A Priori Regularity of Parabolic Partial Differential Equations

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ABSTRACT

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In this thesis, we consider parabolic partial differential equations such as the heat equation, the Fokker-Planck equation, and the porous media equation. Our aim is to develop methods that provide *a priori* estimates for solutions with singular initial data. These estimates are obtained by understanding the time decay of norms of solutions.

First, we derive regularity results for the heat equation by estimating the decay of Lebesgue norms. Then, we apply similar methods to the Fokker-Planck equation with suitable assumptions on the advection and diffusion. Finally, we conclude by extending our techniques to the porous media equation. The sharpness of our results is confirmed by examining known solutions of these equations.

The main contribution of this thesis is the use of functional inequalities to express decay of norms as differential inequalities. These are then combined with ODE methods to deduce estimates for the norms of solutions and their derivatives.

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TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

Parabolic partial differential equations model a variety of physical phenomena. These equations are often used to describe the diffusion of mass, momentum or heat through a material. Here, we consider the heat equation, the Fokker-Planck equation, and the porous media equation.

One of the classical parabolic PDE is the heat equation, which, for a function $u : \mathbb{R}^d \times [0, T] \to \mathbb{R}$, is given by

$$u_t(x, t) = \Delta u(x, t),$$

where $u_t := \frac{d}{dt}u$ and $\Delta = \Delta_x$ stands for the Laplace operator.

This equation describes the process of thermal propagation (or distribution of heat) in a region over time and marks a unique point in the history of modern mathematical physics. Since first introduced in Fourier's essay *Théorie de la Propagation de la Chaleur dans les Solides* in 1822, the heat equation has motivated a continuous growth in areas such as Fourier analysis, spectral theory, set theory, operator theory, as well as the analysis of differential equations involving many physical, biological and social systems.

The derivation of the heat equation is based on conservation laws. A conservation law, simply put, says that the rate at which a quantity changes in a given domain is equal to the rate at which the quantity flows across the boundary of that domain plus the rate at which the quantity is created or destroyed inside the domain. In the particular case of the heat equation, this is done using an energy balance on a differential control volume motivated by experimental laws of physics.

The Fokker-Planck equation is given by

$$u_t(x,t) = \operatorname{div}(b(x,t)u(x,t)) + \operatorname{div}(a(x,t)\nabla u(x,t)),$$

where a is a positive scalar diffusion coefficient and b is a smooth vector field, modeling the advection, which is the transport of a substance by bulk motion. This second-order equation, also known as the Kolmogorov forward equation, models the behavior of a particle under the effect of drag and random forces.

This equation is of particular interest in stochastic differential equations, providing an alternate formulation for certain stochastic processes, such as the Wiener and the Ornstein-Uhlenbeck processes. In physics, the Fokker-Planck equation has numerous applications in polymer fluids, plasma, and surface physics.

Our approach to the Fokker-Planck equation relies on a set of generalizing assumptions regarding the regularity of both diffusion and advection. We do this by considering the advection-free scenario, as well as particular Lebesgue spaces for a and b.

Finally, the porous media equation emerges as a nonlinear generalization of the heat equation, given by

$$u_t(x,t) = \Delta(u(x,t)^m)$$

for $m \geq 1$. This equation models diffusion processes and fluid flow through porous media (sponge or wood, for example) and has applications in mathematical biology, lubrication, and boundary layer theory, with the cases $d = 1, 2, 3$ of most interest to the applied scientist.

Such generalization yields several properties that differ strongly from the heat equation theory, leading in many cases to the development of nonlinear analytical tools for the more general classes of nonlinear equations that arise in both pure and applied mathematics.

In our work, we consider the previous three equations with solutions in $C^\infty(\mathbb{R}^d \times [0,T])$ and initial data in $L^1(\mathbb{R}^d)$. Then, assuming enough decay of the solution, we obtain bounds that depend only on the L^1-norm of the initial data. For the heat equation, using a weak convergence argument, we can also consider singular data $u(x,0) = \delta_{x_0}$. In particular, some of the results regarding the heat equation and the porous media equation are also extended to the d-dimensional torus $\mathbb{T}^d$. The torus is the product of d circles; that is, $\mathbb{T}^d = S^1 \times \cdots \times S^1$. It can also be described as a quotient of $\mathbb{R}^d$ under integral shifts in any coordinate; that is, $\mathbb{T}^d = \mathbb{R}^d \backslash \mathbb{Z}^d$. Solutions on the torus are naturally periodic.

We are particularly interested in the Lebesgue norms of solutions, L^p-norms, which, for a measurable function $u : \Omega \to \mathbb{R}$, are given by

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}},$$

where Ω is either $\mathbb{R}^d$ or $\mathbb{T}^d$. We study the time decay of L^p-norms of solutions and their derivatives. In general, L^p-norms are a way to measure the size of elements of a vector space. The cases $p = 1, 2$ arise in physics when measuring mass or energy, while L^∞ norms describe the essential supremum of functions. For other values of p, these provide alternative descriptions of the size of functions.

The main contribution of this thesis is the use of functional inequalities and a differential argument to derive a method to prove estimates for norms of solutions of linear and non-linear parabolic equations. This method systematizes techniques to infer estimates for solutions of parabolic PDE. We further compare such techniques to prior approaches, namely the entropy method and hypercontractivity.

Similar techniques were studied in [1, 2, 3] regarding both the smoothing effect and the time decay of solutions of the heat equation and the porous media equation. Entropy methods were discussed in [4], and hypercontractivity was examined in [5, 6, 7, 8, 9]. Some of the technical results we use are based on [10, 11, 12, 13, 14, 15].

A method comparable to ours was studied in [16, 17]. There, the regularizing effect and the long and short time decay were studied for the parabolic Cauchy-Dirichlet problem and the viscous Hamilton-Jacobi equation with a superlinear Hamiltonian.

For a solution $u : \Omega \times [0,T] \to \mathbb{R}$ to a parabolic PDE, under assumptions depending on the specific equation, we obtain estimates of the type

$$\|D^k u\|_{L^p(\Omega)} \leq Ct^{-g(p,d,k)},$$

where $k \in \mathbb{N}_0$, $g \geq 0$ is a function of the dimension d, k and p, and C is some non-negative constant depending on the space and the given parameters. Moreover, these estimates depend only on the L^1-norm of the initial data. We analyze our mechanism and estimates by contrasting them with some of the work mentioned above. We emphasize that our results are equally valid in some particular cases.

There are three key techniques used in the proofs of our results. First, we expand the time derivative of the L^p-norms and use integration by parts to get dissipation of these norms. Then, we combine Gagliardo-Nirenberg and Sobolev inequalities with the conservation of L^1-norms to obtain a non-linear dissipation estimate. Finally, we apply a non-linear Grönwall-type estimate to get short-time decay.

We end this introduction with an outline of the thesis. In Chapter 2, we present our main assumptions and estimates for the heat equation. We conclude the chapter by discussing entropy methods and hypercontractivity. Chapter 3 deals with the Fokker-Planck equation, under assumptions such as L^q-regularity of the advection, constant diffusion, and divergence-free advection. Finally, Chapter 4 focuses on the porous media equation.

Chapter 2

Heat equation estimates

In this chapter, we are interested in regularity estimates for solutions of

$$\begin{cases} u_t(x,t) + \Delta u(x,t) = f & \text{in } \Omega \times [0,T) \\ u(x,T) = u_T(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$, $u_T \in W^{1,\infty}(\Omega)$ and $f \in L^p([0,T], L^b(\Omega))$. In the following, we assume $u \geq 0$. We begin by proving an estimate for the L^∞-norm of the solution of the nonhomogeneous problem using the non-linear adjoint method from [10, 12, 13]. Next, using Gagliardo-Nirenberg estimates and a differential inequality method, we deduce an L^2 estimate for the adjoint solution and its derivatives. We further discuss the general L^p bounds and study space-time regularity. Using a weak convergence argument, we extend these estimates to the adjoint problem with singular initial data.

The sharpness of our results is illustrated with a comparison with the fundamental solution. Finally, we conclude the chapter by examining alternative approaches using entropy methods, based on [4], and hypercontractivity, based on [5, 6, 7, 8, 9].

2.1 Motivation

We begin by deducing an estimate involving the solution u and the function f. This estimate illustrates the techniques used later to prove the main estimates.

2.1.1 The non-linear adjoint method setup

Our first goal is to develop tools to prove estimates for solutions of (2.1) such as

$$\|u\|_{L^\infty(\Omega\times[0,T])} \leq C\|f\|_{L^p([0,T],L^b(\Omega))}.$$

For that, we use the nonlinear adjoint method that we describe next. Based on [10, 12, 13], the adjoint equation to (2.1) is

$$\begin{cases} \rho_t(x,t) = \Delta\rho(x,t) & \text{in } \Omega\times(0,T] \\ \rho(x,0) = \delta_{x_0} & \text{in } \Omega, \end{cases} \tag{2.2}$$

where $\Omega = \mathbb{R}^d$ or $\mathbb{T}^d$ and $x_0 \in \mathbb{R}^d$. We further assume that $\rho(\cdot,t) \in L^1(\Omega)$. The central idea of this method is to derive a representation formula for solutions of (2.1) in terms of solutions of (2.2).

The introduction of (2.2) is mainly motivated by weak Kolmogorov-Arnold-Moser (KAM) theory, in a more general case presented in [13] for Lagrangian and Hamiltonian dynamics. In [10], it was shown that the solution to (2.1) can be written in terms of a solution of (2.2). In particular, we have

$$u(x_0,0) = \int_0^T\int_\Omega f(x,t)\rho(x,t)\,dxdt + \int_\Omega u_T(x)\rho(x,T)\,dx. \tag{2.3}$$

Then, it follows that

$$|u(x_0,0)| \leq \int_0^T\int_\Omega |f(x,t)\rho(x,t)|\,dxdt + \int_\Omega |u_T(x)\rho(x,T)|\,dx. \tag{2.4}$$

Therefore, to estimate the left-hand side, it is enough to bound each of the two terms on the right-hand side of the prior inequality. For the second term on the right-hand side, we have that, by Hölder's inequality,

$$\int_{\Omega} |u_T(x)\rho(x,T)|\,dx \leq \|u_T\|_{L^\infty(\Omega)}\|\rho(x,T)\|_{L^1(\Omega)} = \|u_T\|_{L^\infty(\Omega)} \leq C,$$

since $u_T \in W^{1,\infty}(\Omega)$. For the first term in (2.4), we apply Hölder's inequality twice to conclude that

$$\begin{aligned}\int_0^T \int_{\Omega} |f\rho|\,dxdt &= \int_0^T \int_{\Omega} |f||\rho|\,dxdt \\ &\leq \int_0^T \|f\|_{L^b(\Omega)}\|\rho\|_{L^c(\Omega)}\,dt \\ &\leq \|f\|_{L^p([0,T],L^b(\Omega))}\|\rho\|_{L^q([0,T],L^c(\Omega))},\end{aligned} \tag{2.5}$$

where b, c, p, q satisfy

$$\frac{1}{b} + \frac{1}{c} = 1 = \frac{1}{p} + \frac{1}{q}. \tag{2.6}$$

Thus, we see that by getting bounds for ρ, we can convert them into bounds for $u(x_0, 0)$ and so, since (2.1) is a backward heat equation, we also get bounds for u. Therefore, all the estimates in this chapter are presented with respect to the homogeneous heat equation (2.2).

2.1.2 A particular estimate

We now discuss estimates for the Fokker-Planck equation with singular initial conditions. For $\mathbb{T}^d$, these estimates appeared in [10]. Here, we extend them to $\mathbb{R}^d$.

Proposition 2.1.1. *Let ρ solve* (2.2)*. Then, for any $0 < \nu < 1$, there exists a constant $C > 0$ that does not depend on the solution, such that*

$$\int_0^T \int_{\Omega} |D(\rho^{\frac{\nu}{2}})|^2\,dxdt \leq C. \tag{2.7}$$

Remark 2.1.2. In the $\mathbb{R}^d$ case, we could use the fundamental solution, but our proof below avoids cumbersome estimates.

Proof. We begin by noticing that, using (2.2),

$$\begin{aligned}\frac{d}{dt}\int_\Omega \rho^\nu\,dx &= \nu\int_\Omega \rho^{\nu-1}\Delta\rho\,dx\\ &= -\nu(\nu-1)\int_\Omega \rho^{\nu-2}|\nabla\rho|^2\,dx\\ &= C_\nu\int_\Omega |\nabla(\rho^{\frac{\nu}{2}})|^2\,dx,\end{aligned}$$

where

$$C_\nu = \frac{4(1-\nu)}{\nu} > 0.$$

Hence, it follows that

$$C_\nu\int_0^T\int_\Omega |\nabla(\rho^{\frac{\nu}{2}})|^2\,dxdt = \int_\Omega \rho(x,T)^\nu\,dx - \int_\Omega \rho(x,0)^\nu\,dx. \tag{2.8}$$

Now, the $\Omega = \mathbb{T}^d$ case was shown in [10]. For $\Omega = \mathbb{R}^d$, we must bound $\int_{\mathbb{R}^d}\rho(x,T)^\nu\,dx$. Let $k>1$ and $j\in\mathbb{N}$. Assume further that $d>2j$. Then, by Young's inequality, the following holds

$$\begin{aligned}\int_{\mathbb{R}^d}\rho^{\frac{1}{k}}\,dx &= \int_{\mathbb{R}^d}\rho^{\frac{1}{k}}(|x|^{2j}+1)^{\frac{1}{k}}\frac{1}{(|x|^{2j}+1)^{\frac{1}{k}}}\,dx\\ &\le C\int_{\mathbb{R}^d}(|x|^{2j}+1)\rho\,dx + C\int_{\mathbb{R}^d}\frac{1}{(|x|^{2j}+1)^{\frac{1}{k-1}}}\,dx.\end{aligned} \tag{2.9}$$

Regarding the term on the right-hand side of (2.9) we note that, using two changes of variables,

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{1}{(|x|^{2j}+1)^{\frac{1}{k-1}}}\,dx &= \omega_{d-1}\int_0^\infty \frac{r^{d-1}}{(r^{2j}+1)^{\frac{1}{k-1}}}\,dr\\ &= C\omega_{d-1}\int_1^\infty \frac{(y-1)^{\frac{d}{2j}-1}}{y^{\frac{1}{k-1}}}\,dy\end{aligned}$$

$$\leq C\omega_{d-1} \int_1^\infty \frac{y^{\frac{d}{2j}-1}}{y^{\frac{1}{k-1}}}\, dy,$$

where ω_{d-1} is the surface area of the unit ball of $\mathbb{R}^d$. Thus, we have integrability for $j > (k-1)d/2$. For the other term, we have that

$$\frac{d^j}{dt^j} \int_{\mathbb{R}^d} (|x|^{2j}+1)\rho\, dx = \int_{\mathbb{R}^d} (|x|^{2j}+1)\Delta^j \rho\, dx = C \int_{\mathbb{R}^d} \rho\, dx = C.$$

Thus

$$\int_{\mathbb{R}^d} (|x|^{2j}+1)\rho \leq C + Ct^j.$$

Therefore, we get the following estimate

$$\int_{\mathbb{R}^d} \rho(x,T)^{\frac{1}{k}}\, dx \leq C + CT^j \leq C.$$

Given $0 < \nu < 1$, we select k such that $\nu = \frac{1}{k} < 1$. Since $\rho(x,T)^{\frac{1}{k}} \in L^1(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \rho(x,T)^\nu = \int_{\mathbb{R}^d} \rho(x,T)^{\frac{1}{k}}\, dx \leq C.$$

The prior estimate combined with (2.8) yields (2.7). □

2.1.3 Estimate construction

We are now interested in determining the region in the parameter space, (q,c), such that ρ can be bounded in $L^q([0,T], L^c(\Omega))$. The next proposition yields the necessary relation between q and c.

