EXISTENCE OF WEAK SOLUTIONS TO FIRST-ORDER STATIONARY MEAN-FIELD GAMES WITH DIRICHLET CONDITIONS

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Abstract. In this paper, we study first-order stationary monotone mean-field games (MFGs) with Dirichlet boundary conditions. Whereas Dirichlet conditions may not be satisfied for Hamilton–Jacobi equations, here, we establish the existence of solutions to MFGs that satisfy those conditions. To construct these solutions, we introduce a monotone regularized problem. Applying Schaefer's fixed-point theorem and using the monotonicity of the MFG, we verify that there exists a unique weak solution to the regularized problem. Finally, we take the limit of the solutions of the regularized problem and, using Minty's method, we show the existence of weak solutions to the original MFG.

1. Introduction

Mean-field games (MFGs) were introduced in the mathematical community in [28], [29], and [30] and, independently, around the same time, in the engineering community in [26] and [27]. These games model the behavior of large populations of rational agents who seek to optimize an individual utility. Here, we consider the following first-order stationary MFG with a Dirichlet boundary condition.

Problem 1. *Suppose that $\Omega \subset \mathbb{R}^d$ is an open and bounded set with smooth boundary, $\partial\Omega$, $\gamma > 1$, and $k \in \mathbb{N}$ is such that $2k > \frac{d}{2} + 2$. Let V, $\phi \in L^\infty(\Omega) \cap C(\Omega)$, $h \in C^{4k}(\overline{\Omega})$, $g \in C^1(\mathbb{R}_0^+)$, and $H \in C^2(\overline{\Omega} \times \mathbb{R}^d)$ be such that g is increasing, $\phi \geqslant 0$ in Ω, and $\int_\Omega \phi \,\mathrm{d}x = 1$. Find $(m, u) \in L^1(\Omega) \times W^{1,\gamma}(\Omega)$ satisfying $m \geqslant 0$ and*

$$\begin{cases} -u - H(x, Du) + g(m) - V(x) = 0 & \text{in } \Omega, \\ m - \operatorname{div}\big(mD_pH(x, Du)\big) - \phi = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega. \end{cases}$$

In Problem 1, H is the Hamiltonian and its Legendre transform, given by

$$L(v) = \sup_p\{-p \cdot v - H(x, p)\},$$

gives the agent's cost of movement at speed v; the potential, V, determines spatial preferences of each agent, and the coupling, g, encodes the interactions between agents and the mean field. When agents leave the domain through a point $x \in \partial\Omega$, they incur a charge $h(x)$. Agents can leave the domain either through $\partial\Omega$ (at time $T_{\partial\Omega}$) or leave to a graveyard state at an exponential rate; this rate is encoded in unit discount present in the term $-u$ in the Hamilton–Jacobi equation and in the term m in the transport equation. Thus, each agent seeks to find a trajectory $\mathbf{x} : [0, T_{\partial\Omega}) \to \Omega$ that minimizes

$$\int_0^{T_{\partial\Omega}} e^{-t}\,[L(\mathbf{x}, \dot{\mathbf{x}}) + g(m(\mathbf{x}(t))) - V(\mathbf{x}(t))]\,dt.$$

The source, ϕ, represents an incoming flow of agents replacing the ones leaving. We note that m is the density of the distribution of the agents and $u(x)$ the value function of an agent in the state x.

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In general, Hamilton–Jacobi equations with Dirichlet boundary conditions do not admit continuous solutions up to the boundary. This fact can be illustrated by the equation

$$u'(x) = 0$$

for $0 < x < 1$ with $u(0) = 0$ and $u(1) = 1$. However, by keeping the boundary conditions and coupling the previous equation with a transport equation

$$\begin{cases} u'(x) = m(x) \\ m'(x) = 0, \end{cases}$$

we obtain a model that has a unique solution continuous up to the boundary, $(u, m) = (x, 1)$. This model motivates our main result, the existence of solutions for Problem 1 satisfying the Dirichlet boundary condition in the trace sense.

MFGs have been studied extensively, as shown by the great variety of publications and results on this topic (see, e.g., the monograph [23], the surveys [1], [25], the note [7], and the lectures [31]). Because there are few known explicit solutions for stationary MFGs [21], [20], [14], significant efforts have been placed on developing numerical methods [3], [6], finding special transformations [9], and establishing the existence of solutions. The existence of solutions for second-order, stationary MFGs without congestion was investigated in [24], [22], [35], [5], and [11]; problems with congestion were examined in [13], [19]. The theory for first-order MFGs is less developed. The existence of solutions for first or second-order stationary MFGs was examined in [17] (also see [2]) using monotone operators; certain first-order MFGs with congestion were examined in [15] using a variational approach. A regularization of the Hamilton–Jacobi equation that resembles ours was used in [34] to study the existence of solutions of non-monotone MFGs through variational methods.

For first-order MFGs, and often for second-order MFGs, existing publications only consider periodic boundary conditions. However, MFGs with boundary conditions are quite natural: for example, Dirichlet boundary conditions arise when agents can leave the domain and are charged an exit fee, for example, in minimal time [32] and optimal stopping models [4]. Here, our goal is to prove the existence of weak solutions for the first-order stationary monotone MFG with Dirichlet boundary conditions. Thus, we introduce a notion of weak solutions to Problem 1 similar to that considered in [17]. Here, however, we account for the Dirichlet boundary conditions. We recall that in [17], the authors consider stationary monotone MFGs with periodic boundary conditions, and their notion of weak solutions is induced by monotonicity. Monotonicity plays an essential role in the uniqueness of solutions [31]; in its absence, the theory becomes substantially harder, as shown in [21], as well as in the case of non-monotone second-order MFGs addressed in [12], [10], and [33].

Throughout this paper, $\mathcal{A}$ and $H^{2k}_h(\Omega)$ are the sets given, respectively, by

$$\mathcal{A} := \Big\{ m \in H^{2k}(\Omega) \mid m \geqslant 0 \Big\} \tag{1.1}$$

and

$$H^{2k}_h(\Omega) := \Big\{ w \in H^{2k}(\Omega) \mid w - h \in H^{2k}_0(\Omega) \Big\}. \tag{1.2}$$

Definition 1.1. *A weak solution to Problem 1 is a pair* $(m, u) \in L^1(\Omega) \times W^{1,\gamma}(\Omega)$ *satisfying*

(D1) $u = h$ *on* $\partial\Omega$, $m \geqslant 0$ *in* Ω,

(D2) $\left\langle F \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m \\ u \end{bmatrix} \right\rangle \geqslant 0$ *for all* $(\eta, v) \in \mathcal{A} \times H^{2k}_h(\Omega)$,

where, for $(\eta, v) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ *fixed,* $F[\eta, v] : L^1(\Omega) \times L^1(\Omega) \to \mathbb{R}$ *is the functional given by*

$$\begin{aligned} \left\langle F \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle := &\int_\Omega \big(- v - H(x, Dv) + g(\eta) - V \big) w_1 \, \mathrm{d}x \\ &+ \int_\Omega \Big(\eta - \operatorname{div}\big(\eta D_p H(x, Dv)\big) - \phi \Big) w_2 \, \mathrm{d}x. \end{aligned} \tag{1.3}$$

In some cases, it is possible to improve the regularity of weak solutions and obtain a more typical notion. We refer to [17] where this is done in detail for a particular case. Next, we state our main theorem that, under the assumptions detailed in Section 2, establishes the existence of weak solutions to Problem 1.

Theorem 1.2. Consider Problem 1 and suppose that Assumptions 1–7 hold. Then, there exists a weak solution $(m,u) \in L^1(\Omega) \times W^{1,\gamma}(\Omega)$ to Problem 1 in the sense of Definition 1.1.

In [17], the method of continuity was used to prove the existence of a weak solution to stationary monotone MFGs with periodic boundary conditions. Here, we use a different approach: we apply Schaeffer's fixed-point theorem and extend the results in [17] to Dirichlet boundary conditions.

To prove Theorem 1.2, we introduce a regularized problem, Problem 2, that we believe to be of interest on its own. This regularized problem preserves the monotonicity of the original MFG in the sense of Assumption 7 (see Section 2 and Lemma 7.2). Note that the choice of boundary conditions is critical to preserve monotonicity; in general, with arbitrary boundary conditions the MFG may fail to be monotone. Moreover, because of the regularizing terms (see the ϵ-terms in (1.4) below), it is simpler to prove the existence and uniqueness of weak solutions to this problem. Then, by letting $\epsilon \to 0$, we can construct a weak solution to Problem 1 (see Section 7).

