(x, m, h(\boldsymbol{m})) + \sigma^2 \Delta u, \varphi \right\rangle_{\mathcal{D}'(\mathbb{T}^d), C^\infty(\mathbb{T}^d)} \geqslant 0,
$$

and so, (6.11) holds since $\varphi \in C^\infty(\mathbb{T}^d)$ with $\varphi \geqslant 0$ is arbitrary. ☐

Next, we address the deterministic case, $\sigma = 0$, and improve the supersolution property from Proposition 5.5.

Corollary 6.3. *Suppose that assumptions* (h1), (h2), (g1), *and* (H5) *hold with* $\sigma = 0$. *If* $d \geqslant 8$, *assume further that* $\alpha > \frac{d-4}{2}$ *in assumption* (H5). *Then, the weak solution,* (m, u), *of* (6.2) *given by Theorem* 6.2 *satisfies*

$$
\begin{cases}
\left(-u - \dfrac{|Du|^2}{2} + V(x, m, h(\boldsymbol{m}))\right) m = 0, \\
m - \operatorname{div}(mDu) - 1 = 0
\end{cases}
$$

a.e. in $\mathbb{T}^d$.

Proof. Assume that $\sigma = 0$. In view of Theorem 6.2, we are left to prove that

$$
\left(-u - \frac{|Du|^2}{2} + V(x, m, h(\boldsymbol{m}))\right) m = 0 \quad \text{a.e. in } \mathbb{T}^d. \tag{6.34}
$$

In what follows, either $d \leqslant 7$ and $\alpha > 0$ or $d \geqslant 8$ and $\alpha > \frac{d-4}{2}$. Let $\varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, 1)^2$ and $(m^\varepsilon, u^\varepsilon) \in C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ with $\inf_{\mathbb{T}^d} m^\varepsilon > 0$ be as in the proof of

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Theorem 6.2. The arguments in Step 4 of that proof show that by letting $\varepsilon_1 \to 0^+$ in (6.33), we have

$$\int_{\mathbb{T}^d} \left(-u^{\varepsilon_2} - \frac{|Du^{\varepsilon_2}|^2}{2} + V(x, m^{\varepsilon_2}, h(\boldsymbol{m}^{\varepsilon_2})) + \varepsilon_2 m^{\varepsilon_2} - \varepsilon_2 \Delta m^{\varepsilon_2} \right) \varphi \, \mathrm{d}x \geqslant 0$$

for all $\varphi \in C^\infty(\mathbb{T}^d)$ with $\varphi \geqslant 0$. Hence, because $m^{\varepsilon_2} \geqslant 0$ a.e. in $\mathbb{T}^d$, we conclude that

$$\left(-u^{\varepsilon_2} - \frac{|Du^{\varepsilon_2}|^2}{2} + V(x, m^{\varepsilon_2}, h(\boldsymbol{m}^{\varepsilon_2})) + \varepsilon_2 m^{\varepsilon_2} - \varepsilon_2 \Delta m^{\varepsilon_2} \right) m^{\varepsilon_2} \geqslant 0 \tag{6.35}$$

a.e. in $\mathbb{T}^d$.

Next, we prove the converse inequality. Let $\varphi \in C^\infty(\mathbb{T}^d)$ with $\varphi \geqslant 0$. Multiplying (6.3) by $m^\varepsilon \varphi$, integrating the resulting equality over $\mathbb{T}^d$, and using integration by parts, we get