Proposition 2.1.3. *Let ρ solve (2.2). Then, for (q,c,d) such that $q \geq 1$, $d > 2$ and*

$$1 \leq c < \frac{qd}{qd-2}, \tag{2.10}$$

we have that

$$\|\rho\|_{L^q([0,T],L^c(\Omega))} \leq C.$$

Remark 2.1.4. Figure 2.1 illustrates the region for $d = 3$.

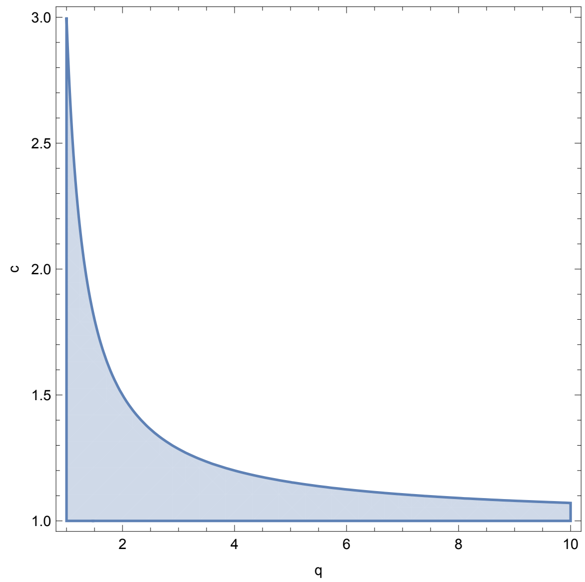

Figure 2.1: Integrability region for norms of the adjoint equation

Proof. Sobolev's inequality yields

$$\|\rho^{\frac{\nu}{2}}\|_{L^{2^*}(\Omega)} \leq C\|D(\rho^{\frac{\nu}{2}})\|_{L^2(\Omega)} + C', \tag{2.11}$$

where

$$2^* = \frac{2d}{d-2}.$$

In the $\Omega = \mathbb{R}^d$ case, we may take $C' = 0$, which corresponds to the Gagliardo-Nirenberg-Sobolev inequality. By Section 2.1.2, we control the time integral of

$$\int_\Omega |D(\rho^{\frac{\nu}{2}})|^2 \, dx$$

and, thus, we also control

$$\int_0^T \|D(\rho^{\frac{\nu}{2}})\|^2_{L^2(\Omega)}\,dt = \int_0^T \int_\Omega |D(\rho^{\frac{\nu}{2}})|^2\,dxdt \leq C. \tag{2.12}$$

Recall the following interpolation inequality

$$\|\rho\|_{L^c(\Omega)} \leq \|\rho\|^{1-\lambda}_{L^s(\Omega)} \|\rho\|^{\lambda}_{L^r(\Omega)} \tag{2.13}$$

for $1 \leq s \leq c \leq r \leq \infty$, where

$$\frac{1}{c} = \frac{1-\lambda}{s} + \frac{\lambda}{r}. \tag{2.14}$$

Fix

$$r = \frac{2^*\nu}{2}, \quad s = 1. \tag{2.15}$$

Then, using (2.13),

$$\begin{aligned}
\|\rho\|_{L^c(\Omega)} &\leq \|\rho\|^{1-\lambda}_{L^1(\Omega)} \|\rho\|^{\lambda}_{L^{2^*\nu/2}(\Omega)} \\
&= \left(\int_\Omega \rho\,dx\right)^{1-\lambda} \left(\int_\Omega \rho^{\frac{2^*\nu}{2}}\,dx\right)^{\frac{2\lambda}{2^*\nu}} \\
&= \|\rho^{\frac{\nu}{2}}\|^{\frac{2\lambda}{\nu}}_{L^{2^*}(\Omega)},
\end{aligned}$$

where

$$\frac{1}{c} = 1 - \lambda + \frac{2\lambda}{2^*\nu} \Leftrightarrow \lambda = \frac{2^*\nu(c-1)}{c(2^*\nu - 2)}, \tag{2.16}$$

and $1 \leq s \leq c \leq r \leq \infty$. Getting back to the original domain $\Omega \times [0,T]$, it follows that

$$\|\rho\|^q_{L^q([0,T],L^c(\Omega))} = \int_0^T \|\rho\|^q_{L^c(\Omega)}\,dt$$

$$\leq \int_0^T \|\rho^{\frac{\nu}{2}}\|_{L^{2^*}(\Omega)}^{q\frac{2\lambda}{\nu}}\, dt.$$

Thus, for $q\lambda = \nu$, we apply (2.11) and (2.12), to get

$$\begin{aligned}\int_0^T \|\rho^{\frac{\nu}{2}}\|_{L^{2^*}(\Omega)}^{q\frac{2\lambda}{\nu}}\, dt &= \int_0^T \|\rho^{\frac{\nu}{2}}\|_{L^{2^*}(\Omega)}^{2}\, dt \\ &\leq C\int_0^T \|D(\rho^{\frac{\nu}{2}})\|_{L^2(\Omega)}^2\, dt + C \\ &= C\int_0^T \int_\Omega |D(\rho^{\frac{\nu}{2}})|^2\, dxdt + C \\ &\leq C.\end{aligned}$$

Therefore, $\|\rho\|_{L^q([0,T],L^c(\Omega))} \leq C$. Notice also that the condition (2.10) is merely a simplification of the parameters r, s, λ and ν satisfying (2.14), (2.15) and (2.16) and such that $1 \leq s \leq c \leq r \leq \infty$ and $0 < \nu < 1$. That condition only holds for $d > 2$. □

As an immediate result, we get a bound for the nonhomogeneous case.

Corollary 2.1.5. *Let u solve* (2.1). *Then, for (p, b, d) such that $p > 1$, $b \geq 1$, $d > 2$ and*

$$b > \frac{pd}{2p-2}$$

we have that

$$\|u\|_{L^\infty(\Omega\times[0,T])} \leq C\|f\|_{L^p([0,T],L^b(\Omega))}.$$

Remark 2.1.6. Figure 2.2 illustrates the region for $d = 3$.

Proof. From (2.6) and (2.10), we have that

$$\begin{aligned}&\frac{b}{b-1} < \frac{\frac{p}{p-1}d}{\frac{p}{p-1}d - 2} \\ &\Leftrightarrow b > \frac{pd}{2p-2}.\end{aligned}$$

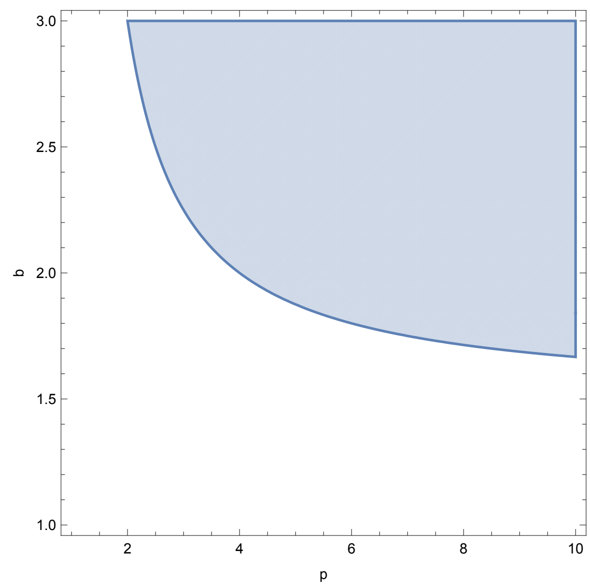

Figure 2.2: Integrability region for norms of the nonhomoegenous heat equation

Then, by (2.5), the estimate follows. □

Next, we study the space-time norm of the fundamental solution of the heat equation to compare its associated parameter region with the one presented in Proposition 2.1.3.

2.1.4 The fundamental solution case

Here, we investigate $L^q([0,T], L^c(\mathbb{R}^d))$ estimates for the fundamental solution of the heat equation. We use these to show that the estimates in the prior section are sharp. The fundamental solution of the heat equation (2.2) is

$$\Phi(x,t) \coloneqq \begin{cases} \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}} & (x \in \mathbb{R}^d,\ t > 0) \\ 0 & (x \in \mathbb{R}^d,\ t < 0). \end{cases}$$

Next, we compute $\|\Phi\|_{L^q([0,T],L^c(\mathbb{R}^d))}$ and determine for which pairs (q,c) this norm is finite. We have

$$\begin{aligned}
\|\Phi\|^q_{L^q([0,T],L^c(\mathbb{R}^d))} &= \int_0^T \left(\int_{\mathbb{R}^d} \left(\frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}} \right)^c dx \right)^{\frac{q}{c}} dt \\
&= (4\pi)^{-\frac{qd}{2}} \int_0^T t^{-\frac{qd}{2}} \left(\int_{\mathbb{R}^d} e^{-\frac{c|x|^2}{4t}} dx \right)^{\frac{q}{c}} dt \\
&= (4\pi)^{-\frac{qd}{2}} \int_0^T t^{-\frac{qd}{2}} \left(\frac{4\pi t}{c} \right)^{\frac{qd}{2c}} dt \\
&= C_{c,q,d} \int_0^T t^{\frac{qd(1-c)}{2c}} dt,
\end{aligned}$$

where $C_{c,q,d} \equiv (4\pi)^{\frac{qd(1-c)}{2c}} c^{-\frac{qd}{c}} > 0$. Thus, $\|\Phi\|_{L^q([0,T],L^c(\mathbb{R}^d))} < \infty$ if and only if

$$\frac{qd(1-c)}{2c} > -1.$$

Hence, for $d > 2$ and $q \geq 1$, we get

$$c < \frac{qd}{qd-2}.$$

This range is the same as in (2.10). Therefore, the estimates from the preceding section are sharp.

2.2 Preliminaries

This section includes some results which are used throughout this chapter. In the computations ahead, we are interested in the solution of the homogeneous heat equation

$$\begin{cases} \rho_t(x,t) = \Delta\rho(x,t) & \text{in } \Omega \times (0,T] \\ \rho(x,0) \in L^1(\Omega) & \text{in } \Omega, \end{cases} \tag{2.17}$$

for all $t > 0$. Then, in Section 2.5, we extend the obtained integrability to the singular data version of the problem, (2.2).

2.2.1 Gagliardo-Nirenberg estimates on bounded domains

One of the main inequalities used throughout this thesis is the Gagliardo-Nirenberg-Sobolev inequality. Here, we present its version for bounded domains for weak derivatives, based on [14]. First, recall the Sobolev norm.

Definition 2.2.1. *The Sobolev norm is defined, for a function $\rho : \Omega \to \mathbb{R}$ in $L^p(\Omega)$, by*

$$\|\rho\|_{W^{k,p}(\Omega)} = \left(\sum_{i=0}^{k} \int_\Omega |D^i \rho|^p \, dx \right)^{\frac{1}{p}}.$$

Remark 2.2.2. $L^p(\Omega) = W^{0,p}(\Omega)$.

Theorem 2.2.3. *Let Ω be a bounded domain with $\partial\Omega$ in C^m, and let ρ be any function in $W^{m,r}(\Omega) \cap L^q(\Omega)$, $1 \le p, r, q \le \infty$. For any integer j, $0 \le j < m$, and for any number a in the interval $j/m \le \alpha \le 1$, set*

$$\frac{1}{p} = \frac{j}{d} + \left(\frac{1}{r} - \frac{m}{d} \right) \alpha + \frac{1-\alpha}{q}.$$

If $m - j - d/r$ is a negative integer, then

$$\|D^j \rho\|_{W^{0,p}(\Omega)} \le C \|\rho\|^{\alpha}_{W^{m,r}(\Omega)} \|\rho\|^{1-\alpha}_{W^{0,q}(\Omega)}. \tag{2.18}$$

If $m - j - d/r$ is a nonnegative integer, then (2.18) *holds for $\alpha = j/m$. The constant C depends only on $\Omega, r, q, m, j, \alpha$. The derivatives that occur in* (2.18) *are weak derivatives.*

In our work, we are particularly interested in the prior theorem for the case of $\mathbb{T}^d$, which can be roughly understood as a bounded domain in $\mathbb{R}^d$. For $\mathbb{R}^d$, we use the usual Gagliardo-Nirenberg inequality, which is stated in the proofs below.

2.2.2 Differential inequalities

Here, we present some of the estimates regarding differential inequalities present throughout our work.

Lemma 2.2.4. *Let* $z : [0, \infty) \to [0, \infty)$ *be a differentiable function satisfying the differential inequality*

$$z'(t) \leq -Cz(t)^{\beta} \tag{2.19}$$

for some constant $C > 0$ *and* $\beta > 1$*. Then,* z *satisfies*

$$z(t) \leq C_{\beta} t^{\frac{1}{1-\beta}}.$$

Proof. Let $z \equiv z(t)$ and $\dot{z} \equiv z'(t)$. Since $\beta - 1 > 0$, multiplying both sides of (2.19) by $-(\beta-1)z^{-\beta}$ leads to

$$-(\beta-1)z^{-\beta}\dot{z} \geq (\beta-1)C.$$

Next, we observe that the left-hand side in the prior equation is $\frac{d}{dt}(z(t)^{1-\beta})$. Hence, integrating in time, we get

$$\begin{aligned} & z(t)^{1-\beta} \geq z(0)^{1-\beta} + (\beta-1)Ct \\ \Leftrightarrow\ & z(t)^{1-\beta} \geq z(0)^{1-\beta}(1 + z(0)^{\beta-1}(\beta-1)Ct). \end{aligned}$$

Therefore,

$$z(t) \leq \frac{z(0)}{(1 + z(0)^{\beta-1}(\beta-1)Ct)^{\frac{1}{\beta-1}}}.$$

Hence, since $0 \leq z(0) < \infty$, z satisfies

$$z(t) \leq Ct^{\frac{1}{1-\beta}}$$

for some constant $C > 0$ depending on β. $\square$

Figure 2.3 shows the region of $z(t)$ with $\beta = 10$.

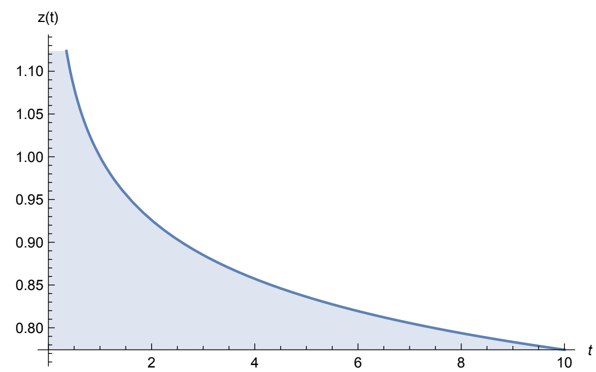

Figure 2.3: $z(t) \leq Ct^{\frac{1}{1-\beta}}$

Lemma 2.2.5. *Let $z : (0, \infty) \to (0, \infty)$ be a differentiable function satisfying the differential inequality*

$$\dot{z} \leq C_1 z^\theta - C_2 z^\beta$$

for constants $C_1, C_2 > 0$, and $1 \leq \theta < \beta$. Then, there exists $T > 0$ such that

$$z(t) \leq C_\beta t^{\frac{1}{1-\beta}}$$

for $t \in (0, T)$. Moreover, for $t > T$, $z(t) \leq C_\beta T^{\frac{1}{1-\beta}}$.