Problem 2. *Let Ω be an open and bounded set with smooth boundary, $\partial\Omega$, and outward pointing unit normal vector $\mathbf{n}$. Let $k \in \mathbb{N}$ be such that $2k > \frac{d}{2} + 2$ and let V, $\phi \in L^\infty(\Omega) \cap C(\Omega)$, h, $\xi \in C^{4k}(\overline{\Omega})$, $g \in C^1(\mathbb{R}_0^+)$, and $H \in C^2(\overline{\Omega} \times \mathbb{R}^d)$ be such that g is increasing, $\phi \geqslant 0$ in Ω, and $\int_\Omega \phi \,\mathrm{d}x = 1$. Fix $\epsilon \in (0,1)$. Find $(m,u) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ satisfying $m \geqslant 0$ and*

$$\begin{cases} -u - H(x,Du) + g(m) - V(x) + \epsilon\big(m + \Delta^{2k} m\big) = 0 \quad \text{in } \Omega, \\ m - \operatorname{div}\big(mD_pH(x,Du)\big) - \phi + \epsilon\big(u + \xi + \Delta^{2k}(u+\xi)\big) = 0 \quad \text{in } \Omega, \\ \frac{\partial}{\partial \mathbf{n}} \partial^\alpha \Delta^i m = 0 \text{ on } \partial\Omega \text{ for all } \alpha \in \mathbb{N}_0^d \text{ and } i \in \mathbb{N}_0 \text{ such that } |\alpha| + i = 2k-1, \\ \partial^\beta u = \partial^\beta h \quad \text{on } \partial\Omega \text{ for all } \beta \in \mathbb{N}_0^d \text{ and } \textit{such that } |\beta| \leqslant 2k-1. \end{cases} \tag{1.4}$$

In the preceding problem, ξ is a technical term used to cancel the boundary conditions in u so that we can work with vanishing Dirichlet boundary conditions, see Section 6. Since the regularizing terms are of order greater than two, we cannot apply the maximum principle. Thus, Problem 2 may not have classical solutions with $m \geqslant 0$. Hence, in the following definition, we introduce a notion of weak solution to Problem 2 that requires positivity and relaxes the equality in the Hamilton–Jacobi equation. This definition is related those in [8] and in [17], where u is only required to be a subsolution of the Hamilton–Jacobi equation.

Definition 1.3. *A weak solution to Problem 2 is a pair $(m,u) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ satisfying, for all $w \in \mathcal{A}$ and $v \in H_0^{2k}(\Omega)$,*

(E1) $u \in H_h^{2k}(\Omega)$, $m \geqslant 0$ *in* Ω,

(E2)
$$\int_\Omega \big(-u - H(x,Du) + g(m) - V\big)(w-m)\,\mathrm{d}x + \int_\Omega \Big[\epsilon m(w-m) + \epsilon \sum_{|\alpha|=2k} \partial^\alpha m(\partial^\alpha w - \partial^\alpha m)\Big]\,\mathrm{d}x \geqslant 0,$$

(E3)
$$\int_\Omega \big(m - \operatorname{div}\big(mD_pH(x,Du)\big) - \phi\big)v\,\mathrm{d}x + \int_\Omega \Big[\epsilon\Big(uv + \sum_{|\alpha|=2k} \partial^\alpha u \partial^\alpha v\Big) + \epsilon(\xi + \Delta^{2k}\xi)v\Big]\,\mathrm{d}x = 0.$$

Remark 1.4. We observe that under the monotonicity property stated in Assumption 7, a weak solution in the sense of Definition 1.3 with $\epsilon = 0$ is also a weak solution in the sense of Definition 1.1. To see this, it suffices to observe that for all $(\eta,\tilde v)\in\mathcal{A}\times H^{2k}_h(\Omega)$, we have

$$\begin{aligned}\left\langle F\begin{bmatrix}\eta\\ \tilde v\end{bmatrix},\begin{bmatrix}\eta\\ \tilde v\end{bmatrix}-\begin{bmatrix}m\\ u\end{bmatrix}\right\rangle=&\left\langle F\begin{bmatrix}\eta\\ \tilde v\end{bmatrix}-F\begin{bmatrix}m\\ u\end{bmatrix},\begin{bmatrix}\eta\\ \tilde v\end{bmatrix}-\begin{bmatrix}m\\ u\end{bmatrix}\right\rangle\\ &+\int_\Omega\big(-u-H(x,Du)+g(m)-V\big)(\eta-m)\,\mathrm{d}x\\ &+\int_\Omega\big(m-\operatorname{div}\big(mD_pH(x,Du)\big)-\phi\big)(\tilde v-u)\,\mathrm{d}x.\end{aligned}$$

Hence, Assumption 7, (E2), and (E3), with $\eta=w$ and $v=\tilde v-u$, yield (D2).

Remark 1.5. Assume that (m,u) is a weak solution to Problem 2. Let $\Omega':=\{x\in\Omega\mid m(x)>0\}$, and fix $w_1\in C^\infty_c(\Omega')$. For all $\tau\in\mathbb{R}$ with $|\tau|$ sufficiently small, we have $w=m+\tau w_1\in\mathcal{A}$. Then, from (E2), we obtain

$$\tau\int_\Omega\big(-u-H(x,Du)+g(m)-V\big)w_1\,\mathrm{d}x+\tau\int_\Omega\Big(\epsilon m w_1+\epsilon\sum_{|\alpha|=2k}\partial^\alpha m\partial^\alpha w_1\Big)\,\mathrm{d}x\geqslant 0.$$

Because the sign of τ is arbitrary, we conclude that m satisfies

$$-u-H(x,Du)+g(m)-V(x)+\epsilon(m+\Delta^{2k}m)=0\ \text{ pointwise in }\Omega'.$$

Moreover, let $w_2\in C^\infty_c(\Omega)$ be such that $w_2\geqslant 0$; taking $w=m+\tau w_2\in\mathcal{A}$ in (E2) and integrating by parts, we obtain

$$\tau\int_\Omega\big(-u-H(x,Du)+g(m)-V(x)\big)w_2\,\mathrm{d}x+\tau\int_\Omega\big(\epsilon m w_2+\epsilon\Delta^{2k}m w_2\big)\,\mathrm{d}x\geqslant 0.$$

Thus,

$$-u-H(x,Du)+g(m)-V(x)+\epsilon(m+\Delta^{2k}m)\geqslant 0\ \text{ in the sense of distributions in }\Omega.$$

Also, by (E3), we have

$$m-\operatorname{div}\big(mD_pH(x,Du)\big)-\phi+\epsilon\big(u+\xi+\Delta^{2k}(u+\xi)\big)=0\ \text{ in the sense of distributions in }\Omega.$$

Under the same assumptions of Theorem 1.2, we prove the existence and uniqueness of the weak solutions to Problem 2.

Theorem 1.6. Consider Problem 2 and suppose that Assumptions 1–5 and 7 hold for some $\gamma>1$. Then, there exists a unique weak solution $(m,u)\in H^{2k}(\Omega)\times H^{2k}(\Omega)$ to Problem 2 in the sense of Definition 1.3.

We begin by addressing Theorem 1.6. First, in Section 3, we prove a priori estimates for classical and weak solutions to Problem 2. Second, in Sections 4 and 5, we introduce two auxiliary problems: a variational problem whose Euler–Lagrange equation is the first equation in Problem 2 and a problem associated with a bilinear form corresponding to the second equation in that problem. In each of these two sections, we show that there exists a unique solution. Finally, in Section 6, we prove Theorem 1.6 using Schaefer's fixed-point theorem together with the results established in Sections 3–5. Finally, the proof of Theorem 1.2 is given in Section 7 using Minty's method. We observe that some of the boundary conditions from Problem 2 are lost in the limiting process. This is due to the a priori estimates that are available, which do not give continuity for the trace of high-order derivatives on the boundary.

2. Assumptions

To prove our main results, we need the following assumptions on the functions that arise in Problems 1 and 2. The first three assumptions prescribe standard growth conditions on the Hamiltonian, H. For instance,

$$H(x,p)=a(x)(1+|p|^2)^{\frac{\gamma}{2}}+b(x)\cdot p,$$

where $a\in C(\overline{\Omega})$, $a(x)>0$ for all $x\in\overline{\Omega}$, and $b:\overline{\Omega}\to\mathbb{R}^d$ is a C^∞ function satisfies our assumptions.