$$\begin{aligned} &- \int_{\mathbb{T}^d} u^\varepsilon m^\varepsilon \varphi \, \mathrm{d}x - \int_{\mathbb{T}^d} \frac{|Du^\varepsilon|^2}{2} m^\varepsilon \varphi \, \mathrm{d}x + \int_{\mathbb{T}^d} V(x, m^\varepsilon, h(\boldsymbol{m}^\varepsilon)) m^\varepsilon \varphi \, \mathrm{d}x \\ &\quad + \varepsilon_1 \int_{\mathbb{T}^d} (m^\varepsilon)^2 \varphi \, \mathrm{d}x + \varepsilon_1 \int_{\mathbb{T}^d} \Delta^p m^\varepsilon \Delta^p (m^\varepsilon \varphi) \, \mathrm{d}x + \int_{\mathbb{T}^d} \beta_{\varepsilon_1}(m^\varepsilon) m^\varepsilon \varphi \, \mathrm{d}x \\ &\quad + \varepsilon_2 \int_{\mathbb{T}^d} (m^\varepsilon)^2 \varphi \, \mathrm{d}x - \varepsilon_2 \int_{\mathbb{T}^d} \Delta m^\varepsilon m^\varepsilon \varphi \, \mathrm{d}x = 0. \end{aligned} \tag{6.36}$$

Using (6.13) and (6.14) together with the embeddings mentioned at the beginning of Step 1 of the previous proof, we have

$$\begin{aligned} &\lim_{\varepsilon_1 \to 0^+} \left(- \int_{\mathbb{T}^d} u^\varepsilon m^\varepsilon \varphi \, \mathrm{d}x + \varepsilon_2 \int_{\mathbb{T}^d} (m^\varepsilon)^2 \varphi \, \mathrm{d}x - \varepsilon_2 \int_{\mathbb{T}^d} \Delta m^\varepsilon m^\varepsilon \varphi \, \mathrm{d}x \right) \\ &\quad = - \int_{\mathbb{T}^d} u^{\varepsilon_2} m^{\varepsilon_2} \varphi \, \mathrm{d}x + \varepsilon_2 \int_{\mathbb{T}^d} (m^{\varepsilon_2})^2 \varphi \, \mathrm{d}x - \varepsilon_2 \int_{\mathbb{T}^d} \Delta m^{\varepsilon_2} m^{\varepsilon_2} \varphi \, \mathrm{d}x. \end{aligned} \tag{6.37}$$

From assumptions (H5ai), (g1a), and (H5bi), it follows that

$$V(x, m^\varepsilon, h(\boldsymbol{m}^\varepsilon)) \geqslant \frac{1}{\kappa_2} (m^\varepsilon)^\alpha - \kappa_2$$

for some positive constant, κ_2, independent of ε. Thus, because $m^\varepsilon \varphi \geqslant 0$ a.e. in $\mathbb{T}^d$, Fatou's lemma, the continuity of V, and the convergences (6.15) and (6.22) imply that

$$\int_{\mathbb{T}^d} V(x, m^{\varepsilon_2}, h(\boldsymbol{m}^{\varepsilon_2})) m^{\varepsilon_2} \varphi \, \mathrm{d}x \leqslant \liminf_{\varepsilon_1 \to 0^+} \int_{\mathbb{T}^d} V(x, m^\varepsilon, h(\boldsymbol{m}^\varepsilon)) m^\varepsilon \varphi \, \mathrm{d}x. \tag{6.38}$$

By definition, $\beta_{\varepsilon_1}(\cdot) \leqslant 0$ and $\beta_{\varepsilon_1}(s) = 0$ if $s \geqslant \varepsilon_1$ and, thus,

$$\left| \int_{\mathbb{T}^d} \beta_{\varepsilon_1}(m^\varepsilon) m^\varepsilon \varphi \, \mathrm{d}x \right| = \left| \int_{\{x \in \mathbb{T}^d \colon m^\varepsilon(x) \leqslant \varepsilon_1\}} \beta_{\varepsilon_1}(m^\varepsilon) m^\varepsilon \varphi \, \mathrm{d}x \right| \leqslant \varepsilon_1 \|\varphi\|_{L^\infty(\mathbb{T}^d)} C,$$

where C is the constant given by Lemma 6.1. Hence,

$$\lim_{\varepsilon_1 \to 0^+} \left(\varepsilon_1 \int_{\mathbb{T}^d} (m^\varepsilon)^2 \varphi \, \mathrm{d}x + \varepsilon_1 \int_{\mathbb{T}^d} \Delta^p m^\varepsilon \Delta^p (m^\varepsilon \varphi) \, \mathrm{d}x + \int_{\mathbb{T}^d} \beta_{\varepsilon_1}(m^\varepsilon) m^\varepsilon \varphi \, \mathrm{d}x \right) = 0, \tag{6.39}$$

where we also used (6.17).

