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ON THE UNIQUENESS OF MINIMIZERS FOR A CLASS OF VARIATIONAL PROBLEMS WITH POLYCONVEX INTEGRAND.

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Abstract. We prove existence and uniqueness of minimizers for a family of energy functionals that arises in Elasticity and involves polyconvex integrands over a certain subset of displacement maps. This work extends previous results by Awi and Gangbo to a larger class of integrands. First, we study these variational problems over displacements for which the determinant is positive. Second, we consider a limit case in which the functionals are degenerate. In that case, the set of admissible displacements reduces to that of incompressible displacements which are measure preserving maps. Finally, we establish that the minimizer over the set of incompressible maps may be obtained as a limit of minimizers corresponding to a sequence of minimization problems over general displacements provided we have enough regularity on the dual problems. We point out that these results defy the direct methods of the calculus of variations.

1. Introduction

We are interested in Euler-Lagrange equations, existence and uniqueness of minimizers for some problems in the vectorial calculus of variations emanating from elasticity theory. These variational problems are related to an open problem in Partial Differential Equations that we describe as follows : let $T > 0$ and let Ω and Λ be two open subsets of $\mathbb{R}^d$; suppose that $\mathbf{u}_0$ is a diffeomorphism between Ω and Λ; we seek $\mathbf{u} : \Omega \times (0,T) \longrightarrow \mathbb{R}^d$ such that $\mathbf{u}(\cdot,t)(\Omega) = \Lambda$ and

$$(1.1)\qquad \begin{cases} \mathbf{u}_t = \mathrm{div}_x D_\xi L(\nabla \mathbf{u}) & \text{on } \Omega \times (0,T), \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 & \text{on } \Omega, \end{cases}$$

in the sense of distributions. In (1.1), we assume that the map $\mathbb{R}^{d\times d} \ni \xi \mapsto L(\xi)$ is quasiconvex. We refer the reader to [6], [7], [11], [3], [13] and [2] for further details on these gradient flows. Understanding variational problems associated to the time-discretization of (1.1) is arguably an important step toward the construction of a solution. In that regard, several partial results are available in the literature (See for instance [6] and [7]).

In [3], the authors have focused on a class of Lagrangians that arises in elastic material. More precisely, they have considered polyconvex Lagrangians of the form $\xi \mapsto L(\xi) = f(\xi) + H(\det \xi)$. Here f is a $C^1(\mathbb{R}^d)$ strictly convex function with p-th order growth, and

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the map H is a $C^1(0,\infty)$ convex function that satisfies

$$\lim_{t\to 0^+} H(t) = \lim_{t\to\infty} \frac{H(t)}{t} = +\infty. \tag{1.2}$$

As a result, a variational problem emerges from the time discretization and has a relaxation that takes the general form :

$$\min\left\{\int_\Omega (f(\nabla u) + H(\beta) - F\cdot u)\,dx;\ (u,\beta)\in\mathscr{U}\right\} \tag{1.3}$$

where $F\in L^1(\Omega,\mathbb{R}^d)$ and

(1.4)
$$\mathscr{U} = \left\{(u,\beta) : u\in W^{1,p}(\Omega,\bar\Lambda),\ \beta:\Omega\to[0,\infty);\ \int_\Omega l(u)\beta dx = \int_\Lambda l(y)dy; \forall l\in C_c(\mathbb{R}^d)\right\}.$$

Although the existence of minimizers in (1.3) follows from the direct methods in the calculus of variations, the uniqueness is a rather challenging problem. Indeed, because of (1.2) and the non-convexity of the integrand, standard techniques in calculus of variations do not apply.