Assumption 1. *There exists a constant, $C > 0$, such that, for all $(x, p) \in \Omega \times \mathbb{R}^d$,*

$$-H(x,p) + D_pH(x,p)\cdot p \geqslant \frac{1}{C}|p|^\gamma - C.$$

Assumption 2. *There exists a constant, $C > 0$, such that, for all $(x, p) \in \Omega \times \mathbb{R}^d$,*

$$H(x,p) \geqslant \frac{1}{C}|p|^\gamma - C.$$

Assumption 3. *There exists a constant, $C > 0$, such that, for all $(x, p) \in \Omega \times \mathbb{R}^d$,*

$$|D_pH(x,p)| \leqslant C|p|^{\gamma-1} + C.$$

The next three assumptions impose growth conditions on g. For example, these are satisfied for $g(m) = m^\alpha, \alpha > 0$.

Assumption 4. *The function g is increasing in $\mathbb{R}_0^+$. Moreover, for all $\delta > 0$, there exists a constant, $C_\delta > 0$, such that, for all $m \in L^1(\Omega)$ with $m \geqslant 0$ a.e. in Ω,*

$$\max\left\{\int_\Omega |g(m)|\,\mathrm{d}x, \int_\Omega m\,\mathrm{d}x\right\} \leqslant \delta\int_\Omega mg(m)\mathrm{d}x + C_\delta.$$

Assumption 5. *There exists a constant, $C > 0$, such that, for all $m \in L^1(\Omega)$ with $m \geqslant 0$ a.e. in Ω,*

$$\int_\Omega mg(m)\,\mathrm{d}x \geqslant -C.$$

Assumption 6. *If $\{m_j\}_{j=1}^\infty \subseteq L^1(\Omega)$ is a sequence of nonnegative functions satisfying*

$$\sup_{j\in\mathbb{N}}\int_\Omega m_j g(m_j)\,\mathrm{d}x < +\infty,$$

then there exists a subsequence of $\{m_j\}_{j=1}^\infty$ that converges weakly in $L^1(\Omega)$.

Remark 2.1. If g is an increasing function with $g \geqslant 0$ and $\lim_{t\to\infty} g(t) = \infty$, then g satisfies Assumption 6. This fact is a consequence of the *De la Vallée Poussin* lemma together with the Dunford–Pettis theorem; see, for instance, Theorems 2.29 and 2.54 in [18].

Our final assumption concerns the monotonicity of the functional F introduced in Definition 1.1. Monotonicity is the key ingredient in the proof of Theorem 1.2 from Theorem 1.6 through Minty's method.

Assumption 7. *The functional F introduced in Definition 1.1 is monotone with respect to the $L^2 \times L^2$-inner product; that is, for all (η_1, v_1), $(\eta_2, v_2) \in \mathcal{A} \times H_h^2(\Omega)$, F satisfies*

$$\left\langle F\begin{bmatrix}\eta_1\\v_1\end{bmatrix} - F\begin{bmatrix}\eta_2\\v_2\end{bmatrix}, \begin{bmatrix}\eta_1\\v_1\end{bmatrix} - \begin{bmatrix}\eta_2\\v_2\end{bmatrix}\right\rangle \geqslant 0.$$

3. Properties of weak solutions

In this section, we investigate properties of weak solutions, (m, u), to Problem 2. First, we prove an a priori estimate for classical solutions. We then verify that this estimate also holds for weak solutions. Finally, we show that u is bounded in $W^{1,\gamma}(\Omega)$, and that $(\sqrt{\epsilon}m, \sqrt{\epsilon}u)$ is bounded in $H^{2k}(\Omega) \times H^{2k}(\Omega)$. In Section 6, we combine these estimates with Schaefer's fixed-point theorem to prove the existence of weak solutions to Problem 2.

To simplify the notation, throughout this section, we use the same letter C to denote any positive *constant that depends only on the problem data*; that is, may depend on Ω, d, γ, H, V, ϕ, ξ, and h, on the constants in the Assumptions 1–5, or on universal constants such as the constant in Morrey's theorem or in the Gagliardo–Nirenberg interpolation inequality. In particular, these constants do not depend on the particular choice of solutions to Problem 2 nor they depend on ϵ.

Proposition 3.1. Consider Problem 2 and suppose that Assumptions 1–4 hold for some $\gamma > 1$. Then, there exists a positive constant, C, depending only on the problem data, such that any solution (m, u) to Problem 2 in the classical sense satisfies

$$\int_\Omega \Big(mg(m)+\frac{1}{C}m|Du|^\gamma+\frac{1}{C}\phi|Du|^\gamma\Big)\,\mathrm{d}x+\epsilon\int_\Omega\Big[m^2+u^2+\sum_{|\alpha|=2k}\big((\partial^\alpha m)^2+(\partial^\alpha u)^2\big)\Big]\,\mathrm{d}x \leqslant C. \tag{3.1}$$

Proof. Multiplying the first equation in (1.4) by $(m-\phi)$ and the second one by $(u-h)$, adding and integrating over Ω, and then using integration by parts and the boundary conditions, we have

$$\begin{aligned}\int_\Omega \Big[&mg(m)+m\big(-H(x,Du)+D_pH(x,Du)\cdot Du\big)+\phi H(x,Du)\\ &+\epsilon\Big(m^2+u^2+\sum_{|\alpha|=2k}\big((\partial^\alpha m)^2+(\partial^\alpha u)^2\big)\Big)\Big]\,\mathrm{d}x\\ =\int_\Omega\Big[&\phi g(m)+(\epsilon\phi+V+h)m+(\epsilon\xi h-V\phi-\phi h)+mD_pH(x,Du)\cdot Dh\\ &+\epsilon\Big(u(h-\xi)+\sum_{|\alpha|=2k}\big(\partial^\alpha\phi\partial^\alpha m+\partial^\alpha u(\partial^\alpha h-\partial^\alpha\xi)+\partial^\alpha\xi\partial^\alpha h\big)\Big)\Big]\,\mathrm{d}x.\end{aligned} \tag{3.2}$$

By Assumptions 1–3, Young's inequality, and the positivity of m and ϕ, we have

$$\begin{aligned}&\int_\Omega m\big(-H(x,Du)+D_pH(x,Du)\cdot Du\big)\,\mathrm{d}x\geqslant\int_\Omega\Big(\frac{m|Du|^\gamma}{C}-Cm\Big)\,\mathrm{d}x,\\ &\int_\Omega\phi H(x,Du)\,\mathrm{d}x\geqslant\int_\Omega\Big(\frac{\phi|Du|^\gamma}{C}-C\phi\Big)\,\mathrm{d}x,\\ &\int_\Omega mD_pH(x,Du)\cdot Dh\,\mathrm{d}x\leqslant\int_\Omega Cm(|Du|^{\gamma-1}+1)\leqslant\int_\Omega\Big(\frac{m|Du|^\gamma}{2C}+Cm\Big)\,\mathrm{d}x.\end{aligned}$$

Therefore, from Assumption 4, Young's inequality, and (3.2), we obtain

$$\begin{aligned}&\int_\Omega\Big[mg(m)+\frac{m|Du|^\gamma}{C}+\frac{\phi|Du|^\gamma}{C}+\frac{\epsilon}{2}\Big(m^2+u^2+\sum_{|\alpha|=2k}\big((\partial^\alpha m)^2+(\partial^\alpha u)^2\big)\Big)\Big]\,\mathrm{d}x\\ &\leqslant\int_\Omega\Big(\phi g(m)+Cm+\frac{m|Du|^\gamma}{2C}\Big)\,\mathrm{d}x+C\leqslant\frac{1}{2}\int_\Omega\Big(mg(m)+\frac{m|Du|^\gamma}{C}\Big)\,\mathrm{d}x+C,\end{aligned}$$

from which Proposition 3.1 follows. □

Corollary 3.2. Consider Problem 2, suppose that Assumptions 1–4 hold for some $\gamma > 1$, and let (m, u) be a weak solution to Problem 2 in the sense of Definition 1.3. Then, (m, u) satisfies the estimate (3.1).