Proof. The function

$$z \mapsto C_1 z^\theta - C_2 z^\beta$$

has one zero $\overline{z}$. Then, since, for any $\epsilon > 1$,

$$C_1(\epsilon\overline{z})^\theta - C_2(\epsilon\overline{z})^\beta < 0,$$

we have that $z(t') \leq \epsilon\overline{z}$, for all $t' \geq t$. On the other hand, if $\lim_{t\to 0^+} z(t) > \epsilon\overline{z}$, there exists $T > 0$ such that

$$\dot{z} \leq C_1 z^\theta - C_2 z^\beta \leq -\tilde{C}_2 z^\beta$$

for all $t \in (0,T)$. Then, by Lemma 2.2.4, $z(t) \leq Ct^{\frac{1}{1-\beta}}$ for all $t \in (0,T)$ and $z(t) \leq CT^{\frac{1}{1-\beta}}$ for $t > T$.

□

2.3 L^2-regularity of the adjoint variable

Now, we examine higher integrability for the adjoint variable. These results are key to understanding the homogeneous heat equation. As an example, consider the homogeneous version of (2.1); that is, $u_t = -\Delta u$. Then, (2.3) and Hölder's inequality yield

$$u(x_0,0) = \int_\Omega u_T(x)\rho(x,T)\,dx \leq C\left(\int_\Omega \rho(x,T)^2\,dx\right)^{\frac{1}{2}},$$

since $u_T \in L^2(\Omega)$. Therefore, we seek to estimate $\rho(x,T)$ in $L^2(\Omega)$. For that, we take a solution ρ of the adjoint equation $\rho_t = \Delta\rho$. Integration by parts leads to

$$\frac{d}{dt}\int_\Omega \rho^2\,dx = -2\int_\Omega |D\rho|^2\,dx. \tag{2.20}$$

Integrating over time, we get

$$\int_\Omega \rho(x,T)^2\,dx - \int_\Omega \rho(x,0)^2\,dx = -2\int_0^T\int_\Omega |D\rho|^2\,dxdt \leq 0. \tag{2.21}$$

Hence, L^2-regularity for ρ is at least preserved.

Next, we show that (2.21) yields a stronger gain of regularity.

Proposition 2.3.1. *Let ρ be a solution to the d-dimensional heat equation* (2.17) *with $\rho \in C^\infty(\Omega \times [0,\infty))$ and $\rho(x,0) \in L^1(\Omega)$. Then, there exists $T > 0$ such that the following estimate holds*

$$\|\rho\|_{L^2(\Omega)} \leq t^{-\frac{d}{4}}, \tag{2.22}$$

for all $t > 0$ with $\Omega = \mathbb{R}^d$ and for $t \in [0,T)$ with $\Omega = \mathbb{T}^d$.

Proof. In $\mathbb{R}^d$, the Gagliardo-Nirenberg interpolation inequality yields that, for some constant C,

$$\|D^k\rho\|_{L^p(\mathbb{R}^d)} \leq C\|D^m\rho\|^{\alpha}_{L^r(\mathbb{R}^d)}\|\rho\|^{1-\alpha}_{L^q(\mathbb{R}^d)} \tag{2.23}$$

whenever

$$\frac{1}{p} = \frac{k}{d} + \left(\frac{1}{r} - \frac{m}{d}\right)\alpha + \frac{1-\alpha}{q}$$

and

$$\frac{k}{m} \leq \alpha \leq 1.$$

According to (2.23), it follows that

$$\|\rho\|_{L^2(\mathbb{R}^d)} \leq C\|D\rho\|^{\alpha}_{L^2(\mathbb{R}^d)}\|\rho\|^{1-\alpha}_{L^1(\mathbb{R}^d)} \leq C\left(\int_{\mathbb{R}^d} |D\rho|^2\,dx\right)^{\frac{\alpha}{2}},$$

where α satisfies

$$\frac{1}{2} = \left(\frac{1}{2} - \frac{1}{d}\right)\alpha + 1 - \alpha \Leftrightarrow \alpha = \frac{d}{2+d}. \tag{2.24}$$

Note that $0 \leq \alpha \leq 1$. According to (2.20), we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} \rho^2\,dx = -2\int_{\mathbb{R}^d} |D\rho|^2\,dx \leq -C\left(\int_{\mathbb{R}^d} \rho^2\,dx\right)^{\frac{1}{\alpha}}.$$

Fixing $z(t) := \int_{\mathbb{R}^d} \rho^2\,dx$, the previous inequality can be rewritten as the following time-dependent differential inequality

$$\dot{z} \le -Cz^\beta$$

with $\beta = 1/\alpha > 1$. By Lemma 2.2.4, we have that z satisfies

$$z(t) \le C_\beta t^{\frac{1}{1-\beta}}.$$

In other words,

$$\|\rho\|^2_{L^2(\mathbb{R}^d)} \le Ct^{\frac{1}{1-\frac{1}{\alpha}}} = Ct^{\frac{1}{1-\frac{2+d}{d}}} = Ct^{-\frac{d}{2}}.$$

and, thus, we get (2.22). For $\mathbb{T}^d$, the Gagliardo-Nirenberg inequality yields a slightly different result. Recalling the remarks on Section 2.2.1, we have that for a bounded domain such as the torus, the Gagliardo-Nirenberg inequality yields, in this case,

$$\begin{aligned}
\|\rho\|_{L^2(\mathbb{T}^d)} &= \|\rho\|_{W^{0,2}(\mathbb{T}^d)} \\
&\le C\|\rho\|^\alpha_{W^{1,2}(\mathbb{T}^d)}\|\rho\|^{1-\alpha}_{W^{0,1}(\mathbb{T}^d)} \\
&= C\left(\int_{\mathbb{T}^d} \rho^2\,dx + \int_{\mathbb{T}^d} |D\rho|^2\,dx\right)^{\frac{\alpha}{2}} \\
&= C\left(\int_{\mathbb{T}^d} \rho^2\,dx - C\frac{d}{dt}\int_{\mathbb{T}^d} \rho^2\,dx\right)^{\frac{\alpha}{2}},
\end{aligned}$$

where α is the same as before. Hence, with z as before, we get a differential inequality of the type

$$\dot{z} \le C_1 z - C_2 z^{\frac{1}{\alpha}}.$$

Since $1/\alpha > 1$, we have that, by Lemma 2.2.5, there exists $T > 0$ such that z satisfies the short-time estimate

$$z(t) \le C_\beta t^{\frac{1}{1-\beta}}$$

for $t \in [0, T)$. Thus, the same estimate follows for the torus. □

The previous techniques can be extended to deduce estimates for derivatives of any order.

Proposition 2.3.2. *Let ρ be a solution to the d-dimensional heat equation* (2.17) *with $\rho \in C^\infty(\Omega \times [0,\infty))$ and $\rho(x,0) \in L^1(\Omega)$. Then, there exists $C > 0$ and $T > 0$ such that, for any $k \in \mathbb{N}_0$, the following estimate holds*

$$\|D^k \rho\|_{L^2(\Omega)} \le C t^{-\left(\frac{d}{4}+\frac{k}{2}\right)}, \tag{2.25}$$

for all $t > 0$ with $\Omega = \mathbb{R}^d$ and for $t \in [0, T)$ with $\Omega = \mathbb{T}^d$.

Proof. We have that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |D^k\rho|^2\,dx &= \int_\Omega D^k\rho (D^k\rho)_t\,dx \\ &= -\int_\Omega |D^{k+1}\rho|^2\,dx. \end{aligned}$$

On the other hand, for $\mathbb{R}^d$, using (2.23), it follows that

$$\begin{aligned} \|D^k\rho\|_{L^2(\mathbb{R}^d)} &\le C\|D^{k+1}\rho\|^{\alpha}_{L^2(\mathbb{R}^d)}\|\rho\|^{1-\alpha}_{L^1(\mathbb{R}^d)} \\ &\le C\left(\int_{\mathbb{R}^d}|D^{k+1}\rho|^2\,dx\right)^{\frac{\alpha}{2}}, \end{aligned}$$

where α satisfies

$$\frac{1}{2} = \frac{k}{d} + \left(\frac{1}{2} - \frac{k+1}{d}\right)\alpha + 1 - \alpha \Leftrightarrow \alpha = \frac{2k+d}{2+2k+d}. \tag{2.26}$$

The prior choice of α should be compared with (2.24). Hence, as before, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d}|D^k\rho|^2\,dx = -2\int_{\mathbb{R}^d}|D^{k+1}\rho|^2\,dx \le -C\left(\int_{\mathbb{R}^d}|D^k\rho|^2\,dx\right)^{\frac{1}{\alpha}}. \tag{2.27}$$

We notice again that, with $z(t) \coloneqq \int_{\mathbb{R}^d} |D^k\rho|^2\,dx$ for $k \in \mathbb{N}_0$, (2.27) yields

$$\dot{z} \leq -Cz^{\beta}, \tag{2.28}$$

where $\beta = 1/\alpha > 1$. We have seen that a solution of (2.28) satisfies

$$z(t) \leq Ct^{\frac{1}{1-\beta}}. \tag{2.29}$$

Hence, by (2.26) and (2.29),

$$\|D^k\rho\|^2_{L^2(\mathbb{R}^d)} \leq Ct^{\frac{1}{1-1/\alpha}} = Ct^{\frac{1}{1-\frac{2+2k+d}{2k+d}}} = Ct^{-\frac{d}{2}-k}$$

which yields (2.25). The proof for the torus follows the same reasoning as in the proof of Proposition 2.3.1. □

2.3.1 Comparison with the fundamental solution

We now show that the estimates in the previous subsections are sharp by comparing them with the fundamental solution; that is, no better estimates are possible.

As a first study case, let $k = 0$. Recall that, for $a, t > 0$,

$$\int_{\mathbb{R}^d} e^{-a\frac{|x|^2}{t}}\,dx = C_{a,d}t^{\frac{d}{2}}.$$

Then, with $\rho = \Phi$,

$$\begin{aligned} z(t) = \int_{\mathbb{R}^d} \Phi^2\,dx &= \int_{\mathbb{R}^d} \left(\frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}} \right)^2 dx \\ &= Ct^{-d} \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2t}}\,dx \\ &= Ct^{-\frac{d}{2}} \end{aligned}$$

which corresponds to the estimate in Proposition 2.3.1. Hence, our estimate is sharp. To study the sharpness of general k, we prove an estimate presented in [15] regarding the derivatives of the fundamental solution of the heat equation.

Proposition 2.3.3. *In $\mathbb{R}^d$, the fundamental solution of the heat equation satisfies*

$$|D_t^r D_x^k \Phi(x,t)| \leq C_{r,k} t^{-\frac{d}{2}-r-\frac{k}{2}} e^{-C\frac{|x|^2}{t}} \tag{2.30}$$

for each $r, k \in \mathbb{N}_0$ and all $t > 0$.

Proof. First, we notice that

$$|D_x \Phi(x,t)| \leq \frac{C|x|}{(4\pi t)^{\frac{d}{2}+1}} e^{-C\frac{|x|^2}{t}}$$

Since Φ solves the heat equation, an induction argument gives the generalized result

$$\begin{aligned} \left|D_x^k \Phi(x,t)\right| &\leq -\frac{C|x|^k}{t^{\frac{d}{2}+k}} e^{-C\frac{|x|^2}{t}}, \\ |D_t^r \Phi(x,t)| = \left|D_x^{2r}\Phi(x,t)\right| &\leq \frac{C|x|^{2r}}{t^{\frac{d}{2}+2r}} e^{-C\frac{|x|^2}{t}}, \end{aligned}$$

For $a > 0$, the function $y^a e^{-\frac{y}{2}}$ is bounded on $[0, \infty)$ and so

$$y^a e^{-y} = y^a e^{-\frac{y}{2}} e^{-\frac{y}{2}} \leq C e^{-\frac{y}{2}}.$$

Hence,

$$\frac{|x|^a}{t^{\frac{a}{2}}} e^{-C\frac{|x|^2}{t}} \leq C e^{-\frac{C|x|^2}{2t}} \equiv C e^{-\nu\frac{|x|^2}{t}}.$$

Therefore, for $a = 2r$ and $a = k$, we get

$$\begin{aligned} |D_x^k \Phi(x,t)| &\leq \frac{C|x|^k}{(4\pi t)^{\frac{d}{2}+k}} e^{-\nu\frac{|x|^2}{t}} = \frac{C}{t^{\frac{d}{2}+\frac{k}{2}}} \frac{|x|^k}{t^{\frac{k}{2}}} e^{-\nu\frac{|x|^2}{t}} \leq C \frac{e^{-\nu\frac{|x|^2}{t}}}{t^{\frac{d}{2}+\frac{k}{2}}}, \\ |D_t^r \Phi(x,t)| &\leq \frac{C|x|^{2r}}{t^{\frac{d}{2}+2r}} e^{-\nu\frac{|x|^2}{t}} = \frac{C}{t^{\frac{d}{2}+r}} \frac{|x|^2}{t} e^{-\nu\frac{|x|^2}{t}} \leq C \frac{e^{-\nu\frac{|x|^2}{t}}}{t^{\frac{d}{2}+r}}, \end{aligned}$$

where $\nu > 0$. Combining these two inequalities we get (2.30). $\square$

Then, for any $k \in \mathbb{N}_0$, using (2.30) with $r = 0$ yields

$$z(t) = \int_{\mathbb{R}^d} |D^k \Phi|^2 \, dx \leq C t^{-d-k} \int_{\mathbb{R}^d} e^{-2\nu \frac{|x|^2}{t}} \, dx = C t^{-d-k} t^{\frac{d}{2}} = C t^{-\frac{d}{2}-k}$$

which is the same estimate as in Proposition 2.3.2. Hence, our estimate is sharp. Our goal is now to deduce a generalization to L^p-norms.

2.4 Generalization to L^p-norms

We now generalize the estimates in Section 2.3 and Proposition 2.3.2 to the L^p-norms of the solution of the heat equation.

2.4.1 First-order estimate

We begin by considering the L^p-norm of the solution.

Proposition 2.4.1. *Let ρ be a solution to the d-dimensional heat equation* (2.17) *with $\rho \in C^\infty(\Omega \times [0, \infty))$ and $\rho(x, 0) \in L^1(\Omega)$. Then, for $p \geq 1$, there exists $T > 0$ such that the following estimate holds*

$$\|\rho\|_{L^p(\Omega)} \leq C t^{-\frac{1}{2p} d(p-1)}, \tag{2.31}$$

for all $t > 0$ with $\Omega = \mathbb{R}^d$ and for $t \in [0, T)$ with $\Omega = \mathbb{T}^d$.