To bypass these difficulties, the authors of [3] have introduced a pseudo-projected gradient operator $\mathscr{U}_\mathcal{S}\ni u\mapsto\nabla_\mathcal{S} u$ defined as follows : for a given $u\in\mathscr{U}_\mathcal{S}$, the map $\nabla_\mathcal{S} u$ is the unique minimizer of

$$\int_\Omega f(G)dx$$

over

$$\mathcal{G}_\mathcal{S}(u) := \left\{G\in L^p(\Omega,\mathbb{R}^{d\times d}) : \int_\Omega u\operatorname{div}\varphi = -\int_\Omega\langle G,\varphi\rangle\ \forall\varphi\in\mathcal{S}\right\}.$$

Here, $\mathcal{S}$ is a finite-dimensional subspace of $W_0^{1,q}(\Omega,\mathbb{R}^{d\times d})$, q is the conjugate of p, $\mathscr{U}_\mathcal{S}$ is the set of all $u:\Omega\to\bar\Lambda$ measurable such that there exists a $c=c(u,\Omega,\Lambda)>0$ satisfying :

$$\left|\int_\Omega u\cdot\operatorname{div}\varphi\right| \le c\|\varphi\|_{L^q(\Omega,\mathbb{R}^{d\times d})}, \qquad \forall\varphi\in\mathcal{S}. \tag{1.5}$$

We point out that the pseudo-projected gradient operator depends also on f, though the dependance is not exhibited in its notation. As a first step to approaching (1.3), they have considered the following perturbed problem:

$$\inf\left\{\int_\Omega (f(\nabla_S u) + H(\beta) - F\cdot u)\,dx;\ (u,\beta)\in\mathscr{U}\right\}. \tag{1.6}$$

The choice of problem (1.6) is justified by the construction of a family of finite dimensional subspaces $\{\mathcal{S}^\tau\}_{\tau>0}$ dense in $W_0^{1,q}(\Omega,\mathbb{R}^{d\times d})$ such that for $u\in W^{1,p}(\Omega,\mathbb{R}^d)$, one has

$$\lim_{\tau\to\infty}\int_\Omega f(\nabla_{\mathcal{S}^\tau} u) = \int_\Omega f(\nabla u). \tag{1.7}$$

We note that a $L^p(\Omega,\ \mathbb{R}^d)-$bounded subset of $\mathscr{U}_\mathcal{S}$ whose image by the operator $\nabla_\mathcal{S}$ is bounded in $L^p(\Omega,\ \mathbb{R}^{d\times d})$ is not in general strongly pre-compact with respect to the $L^p(\Omega,\ \mathbb{R}^d)$ topology. As a result, compactness of level subsets of the functional in (1.6) can not be guaranteed. Nevertheless, the authors of [3] have successfully shown existence and, more importantly, uniqueness in (1.6) under the assumption that F is non-degenerate (see definition below). This condition of non-degeneracy for uniqueness is crucial in a similar problem, the so called

Brenier polar factorization, and more generally, in optimal transport problems. Confer [1], [4], [12], [9], [10] and [15].

In this paper, we investigate the role played by the strict convexity of f and the non-degeneracy of F in problem (1.6). More precisely, we impose less stringent conditions so that the map F is allowed to be degenerate or f is allowed to be merely convex. To deal with these weaker assumptions, we introduce a family of operators $\left\{V_{\mathcal{S}}^f : \mathcal{S} \subset W_0^{1,q}(\Omega, \mathbb{R}^d),\ f \text{ convex}\right\}$ defined by

$$W^{1,p}(\Omega, \mathbb{R}^d) \ni u \mapsto V_{\mathcal{S}}^f[u] := \sup_{\varphi \in \mathcal{S}} \int_\Omega \left(-u \operatorname{div} \varphi - f^*(\varphi)\right). \tag{1.8}$$

We note that the operator $V_{\mathcal{S}}^f$ is actually well defined on the set of measurable functions u defined from Ω to $\bar{\Lambda}$ when the set $\mathcal{S}$ is a finite dimensional nonempty set and the function f satisfies appropriate growth conditions. As a family, these operators extend the pseudo-projected gradient operators and the distributional gradient. Indeed, $V_{\mathcal{S}}^f[u] = \displaystyle\int_\Omega f(\nabla_{\mathcal{S}} u)$ if $\mathcal{S}$ is a finite dimensional subspace of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and $u \in \mathscr{U}_{\mathcal{S}}$ and furthermore $V_{\mathcal{S}}^f[u] = \displaystyle\int_\Omega f(\nabla u)$ if $\mathcal{S} = W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and $u \in W^{1,p}(\Omega, \mathbb{R}^d)$. These extensions are only valid under appropriate conditions on f. It is worth pointing out that if $f(\xi) = |\xi|$ and $\mathcal{S} = W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ then $V_{\mathcal{S}}^f(u)$ is nothing but the total variation of u on the set Ω. We show that for a collection of sets $\{\mathcal{S}_n\}_{n=1}^\infty$ of $W_0^{1,q}(\Omega, \mathbb{R}^d)$ satisfying *Hypothesis (H1)* or *Hypothesis (H2)* (see section 2), we have a convergence result in the same spirit as (1.7):