Proof. Let (m, u) be a weak solution to Problem 2 in the sense of Definition 1.3. By taking $v = u - h \in H_0^{2k}(\Omega)$ and $w = \phi \in \mathcal{A}$ in (E2) and (E3), in Definition 1.3, and then summing the resulting inequalities, we obtain (3.2) with "=" replaced by "⩽". Thus, arguing as in Proposition 3.1, we conclude that (m, u) satisfies (3.1). □

Corollary 3.3. Consider Problem 2 and suppose that Assumptions 1–5 hold for some $\gamma > 1$. Then, there exists a positive constant, C, depending only on the problem data, such that any weak solution (m, u) to Problem 2 satisfies $\|u\|_{W^{1,\gamma}(\Omega)} \leqslant C$.

Proof. By Corollary 3.2, the positivity of m and ϕ, and Assumption 5, we have

$$\Big|\int_\Omega mg(m)\,\mathrm{d}x\Big|\leqslant C.$$

On the other hand, using (E2) in Definition 1.3 with $w := m+1 \in \mathcal{A}$, $\epsilon \leqslant 1$, Assumption 4, the previous estimate, and a weighted Young's inequality, we obtain

$$\int_\Omega H(x, Du)\,\mathrm{d}x \leqslant \int_\Omega \big(\epsilon m - u + g(m) - V\big)\,\mathrm{d}x \leqslant \sigma \int_\Omega |u|^\gamma\,\mathrm{d}x + C_\sigma,$$

where $\sigma > 0$ is arbitrary. Moreover, because $u - h = 0$ on $\partial\Omega$, Poincaré's inequality yields

$$\int_\Omega |u|^\gamma\,\mathrm{d}x \leqslant C \int_\Omega |Du|^\gamma\,\mathrm{d}x + C. \tag{3.3}$$

By invoking Assumption 2 and using the above estimates with σ chosen appropriately, we obtain

$$\int_\Omega |Du|^\gamma\,\mathrm{d}x \leqslant C.$$

Using (3.3) once more, we conclude that $\|u\|_{W^{1,\gamma}(\Omega)} \leqslant C$, where C is a positive constant that depends only on the problem data. □

Corollary 3.4. Consider Problem 2 and suppose that Assumptions 1–5 hold for some $\gamma > 1$. Then, there exists a positive constant, C, depending only on the problem data, such that any weak solution (m, u) to Problem 2 satisfies $\|\sqrt{\epsilon} m\|_{H^{2k}(\Omega)} + \|\sqrt{\epsilon} u\|_{H^{2k}(\Omega)} \leqslant C$.

Proof. Using Corollary 3.2, Assumption 5, and the positivity of m and ϕ, we obtain

$$\epsilon \int_\Omega \Big[m^2 + u^2 + \sum_{|\alpha|=2k} \big((\partial^\alpha m)^2 + (\partial^\alpha u)^2\big)\Big]\,\mathrm{d}x \leqslant C,$$

where C is a positive constant that depends only on the problem data. The conclusion follows from the Gagliardo–Nirenberg interpolation inequality. □

4. An auxiliary variational problem

Here, we introduce a variational problem whose Euler–Lagrange equation is the first equation in (1.4). We prove the existence of a unique minimizer, m, to this variational problem. We then investigate properties of m that enable us to prove the existence and uniqueness of a weak solution to Problem 2 in Section 6.

Given $(m, u) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m \geqslant 0$, let $I_{(m,u)} : H^{2k}(\Omega) \to \mathbb{R}$ be the functional defined, for $w \in H^{2k}(\Omega)$, by

$$I_{(m,u)}[w] := \int_\Omega \Big[\frac{\epsilon}{2}\Big(w^2 + \sum_{|\alpha|=2k} (\partial^\alpha w)^2\Big) + \big(-u - H(x, Du) + g(m) - V\big)w\Big]\,\mathrm{d}x. \tag{4.1}$$

Note that by Morrey's embedding theorem, $H^{2k-2}(\Omega)$ is compactly embedded in $C^{0,l}(\overline{\Omega})$ for some $l \in (0, 1)$. In particular, there exists a positive constant, $C = C(\Omega, k, d, l)$, such that, for all $\vartheta \in H^{2k-2}(\Omega)$, we have

$$\|\vartheta\|_{C^{0,l}(\overline{\Omega})} \leqslant C\|\vartheta\|_{H^{2k-2}(\Omega)}. \tag{4.2}$$

Next, we fix $(m_0, u_0) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m_0 \geqslant 0$, and set $I_0 = I_{(m_0,u_0)}$. We address the variational problem of finding $m \in \mathcal{A}$ such that

$$I_0[m] = \inf_{w\in\mathcal{A}} I_0[w], \tag{4.3}$$

where $\mathcal{A}$ is defined in (1.1).

Proposition 4.1. Let H, g, and V be as in Problem 2, and fix $(m_0, u_0) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m_0 \geqslant 0$. Then, there exists a unique $m \in \mathcal{A}$ satisfying (4.3).

Proof. Using Young's inequality, we obtain that $I_0[\cdot]$ is bounded from below by a constant only depending on ϵ, m, and u; hence, the infimum in (4.3) is finite.

Let $\{m_n\}_{n=1}^\infty \subset \mathcal{A}$ be a minimizing sequence for (4.3), and fix $\delta \in (0, 1)$. Then, there exists $N \in \mathbb{N}$ such that, for all $n \geqslant N$,

$$I_0[m_n] < \inf_{w\in\mathcal{A}} I_0[w] + \delta \leqslant I_0[0] + 1 = 1. \tag{4.4}$$

By (4.2), we have $m_0 \in C^{0,l}(\overline{\Omega})$ and $u_0 \in C^{1,l}(\overline{\Omega})$ for some $l \in (0,1)$, and $C_0 := \|-u_0 - H(x,Du_0)+g(m_0)-V\|_{L^\infty(\Omega)} \in \mathbb{R}$. Then, using Young's inequality and (4.4), for all $n \geqslant N$, we have

$$\frac{\epsilon}{2}\int_\Omega \Big[m_n^2 + \sum_{|\alpha|=2k}(\partial^\alpha m_n)^2\Big]\,\mathrm{d}x \leqslant \int_\Omega C_0 m_n\,\mathrm{d}x + 1 \leqslant \frac{\epsilon}{4}\int_\Omega m_n^2\,\mathrm{d}x + \frac{C_0^2}{\epsilon} + 1. \tag{4.5}$$

By the Gagliardo–Nirenberg interpolation inequality, we have

$$\|\partial^\alpha m_n\|^2_{L^2(\Omega)} \leqslant C\big(\|m_n\|^2_{L^2(\Omega)} + \|D^{2k}m_n\|^2_{L^2(\Omega)}\big), \tag{4.6}$$

where α is any multi-index such that $|\alpha| \leqslant 2k$. Hence, from (4.5) and (4.6), we conclude that $\{m_n\}_{n=1}^\infty$ is bounded in $H^{2k}(\Omega)$. Consequently, $m_n \rightharpoonup m$ weakly in $H^{2k}(\Omega)$ for some $m \in H^{2k}(\Omega)$, extracting a subsequence if necessary. Because $m_n \geqslant 0$, also $m \geqslant 0$; so, $m \in \mathcal{A}$.

Moreover, $m_n \to m$ in $L^2(\Omega)$ and $\|D^{2k}m\|^2_{L^2(\Omega)} \leqslant \liminf_{n\to\infty}\|D^{2k}m_n\|^2_{L^2(\Omega)}$; hence, $I_0[m] \leqslant \liminf_n I_0[m_n] = \inf I_0[w] \leqslant I_0[m]$, which shows that m is a minimizer of I_0 over $\mathcal{A}$.