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Next, we prove that

$$\lim_{\varepsilon_1\to 0^+}\int_{\mathbb{T}^d}\frac{|Du^\varepsilon|^2}{2}m^\varepsilon\varphi\,\mathrm{d}x=\int_{\mathbb{T}^d}\frac{|Du^{\varepsilon_2}|^2}{2}m^{\varepsilon_2}\varphi\,\mathrm{d}x. \tag{6.40}$$

In view of the embeddings mentioned at the beginning of Step 1 of the previous proof, we have

$$m^\varepsilon\to m^{\varepsilon_2}\text{ in }L^p(\mathbb{T}^d)\text{ as }\varepsilon_1\to 0^+\text{ and for all }p\in(1,(2^*)^*), \tag{6.41}$$

$$m^\varepsilon\to m^{\varepsilon_2}\text{ in }L^r(\mathbb{T}^d)\text{ as }\varepsilon_1\to 0^+\text{ and for all }r\in\left(1,\frac{(\alpha+1)2^*}{2}\right), \tag{6.42}$$

$$|Du^\varepsilon|^2\to|Du^{\varepsilon_2}|^2\text{ in }L^s(\mathbb{T}^d)\text{ as }\varepsilon_1\to 0^+\text{ and for all }s\in\left(1,\frac{2^*}{2}\right). \tag{6.43}$$

If $d\leqslant 7$, then $(\frac{2^*}{2})'<(2^*)^*$. Accordingly, for some $p<\frac{2^*}{2}$ in (6.41) and some $s<\frac{2^*}{2}$ in (6.43), we get $|Du^\varepsilon|^2m^\varepsilon\to|Du^{\varepsilon_2}|^2m^{\varepsilon_2}$ strongly in $L^q(\mathbb{T}^d)$ with $\frac{1}{q}=\frac{1}{r}+\frac{1}{s}$ and $q>1$. Moreover, if $d\geqslant 8$ and $\alpha>\frac{d-4}{2}$, then there is $q>1$ such that $\frac{1}{q}=\frac{1}{r}+\frac{1}{s}$, where r and s are as in (6.42) and (6.43), respectively. Therefore, we get again $|Du^\varepsilon|^2m^\varepsilon\to|Du^{\varepsilon_2}|^2m^{\varepsilon_2}$ strongly in $L^q(\mathbb{T}^d)$.

Finally, by (6.36), (6.37), (6.38), (6.39), and (6.40), we have

$$\int_{\mathbb{T}^d}\left(-u^{\varepsilon_2}-\frac{|Du^{\varepsilon_2}|^2}{2}+V(x,m^{\varepsilon_2},h(\boldsymbol{m}^{\varepsilon_2}))+\varepsilon_2m^{\varepsilon_2}-\varepsilon_2\Delta m^{\varepsilon_2}\right)m^{\varepsilon_2}\varphi\,\mathrm{d}x\leqslant 0.$$

Because $\varphi\in C^\infty(\mathbb{T}^d)$ with $\varphi\geqslant 0$ is arbitrary, recalling (6.35), it follows that

$$\left(-u^{\varepsilon_2}-\frac{|Du^{\varepsilon_2}|^2}{2}+V(x,m^{\varepsilon_2},h(\boldsymbol{m}^{\varepsilon_2}))+\varepsilon_2m^{\varepsilon_2}-\varepsilon_2\Delta m^{\varepsilon_2}\right)m^{\varepsilon_2}=0\quad\text{a.e. in }\mathbb{T}^d. \tag{6.44}$$

Let $A:=\{x\in\mathbb{T}^d\colon \text{(6.32) and } m(x)\geqslant 0\text{ hold}\}$. Then, $\mathcal{L}^d(\mathbb{T}^d\backslash A)=0$.