Proof. Notice that

$$\begin{aligned} \frac{d}{dt} \int_\Omega \rho^p \, dx &= p \int_\Omega \rho^{p-1} \Delta \rho \, dx \\ &= -p(p-1) \int_\Omega \rho^{p-2} |\nabla \rho|^2 \, dx. \end{aligned} \tag{2.32}$$

Fixing $\gamma = p/2$ leads to

$$\frac{d}{dt}\int_\Omega \rho^p\,dx = -C\int_\Omega \rho^{2\gamma-2}|\nabla\rho|^2\,dx$$
$$= -C\int_\Omega |\nabla(\rho^\gamma)|^2\,dx.$$

For $\mathbb{R}^d$, by Sobolev's inequality, we have that

$$\left(\int_{\mathbb{R}^d}\rho^{2^*\gamma}\,dx\right)^{\frac{1}{2^*}} \le C\left(\int_{\mathbb{R}^d}|\nabla(\rho^\gamma)|^2\,dx\right)^{\frac{1}{2}}. \tag{2.33}$$

Using the interpolation inequality with $\|\rho\|_{L^1(\mathbb{R}^d)} = 1$ and $0 < \lambda < 1$, we have that

$$\left(\int_{\mathbb{R}^d}\rho^{2^*\gamma}\,dx\right)^{\frac{\lambda}{2^*}} = \|\rho\|^{\gamma\lambda}_{L^{2^*\gamma}(\mathbb{R}^d)} = \|\rho\|^{\gamma\lambda}_{L^{2^*\gamma}(\mathbb{R}^d)}\|\rho\|^{\gamma(1-\lambda)}_{L^1(\mathbb{R}^d)} \ge \|\rho\|^{\gamma}_{L^p(\mathbb{R}^d)}, \tag{2.34}$$

where

$$\frac{1}{p} = 1-\lambda+\frac{\lambda}{2^*\gamma} \Leftrightarrow \lambda = \frac{d(p-1)}{2+d(p-1)}.$$

Hence, (2.32), (2.33), and (2.34) lead to

$$\frac{d}{dt}\int_{\mathbb{R}^d}\rho^p\,dx \le -C\left(\int_{\mathbb{R}^d}\rho^p\,dx\right)^{\frac{2\gamma}{\lambda p}}. \tag{2.35}$$

Let $z(t) \coloneqq \int_{\mathbb{R}^d}\rho^p\,dx$. Then, the previous inequality can be written as

$$\dot z \le -Cz^\beta,$$

where $\beta = 2\gamma/(\lambda p) = 1/\lambda > 1$. As before, we get the following time estimate

$$z(t) \le Ct^{\frac{1}{1-\beta}} = Ct^{\frac{1}{2}d(1-p)},$$

and thus (2.31) follows. For the torus, we have that

$$\begin{aligned}\left(\int_{\mathbb{T}^d} \rho^p\, dx\right)^{\frac{1}{2}} &= \left(\int_{\mathbb{T}^d} \rho^{2\gamma}\, dx\right)^{\frac{1}{2}} \\ &\leq C\left(\int_{\mathbb{T}^d} \rho^{2\gamma}\, dx + \int_{\mathbb{T}^d} |D(\rho^\gamma)|^2\, dx\right)^{\frac{\alpha}{2}} \left(\int_{\mathbb{T}^d} \rho\, dx\right)^{\frac{p(1-\alpha)}{2}} \\ &= C\left(\int_{\mathbb{T}^d} \rho^p\, dx - C\frac{d}{dt}\int_{\mathbb{T}^d} \rho^p\, dx\right)^{\frac{\alpha}{2}},\end{aligned}$$

where α is such that

$$\frac{1}{2} = \left(\frac{1}{2} - \frac{1}{d}\right)\alpha + \frac{p(1-\alpha)}{2} \Leftrightarrow \alpha = \frac{d(p-1)}{2+d(p-1)}.$$

Then, fixing $z(t) \coloneqq \int_{\mathbb{T}^d} \rho^p\, dx$, we get the following differential inequality

$$\dot{z} \leq C_1 z - C_2 z^{\frac{1}{\alpha}}$$

Hence, following the proof of Proposition 2.3.1, there exists $T > 0$ such that z satisfies

$$z(t) \leq Ct^{\frac{1}{1-1/\alpha}} = Ct^{\frac{1}{2}d(1-p)}$$

for $t \in [0, T)$. Thus, we get a similar estimate for $\mathbb{T}^d$.

□

2.4.2 k^{th}-order derivative

We now prove the main result in this chapter which gives an estimate for the L^p-norm of a derivative of any order of the solution of the heat equation.

Theorem 2.4.2. *Let ρ be a solution to the d-dimensional adjoint equation* (2.17) *with $\rho \in C^\infty(\Omega \times [0, \infty))$ and $\rho(x, 0) \in L^1(\Omega)$. Then, there exists $T > 0$ such that,* *for any $k \in \mathbb{N}_0$ and $p \geq 1$, the following estimate holds*

$$\|D^k\rho\|_{L^p(\Omega)} \leq Ct^{-\frac{dp+kp-d}{2p}}, \tag{2.36}$$

for all $t > 0$ with $\Omega = \mathbb{R}^d$ and for $t \in [0, T)$ with $\Omega = \mathbb{T}^d$.

Remark 2.4.3. We note that (2.31) corresponds to (2.36) with $k = 0$.

Proof. Fix $\gamma = p/2$. Applying the same reasoning as the prior section leads to

$$\begin{aligned}\frac{d}{dt}\int_\Omega |D^k\rho|^p\,dx &= -C\int_\Omega |D^k\rho|^{p-2}|D^{k+1}\rho|^2\,dx \\ &= -C\int_\Omega |\nabla(|D^k\rho|^\gamma)|^2\,dx.\end{aligned}$$

For $\mathbb{R}^d$, by Sobolev and Gagliardo-Nirenberg inequalities, we have that

$$\begin{aligned}C\left(\int_{\mathbb{R}^d} |\nabla(|D^k\rho|^\gamma)|^2\,dx\right)^{\frac{\lambda}{2}} &\geq \left(\int_{\mathbb{R}^d} |D^k\rho|^{2^*\gamma}\,dx\right)^{\frac{\lambda}{2^*}} \\ &= \|D^k\rho\|^{\gamma\lambda}_{L^{2^*\gamma}(\mathbb{R}^d)} \\ &\geq \|D^k\rho\|^{\gamma}_{L^p(\mathbb{R}^d)}.\end{aligned}$$

Following the procedure in Sections 2.3 and 2.4.1 and using the condition

$$\frac{1}{p} = \frac{k}{d} + \left(\frac{1}{2^*\gamma} - \frac{k}{d}\right)\lambda + 1 - \lambda \Leftrightarrow \lambda = \frac{d(p-1)+kp}{2+d(p-1)+kp},$$

we have that (2.36) follows. The torus case follows the same reasoning as the proof of Proposition 2.31. □

Remark 2.4.4. Comparing again with the fundamental solution, we have that, by the remarks on Section 2.3.1,

$$\int_{\mathbb{R}^d} |D^k\Phi|^p\,dx \leq Ct^{-\frac{dp}{2}-\frac{kp}{2}}\int_{\mathbb{R}^d} e^{-C\frac{p|x|^2}{t}}\,dx$$

$$= Ct^{-\frac{dp+kp-d}{2}},$$

which yields the same estimate as (2.36). Hence, our estimates remain as sharp as possible.

Finally, we conclude by introducing the time variable and studying its effect on the regularity.

2.4.3 Space-time regularity

Similarly to Sections 2.1.1 and 2.1.3, we are now interested in studying the space-time regularity of derivatives of any order of the solution u using the regularity results from Section 2.3.

Proposition 2.4.5. *Let u solve* (2.1)*. Assume also that $u(x,0) \in L^1(\mathbb{R}^d)$. Then, for (k,q,d) such that $k \in \mathbb{N}_0$, $q > 1$, $1 \leq d \leq 3$ and*

$$k < \frac{4-d}{2}, \tag{2.37}$$

we have that

$$\|D^k u\|_{L^p([0,T],L^q(\mathbb{R}^d))} \leq C.$$

Remark 2.4.6. The previous result yields regularity only up to the first order derivative of the solution, but only requires L^1 initial data.

Proof. Young's convolution inequality yields that, for $m,n,r \geq 1$ such that

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{r} = 2,$$

then

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(x-y)h(y)\,dxdy \le \left(\int_{\mathbb{R}^d} |f|^m\,dx\right)^{\frac{1}{m}} \left(\int_{\mathbb{R}^d} |g|^n\,dx\right)^{\frac{1}{n}} \left(\int_{\mathbb{R}^d} |h|^r\,dx\right)^{\frac{1}{r}}.$$

Notice that the solution to the nonhomogeneous problem yields

$$D^k u(x,t) = \int_0^t \int_{\mathbb{R}^d} D^k\rho(x-y,t-s)f(y,s)\,dyds$$

for $k \in \mathbb{N}_0$. We introduce the notation $L^pL^q := L^p([0,T], L^q(\mathbb{R}^d))$. Using duality and Young's convolution inequality, we have, for $\varphi \in C^\infty([0,T]\times\mathbb{R}^d)$, that

$$\begin{aligned}
\|D^k u\|_{L^pL^q} &= \sup_{\|\varphi\|_{L^{p'}L^{q'}}\le 1} \left|\int_0^T \int_{\mathbb{R}^d} \varphi(x,t)D^k u(x,t)\,dxdt\right| \\
&\le \sup_{\|\varphi\|_{L^{p'}L^{q'}}\le 1} \int_0^T\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\varphi(x,t)D^k\rho(x-y,t-s)f(y,s)\right|\,dxdydsdt \\
&\le \sup_{\|\varphi\|_{L^{p'}L^{q'}}\le 1} \int_0^T \|\varphi(\cdot,t)\|_{L^{q'}(\mathbb{R}^d)} \int_0^t \|D^k\rho(\cdot,t-s)\|_{L^n(\mathbb{R}^d)}\|f(\cdot,s)\|_{L^{n'}(\mathbb{R}^d)}\,dsdt,
\end{aligned}$$

where

$$1 = \frac{1}{p}+\frac{1}{p'} = \frac{1}{q}+\frac{1}{q'} = \frac{1}{q'}+\frac{1}{n}+\frac{1}{n'}-1.$$

Concerning the spatial L^2-norm, $n=2$, we have from Proposition 2.3.2 that

$$\|D^k\rho(\cdot,t-s)\|_{L^2(\mathbb{R}^d)} \le C(t-s)^{-\frac{d}{4}-\frac{k}{2}}.$$

Assume $1/m + 1/m' = 1$ and $f \in L^{m'}L^{n'}$. Hence, using convolution and Hölder's inequality,

$$\begin{aligned}
\|D^k u\|_{L^pL^q} &\le \sup_{\|\varphi\|_{L^{p'}L^{q'}}\le 1} C\int_0^T \|\varphi(\cdot,t)\|_{L^{q'}(\mathbb{R}^d)} \int_0^t (t-s)^{-\frac{d}{4}-\frac{k}{2}}\|f(\cdot,s)\|_{L^{n'}(\mathbb{R}^d)}\,dsdt \\
&\le \sup_{\|\varphi\|_{L^{p'}L^{q'}}\le 1} C\int_0^T \|\varphi(\cdot,t)\|_{L^{q'}(\mathbb{R}^d)} \left(\int_0^t (t-s)^{-\frac{md}{4}-\frac{mk}{2}}\,ds\right)^{\frac{1}{m}} \|f\|_{L^{m'}L^{n'}}\,dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{\|\varphi\|_{L^{p'}L^{q'}}\leq 1} C\int_0^T \|\varphi(\cdot,t)\|_{L^{q'}(\mathbb{R}^d)} t^{\frac{1}{m}-\frac{d}{4}-\frac{k}{2}}\,dt \\
&\leq \sup_{\|\varphi\|_{L^{p'}L^{q'}}\leq 1} C\|\varphi(\cdot,t)\|_{L^{p'}L^{q'}} \left(\int_0^T t^{q'\left(\frac{1}{m}-\frac{d}{4}-\frac{k}{2}\right)}\,dt\right)^{\frac{1}{q'}} \\
&\leq C\|t^{\frac{1}{m}-\frac{d}{4}-\frac{k}{2}}\|_{L^{q'}([0,T])},
\end{aligned}
\tag{2.38}
$$

which holds if

$$\frac{md}{4}+\frac{mk}{2}<1 \Leftrightarrow k<\frac{4-dm}{2m}.$$

Moreover, (2.38) is finite if

$$q'\left(\frac{1}{m}-\frac{d}{4}-\frac{k}{2}\right)>-1 \Leftrightarrow k<\frac{2}{q'}+\frac{2}{m}-\frac{d}{2}.$$

Hence, reducing the necessary conditions for integrability, we get (2.37) and the dimensional constrictions. □

2.5 A weak convergence argument

Here, we show that by obtaining estimates for the heat equation for initial data in L^1, we can extend them for singular initial data. This is possible because of the linearity of the heat equation. In particular, we get estimates for the singular initial data; that is, $\rho(x,0)=\delta_{x_0}$, as presented in (2.17). The main idea is to study the weak convergence of a sequence of solutions of the heat equation and understand its limit as a weak solution of the singular data case.

Definition 2.5.1 (Weak convergence)**.** *A sequence* $\{u^n\}_{n=1}^{\infty}$ *in the* $L^p(\Omega)$ converges weakly *(or in the sense of distributions) to* $u\in L^p(\Omega)$*, denoted by*

$$u^n \rightharpoonup u,$$

if

$$\int_\Omega u^n(x)\varphi(x)\,dx \to \int_\Omega u(x)\varphi(x)\,dx$$

for every smooth function φ with compact support in Ω.

Remark 2.5.2. The Dirac delta can be interpreted as a limit, in the weak sense, of a sequence of functions $\{f_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty}\int_\Omega f_n(x)\varphi(x)\,dx = \varphi(0)$$

for every $\varphi \in C_c^\infty(\Omega)$.

Now, let $\{\rho^n(\cdot,t)\}_{n=1}^\infty$ be a sequence in $L^p(\Omega)$ satisfying

$$\begin{cases} \rho^n_t = \Delta\rho^n & \text{in } \Omega\times(0,T] \\ \rho^n(x,0) = f_n(x) & \text{in } \Omega, \end{cases} \tag{2.39}$$

where $f_n \in L^1(\Omega)$ for all $n \geq 1$. We now want to show that there exists a function ρ with singular initial data in the weak closure of $L^1(\Omega)$ which is a solution of (2.17).

Proposition 2.5.3. *Let ρ^n solve* (2.39)*. Then, there exists a weak solution of* (2.17) *such that*

$$\rho^n \rightharpoonup \rho,$$

as $n \to \infty$.

Proof. First, we show that δ_{x_0} is in the weak closure of $L^1(\Omega)$, i.e, there exists a sequence $\{f_n\}_{n=1}^\infty \subset L^1(\Omega)$ such that $f_n \rightharpoonup \delta_{x_0}$. Without loss of generality, we assume $x_0 = 0$. Consider the class of functions defined by

$$f_n(x) = n^{-d} f\left(\frac{x}{n}\right),$$

where f is any function such that $\|f\|_{L^1(\Omega)} = 1$ and $\operatorname{supp} f \subset \{x : |x| < 1\}$, for example, a characteristic function. Then, for any $n \geq 1$, we also have $\|f_n\|_{L^1(\Omega)} = 1$ and $\operatorname{supp} f_n \subset \{x : |x| < 1/n\}$. Now, let $\sigma > 0$ and $\varphi \in C_c^\infty(\Omega)$. Since φ is continuous, we can choose n large enough such that $|x| \leq 1/n$ implies $|\varphi(x) - \varphi(0)| < \sigma$, for some $\sigma > 0$. Thus, we have that

$$\begin{aligned}
\left|\varphi(0) - \int_\Omega f_n(x)\varphi(x)\,dx\right| &= \left|\int_\Omega f_n(x)(\varphi(0) - \varphi(x))\,dx\right| \\
&= \left|\int_{|x|<1/n} f_n(x)(\varphi(0) - \varphi(x))\,dx\right| \\
&\leq \sigma \|f_n\|_{L^1(\Omega)} \\
&= \sigma.
\end{aligned}$$

Then, since σ is arbitrary, we have that $f_n \rightharpoonup \delta_{x_0}$. Now, let ρ be the weak limit of ρ^n. Then, we have that $\rho(x, 0) = \delta_{x_0}$. Furthermore, since $\rho^n_t = \Delta\rho^n$, for every $\psi \in C_c^\infty(\mathbb{R}^d \times [0, \infty))$, we have that

$$\begin{aligned}
0 &= \int_0^\infty \int_{\mathbb{R}^d} \psi(\rho^n_t - \Delta\rho^n)\,dxdt \\
&= \int_0^\infty \int_{\mathbb{R}^d} (-\psi_t - \Delta\psi)\rho^n\,dxdt + \int_{\mathbb{R}^d} \varphi f_n\,dx \\
&\to \int_0^\infty \int_{\mathbb{R}^d} (-\psi_t - \Delta\psi)\rho\,dxdt + \varphi(0),
\end{aligned}$$

where $\varphi(x) = \psi(x, 0)$. Hence, ρ is a weak solution of (2.17). For the torus case, the reasoning is similar. □

Recall the following lower semi-continuity property for weak convergent sequences.