We now prove uniqueness. Assume that $m, \widetilde{m} \in \mathcal{A}$ are minimizers of I_0 over $\mathcal{A}$ with $m \neq \widetilde{m}$. Then, $\frac{m+\widetilde{m}}{2} \in \mathcal{A}$ and, recalling that $m - \widetilde{m} \in C^0(\overline{\Omega})$, $\int_\Omega (m-\widetilde{m})^2\,\mathrm{d}x > 0$. Moreover,

$$\begin{aligned}
I_0\left[\frac{m+\widetilde{m}}{2}\right] &= \int_\Omega \bigg[\frac{\epsilon}{2}\Big(\Big(\frac{m+\widetilde{m}}{2}\Big)^2 + \sum_{|\alpha|=2k}\Big(\frac{\partial^\alpha m + \partial^\alpha \widetilde{m}}{2}\Big)^2\Big)\\
&\qquad + \big(-u_0 - H(x,Du_0) + g(m_0) - V\big)\Big(\frac{m+\widetilde{m}}{2}\Big)\bigg]\,\mathrm{d}x\\
&= \frac{1}{2}\int_\Omega \bigg[\frac{\epsilon}{2}\Big(m^2 + \sum_{|\alpha|=2k}(\partial^\alpha m)^2\Big) + \big(-u_0 - H(x,Du_0)+g(m_0)-V\big)m\bigg]\,\mathrm{d}x\\
&\qquad + \frac{1}{2}\int_\Omega \bigg[\frac{\epsilon}{2}\Big(\widetilde{m}^2 + \sum_{|\alpha|=2k}(\partial^\alpha \widetilde{m})^2\Big) + \big(-u_0 - H(x,Du_0)+g(m_0)-V\big)\widetilde{m}\bigg]\,\mathrm{d}x\\
&\qquad - \frac{\epsilon}{8}\int_\Omega \Big[(m-\widetilde{m})^2 + \sum_{|\alpha|=2k}(\partial^\alpha m - \partial^\alpha \widetilde{m})^2\Big]\,\mathrm{d}x\\
&< \frac{1}{2}I_0[m] + \frac{1}{2}I_0[\widetilde{m}] = \min_{w\in\mathcal{A}} I_0[w],
\end{aligned} \tag{4.7}$$

which contradicts the fact that $\frac{m+\widetilde{m}}{2} \in \mathcal{A}$. Thus, $m = \widetilde{m}$. □

Corollary 4.2. Let H, g, and V be as in Problem 2, fix $(m_0, u_0) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m_0 \geqslant 0$, and let $m \in \mathcal{A}$ be the unique solution to (4.3). Set $C_0 := \|u_0 - H(x,Du_0) + g(m_0) - V\|_{L^\infty(\Omega)}$. Then, there exists a positive constant, C, depending only on the problem data and on C_0, such that $\|m\|_{H^{2k}(\Omega)} \leqslant C$.

Proof. Because $I_0[m] \leqslant I_0[0]$, (4.5) and (4.6) hold with m_n replaced by m, from which Corollary 4.2 follows. □

Proposition 4.3. Let H, g, and V be as in Problem 2, fix $(m_0, u_0) \in H^{2k-2}(\Omega)\times H^{2k-1}(\Omega)$ with $m_0 \geqslant 0$, and let $m \in \mathcal{A}$ be the unique solution to (4.3). Then, for any $w \in \mathcal{A}$, m satisfies

$$\begin{aligned}
&\int_\Omega \big(-u_0 - H(x,Du_0) + g(m_0) - V\big)(w-m)\,\mathrm{d}x\\
&\qquad + \int_\Omega \Big[\epsilon m(w-m) + \epsilon \sum_{|\alpha|=2k}\partial^\alpha m(\partial^\alpha w - \partial^\alpha m)\Big]\,\mathrm{d}x \geqslant 0.
\end{aligned} \tag{4.8}$$

Proof. Let $w \in \mathcal{A}$. If $\tau \in [0,1]$, then $m + \tau(w-m) = (1-\tau)m + \tau w \in \mathcal{A}$; hence, the mapping $i : [0,1] \to \mathbb{R}$ given by

$$i[\tau] := I_0\big[m + \tau(w-m)\big]$$

is a well-defined C^1-function.

Because $i(0) \leqslant i(\tau)$ for all $0 \leqslant \tau \leqslant 1$, we have $i'(0) \geqslant 0$. On the other hand, for $0 < \tau \leqslant 1$, we have

$$\frac{1}{\tau}\big(i(\tau) - i(0)\big) = \int_\Omega \big(- u_0 - H(x, Du_0) + g(m_0) - V\big)(w - m)\,\mathrm{d}x + \epsilon \int_\Omega \Big[m(w - m) + \sum_{|\alpha|=2k} \partial^\alpha m(\partial^\alpha w - \partial^\alpha m)\Big]\,\mathrm{d}x + \tau\frac{\epsilon}{2} \int_\Omega \Big[(w - m)^2 + \sum_{|\alpha|=2k} (\partial^\alpha w - \partial^\alpha m)^2\Big]\,\mathrm{d}x.$$

Consequently, by letting $\tau \to 0^+$ in this equality and using $i'(0) \geqslant 0$, we obtain (4.8). □

Proposition 4.4. Let H, g, and V be as in Problem 2, fix $(m_0, u_0) \in H^{2k-2}(\Omega)\times H^{2k-1}(\Omega)$ with $m_0 \geqslant 0$, and let m be the unique solution of (4.3). Set $\Omega_1 = \{x \in \Omega \mid m(x) > 0\}$; then, m satisfies

$$-u_0 - H(x, Du_0) + g(m_0) - V + \epsilon(m + \Delta^{2k} m) = 0 \quad \text{pointwise in } \Omega_1$$

and

$$-u_0 - H(x, Du_0) + g(m_0) - V + \epsilon(m + \Delta^{2k} m) \geqslant 0 \quad \text{in the sense of distributions in } \Omega.$$

Proof. To prove Proposition 4.4, it suffices to argue as in Remark 1.5, invoking (4.8) instead of (E2) and recalling the embedding $H^{2k-2}(\Omega) \hookrightarrow C^{0,l}(\overline{\Omega})$ for some $l \in (0, 1)$. □

5. A problem given by a bilinear form

Here, we consider an auxiliary problem determined by a bilinear form related to the second equation in (1.4). Using the Lax–Milgram Theorem, we prove the existence and uniqueness of a solution, u, to this auxiliary problem, and we establish a uniform bound on u. These results are used in Section 6 to study the existence and uniqueness of a weak solution to Problem 2.

With H, ϕ, and ξ as in Problem 2, given $(m, u) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m \geqslant 0$, we define a bilinear form, $B : H_0^{2k}(\Omega) \times H_0^{2k}(\Omega) \to \mathbb{R}$, and a continuous linear functional, $f_{(m,u)} : L^2(\Omega) \to \mathbb{R}$, by setting, for $v_1, v_2 \in H_0^{2k}(\Omega)$ and $v \in L^2(\Omega)$,

$$\begin{aligned} B[v_1, v_2] &:= \int_\Omega \epsilon\Big(v_1 v_2 + \sum_{|\alpha|=2k} \partial^\alpha v_1 \partial^\alpha v_2\Big)\,\mathrm{d}x, \\ \big\langle f_{(m,u)}, v\big\rangle &:= \int_\Omega \big[- m + \operatorname{div}\big(mD_pH(x, Du)\big) + \phi - \epsilon(\xi + \Delta^{2k}\xi)\big]v\,\mathrm{d}x. \end{aligned} \tag{5.1}$$

Fix $(m_0, u_0) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m_0 \geqslant 0$, and set $f_0 := f_{(m_0,u_0)}$. We address next the problem of finding $u \in H_0^{2k}(\Omega)$ satisfying

$$B[u, v] = \langle f_0, v\rangle \quad \text{for all } v \in H_0^{2k}(\Omega). \tag{5.2}$$

Proposition 5.1. Let H, ϕ, and ξ be as in Problem 2, and fix $(m_0, u_0) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m_0 \geqslant 0$. Then, there exists a unique solution, $u \in H_0^{2k}(\Omega)$, to (5.2).

Proof. Because $(m_0, u_0) \in \big(H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)\big) \cap \big(C^{0,l}(\overline{\Omega}) \times C^{1,l}(\overline{\Omega})\big)$ for some $l \in (0, 1)$ (see (4.2)), we have $\big(-m_0 + \operatorname{div}\big(m_0D_pH(x, Du_0)\big) + \phi - \epsilon(\xi + \Delta^{2k}\xi)\big) \in L^2(\Omega)$. Hence, by Hölder's inequality, f_0 is bounded in $L^2(\Omega)$.

Next, we observe that by the Poincaré–Friedrichs inequality, we may endow $H_0^{2k}(\Omega)$ with the equivalent norm involving the $2k$-order derivatives only. Then, using Hölder's and Poincaré–Friedrichs's inequalities, we have $|B[v_1, v_2]| \leqslant c\|v_1\|_{H_0^{2k}(\Omega)}\|v_2\|_{H_0^{2k}(\Omega)}$ for all $v_1, v_2 \in H_0^{2k}(\Omega)$, where $c > 0$ is a constant independent of v_1 and v_2. Moreover, we clearly have $B[v_1, v_1] \geqslant \epsilon\|v_1\|^2_{H_0^{2k}(\Omega)}$ for all $v_1 \in H_0^{2k}(\Omega)$.