If $x\in A$ is such that $m(x)=0$, then clearly

$$\left(-u(x)-\frac{|Du(x)|^2}{2}+V(x,m(x),h(\boldsymbol{m})(x))\right)m(x)=0. \tag{6.45}$$

If $x\in A$ is such that $m(x)>0$, then $m^{\varepsilon_2}(x)>0$ for all sufficiently small ε_2; moreover,

$$\begin{aligned}\lim_{\varepsilon_2\to 0^+}|Du^{\varepsilon_2}(x)|^2m^{\varepsilon_2}(x)&=\lim_{\varepsilon_2\to 0^+}|Du^{\varepsilon_2}(x)|^2(m^{\varepsilon_2})^{\alpha+1}(x)(m^{\varepsilon_2})^{1-\alpha}(x)\\&=|Du(x)|^2m(x),\end{aligned}$$

which, together with (6.32) and (6.44), proves that (6.45) holds for all such x. This completes the proof of (6.34). ∎

7. Final remarks. The monotonicity method developed here is a powerful and flexible way to study MFGs. Several extensions can be considered using similar ideas and are briefly described here: fully nonlinear Hamiltonians, nonsmooth Hamiltonians, and Hamiltonians that are not monotone in the nonlocal term.

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7.1. Fully nonlinear Hamiltonians. The way we state assumption (H1b), the case of fully nonlinear Hamiltonians is excluded. However, as in Remark 3.5, let $\hat{H}(x,p,M,m,\theta) := H(x,p,M,m,\theta) + H^0(M)$, where H satisfies (h1), (g1), (g2), and (H1)–(H3) and $H^0(D^2u)$ is a C^∞ convex fully nonlinear operator. If we consider the MFG

$$\begin{cases} -u - H(x,Du,D^2u,m,h(\boldsymbol{m})) - H^0(D^2u) = 0, \\ m - \operatorname{div}(mD_pH(x,Du,D^2u,m,h(\boldsymbol{m}))) + \left(mD_{M_{ij}}H^0(D^2u)\right)_{x_ix_j} = 1, \end{cases} \tag{7.1}$$

the bounds in Lemma 3.3 and Remark 3.4 remain unchanged. In addition, a similar argument yields the existence of a smooth solution for the regularized MFG (3.1). Thus, the corresponding existence result in Proposition 3.1 still holds. Accordingly, in view of Remark 4.3, we can apply Minty's method and obtain a weak solution for (7.1).

Aubry–Mather theory can be seen as a precursor of MFGs: Mather measures satisfy a transport equation coupled with a Hamilton–Jacobi equation. A number of extensions to the nonlinear setting were considered in [21], [22], [45], and [46]. This duality framework might be applicable for fully nonlinear MFGs.

7.2. Nonsmooth Hamiltonians. The existence of solutions for (3.1) by the continuation method requires a high degree of smoothness for the Hamiltonian. This is reflected in the C^∞ requirement in assumption (H2). However, by a regularization argument, it is possible to reduce substantially the regularity requirements. For example, let H be a continuous Hamiltonian, C^1 in the (p,M) variables, and such that D_pH is C^1 and D_MH is C^2. Suppose that H satisfies assumptions (h1), (g1), (g2), (H1), and (H3). Assume that there exists a Hamiltonian H^δ, $\delta > 0$, satisfying assumption (H2) and all the preceding assumptions with constants that are uniform in δ. Let A^δ be as in (1.7) for $H = H^\delta$ and assume that

$$A^\delta \begin{bmatrix} \eta \\ v \end{bmatrix} \to A \begin{bmatrix} \eta \\ v \end{bmatrix} \tag{7.2}$$

for all $v, \eta \in C^\infty$ with $\eta > 0$. Then, by the discussion above there exists a weak solution (m^δ, u^δ). By weak compactness, we can extract a weak limit (m,u) as $\delta \to 0$. Using the Minty method once more, we obtain that (m,u) is a weak solution of the limit problem.