Lemma 2.5.4. *Let $u^n \rightharpoonup u$ in $L^p(\Omega)$. Then*

$$\|u\|_{L^p(\Omega)} \leq \liminf_{n\to\infty} \|u^n\|_{L^p(\Omega)}.$$

Proof. Assume p, p' are conjugate exponents. Then, using duality and Hölder's inequality, we have that, for any $\varphi \in C_c^\infty(\Omega)$,

$$\begin{aligned}
\|u\|_{L^p(\Omega)} &= \sup_{\|\varphi\|_{L^{p'}(\Omega)} \leq 1} \int_\Omega u(x)\varphi(x)\,dx \\
&= \sup_{\|\varphi\|_{L^{p'}(\Omega)} \leq 1} \lim_{n\to\infty} \int_\Omega u^n(x)\varphi(x)\,dx \\
&\leq \liminf_{n\to\infty} \|u^n\|_{L^p(\Omega)} \sup_{\|\varphi\|_{L^{p'}(\Omega)} \leq 1} \|\varphi\|_{L^{p'}(\Omega)} \\
&\leq \liminf_{n\to\infty} \|u^n\|_{L^p(\Omega)}.
\end{aligned}$$

□

Furthermore, we may extend the previous results to any derivative of ρ.

Lemma 2.5.5. *Let ρ^n solve* (2.39). *Then, Proposition 2.5.3 and Lemma 2.5.4 still hold for $D^k(\rho^n)$, $k \in \mathbb{N}_0$.*

Proof. Consider the heat operator

$$A[u] = u_t - \Delta u.$$

A is linear. Moreover, the heat equation is translation invariant, i.e,

$$A[u(x)] = 0 \Rightarrow A[u(x+h)] = 0,$$

for any h. In particular,

$$A\left[\frac{u(x+h) - u(x)}{h}\right] = 0.$$

Hence, any derivative of u solves the heat equation. Let $u(\cdot) = \rho^n(\cdot, t)$. Then, the results in Proposition 2.5.3 and Lemma 2.5.4 still hold for $D^k(\rho^n)$. □

We now state the main result of this section.

Theorem 2.5.6. *Let ρ^n solve* (2.39)*. Assume that ρ^n satisfies, for some $p \geq 1$ and all $n \geq 1$,*

$$\|D^k(\rho^n)\|_{L^p(\Omega)} \leq Ct^{-g(p,d)},$$

where $g(p,d) \geq 0$ and $k \in \mathbb{N}_0$. Then, a weak solution of (2.17) *satisfies the same estimate.*

Proof. Let ρ^n define a sequence in $L^p(\Omega)$. Then, by Proposition 2.5.3 and Lemma 2.5.5, there exists $\rho \in L^p(\Omega)$ which is a weak solution of (2.17) and such that $D^k(\rho^n) \rightharpoonup D^k\rho$. Then, by Lemma 2.5.4, we have that

$$\|D^k\rho\|_{L^p(\Omega)} \leq \liminf_{n\to\infty} \|D^k(\rho^n)\|_{L^p(\Omega)} \leq Ct^{-g(p,d)}$$

for all $k \geq 0$. Thus, the estimate is preserved when considering singular initial data. □

2.6 Alternative approaches

In this section, we compare the methods in Section 2.3 with two alternative approaches: the entropy and hypercontractivity methods.

2.6.1 Entropy methods

We now present the d-dimensional version of the entropy methods presented in [4] for the Fokker-Planck equations in the particular case of heat equation. Regarding the heat equation (2.1) in one dimension, we define the entropy

$$H(t) = \int_{\mathbb{R}^d} \phi(\rho)\, dx,$$

where ρ is the solution of the homogeneous the heat equation and ϕ is a convex function. Integration by parts yields

$$\dot{H}(t) = \frac{d}{dt}\int_{\mathbb{R}^d} \phi(\rho)\,dx = -\int_{\mathbb{R}^d} \phi''(\rho)|\nabla\rho|^2\,dx \leq 0.$$

Furthermore,

$$\ddot{H}(t) = -\int_{\mathbb{R}^d} \phi^{(3)}(\rho)\rho_t|\nabla\rho|^2 + 2\phi''(\rho)|\nabla\rho||\nabla(\rho_t)|\,dx =: I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &= -\int_{\mathbb{R}^d} \phi^{(3)}(\rho)\rho_t|\nabla\rho|^2\,dx \\
&= -\int_{\mathbb{R}^d} \phi^{(3)}(\rho)\Delta\rho|\nabla\rho|^2\,dx \\
&= \int_{\mathbb{R}^d} |\nabla(\phi^{(3)}(\rho)|\nabla\rho|^2)||\nabla\rho|\,dx \\
&= \int_{\mathbb{R}^d} \phi^{(4)}(\rho)|\nabla\rho|^4 + 2\phi^{(3)}(\rho)\Delta\rho|\nabla\rho|^2\,dx
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= -2\int_{\mathbb{R}^d} \phi''(\rho)|\nabla\rho||\nabla(\rho_t)|\,dx \\
&= -2\int_{\mathbb{R}^d} \phi''(\rho)|\nabla\rho||D^3\rho|\,dx \\
&= -2\int_{\mathbb{R}^d} \phi''(\rho)(|\nabla(\Delta\rho|\nabla\rho|)| - (\Delta\rho)^2)\,dx \\
&= -2\int_{\mathbb{R}^d} \phi''(\rho)|\nabla(\Delta\rho|\nabla\rho|)|\,dx + 2\int_{\mathbb{R}^d} \phi''(\rho)(\Delta\rho)^2\,dx \\
&= 2\int_{\mathbb{R}^d} \phi^{(3)}(\rho)\Delta\rho|\nabla\rho|^2 + \phi''(\rho)(\Delta\rho)^2\,dx.
\end{aligned}$$

Hence,

$$\ddot{H}(t) = \int_{\mathbb{R}^d} \phi^{(4)}(\rho)|\nabla\rho|^4 + 4\phi^{(3)}(\rho)\Delta\rho|\nabla\rho|^2 + 2\phi''(\rho)(\Delta\rho)^2\,dx.$$

We now set $\phi(\rho) = \rho^2$. Because $\phi^{(k)}(\rho) = 0$ for all $k \geq 3$, the second derivative of the entropy is given by

$$\ddot{H}(t) = 4\int_{\mathbb{R}^d} (\Delta\rho)^2\,dx.$$

Hence, for some constant $C > 0$, the Gagliardo-Nirenberg inequality yields

$$\ddot{H}(t) = 4\int_{\mathbb{R}^d} (\Delta\rho)^2\,dx \leq C\left(\int_{\mathbb{R}^d} |\nabla\rho|^2\,dx\right)^\alpha = -C\dot{H}(t)^\alpha,$$

where α satisfies

$$\frac{1}{2} = \frac{2}{d} + \left(\frac{1}{2} - \frac{1}{d}\right)\alpha + 1 - \alpha \Leftrightarrow \alpha = \frac{4+d}{2+d}.$$

Hence, the entropy satisfies the following differential inequality

$$\ddot{H}(t) \leq -C\dot{H}(t)^\alpha.$$

Hence, as before, $\dot{H}$ satisfies

$$\dot{H}(t) \leq -Ct^{\frac{1}{1-\alpha}}$$

and thus, for some C depending on α,

$$\int_{\mathbb{R}^d} \rho^2\,dx = H(t) \leq Ct^{1+\frac{1}{1-\alpha}} = Ct^{-\frac{d}{2}},$$

which is the same estimate as the one obtained from Proposition 2.3.2 with $k = 0$. We have then shown that our technique gives similar results to entropy methods.

2.6.2 On logarithmic Sobolev inequalities and hypercontractivity

The gain of regularity in time can be also understood using the results in [3, 5, 6] on logarithmic Sobolev inequalities and hypercontractivity. Contractivity principles, which appear in quantum field theory, are often used to describe operators as contractions between Lebesgue spaces, being of particular interest the case from L^p to L^q when $p \leq q$. Here, however, we focus on the practical use of logarithmic Sobolev inequalities that arise in hypercontractivity theory.

Next, we state a result from [9] that yields a generalization of the logarithmic Sobolev inequality presented in [5]. First, we recall a useful definition.

Definition 2.6.1 (Fenchel-Legendre transform)**.** *In $\mathbb{R}^d$, the Fenchel-Legendre transform of a convex function φ is the function $\varphi^* : (\mathbb{R}^d)^* \to \mathbb{R}$ defined by*

$$\varphi^*(\mu) = \sup_{x \in \mathbb{R}^d} \{\mu \cdot x - \varphi(x)\}, \tag{2.40}$$

where $(\mathbb{R}^d)^*$ is the dual space of $\mathbb{R}^d$.

Remark 2.6.2. The right-hand side of (2.40) is maximized for x determined implicitly by

$$\mu = \nabla \varphi(x).$$

Proposition 2.6.3. *Let φ be a C^1 strictly convex function on $\mathbb{R}^d$ such that*

$$\lim_{|x| \to \infty} \frac{\varphi(x)}{\|x\|} = \infty.$$

Then, for all $\lambda > 0$ and for any smooth function g on $\mathbb{R}^d$, we have that the following

Euclidean logarithmic Sobolev inequality holds

$$\int_{\mathbb{R}^d} e^g \log\left(\frac{e^g}{\int_{\mathbb{R}^d} e^g\,dx}\right) dx \leq -d\log(\lambda e)\int_{\mathbb{R}^d} e^g\,dx + \int_{\mathbb{R}^d} \varphi^*(-\lambda\nabla g)e^g\,dx, \tag{2.41}$$

where φ^* *denotes the Fenchel-Legendre transform of* φ*.*

We begin by considering a time-dependent Lebesgue norm. More specifically, we are interested in the decay of

$$\|\rho\|_{L^{s(t)}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \rho^{s(t)}\,dx\right)^{\frac{1}{s(t)}},$$

where $1 \leq s(t) < \infty$.

Proposition 2.6.4. *Let* ρ *be a solution to the* d*-dimensional heat equation* (2.17)*. Assume that* $1 \leq s(t) < \infty$ *is a nondecreasing function, with* $s(0) = 1$ *and such that*

$$s(t) \leq 1 + Ce^{\frac{t}{\lambda^2}} \tag{2.42}$$

for some constants $C, \lambda > 0$*. Assume further that* $\rho \in L^1(\mathbb{R}^d)$*. Then, the following estimate holds*

$$\|\rho\|_{L^{s(t)}(\mathbb{R}^d)} \leq \|\rho(x,0)\|_{L^1(\mathbb{R}^d)} \tag{2.43}$$

for all $t \geq 0$*. In particular,* $\rho(x,t) \in L^{s(t)}(\mathbb{R}^d)$*.*

Proof. Let $s \equiv s(t)$. As before, we have that

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} \rho^s\,dx &= \frac{d}{dt}\int_{\mathbb{R}^d} e^{s\log\rho}\,dx \\ &= \int_{\mathbb{R}^d} \rho^s\left(\frac{s\rho_t}{\rho} + \dot{s}\log\rho\right) dx \\ &= s\int_{\mathbb{R}^d} \rho^{s-1}\Delta\rho\,dx + \dot{s}\int_{\mathbb{R}^d} \rho^s\log\rho\,dx. \end{aligned} \tag{2.44}$$

For the term on the left-hand side of (2.44), we have that

$$s\int_{\mathbb{R}^d}\rho^{s-1}\Delta\rho\,dx = -s(s-1)\int_{\mathbb{R}^d}\rho^{s-2}|\nabla\rho|^2\,dx$$
$$= -\frac{4(s-1)}{s}\int_{\mathbb{R}^d}|\nabla(\rho^{\frac{s}{2}})|^2\,dx \le 0. \tag{2.45}$$

Fix $g = \log(\rho^s)$ in (2.41) to get

$$\int_{\mathbb{R}^d}\rho^s\log\left(\frac{\rho^s}{\int_{\mathbb{R}^d}\rho^s\,dx}\right)dx \le -d\log(\lambda e)\int_{\mathbb{R}^d}\rho^s\,dx + \int_{\mathbb{R}^d}\varphi^*(-\lambda\nabla\log(\rho^s))\rho^s\,dx.$$

We may take $\lambda \ge e^{-1}$ to rewrite (2.44) as

$$\dot{s}\int_{\mathbb{R}^d}\rho^s\log(\rho^s) - \rho^s\log\left(\int_{\mathbb{R}^d}\rho^s\,dx\right)dx \le \dot{s}\int_{\mathbb{R}^d}\varphi^*(-\lambda\nabla\log(\rho^s))\rho^s\,dx$$
$$\Leftrightarrow\ \dot{s}\int_{\mathbb{R}^d}\rho^s\log\rho\,dx \le \frac{\dot{s}}{s}\left(\int_{\mathbb{R}^d}\varphi^*(-\lambda\nabla\log(\rho^s))\rho^s\,dx + \log\left(\int_{\mathbb{R}^d}\rho^s\,dx\right)\int_{\mathbb{R}^d}\rho^s\,dx\right) \tag{2.46}$$

Now, for notation simplicity, let $d = 1$. Fix

$$\varphi(x) = \frac{x^\gamma}{\gamma}$$

for some $\gamma > 1$. Then, by the remark on the Fenchel-Legendre transform, we have that, for μ fixed, the function of x,

$$\mu x - \varphi(x) = \mu x - \frac{x^\gamma}{\gamma}$$

has the first derivative $\mu - x^{\gamma-1}$ and second derivative $-(\gamma-1)x^{\gamma-2}$. Thus, there is one stationary point at $x = \mu^{\frac{1}{\gamma-1}}$, which is a maximum. Therefore,

$$\varphi^*(\mu) = \mu\mu^{\frac{1}{\gamma-1}} - \frac{\mu^{\frac{\gamma}{\gamma-1}}}{\gamma}$$
$$= \frac{\gamma-1}{\gamma}\mu^{\frac{\gamma}{\gamma-1}}$$

Fixing β such that $\frac{1}{\gamma}+\frac{1}{\beta}=1$, we have that the associated Fenchel-Legendre transform is

$$\varphi^*(\mu) = \frac{\mu^\beta}{\beta}.$$

Now, we may go back to the $\mathbb{R}^d$ case by using the ℓ^1-norm to define φ and considering $|x| = \|x\|_{\ell^1(\mathbb{R}^d)} = \sum_{i=1}^d x_i$. Hence, the first term in (2.46) satisfies

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi^*(-\lambda\nabla\log(\rho^s))\rho^s\,dx &= \int_{\mathbb{R}^d} \varphi^*\left(-\lambda s\rho^{-1}\nabla\rho\right)\rho^s\,dx \\ &= (-\lambda s)^\beta \int_{\mathbb{R}^d} |\nabla\rho|^\beta \rho^{s-\beta}\,dx.\end{aligned}$$

Fixing $\beta = 2$, we get

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi^*(-\lambda\nabla\log(\rho^s))\rho^s\,dx &= (\lambda s)^2 \int_{\mathbb{R}^d} |\nabla\rho|^2\rho^{s-2}\,dx \\ &= 4\lambda^2 \int_{\mathbb{R}^d} |\nabla(\rho^{\frac{s}{2}})|^2\,dx. \end{aligned} \tag{2.47}$$