Therefore, by the Lax–Milgram Theorem, there exists a unique $u \in H_0^{2k}(\Omega)$ satisfying (5.2). □

Lemma 5.2. Let H, ϕ, and ξ be as in Problem 2, fix $(m_0, u_0) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m_0 \geqslant 0$, and let u be the unique solution to (5.2) in $H_0^{2k}(\Omega)$. Then, there exists a positive constant, C, depending only on the problem data, on $\|m_0\|_{H^{2k-2}(\Omega)}$, and $\|u_0\|_{H^{2k-1}(\Omega)}$, such that $\|u\|_{H^{2k}(\Omega)} \leqslant C$.

Proof. Arguing as in the proof of the Proposition 5.1, we have $c_0 := \|-m_0 + \operatorname{div}\big(m_0 D_p H(x, Du_0)\big) + \phi - \epsilon(\xi + \Delta^{2k}\xi)\|^2_{L^2(\Omega)} \in \mathbb{R}_0^+$. Then, using Young's inequality, we obtain

$$\epsilon\left(\|u\|^2_{L^2(\Omega)} + \|u\|^2_{H_0^{2k}(\Omega)}\right) = B[u,u] = \langle f_0, u\rangle \leqslant \epsilon\|u\|^2_{L^2(\Omega)} + \frac{c_0}{4\epsilon}.$$

Hence, $\|u\|^2_{H_0^{2k}(\Omega)} \leqslant c_0/(4\epsilon^2)$, and the conclusion follows by Poincaré's inequality. □

6. Proof of Theorem 1.6

This section is devoted to the proof of Theorem 1.6. First, using Schaefer's fixed-point theorem, we prove existence and uniqueness of a weak solution to (1.4) with $h \equiv 0$. Next, we apply this result to address the case of an arbitrary $h \in C^\infty(\overline{\Omega})$.

Let $\widetilde{\mathcal{A}}$ be the subset of $\mathcal{A}$ (see (1.1)) given by

$$\widetilde{\mathcal{A}} := \{w \in H^{2k-2}(\Omega) \mid w \geqslant 0\},$$

and consider the mapping $A : \widetilde{\mathcal{A}} \times H^{2k-1}(\Omega) \to \widetilde{\mathcal{A}} \times H^{2k-1}(\Omega)$ defined, for $(m_0, u_0) \in \tilde{A} \times H^{2k-1}(\Omega)$, by

$$A\begin{bmatrix} m_0 \\ u_0 \end{bmatrix} := \begin{bmatrix} m_0^* \\ u_0^* \end{bmatrix}, \tag{6.1}$$

where $m_0^* \in \mathcal{A}$ is the unique solution to (4.3) and $u_0^* \in H_0^{2k}(\Omega)$ is the unique solution to (5.2). In the following proposition, we show that A is continuous and compact.

Proposition 6.1. Let H, g, V, ϕ, and ξ be as in Problem 2. Then, the mapping $A : \widetilde{\mathcal{A}} \times H^{2k-1}(\Omega) \to \widetilde{\mathcal{A}} \times H^{2k-1}(\Omega)$ defined by (6.1) is continuous and compact.

Proof. We start by proving that A is continuous. Let (m_0, u_0), $(m_n, u_n) \in \widetilde{\mathcal{A}} \times H^{2k-1}(\Omega)$ be such that $m_n \to m_0$ in $H^{2k-2}(\Omega)$ and $u_n \to u_0$ in $H^{2k-1}(\Omega)$. We want to show that $m_n^* \to m_0^*$ in $H^{2k-2}(\Omega)$ and $u_n^* \to u_0^*$ in $H^{2k-1}(\Omega)$, where

$$\begin{bmatrix} m_0^* \\ u_0^* \end{bmatrix} = A\begin{bmatrix} m_0 \\ u_0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} m_n^* \\ u_n^* \end{bmatrix} = A\begin{bmatrix} m_n \\ u_n \end{bmatrix}.$$

Recalling (4.1) and (5.1), we set $I_n := I_{(m_n, u_n)}$ and $f_n := f_{(m_n, u_n)}$. By the definition of A, we have that (m_0^*, u_0^*) and (m_n^*, u_n^*) belong to $\mathcal{A} \times H_0^{2k}(\Omega)$ and satisfy, for all $v \in L^2(\Omega)$,

$$I_0[m_0^*] = \min_{w \in \mathcal{A}} I_0[w], \quad I_n[m_n^*] = \min_{w \in \mathcal{A}} I_n[w], \quad B[u_0^*, v] = \langle f_0, v\rangle, \quad B[u_n^*, v] = \langle f_n, v\rangle.$$

Because of (4.2), there exists a positive constant, $c > 0$, independent of $n \in \mathbb{N}$, such that

$$\sup_{n\in\mathbb{N}}(\|m_0\|_{L^\infty(\Omega)} + \|m_n\|_{L^\infty(\Omega)} + \|u_0\|_{W^{1,\infty}(\Omega)} + \|u_n\|_{W^{1,\infty}(\Omega)}) < c. \tag{6.2}$$

Then, because H, D_pH, and g are locally Lipschitz functions, we have

$$\tilde{c} := \max\left\{Lip(H; \Omega \times B(0,c)), Lip(D_pH; \Omega \times B(0,c)), Lip(g; B(0,c))\right\} \in \mathbb{R}_0^+. \tag{6.3}$$

Using the fact that m_n^* and m_0^* are minimizers, we have

$$I_0[m_0^*] + I_n[m_n^*] \leqslant I_0\left[\frac{m_0^* + m_n^*}{2}\right] + I_n\left[\frac{m_0^* + m_n^*}{2}\right].$$

Then, by exploiting the second equality in (4.7) in the preceding estimate first, and then using Young's inequality and (6.3), we obtain

$$\begin{aligned}&\int_\Omega \frac{\epsilon}{4}\Big[\big(m_0^*-m_n^*\big)^2+\sum_{|\alpha|=2k}\big(\partial^\alpha m_0^*-\partial^\alpha m_n^*\big)^2\Big]\,\mathrm{d}x\\ \leqslant&\int_\Omega \frac{1}{2}|m_0^*-m_n^*|\,\big(|u_0-u_n|+\big|H(x,Du_0)-H(x,Du_n)\big|+|g(m_0)-g(m_n)|\big)\,\mathrm{d}x\\ \leqslant&\int_\Omega\left[\frac{\epsilon}{8}(m_0^*-m_n^*)^2+\frac{6}{\epsilon}\big((u_0-u_n)^2+\tilde c^2|Du_0-Du_n|^2+\tilde c^2(m_0-m_n)^2\big)\right]\mathrm{d}x.\end{aligned}\tag{6.4}$$

Because $m_n\to m_0$ in $H^{2k-2}(\Omega)$ and $u_n\to u_0$ in $H^{2k-1}(\Omega)$, (6.4) yields

$$\lim_{n\to\infty}\|m_0^*-m_n^*\|_{L^2(\Omega)}=0,\qquad \lim_{n\to\infty}\sum_{|\alpha|=2k}\|\partial^\alpha m_0^*-\partial^\alpha m_n^*\|_{L^2(\Omega)}=0.$$

Then, by the Gagliardo–Nirenberg interpolation inequality, we have $m_n^*\to m_0^*$ in $H^{2k-2}(\Omega)$.

Next, we show that u_n^* converges to u_0^* in $H^{2k}(\Omega)$. By (5.1), (6.2), and (6.3), we have

$$\begin{aligned}&\epsilon\Big(\|u_0^*-u_n^*\|^2_{L^2(\Omega)}+\sum_{|\alpha|=2k}\|\partial^\alpha u_0^*-\partial^\alpha u_n^*\|^2_{L^2(\Omega)}\Big)\\ =\;&B[u_0^*-u_n^*,u_0^*-u_n^*]=\langle f_0-f_n,u_0^*-u_n^*\rangle\\ \leqslant&\int_\Omega\big[|m_n-m_0||u_0^*-u_n^*|+|m_0D_pH(x,Du_0)-m_nD_pH(x,Du_n)||Du_0^*-Du_n^*|\big]\,\mathrm{d}x\\ \leqslant&\int_\Omega\big[|m_n-m_0||u_0^*-u_n^*|+\tilde c|m_0-m_n||Du_0^*-Du_n^*|+c|Du_0-Du_n||Du_0^*-Du_n^*|\big]\,\mathrm{d}x.\end{aligned}\tag{6.5}$$

Using Gagliardo–Nirenberg interpolation inequality together with Young's inequality, we obtain from (6.5) that

$$\|u_0^*-u_n^*\|^2_{L^2(\Omega)}+\sum_{|\alpha|=2k}\|\partial^\alpha u_0^*-\partial^\alpha u_n^*\|_{L^2(\Omega)}\leqslant C\big(\|m_0-m_n\|^2_{L^2(\Omega)}+\|Du_0-Du_n\|^2_{L^2(\Omega)}\big)$$

for some constant $C>0$ independent of $n\in\mathbb{N}$. Arguing as before, we conclude that $u_n^*\to u_0^*$ in $H^{2k}(\Omega)$.