A problem where such a regularization is possible is the case in which

$$H(x,p,M,m,\theta) = H_0(x,p,M) - g(m) + V(x),$$

where H_0 and g are C^∞ and satisfying the preceding assumptions and V is continuous. In this case, we introduce

$$H^\delta(x,p,M,m,\theta) = H_0(x,p,M) - g(m) + \zeta^\delta * V,$$

where ζ^δ is a smooth standard mollifier. It is easy to see that our assumptions hold with constants that are uniform in δ and that (7.2) is satisfied. Therefore, we have a weak solution of the corresponding MFG. We believe that with regularization methods, many problems can be solved but this discussion is outside the scope of the present paper.

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7.3. Hamiltonians that are not monotone with respect to the nonlocal term. The monotonicity property in assumption (H3) can be relaxed in the following way. Suppose that H is such that for fixed $\bar{\boldsymbol{m}} \in \mathcal{M}_{ac}(\mathbb{T}^d)$ we have that $H(x, p, M, m, h(\bar{\boldsymbol{m}}))$ satisfies assumptions (h1), (g1), (g2), and (H1)–(H3), with constants that are uniform in $\bar{m}$. Suppose further that h is a compact mapping; that is, $\boldsymbol{m}^\varepsilon \rightharpoonup \boldsymbol{m}$ in $\mathcal{M}_{ac}(\mathbb{T}^d)$ implies $h(\boldsymbol{m}^\varepsilon) \to h(\boldsymbol{m})$ in $C(\mathbb{T}^d)$ through some subsequence.

For every $\varepsilon > 0$, there exists a solution of (3.1) for the Hamiltonian $H(x, p, M, m, h(\bar{\boldsymbol{m}}))$. We denote this solution $(\bar{M}^\varepsilon(\bar{\boldsymbol{m}}), \bar{U}^\varepsilon(\bar{\boldsymbol{m}}))$. By the implicit function theorem, the map $\bar{\boldsymbol{m}} \mapsto \bar{M}^\varepsilon(\bar{\boldsymbol{m}})$, regarded as a mapping from $\mathcal{M}_{ac}(\mathbb{T}^d)$ to itself, is continuous and compact. Thus, by Schauder's fixed point theorem, there exists a fixed point $\bar{\boldsymbol{m}}^\varepsilon$. The corresponding solution is denoted by $(\bar{m}^\varepsilon, \bar{u}^\varepsilon)$. Next, by passing to the limit as $\varepsilon \to 0$, we obtain a limiting point $(\bar{m}^0, \bar{u}^0)$. Using Minty's method once more, we see that $(\bar{m}^0, \bar{u}^0)$ is a weak solution for the Hamiltonian $H(x, p, M, m, h(\boldsymbol{m}^0))$.

7.4. Final remarks. Using ideas similar to the ones in this paper, we can study many other problems, including the standard stationary MFG:

$$\begin{cases} H(x, Du, D^2u, m, h(\boldsymbol{m})) = \overline{H}, \\ -\operatorname{div}(mD_pH(x, Du, D^2u, m, h(\boldsymbol{m}))) + \left(mD_{M_{ij}}H(x, Du, D^2u, m, h(\boldsymbol{m}))\right)_{x_ix_j} = 0. \end{cases}$$

In addition, there are several areas where future developments are likely. First, we foresee improvements in the regularity theory for weak solutions. Additional regularity is essential to prove the uniqueness of solutions. Second, the congestion problems examined here may enjoy further regularity properties; this topic was not explored in the present paper. Third, time-dependent MFGs are a natural application of our methods. Here, we need to develop a different regularization method because of the initial-terminal boundary conditions. Moreover, the regularity theory for these problems may differ substantially from the stationary case. Finally, our results may be of independent interest in the calculus of variations as certain MFGs are the Euler–Lagrange equation of integral functionals. These functionals are convex but, in many cases, noncoercive, and their study presents substantial challenges; see, for example, [35].

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