Then, from (2.44), (2.45), (2.46) and (2.47),

$$\frac{d}{dt}\int_{\mathbb{R}^d} \rho^s\,dx \le (g(s)-f(s))\int_{\mathbb{R}^d} |\nabla(\rho^{\frac{s}{2}})|^2\,dx + \frac{\dot{s}}{s}\log\left(\int_{\mathbb{R}^d}\rho^s\,dx\right)\int_{\mathbb{R}^d}\rho^s\,dx,$$

where

$$f(s) = \frac{4(s-1)}{s} \quad \text{and} \quad g(s) = \frac{4\lambda^2\dot{s}}{s}.$$

Now, we select $\dot{s} \ge 0$ such that

$$g(s) - f(s) \le 0 \Leftrightarrow \dot{s} \le \frac{s-1}{\lambda^2}. \tag{2.48}$$

Indeed, from assumption (2.42), we have that

$$\dot{s} \le \frac{C}{\lambda^2}e^{\frac{t}{\lambda^2}} = \frac{1 + Ce^{\frac{t}{\lambda^2}} - 1}{\lambda^2} = \frac{s-1}{\lambda^2}.$$

Hence, for s such that (2.48) holds, we have that

$$\frac{d}{dt}\int_{\mathbb{R}^d}\rho^s\,dx \le \frac{\dot{s}}{s}\log\left(\int_{\mathbb{R}^d}\rho^s\,dx\right)\int_{\mathbb{R}^d}\rho^s\,dx.$$

Fix $z(t) = \int_{\mathbb{R}^d}\rho^s\,dx$ and $h(t) = \frac{\dot{s}}{s} = \frac{d}{dt}\log(s(t))$. Thus, the previous inequality simplifies to

$$\dot{z}(t) \le h(t)\log(z(t))z(t).$$

Rewriting, we get

$$\frac{\dot{z}(t)}{\log(z(t))z(t)} \le h(t)$$

and thus

$$\frac{d}{dt}(\log(\log(z(t)))) \le \frac{d}{dt}\log(s(t)).$$

Finally, with $s(0) = 1$, integrating the prior expression leads to

$$\log(\log(z(t))) \le \log(s(t)) + \log(\log(z(0)))$$

and thus

$$z(t) \le \exp\{\exp\{\log(s(t)) + \log(\log(z(0)))\}\} = z(0)^{s(t)}.$$

Since $s(0) = 1$, we have that $z(0) = \|\rho(x,0)\|_{L^1(\mathbb{R}^d)}$ and thus (2.43) follows. $\square$

Remark 2.6.5. The fundamental solution verifies this estimate. Indeed, recalling the computation in Section 2.1.4, we have that

$$\|\Phi(x,t)\|_{L^{s(t)}(\mathbb{R}^d)} = Ct^{-\frac{d(s(t)-1)}{2s(t)}} < \infty, \tag{2.49}$$

where s is such that (2.42) holds. Figure 2.4 shows the $L^{s(t)}$-norm for the case $s(t) = 1 + Ce^{\frac{t}{\lambda^2}}$, for $d = 1$ and fixed constants.

Furthermore, we may also compare this result to our prior decay results. On one

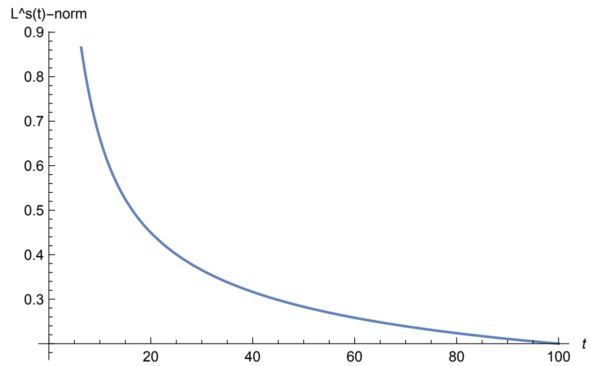

Figure 2.4: An $L^{s(t)}$-norm of the fundamental solution

hand, for $s(t) \geq 2$, interpolating leads to

$$\|\Phi\|_{L^2(\mathbb{R}^d)} \leq \|\Phi\|^{\lambda(t)}_{L^1(\mathbb{R}^d)} \|\Phi\|^{1-\lambda(t)}_{L^{s(t)}(\mathbb{R}^d)},$$

where $\lambda(t)$ is such that

$$\frac{1}{2} = \lambda(t) + \frac{1-\lambda(t)}{s(t)} \Leftrightarrow 1 - \lambda(t) = \frac{s(t)}{2(s(t)-1)}.$$

Then, from (2.49), we get

$$\|\Phi\|_{L^2(\mathbb{R}^d)} \leq \|\Phi\|^{1-\lambda(t)}_{L^{s(t)}(\mathbb{R}^d)} = Ct^{\frac{d(1-s(t))}{2s(t)} \frac{s(t)}{2(s(t)-1)}} = Ct^{-\frac{d}{4}}.$$

On the other hand, from Section 2.3.1, we have that Φ satisfies

$$\|\Phi\|_{L^2(\mathbb{R}^d)} \leq Ct^{-\frac{d}{4}}.$$

Hence, sharpness is still verified.

Chapter 3

Fokker-Planck equations

Consider the Fokker-Planck equation with singular initial data

$$\begin{cases} u_t(x,t) = \operatorname{div}(b(x,t)u(x,t)) + \operatorname{div}(a(x,t)\nabla u(x,t)) & \text{in } \mathbb{R}^d \times (0,\infty) \\ u(x,0) \in L^1(\mathbb{R}^d) & \text{in } \mathbb{R}^d, \end{cases} \tag{3.1}$$

where a is a positive scalar diffusion coefficient and b is a smooth vector field, known as the advection. In this chapter, we are interested in deducing conditions on the integrability of a and b that imply estimates similar to the previous chapters. In particular, we assume a and b to be time independent.

3.1 The advection-free scenario

We begin by taking $b = 0$. Then, (3.1) simplifies to

$$u_t = \operatorname{div}(a(x)\nabla u). \tag{3.2}$$

Let $a \equiv a(x)$. Next, we present an estimate regarding this case.

Proposition 3.1.1. *Let u be the solution to the d-dimensional simplified Fokker-Planck equation* (3.2) *with $u \in C^\infty(\mathbb{R}^d \times [0,\infty))$ and $u(x,0) \in L^1(\mathbb{R}^d)$. Moreover, assume $a \in L^{\frac{1}{1-q}}(\mathbb{R}^d)$ for some $1 < q < 2$. Then, the following estimate holds*

$$\|u\|_{L^2(\mathbb{R}^d)} \le C t^{-\frac{d}{4-2d(q-1)}}, \tag{3.3}$$

for all $t > 0$.

Remark 3.1.2. The exponent on the right-hand side of (3.3) is only negative if $d = 1, 2$, or

$$q < \frac{d+2}{d},$$

and $d \geq 3$.

Proof. Regarding the L^2-norm, we have as before that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} u^2\,dx &= 2\int_{\mathbb{R}^d} uu_t\,dx \\ &= 2\int_{\mathbb{R}^d} u\,\mathrm{div}(a\nabla u)\,dx \\ &= -2\int_{\mathbb{R}^d} a|\nabla u|^2\,dx.\end{aligned}$$

Recall the reverse Hölder inequality, which states that the following inequality holds

$$\|fg\|_{L^1(\mathbb{R}^d)} \geq \|f\|_{L^{\frac{1}{q}}(\mathbb{R}^d)}\|g\|_{L^{\frac{1}{1-q}}(\mathbb{R}^d)}. \tag{3.4}$$

Note that on the right-hand side of (3.4), $\|g\|_{L^{\frac{1}{1-q}}(\mathbb{R}^d)}$ is not a norm, since $q > 1$ and thus $1/(1-q) < 0$. Now, since $a \in L^{\frac{1}{1-q}}(\mathbb{R}^d)$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2\,dx \leq -2\left(\int_{\mathbb{R}^d} a^{\frac{1}{1-q}}\,dx\right)^{1-q}\left(\int_{\mathbb{R}^d} |\nabla u|^{\frac{2}{q}}\,dx\right)^{q}.$$

Using the Gagliardo-Nirenberg inequality we further get that, for some constant C, the following holds

$$C\|\nabla u\|_{L^{\frac{2}{q}}(\mathbb{R}^d)} \geq \|u\|^{\frac{1}{\alpha}}_{L^2(\mathbb{R}^d)},$$

for $q < 2$ and α such that

$$\frac{1}{2} = \left(\frac{q}{2} - \frac{1}{d}\right)\alpha + 1 - \alpha \Leftrightarrow \alpha = \frac{d}{2+2d-dq}.$$

Hence, we have that

$$\frac{d}{dt}\|u\|_{L^2(\mathbb{R}^d)}^2 \leq -C\|u\|_{L^2(\mathbb{R}^d)}^{2\beta}, \tag{3.5}$$

where $\beta = 1/\alpha = (2 + 2d - dq)/d$. Defining $z := \|u\|_{L^2(\mathbb{R}^d)}^2$, we rewrite (3.5) as the differential inequality

$$\dot{z} \leq -Cz^\beta,$$

Then, similar to previous chapters,

$$\int_{\mathbb{R}^d} u^2\,dx \leq Ct^{\frac{1}{1-\beta}} = Ct^{\frac{d}{d(q-1)-2}}$$

and thus (3.3) follows. □

We may now go further and deal with the L^p case, as it shows no essential difference.

Proposition 3.1.3. *Let u be the solution to the d-dimensional simplified Fokker-Planck equation* (3.2) *with $u \in C^\infty(\mathbb{R}^d \times [0,\infty))$ and $u(x,0) \in L^1(\mathbb{R}^d)$. Moreover, assume $a \in L^{\frac{1}{1-q}}(\mathbb{R}^d)$ for some $1 < q < 2$. Then, for any $p > 1$, the following estimate holds*

$$\|u\|_{L^p(\mathbb{R}^d)} \leq Ct^{-\frac{d(p-1)}{p(2-d(q-1))}}, \tag{3.6}$$

for all $t > 0$.

Remark 3.1.4. The exponent on the right-hand side of (3.6) is only negative if $d = 1, 2$ and $1 < q < 2$, or

$$q < \frac{d+2}{d},$$

and $d \geq 3$.

Proof. Similar to Section 2.4, we have that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx &= \int_{\mathbb{R}^d} u^{p-1}\,\mathrm{div}(a\nabla u)\,dx\\ &= -C\int_{\mathbb{R}^d} au^{p-2}|\nabla u|^2\,dx.\end{aligned}$$

Then, since $a \in L^{\frac{1}{1-q}}(\mathbb{R}^d)$, we similarly get

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx \le -C\left(\int_{\mathbb{R}^d} (u^{p-2}|\nabla u|^2)^{\frac{1}{q}}\,dx\right)^q.$$

Fix $\gamma = p/2$. Then, by the Gagliardo-Nirenberg-Sobolev inequality for $q < 2$, it follows that

$$\begin{aligned}\left(\int_{\mathbb{R}^d} (u^{p-2}|\nabla u|^2)^{\frac{1}{q}}\,dx\right)^q &= \left(\int_{\mathbb{R}^d} |\nabla(u^\gamma)|^{\frac{2}{q}}\,dx\right)^q\\ &\ge C\left(\int_{\mathbb{R}^d} u^{\gamma q^*}\,dx\right)^{\frac{2}{q^*}},\end{aligned}\tag{3.7}$$

where

$$\frac{1}{q^*} = \frac{q}{2} - \frac{1}{d} \Leftrightarrow q^* = \frac{2d}{dq-2}.$$

Using the interpolation inequality with $\|u\|_{L^1(\mathbb{R}^d)} = 1$ and $0 < \lambda < 1$, we have that

$$\left(\int_{\mathbb{R}^d} u^{\gamma q^*}\,dx\right)^{\frac{\lambda}{q^*}} = \|u\|^{\gamma\lambda}_{L^{\gamma q^*}(\mathbb{R}^d)} = \|u\|^{\gamma\lambda}_{L^{\gamma q^*}(\mathbb{R}^d)}\|u\|^{\gamma(1-\lambda)}_{L^1(\mathbb{R}^d)} \ge \|u\|^{\gamma}_{L^p(\mathbb{R}^d)},\tag{3.8}$$

where

$$\frac{1}{p} = 1-\lambda+\frac{\lambda}{\gamma q^*} \Leftrightarrow \lambda = \frac{d(p-1)}{2+d(p-q)}.$$

Note that, in (3.8), Remark 3.1.4 yields $\gamma q^* > p$. Combining all the previous we get

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx \leq -C\left(\int_{\mathbb{R}^d} u^p\,dx\right)^{\frac{2\gamma}{\lambda p}},$$

which is similar to (2.35). Therefore, as before,

$$\int_{\mathbb{R}^d} u^p\,dx \leq Ct^{\frac{1}{1-1/\lambda}} = Ct^{\frac{d(p-1)}{d(q-1)-2}},$$

which yields (3.6) and generalizes (3.3). □

3.2 L^2 derivatives

We now generalize the previous technique to L^2-norms of derivatives of order k. First, we consider the special case where $a \in L^\infty(\mathbb{R}^d)$. In this setting, we deduce a similar estimate to the one in Section 2.3. Then, we consider less regularity for a and deduce an estimate for the first derivative.

3.2.1 Odd derivatives

In this section, we present an estimate for the norms of odd derivatives. Here, we assume enough regularity for the diffusion a.

Proposition 3.2.1. *Let u be the solution to the d-dimensional simplified Fokker-Planck equation* (3.2) *with $u \in C^\infty(\mathbb{R}^d \times [0,\infty))$ and $u(x,0) \in L^1(\mathbb{R}^d)$. Moreover, assume that $a \in L^\infty(\mathbb{R}^d)$. Then, for any odd k and some constant $C > 0$, the following estimate holds*

$$\|D^k u\|_{L^2(\mathbb{R}^d)} \leq Ct^{-\left(\frac{d}{4}+\frac{k}{2}\right)}, \tag{3.9}$$

for all $t > 0$.

Proof. Similar to before, integration by parts and Hölder's inequality yield

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} |D^k u|^2 &= 2\int_{\mathbb{R}^d} D^k u D^k(\operatorname{div}(a\nabla u))\,dx \\
&= \int_{\mathbb{R}^d} aD^{2k+1}u\cdot\nabla u\,dx \\
&\le \|a\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d} D^{2k+1}u\cdot\nabla u\,dx \\
&\le -C\int_{\mathbb{R}^d} uD^{2k+2}u\,dx \\
&= -C\int_{\mathbb{R}^d} |D^{k+1}u|^2\,dx.
\end{aligned}$$

Then, using the Gagliardo-Nirenberg inequality, we have that

$$\left(\int_{\mathbb{R}^d}|D^k u|^2\,dx\right)^{\frac{1}{2}} \le C\left(\int_{\mathbb{R}^d}|D^{k+1}u|^2\,dx\right)^{\frac{\alpha}{2}}\left(\int_{\mathbb{R}^d}u\,dx\right)^{1-\alpha},$$

where

$$\frac{1}{2} = \frac{k}{d} + \left(\frac{1}{2} - \frac{k+1}{d}\right)\alpha + 1 - \alpha \Leftrightarrow \alpha = \frac{2k+d}{2+2k+d}.$$

Hence, since $u \in L^1(\mathbb{R}^d)$, we have that

$$\frac{d}{dt}\int_{\mathbb{R}^d}|D^k u|^2\,dx \le -C\left(\int_{\mathbb{R}^d}|D^k u|^2\,dx\right)^{\frac{1}{\alpha}}.$$

Therefore, defining $z(t) = \int_{\mathbb{R}^d}|D^k u|^2\,dx$, we get an inequality of the form

$$\dot{z} \le -Cz^{\frac{1}{\alpha}},$$

which yields

$$\int_{\mathbb{R}^d}|D^k u|^2 \le Ct^{-\frac{d}{2}-k}.$$

Thus, (3.9) follows. □

3.2.2 A gradient estimate

We are now interested in further generality. Here, we assume D^2a to be bounded and deduce an estimate for the L^2-norm of the gradient.