Finally, we address the compactness of A. We want to show that if $\{(m_n,u_n)\}_{n=1}^\infty$ is a bounded sequence in $\widetilde{\mathcal{A}}\times H^{2k-1}(\Omega)$, then $\{A(m_n,u_n)\}_{n=1}$ is pre-compact in $\widetilde{\mathcal{A}}\times H^{2k-1}(\Omega)$. This is an immediate consequence of (4.2), Corollary 4.2, Lemma 5.2, and the compact embedding $H^{2k}(\Omega)\times H^{2k}(\Omega)\hookrightarrow H^{2k-2}(\Omega)\times H^{2k-1}(\Omega)$ due to the Rellich–Kondrachov theorem. □

As we mentioned before, the existence of weak solutions to Problem 2 follows from Schaefer's fixed-point Theorem. We state next the version of this result used here, whose proof is a straightforward adaptation of the proof of Theorem 4, Section 9.2.2, in [16].

Theorem 6.2. *Let X be a convex and closed subset of a Banach space such that $0\in X$. Assume that $A:X\to X$ is a continuous and compact mapping such that the set*

$$\big\{w\in X|\ w=\lambda A[w]\ \text{for some}\ \lambda\in[0,1]\big\}$$

is bounded. Then, A has a fixed point.

Proposition 6.3. Consider Problem 2, let A be the mapping defined in (6.1), and suppose that Assumptions 1–5 hold for some $\gamma>1$. Then, there exists a unique weak solution, $(m,u)\in H^{2k}(\Omega)\times H^{2k}(\Omega)$, to Problem 2 with $h=0$ in the sense of Definition 1.3.

Proof. *(Existence)* Fix $\lambda\in[0,1]$, and let $(m_\lambda,u_\lambda)\in\widetilde{\mathcal{A}}\times H^{2k-1}(\Omega)$ be such that

$$\begin{bmatrix}m_\lambda\\u_\lambda\end{bmatrix}=\lambda A\begin{bmatrix}m_\lambda\\u_\lambda\end{bmatrix}.$$

If $\lambda = 0$, then $(m_\lambda, u_\lambda) = (0,0)$. Assume that $0 < \lambda \leqslant 1$; then, by the definition of A, Proposition 4.1, Proposition 4.3, and Proposition 5.1, we have $\frac{m_\lambda}{\lambda} \in \mathcal{A}$, $\frac{u_\lambda}{\lambda} \in H_0^{2k}(\Omega)$, and

$$\begin{aligned}&\int_\Omega \lambda\big(-u_\lambda - H(x,Du_\lambda) + g(m_\lambda) - V\big)(w - m_\lambda)\,\mathrm{d}x\\&\qquad + \int_\Omega \epsilon m_\lambda(w-m_\lambda) + \epsilon\sum_{|\alpha|=2k}\partial^\alpha m_\lambda(\partial^\alpha w - \partial^\alpha m_\lambda)\,\mathrm{d}x \geqslant 0,\\&\int_\Omega \Big[\lambda\big(m_\lambda - \operatorname{div}\big(m_\lambda D_pH(x,Du_\lambda)\big) - \phi\big)v\Big]\mathrm{d}x\\&\qquad + \int_\Omega \Big[\epsilon\Big(u_\lambda v + \sum_{|\alpha|=2k}\partial^\alpha u_\lambda \partial^\alpha v\Big) + \epsilon(\xi + \Delta^{2k}\xi)v\Big]\,\mathrm{d}x = 0,\end{aligned}$$

for all $w \in \mathcal{A}$ and $v \in H_0^{2k}(\Omega)$. Hence, arguing as in Corollary 3.2, we have

$$\begin{aligned}&\int_\Omega \lambda\left[m_\lambda g(m_\lambda) + m_\lambda|Du_\lambda|^\gamma + \phi|Du_\lambda|^\gamma\right]\,\mathrm{d}x\\&\qquad + \epsilon\int_\Omega \Big[m_\lambda^2 + u_\lambda^2 + \sum_{|\alpha|=2k}\big((\partial^\alpha m_\lambda)^2 + (\partial^\alpha u_\lambda)^2\big)\Big]\,\mathrm{d}x \leqslant C,\end{aligned}$$

where C is a positive constant independent of λ. Consequently, by Assumption 5 and the positivity of m_λ and ϕ, we have

$$\epsilon\int_\Omega \Big[m_\lambda^2 + u_\lambda^2 + \sum_{|\alpha|=2k}\big((\partial^\alpha m_\lambda)^2 + (\partial^\alpha u_\lambda)^2\big)\Big]\,\mathrm{d}x \leqslant C$$

where C is another positive constant independent of λ. Invoking the Gagliardo–Nirenberg interpolation inequality, we conclude that (m_λ, u_λ) is uniformly bounded in $H^{2k}(\Omega)\times H^{2k}(\Omega)$ with respect to λ. This fact and Proposition 6.1 allow us to use Theorem 6.2 to conclude that A has a fixed point, $(m,u) \in \tilde{\mathcal{A}} \times H^{2k-1}(\Omega)$. Finally, as before, by using the definition of A, Proposition 4.1, Proposition 4.3, and Proposition 5.1, we conclude that (m,u) belongs to $H^{2k}(\Omega)\times H^{2k}(\Omega)$ and satisfies (E1)–(E3) with $h = 0$.

(Uniqueness) Assume that there are two fixed points, (m_1,u_1) and (m_2,u_2). Taking $w = m_2$ in (E2) for (u_1,m_1) and $w = m_1$ in (E2) for (u_2,m_2), and then summing the resulting inequalities, we have

$$\begin{aligned}&\int_\Omega \left[u_1 - u_2 + H(x,Du_1) - H(x,Du_2) - (g(m_1) - g(m_2))\right](m_1 - m_2)\,\mathrm{d}x\\&\qquad - \int_\Omega \Big[\epsilon(m_1-m_2)^2 + \epsilon\sum_{|\alpha|=2k}(\partial^\alpha m_1 - \partial^\alpha m_2)^2\Big]\,\mathrm{d}x \geqslant 0.\end{aligned}\tag{6.6}$$

Because $u_1 - u_2 \in H_0^{2k}(\Omega)$, choosing $v = u_1 - u_2$ in (E3) for (u_1,m_1) and (u_2,m_2), and then subtracting the resulting equalities, we obtain

$$\begin{aligned}&\int_\Omega \Big(m_1 - m_2 - \operatorname{div}\big(m_1D_pH(x,Du_1) - m_2D_pH(x,Du_2)\big)\Big)(u_1-u_2)\,\mathrm{d}x\\&\qquad + \int_\Omega \Big[\epsilon(u_1-u_2)^2 + \epsilon\sum_{|\alpha|=2k}(\partial^\alpha u_1 - \partial^\alpha u_2)^2\Big] = 0.\end{aligned}\tag{6.7}$$

Subtracting (6.6) from (6.7) and using Assumption 7, we have

$$\begin{aligned}0 \geqslant &\int_\Omega \Big[\epsilon(m_1-m_2)^2 + \epsilon\sum_{|\alpha|=2k}(\partial^\alpha m_1 - \partial^\alpha m_2)^2 + \epsilon(u_1-u_2)^2 + \epsilon\sum_{|\alpha|=2k}(\partial^\alpha u_1 - \partial^\alpha u_2)^2\Big]\,\mathrm{d}x\\&+ \left\langle F\begin{bmatrix}m_1\\u_1\end{bmatrix} - F\begin{bmatrix}m_2\\u_2\end{bmatrix}, \begin{bmatrix}m_1\\u_1\end{bmatrix} - \begin{bmatrix}m_2\\u_2\end{bmatrix}\right\rangle \geqslant 0.\end{aligned}$$

Invoking Assumption 7 once more, we conclude that the integral in the preceding estimate is equal to zero, from which we the identity $(m_1,u_1) = (m_2,u_2)$ follows. □

Proof of Theorem 1.6. Define, for $x \in \Omega$ and $p \in \mathbb{R}^d$, $\widehat{H}(x,p) := H(x, p + Dh(x))$, $\widehat{V}(x) := V(x) + h(x)$, and $\widehat{\xi}(x) := \xi(x) + h(x)$.