Proposition 3.2.2. *Let u be the solution to the d-dimensional simplified Fokker-Planck equation* (3.2) *with $u \in C^\infty(\mathbb{R}^d \times [0,\infty))$ and $u(x,0) \in L^1(\mathbb{R}^d)$. Moreover, assume $D^2a \in L^q(\mathbb{R}^d)$, $a \in L^{\frac{1}{1-r}}(\mathbb{R}^d)$, with $q > 1$ and $1 < r < 2$. Then, there exists a constant $C > 0$ and a time $T > 0$ such that the following estimate holds*

$$\|\nabla u\|_{L^2(\mathbb{R}^d)} \leq Ct^{-\frac{d+2q+dq}{4q-2dq(r-1)-2d}}, \tag{3.10}$$

for all $t \in [0,T)$.

Remark 3.2.3. Note that the exponent in (3.10) is negative for r, q such that

$$r < \frac{dq+2q-2}{dq}.$$

Proof. We have that

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} |\nabla u|^2 \, dx &= C\int_{\mathbb{R}^d} \nabla u \cdot \nabla(\operatorname{div}(a\nabla u)) \, dx \\ &= -C\int_{\mathbb{R}^d} D^2u(\nabla a \cdot \nabla u + aD^2u) \, dx \\ &= -C\int_{\mathbb{R}^d} D^2u\nabla a \cdot \nabla u \, dx - C\int_{\mathbb{R}^d} a|D^2u|^2 \, dx. \end{aligned} \tag{3.11}$$

The first term of (3.11) satisfies

$$\begin{aligned} -C\int_{\mathbb{R}^d} D^2u\nabla a \cdot \nabla u \, dx &= -C\int_{\mathbb{R}^d} \nabla a \cdot \nabla\left(|\nabla u|^2\right) \, dx \\ &= C\int_{\mathbb{R}^d} D^2a|\nabla u|^2 \, dx. \end{aligned}$$

By Hölder's reverse and Gagliardo-Nirenberg-Sobolev inequalities, we have that, for

$r < 2$, the second term of (3.11) satisfies

$$-C\int_{\mathbb{R}^d} a|D^2u|^2\,dx \leq -C\left(\int_{\mathbb{R}^d}|a|^{\frac{1}{1-r}}\,dx\right)^{1-r}\left(\int_{\mathbb{R}^d}|D^2u|^{\frac{2}{r}}\,dx\right)^{r}$$
$$\leq -C\left(\int_{\mathbb{R}^d}|\nabla u|^{2q'}\,dx\right)^{\frac{1}{q'\alpha}},$$

where α is such that

$$\frac{1}{2q'} = \frac{1}{d} + \left(\frac{r}{2} - \frac{2}{d}\right)\alpha + 1 - \alpha \Leftrightarrow \alpha = \frac{d - 2q' - 2dq'}{q'(dr - 2d - 4)}.$$

Then, we have that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}|\nabla u|^2\,dx &= C\int_{\mathbb{R}^d} D^2a|\nabla u|^2\,dx - C\left(\int_{\mathbb{R}^d}|\nabla u|^{2q'}\,dx\right)^{\frac{1}{q'\alpha}}\\ &\leq C\left(\int_{\mathbb{R}^d}|D^2a|^q\,dx\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^d}|\nabla u|^{2q'}\,dx\right)^{\frac{1}{q'}} - C\left(\int_{\mathbb{R}^d}|\nabla u|^{2q'}\,dx\right)^{\frac{1}{q'\alpha}}\\ &= C\left(\int_{\mathbb{R}^d}|\nabla u|^{2q'}\,dx\right)^{\frac{1}{q'}} - C\left(\int_{\mathbb{R}^d}|\nabla u|^{2q'}\,dx\right)^{\frac{1}{q'\alpha}}, \end{aligned} \tag{3.12}$$

where $1/q + 1/q' = 1$. Defining $y(t) = \int_{\mathbb{R}^d}|\nabla u|^{2q'}\,dx$, we have, in (3.12), an expression of the type

$$C_1 y^{\frac{1}{q'}} - C_2 y^{\frac{1}{q'\alpha}} = C_1 y^{\frac{1}{q'}} - \epsilon C_2 y^{\frac{1}{q'\alpha}} - (1-\epsilon)C_2 y^{\frac{1}{q'\alpha}},$$

for $0 < \epsilon < 1$. We conclude that, since $1/\alpha > 1$, from Lemma 2.2.5, there exists $T > 0$ such that, for all $t \in [0, T)$,

$$C_1 y^{\frac{1}{q'}} - \epsilon C_2 y^{\frac{1}{q'\alpha}} < 0.$$

Now, with $C_\epsilon = (1-\epsilon)C_2$, applying Gagliardo-Nirenberg again leads to

$$-C_\epsilon y^{\frac{1}{q'\alpha}} = -C_\epsilon\left(\int_{\mathbb{R}^d}|\nabla u|^{2q'}\,dx\right)^{\frac{1}{q'\alpha}}$$

$$\leq -C\left(\int_{\mathbb{R}^d} |\nabla u|^2\,dx\right)^{\frac{1}{\alpha}}.$$

Then, with $z \coloneqq \int_{\mathbb{R}^d} |\nabla u|^2\,dx$, we have, for $t < T$, an inequality of the type

$$\dot{z} \leq -Cz^{\frac{1}{\alpha}}$$

Rewriting for q, we get

$$\frac{1}{\alpha} = \frac{q(4 - d(r-2))}{d + 2q + dq}$$

and so

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \leq Ct^{\frac{1}{1-1/\alpha}} = Ct^{\frac{d+2q+dq}{d-2q+dq(r-1)}},$$

from which (3.10) follows. □

3.3 Generalized equation

Let us now assume that $b \equiv b(x) \neq 0$. We are interested in two scenarios. In the first, we will assume integrability on the divergence of b. In the second, we assume integrability on b.

3.3.1 Integrability conditions on the divergence of the advection

The next theorem yields two estimates depending on the assumptions for the divergence of b.

Theorem 3.3.1. *Let u be the solution to the d-dimensional simplified Fokker-Planck equation* (3.1) *with $u \in C^\infty(\mathbb{R}^d \times [0,\infty))$ and $u(x,0) \in L^1(\mathbb{R}^d)$. Let $a > 0$. Moreover, assume $a \in L^{\frac{1}{1-q}}(\mathbb{R}^d)$ for some $1 < q < 2$. Then, for $d \geq 2$, the following holds*

1. If $\operatorname{div} b = 0$ *and* $p > 1$*, then*

$$\|u\|_{L^p(\mathbb{R}^d)} \leq Ct^{-\frac{d(p-1)}{p(2-d(q-1))}}, \tag{3.13}$$

for all $t > 0$*.*

2. If $\operatorname{div} b \in L^r(\mathbb{R}^d)$ *and* p, q *are such that*

$$2 \leq d < 2r \;\; and \;\; 1 < q < \frac{2r + dr - d}{dr}, \tag{3.14}$$

then, there exists $T > 0$ *such that*

$$\|u\|_{L^p(\mathbb{R}^d)} \leq Ct^{-\frac{d(p-1)}{p(2-d(q-1))}}, \tag{3.15}$$

holds for all $t < T$*.*

Remark 3.3.2. The exponent on the right-hand side of (3.13) is only negative if $d = 1, 2$ and $1 < q < 2$, or

$$q < \frac{d+2}{d},$$

and $d \geq 3$.

Proof. 1. We have that

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^p \, dx = p \int_{\mathbb{R}^d} u^{p-1} \operatorname{div}(bu) \, dx + p \int_{\mathbb{R}^d} u^{p-1} \operatorname{div}(a \nabla u) \, dx. \tag{3.16}$$

Then, from the proof of Proposition 3.1.3, we have the following bound

$$\int_{\mathbb{R}^d} u^{p-1} \operatorname{div}(a \nabla u) \, dx \leq -C \left(\int_{\mathbb{R}^d} u^p \, dx \right)^{\beta},$$

where

$$\beta = \frac{2 + d(p-q)}{d(p-1)}.$$

For the other term in (3.16), we have that

$$\begin{aligned}\int_{\mathbb{R}^d} u^{p-1} \operatorname{div}(bu)\, dx &= -\int_{\mathbb{R}^d} u^{p-1} \nabla u \cdot b\, dx \\ &= -C \int_{\mathbb{R}^d} \nabla(u^p) \cdot b\, dx \\ &= C \int_{\mathbb{R}^d} u^p \operatorname{div} b\, dx.\end{aligned}$$

It is then clear that if $\operatorname{div} b = 0$, we have the estimate in (3.13).

2. If we assume that $\operatorname{div} b \in L^r(\mathbb{R}^d)$, Hölder's inequality leads to

$$\int_{\mathbb{R}^d} u^p \operatorname{div} b\, dx \le \left(\int_{\mathbb{R}^d} u^{pr'}\, dx \right)^{\frac{1}{r'}} \left(\int_{\mathbb{R}^d} (\operatorname{div} b)^r\, dx \right)^{\frac{1}{r}},$$

where $1/r' + 1/r = 1$. From (3.7) in the proof of Proposition 3.1.3, we also have

$$\int_{\mathbb{R}^d} u^{p-1} \operatorname{div}(a\nabla u)\, dx \le -C \left(\int_{\mathbb{R}^d} u^{\gamma q^*}\, dx \right)^{\frac{2}{q^*}},$$

where $\gamma = p/2$ and $q^* = 2d/(dq-2)$. Then, we have that

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^p\, dx \le C \left(\int_{\mathbb{R}^d} u^{pr'}\, dx \right)^{\frac{1}{r'}} - C \left(\int_{\mathbb{R}^d} u^{\gamma q^*}\, dx \right)^{\frac{2}{q^*}}. \tag{3.17}$$

Note that, by interpolation,

$$\left(\int_{\mathbb{R}^d} u^{pr'}\, dx \right)^{\frac{1}{r'}} \le \left(\int_{\mathbb{R}^d} u^{\gamma q^*}\, dx \right)^{\frac{p\theta}{\gamma q^*}},$$

where θ is such that

$$\frac{1}{pr'} = \frac{\theta}{\gamma q^*} + 1 - \theta.$$

Note that the previous inequality only holds if

$$pr' < \gamma q^* \Leftrightarrow \frac{pr}{r-1} < \frac{pd}{dq-2},$$

which is true if (3.14) holds. Therefore, with $y(t) = \int_{\mathbb{R}^d} u^{\gamma q^*}\,dx$, we have that the right-hand side of (3.17) is less or equal than

$$C_1 y^{\frac{p\theta}{\gamma q^*}} - C_2 y^{\frac{2}{q^*}} = C_1 y^{\frac{2\theta}{q^*}} - C_2 y^{\frac{2}{q^*}}.$$

Then, since $\theta < 1$, with $z(t) = \int_{\mathbb{R}^d} u^p\,dx$, we have that, by Lemma 2.2.5 and using interpolation again, there exists $T > 0$ such that, for all $t < T$,

$$\begin{aligned} \dot{z} &\leq -C y^{\frac{2}{q^*}} \\ &= -C \left(\int_{\mathbb{R}^d} u^{\gamma q^*}\,dx \right)^{\frac{2}{q^*}} \\ &\leq -C \left(\int_{\mathbb{R}^d} u^p\,dx \right)^{\frac{1}{\lambda}} = -C z^{\frac{1}{\lambda}}, \end{aligned}$$

where λ is such that

$$\frac{1}{p} = \frac{\lambda}{\gamma q^*} + 1 - \lambda \Leftrightarrow \lambda = \frac{d(p-1)}{2 + dp - dq}.$$

Then, we get

$$z(t) \leq C t^{\frac{1}{1-1/\lambda}} = C t^{\frac{d(p-1)}{d(q-1)-2}},$$

and thus (3.15) follows, for all $t < T$.

□

3.3.2 Integrability conditions on the advection

We now consider integrability on the advection itself.

Theorem 3.3.3. *Let u be the solution to the d-dimensional simplified Fokker-Planck equation (3.1) with $u \in C^\infty(\mathbb{R}^d \times [0,\infty))$ and $u(x,0) \in L^1(\mathbb{R}^d)$. Moreover, assume that $a^{-1} \in L^r(\mathbb{R}^d)$ and $|b| \in L^{\frac{2rq}{r-1}}(\mathbb{R}^d)$ for some $q > 1$, $r > 2$. Then, for any $p > 1$ and $d \geq 2$, the following holds*

1. *Let q be such that*

$$q > \frac{d(1-r)}{d-2r}, \quad \text{for } 2 < d < 2r. \tag{3.18}$$

If $a = 1$, there exists $T > 0$ such that

$$\|u\|_{L^p(\mathbb{R}^d)} \leq Ct^{-\frac{qr(p-1)}{p(r+q-1)}}, \tag{3.19}$$

holds for $t < T$.

2. *Let q be such that*

$$q > \frac{d(1-r)}{dr(s-1)+d-2r}, \quad \text{for } \frac{2}{s} < d < \frac{2r}{1+r(s-1)}. \tag{3.20}$$

Moreover, if $a \in L^{\frac{1}{1-s}}(\mathbb{R}^d)$, there also exists $T > 0$ such that

$$\|u\|_{L^p(\mathbb{R}^d)} \leq Ct^{-\frac{qr(p-1)}{p(r+q-1)}}, \tag{3.21}$$

holds for $1 < s < 2$ and $t < T$.

Proof. Let a be arbitrary for now. We have that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx &= p\int_{\mathbb{R}^d} u^{p-1}\operatorname{div}(bu)\,dx + p\int_{\mathbb{R}^d} u^{p-1}\operatorname{div}(a\nabla u)\,dx \\
&= -p\int_{\mathbb{R}^d} u^{p-1}\nabla u\cdot b\,dx - p\int_{\mathbb{R}^d} au^{p-2}|\nabla u|^2\,dx \\
&= -p\int_{\mathbb{R}^d} a^{\frac{1}{2}}u^{\frac{p}{2}-1}\nabla u\cdot bu^{\frac{p}{2}}a^{-\frac{1}{2}}\,dx - p\int_{\mathbb{R}^d} au^{p-2}|\nabla u|^2\,dx.
\end{aligned}$$

Then, reorganizing the previous inequality and using Cauchy's inequality with ϵ, we have that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx + p\int_{\mathbb{R}^d} au^{p-2}|\nabla u|^2\,dx &= -p\int_{\mathbb{R}^d} a^{\frac{1}{2}}u^{\frac{p}{2}-1}\nabla u\cdot bu^{\frac{p}{2}}a^{-\frac{1}{2}}\,dx \\ &\leq \left|p\int_{\mathbb{R}^d} a^{\frac{1}{2}}u^{\frac{p}{2}-1}\nabla u\cdot bu^{\frac{p}{2}}a^{-\frac{1}{2}}\,dx\right| \\ &\leq p\epsilon\int_{\mathbb{R}^d}|a|u^{p-2}|\nabla u|^2\,dx + pC_\epsilon\int_{\mathbb{R}^d}|b|^2u^p|a|^{-1}\,dx,\end{aligned} \tag{3.22}$$

where $C_\epsilon = 1/(4\epsilon)$. For $0 < \eta < 1$, we split the second term on the left-hand side to get

$$p\int_{\mathbb{R}^d} au^{p-2}|\nabla u|^2\,dx = \eta p\int_{\mathbb{R}^d} au^{p-2}|\nabla u|^2\,dx + (1-\eta)p\int_{\mathbb{R}^d} au^{p-2}|\nabla u|^2\,dx.$$

Hence, for $\epsilon < \eta$, rewriting (3.22) leads to

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx \leq -C_\eta\int_{\mathbb{R}^d} au^{p-2}|\nabla u|^2\,dx + pC_\epsilon\int_{\mathbb{R}^d}|b|^2u^p|a|^{-1}\,dx, \tag{3.23}$$

where $C_\eta = p(1-\eta)$. Now, applying Hölder's inequality twice to the last term in the previous inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^d}|b|^2u^p|a|^{-1}\,dx &\leq \left(\int_{\mathbb{R}^d}|b|^{2r'}u^{pr'}\right)^{\frac{1}{r'}}\left(\int_{\mathbb{R}^d}|a|^{-r}\right)^{\frac{1}{r}} \\ &\leq C\left(\int_{\mathbb{R}^d}u^{pr'q'}\right)^{\frac{1}{r'q'}}\left(\int_{\mathbb{R}^d}|b|^{2r'q}\right)^{\frac{1}{r'q}} \\ &\leq C\left(\int_{\mathbb{R}^d}u^{pr'q'}\right)^{\frac{1}{r'q'}},\end{aligned}$$

where

$$\frac{1}{r}+\frac{1}{r'} = 1 = \frac{1}{q}+\frac{1}{q'}$$

and $r'q = \frac{rq}{r-1}$. Hence, defining

$$\gamma = pr'q' = \frac{pqr}{(q-1)(r-1)},$$

we have, from (3.23),

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx \le C_1\left(\int_{\mathbb{R}^d} u^\gamma\,dx\right)^{\frac{p}{\gamma}} - C_2\int_{\mathbb{R}^d} a|\nabla(u^{\frac{p}{2}})|^2\,dx,$$

where $C_1, C_2 > 0$ are constants depending on η and ϵ. Now, we consider the two cases separately.

1. If $a = 1$, then, by Sobolev's inequality, we have that

$$\int_{\mathbb{R}^d} |\nabla(u^{\frac{p}{2}})|^2\,dx \ge C\left(\int_{\mathbb{R}^d} u^{\frac{2^*p}{2}}\,dx\right)^{\frac{2}{2^*}}.$$

Then, using interpolation, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx &\le C_1\left(\int_{\mathbb{R}^d} u^\gamma\,dx\right)^{\frac{p}{\gamma}} - C_2\left(\int_{\mathbb{R}^d} u^{\frac{2^*p}{2}}\,dx\right)^{\frac{2}{2^*}}\\ &\le C_1\left(\int_{\mathbb{R}^d} u^{\frac{2^*p}{2}}\,dx\right)^{\frac{2\theta}{2^*}} - C_2\left(\int_{\mathbb{R}^d} u^{\frac{2^*p}{2}}\,dx\right)^{\frac{2}{2^*}},\end{aligned}$$

where θ is such that

$$\frac{1}{\gamma} = \frac{2\theta}{2^*p} + 1 - \theta.$$

Note that the previous inequality only holds if $\gamma \le 2^*p/2$. This is true for q such that (3.18) holds. Hence, since $\theta < 1$, using Lemma 2.2.5 and interpolation again, there exists $T > 0$ such that, for all $t < T$,

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx \le -C\left(\int_{\mathbb{R}^d} u^p\,dx\right)^{\frac{1}{\lambda}}$$

for some $\lambda > 0$ such that

$$\frac{1}{p} = \frac{\lambda}{\gamma} + 1 - \lambda \Leftrightarrow \lambda = \frac{\gamma(p-1)}{p(\gamma-1)},$$

which yields

$$\lambda = \frac{qr(p-1)}{qr(p-1)+q+r-1}.$$

Hence, setting $z(t) = \int_{\mathbb{R}^d} u^p\, dx$, we get an inequality of the type

$$\dot{z} \leq -Cz^{\frac{1}{\lambda}}.$$

Thus,

$$z(t) \leq Ct^{\frac{1}{1-1/\lambda}} = Ct^{\frac{qr(1-p)}{r+q-1}},$$

which combined with (3.18), yields (3.19), for $t < T$.

2. If $a \in L^{\frac{1}{1-s}}(\mathbb{R}^d)$, by Hölder's reverse inequality, we have that

$$\begin{aligned}\int_{\mathbb{R}^d} a|\nabla(u^{\frac{p}{2}})|^2\, dx &\geq \left(\int_{\mathbb{R}^d} a^{\frac{1}{1-s}}\, dx\right)^{1-s} \left(\int_{\mathbb{R}^d} |\nabla(u^{\frac{p}{2}})|^{\frac{2}{s}}\, dx\right)^{s} \\ &\geq C\left(\int_{\mathbb{R}^d} |\nabla(u^{\frac{p}{2}})|^{\frac{2}{s}}\, dx\right)^{s}.\end{aligned}$$

Then, for $s < 2$, the Gagliardo-Nirenberg-Sobolev inequality yields

$$\left(\int_{\mathbb{R}^d} |\nabla(u^{\frac{p}{2}})|^{\frac{2}{s}}\, dx\right)^{s} \geq C\left(\int_{\mathbb{R}^d} u^{\frac{mp}{2}}\, dx\right)^{\frac{2}{m}}$$

with

$$\frac{1}{m} = \frac{s}{2} - \frac{1}{d} \Leftrightarrow m = \frac{2d}{ds-2}.$$

Furthermore, interpolation yields

$$\left(\int_{\mathbb{R}^d} u^{\gamma}\, dx\right)^{\frac{p}{\gamma}} \leq \left(\int_{\mathbb{R}^d} u^{\frac{mp}{2}}\, dx\right)^{\frac{2\theta}{m}}, \tag{3.24}$$

where θ is such that

$$\frac{1}{\gamma} = \frac{2\theta}{mp} + 1 - \theta.$$

Note that (3.24) holds if $\gamma < mp/2$. This is true for q such that (3.20) holds. Then, following the same steps as before, since $\theta < 1$, we have that there exists $T > 0$ such that, for all $t < T$,

$$\dot{z} \leq -Cz^{\frac{1}{\lambda}}.$$

Thus,

$$z(t) \leq Ct^{\frac{1}{1-1/\lambda}} = Ct^{\frac{qr(1-p)}{r+q-1}},$$

which combined with (3.20), yields (3.21), for $1 < s < 2$ and $t < T$. $\square$

Chapter 4

The porous media equation

The porous media equation (PME) is the following PDE

$$\begin{cases} u_t(x,t) = \Delta(u(x,t)^m) & \text{in } \mathbb{R}^d \times (0,T) \\ u(x,0) \in L^1(\mathbb{R}^d) & \text{in } \mathbb{R}^d, \end{cases} \tag{4.1}$$

for some $m \in [1,\infty)$ and where we take $u \geq 0$. Note that $m = 1$ corresponds to the heat equation. Here, we extend the ideas from Section 2.3 to obtain integrability estimates for the solution of the PME. Next, we examine the Barenblatt solutions to deduce sharpness. We conclude this section by comparing our method with results in [2].

4.1 Estimate methods revisited

We begin by applying our method to (4.1).

Proposition 4.1.1. *Let u be the solution to the d-dimensional porous media equation (4.1) with $u \in C^\infty(\mathbb{R}^d \times [0,\infty))$ and $u(x,0) \in L^1(\mathbb{R}^d)$. Then, for $p \geq 1$, the following estimate holds*

$$\|u\|_{L^p(\mathbb{R}^d)} \leq Ct^{-\frac{d(p-1)}{p(d(m-1)+2)}}, \tag{4.2}$$

for all $t > 0$.

Remark 4.1.2. For $m = 1$, (4.2) is similar to (2.31).

Proof. We begin by noticing that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx &= p\int_{\mathbb{R}^d} u^{p-1}\Delta(u^m)\,dx\\ &= -mp(p-1)\int_{\mathbb{R}^d} u^{m+p-3}|\nabla u|^2\,dx \le 0. \end{aligned}\tag{4.3}$$

Fix $\gamma = (m+p-1)/2$. Then, (4.3) yields

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx &\le -C\int_{\mathbb{R}^d} u^{2\gamma-2}|\nabla u|^2\,dx\\ &= -C\int_{\mathbb{R}^d} |\nabla(u^\gamma)|^2\,dx. \end{aligned}\tag{4.4}$$

By Sobolev inequality, we have that

$$\left(\int_{\mathbb{R}^d} u^{2^*\gamma}\,dx\right)^{\frac{1}{2^*}} \le C\left(\int_{\mathbb{R}^d} |\nabla(u^\gamma)|^2\,dx\right)^{\frac{1}{2}}. \tag{4.5}$$

Using the interpolation inequality with $\|u\|_{L^1(\mathbb{R}^d)} = 1$ and $0 < \lambda < 1$, we have that

$$\left(\int_{\mathbb{R}^d} u^{2^*\gamma}\,dx\right)^{\frac{2\lambda}{2^*}} = \|u\|^{2\gamma\lambda}_{L^{2^*\gamma}(\mathbb{R}^d)} = \|u\|^{2\gamma\lambda}_{L^{2^*\gamma}(\mathbb{R}^d)}\|u\|^{2\gamma(1-\lambda)}_{L^1(\mathbb{R}^d)} \ge \|u\|^{2\gamma}_{L^p(\mathbb{R}^d)}, \tag{4.6}$$

where

$$\frac{1}{p} = 1-\lambda+\frac{\lambda}{2^*\gamma} \Leftrightarrow \lambda = \frac{d(p-1)(m+p-1)}{p(2+d(m+p-2))}.$$

Hence, (4.4), (4.5) and (4.6) lead to

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx \le -C\left(\int_{\mathbb{R}^d} u^p\,dx\right)^{\frac{2\gamma}{\lambda p}}.$$

Let $z(t) := \int_{\mathbb{R}^d} u^p \, dx$. Then, the previous inequality can be written as

$$\dot{z} \leq -Cz^{\beta},$$

where $\beta = 2\gamma/(\lambda p) > 1$. As before, we get the following time estimate

$$z(t) \leq C_\beta t^{\frac{1}{1-\beta}} = Ct^{\frac{d(1-p)}{d(m-1)+2}}$$

and, thus, (4.2) follows. □

Next, we consider an estimate for a known solution to (4.1) and compare it to the prior estimate.

4.1.1 Barenblatt solutions

We now present a similar approach to the one in Section 2.3.1 to verify sharpness of (4.2). The Barenblatt solution of the PME has the following explicit formula, for an arbitrary constant $C > 0$,

$$\mathcal{U}(x,t) = t^{-\alpha}(C - k|x|^2 t^{-2\sigma})_+^{\frac{1}{m-1}},$$

where $(s)_+ = \max\{s, 0\}$ and

$$\alpha = \frac{d}{d(m-1)+2}, \quad \sigma = \frac{\alpha}{d}, \quad k = \frac{\alpha(m-1)}{2md}.$$

Denote the ball centered at the origin with radius $R = (Ct^{2\sigma}/k)^{\frac{1}{2}}$ by B_R. Then, with $u = \mathcal{U}$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{U}^p \, dx &= \int_{B_R} \mathcal{U}^p \, dx \\ &= t^{-p\alpha} \int_{B_R} (C - k|x|^2 t^{-2\sigma})^{\frac{p}{m-1}} \, dx\end{aligned}$$

$$= t^{-p\alpha} \int_{B_R} (C - k|y|^2)^{\frac{p}{m-1}} t^{\sigma d} \, dy$$

$$= C_{m,p,k} t^{-p\alpha + \sigma d} = C_{m,p,k} t^{\alpha(1-p)} = C_{m,p,k} t^{-\frac{d(p-1)}{d(m-1)+2}},$$

where we considered the change of variables $y = x/t^\sigma$, with $dx = t^{\sigma d} \, dy$. Then, by comparison to (4.2), we conclude that our estimate is sharp.

Remark 4.1.3. For $p = 1$, we have the short result $\mathcal{U} \in L^1(\mathbb{R}^d)$.

4.2 Comparison to known work

We now compare our method to known estimates from the literature. In [1], using phase-plane analysis, scaling techniques and self-similarity, it was shown that

$$\|u\|_{L^p(\mathbb{R}^d)} \leq C \|u_0\|_{L^q(\mathbb{R}^d)}^{\sigma(p,q)} t^{-\alpha(p,q)}$$

with

$$\alpha(p,q) = \frac{d(p-q)}{p(d(m-1)+2q)}, \quad \sigma(p,q) = \frac{q(d(m-1)+2p)}{p(d(m-1)+2q)}.$$

In our case, we assume $u_0 \in L^1(\mathbb{R}^d)$ and fix $q = 1$ to get

$$\|u\|_{L^p(\mathbb{R}^d)} \leq C t^{-\alpha(p,q)} = C t^{-\frac{d(p-1)}{p(d(m-1)+2)}}$$

which yields the same estimate as in (4.2). Hence, our technique does not depend on symmetry and so it provides a different method to the one in [1].

4.3 Periodic solutions of the porous media equation

We now the deduce a similar estimate for the porous media equation on $\mathbb{T}^d$.

Proposition 4.3.1. *Let u solve* (4.1) *on the torus with $u \in C^\infty(\mathbb{T}^d \times [0, \infty))$. Then,* *there exists $T > 0$ such that the following holds*

$$\|u\|_{L^p(\mathbb{T}^d)} \leq Ct^{-\frac{d(p-1)}{p(d(m-1)+2)}} \tag{4.7}$$

for all $t \in [0, T)$. For $t > T$, $\|u\|_{L^p(\mathbb{T}^d)} \leq CT^{-\frac{d(p-1)}{p(d(m-1)+2)}}$.

Proof. Fix $\gamma = (m + p - 1)/2$, thus $2\gamma > p$. From (4.4), we have that $\frac{d}{dt} \int_{\mathbb{R}^d} u^p \, dx = -C \int_{\mathbb{R}^d} |\nabla(u^\gamma)|^2 \, dx$. Then,

$$\begin{aligned}
\left(\int_{\mathbb{T}^d} u^p \, dx \right)^{\frac{\gamma}{p}} &\leq C \left(\int_{\mathbb{T}^d} u^{2^*\gamma} \, dx \right)^{\frac{\lambda}{2^*}} \\
&\leq C \left(C + \int_{\mathbb{T}^d} |D(u^\gamma)|^2 \, dx \right)^{\frac{\lambda}{2}} \\
&\leq C \left(C - C \frac{d}{dt} \int_{\mathbb{T}^d} u^p \, dx \right)^{\frac{\lambda}{2}},
\end{aligned}$$

where $\lambda = \frac{d(p-1)(m+p-1)}{p(2+d(m+p-2))}$. Then, fixing $z(t) = \int_{\mathbb{T}^d} u^p \, dx$, we get the following differential inequality $\dot{z} \leq C_1 - C_2 z^\beta$, where $\beta = \frac{2\gamma}{\lambda p}$. Hence, by Lemma 2.2.5, there exists $T > 0$ such that z satisfies

$$z(t) \leq Ct^{\frac{d(1-p)}{d(m-1)+2}}$$

for $t \in [0, T)$, which yields (4.7). For $t > T$, $\|u\|_{L^p(\mathbb{T}^d)} \leq CT^{-\frac{d(p-1)}{p(d(m-1)+2)}}$. $\square$

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