Note that H satisfies Assumptions 1–3 for some $\gamma > 1$ if and only if $\widehat{H}$ satisfies Assumptions 1–3 for the same γ. Moreover, $(u, m) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ satisfies (E1)–(E3) if and only if $(\hat{u}, m) := (u - h, m) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ satisfies (E1)–(E3) with $h = 0$ and with H, V, and ξ replaced by $\widehat{H}$, $\widehat{V}$, and $\widehat{\xi}$, respectively.

To conclude, it suffices to invoke Proposition 6.3, which gives existence and uniqueness of a pair $(\hat{u}, m) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ satisfying (E1)–(E3) with 0 boundary conditions and with H, V, and ξ replaced by $\widehat{H}$, $\widehat{V}$, and $\widehat{\xi}$, respectively. □

7. Proof of Theorem 1.2

Here, we prove Theorem 1.2. First, we study a compactness property of the unique weak solution to Problem 2. Then, we introduce a linear functional, F_ϵ, corresponding to the equations (1.4) in Problem 2 and address its monotonicity. Finally, using Minty's method, we prove the existence of a weak solution to Problem 1.

Lemma 7.1. Consider Problem 2 and suppose that Assumptions 1–6 hold for some $\gamma > 1$. Let $(m_\epsilon, u_\epsilon) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ be the unique weak solution to Problem 2. Then, there exists $(m, u) \in L^1(\Omega) \times W^{1,\gamma}(\Omega)$ such that $m \geqslant 0$, $u = h$ on $\partial\Omega$ in the sense of traces, and (m_ϵ, u_ϵ) converges to (m, u) weakly in $L^1(\Omega) \times W^{1,\gamma}(\Omega)$ as $\epsilon \to 0$, extracting a subsequence if necessary.

Proof. The existence of $u \in W^{1,\gamma}(\Omega)$ as stated follows from the fact that $u_\epsilon = h$ on $\partial\Omega$ in the sense of traces and that, by Corollary 3.3, u_ϵ is uniformly bounded in $W^{1,\gamma}(\Omega)$ with respect to ϵ.

On the other hand, by Corollary 3.2 and the non-negativity of m_ϵ and ϕ, we have

$$\sup_{\epsilon \in (0,1)} \int_\Omega m_\epsilon g(m_\epsilon)\,\mathrm{d}x < \infty.$$

Therefore, by Assumption 6, there exists $m \in L^1(\Omega)$ such that $m_\epsilon \rightharpoonup m$ in $L^1(\Omega)$ as $\epsilon \to 0$, extracting a subsequence if necessary. Because $m_\epsilon \geqslant 0$, we have $m \geqslant 0$. □

Fix $(\eta, v) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$, let $F[\eta, v]$ be the functional introduced in (1.3), and let $F_\epsilon[\eta, v] : H^{2k}(\Omega) \times H^{2k}(\Omega) \to \mathbb{R}$ be the linear functional given by

$$\begin{aligned}\left\langle F_\epsilon \begin{bmatrix}\eta \\ v\end{bmatrix}, \begin{bmatrix}w_1 \\ w_2\end{bmatrix}\right\rangle := \left\langle F \begin{bmatrix}\eta \\ v\end{bmatrix}, \begin{bmatrix}w_1 \\ w_2\end{bmatrix}\right\rangle + \int_\Omega \Big(\epsilon \eta w_1 + \epsilon \sum_{|\alpha|=2k} \partial^\alpha \eta \partial^\alpha w_1\Big)\,\mathrm{d}x \\ + \int_\Omega \Big[\epsilon(v + \xi) w_2 + \epsilon \sum_{|\alpha|=2k} \partial^\alpha (v + \xi) \partial^\alpha w_2\Big]\,\mathrm{d}x.\end{aligned} \tag{7.1}$$

Next, we prove the monotonicity of F_ϵ over $\mathcal{A} \times H^{2k}_h(\Omega)$, where, we recall, $\mathcal{A}$ and $H^{2k}_h(\Omega)$ are given by (1.1)–(1.2).

Lemma 7.2. Let H, g, V, ϕ, ξ, and h be as in Problem 2, let F_ϵ be given by (7.1), and suppose that Assumption 7 holds. Then, for any (η_1, v_1), $(\eta_2, v_2) \in \mathcal{A} \times H^{2k}_h(\Omega)$, we have

$$\left\langle F_\epsilon \begin{bmatrix}\eta_1 \\ v_1\end{bmatrix} - F_\epsilon \begin{bmatrix}\eta_2 \\ v_2\end{bmatrix}, \begin{bmatrix}\eta_1 \\ v_1\end{bmatrix} - \begin{bmatrix}\eta_2 \\ v_2\end{bmatrix}\right\rangle \geqslant 0.$$

Proof. Let (η_1, v_1), $(\eta_2, v_2) \in \mathcal{A} \times H^{2k}_h(\Omega)$. Then, $v_1 - v_2 \in H^{2k}_0(\Omega)$; thus, using Assumption 7, we obtain

$$\begin{aligned}&\left\langle F_\epsilon \begin{bmatrix}\eta_1 \\ v_1\end{bmatrix} - F_\epsilon \begin{bmatrix}\eta_2 \\ v_2\end{bmatrix}, \begin{bmatrix}\eta_1 \\ v_1\end{bmatrix} - \begin{bmatrix}\eta_2 \\ v_2\end{bmatrix}\right\rangle \\ &\geqslant \int_\Omega \epsilon \Big[(\eta_1 - \eta_2)^2 + (v_1 - v_2)^2 + \sum_{|\alpha|=2k} \big((\partial^\alpha \eta_1 - \partial^\alpha \eta_2)^2 + (\partial^\alpha v_1 - \partial^\alpha v_2)^2\big)\Big]\,\mathrm{d}x \geqslant 0.\end{aligned}$$

□

Proof of Theorem 1.2. Let $(m_\epsilon, u_\epsilon) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ be the unique weak solution to Problem 2 in the sense of Definition 1.3. Fix $(\eta, v) \in \mathcal{A} \times H_h^{2k}(\Omega)$. By (E2) and (E3), we have

$$\left\langle F_\epsilon \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix} \right\rangle \geqslant 0.$$

Thus, by Lemma 7.2,

$$0 \leqslant \left\langle F_\epsilon \begin{bmatrix} \eta \\ v \end{bmatrix} - F_\epsilon \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix} \right\rangle \leqslant \left\langle F_\epsilon \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix} \right\rangle = \left\langle F \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix} \right\rangle + c_\epsilon, \tag{7.2}$$

where

$$\begin{aligned} c_\epsilon := & \int_\Omega \Big(\epsilon\eta(\eta - m_\epsilon) + \epsilon \sum_{|\alpha|=2k} \partial^\alpha \eta \partial^\alpha (\eta - m_\epsilon) \Big) \mathrm{d}x \\ & + \int_\Omega \Big[\epsilon(v+\xi)(v-u_\epsilon) + \epsilon \sum_{|\alpha|=2k} \partial^\alpha (v+\xi) \partial^\alpha (v-u_\epsilon) \Big] \mathrm{d}x. \end{aligned}$$

By Hölder's inequality and Corollary 3.4, we conclude that

$$\lim_{\epsilon \to 0} c_\epsilon = 0. \tag{7.3}$$

On the other hand, by Lemma 7.1, there exists $(m, u) \in L^1(\Omega) \times W^{1,\gamma}(\Omega)$ satisfying (D1) and such that (m_ϵ, u_ϵ) converges to (m, u) weakly in $L^1(\Omega) \times W^{1,\gamma}(\Omega)$ as $\epsilon \to 0$, extracting a subsequence if necessary. Then, using the definition of $F[\eta, v]$ (see (1.3)), we get

$$\lim_{\epsilon \to 0} \left\langle F \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix} \right\rangle = \left\langle F \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m \\ u \end{bmatrix} \right\rangle. \tag{7.4}$$

From (7.2), (7.3), and (7.4), we conclude that

$$\left\langle F \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m \\ u \end{bmatrix} \right\rangle \geqslant 0;$$

that is, (m, u) also satisfies (D2). Hence, (m, u) is a weak solution of Problem 1 in the sense of Definition 1.1. □

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