$$\lim_{\tau\to\infty} V_{\mathcal{S}_n}^f[u] = V_{W_0^{1,q}(\Omega,\mathbb{R}^d)}^f[u] \left(= \int_\Omega f(\nabla u)\right) \tag{1.9}$$

for any $u \in W^{1,p}(\Omega, \mathbb{R}^{d\times d})$ and appropriate conditions on f. We thus proceed to study a more general problem :

$$\inf_{(u,\beta)\in \mathscr{U}_{\mathcal{S}}^*} \left\{ V_{\mathcal{S}}^f[u] + \int_\Omega H(\beta) - F\cdot u dx \right\} \tag{1.10}$$

where $\mathcal{S}$ is an element of a collection of sets satisfying *Hypothesis (H1)* or *Hypothesis (H2)*, and

(1.11)

$$\mathscr{U}_{\mathcal{S}}^* = \left\{ (u,\beta) : u \in \mathscr{U}_{\mathcal{S}};\ \beta : \Omega \to [0,\infty);\ \int_\Omega l(u(x))\beta(x)dx = \int_\Omega l(y)dy\ \forall l \in C_c(\mathbb{R}^d) \right\}.$$

Sublevel sets of the integrand in (1.10) are not compact. Nor is f necessarily strictly convex. However, we show existence and uniqueness in Problem (1.10). In fact, this result holds for F non-degenerate as well as for a class of degenerate F provided that the set $\mathcal{S}$ is chosen accordingly (see Corollaries 3.6 and 3.7). Unlike optimal transport theory, this analysis suggests that the non-degeneracy condition is not essential for a uniqueness result in (1.3).

Existence and uniqueness results for Problem (1.10) are established thanks to the discovery of suitable dual problems. Indeed, call $\mathcal{C}$ the set of all functions (k, l) with $k,\ l : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ Borel measurable, finite at least at one point, and satisfying the relation $l \equiv \infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$ and such that

$$k(v) + tl(u) + H(t) \geq u \cdot v \quad \forall u, v \in \mathbb{R}^d, t > 0.$$

Let $\mathscr{A}$ be the set of (k, l, φ) such that $(k, l) \in \mathcal{C}$ and $\varphi \in \mathcal{S}$. Define the following functional over the set $\mathscr{A}$:

$$J(k, l, \varphi) := \int_\Omega k(F + \operatorname{div}\varphi) + \int_\Lambda l + \int_\Omega f^*(\varphi).$$

Next, assume that the map F and the set $\mathcal{S}$ are such that for all $\varphi \in \mathcal{S}$,

$$F + \operatorname{div}\varphi \text{ is non-degenerate.} \tag{1.12}$$

Then $-J$ admits a maximizer (k_0, l_0, φ_0) with k_0 convex and $diam(\Lambda)$-Lipschitz. As a consequence, Problem (1.10) admits a unique minimizer (u_0, β_0) and u_0 satisfies

$$\begin{cases} u_0 = & \nabla k_0(F + \operatorname{div}\varphi_0) \\ \varphi_0 \in & \Phi_{\mathcal{S}}(u_0). \end{cases} \tag{1.13}$$

Here, we have denoted by $\Phi_{\mathcal{S}}(u_0)$, the non-empty set of maximizers of problem (1.8) (see Proposition 2.8). To realize condition (1.12), we consider two distinct situations.

First, we assume that F has a countable range, thus degenerate. If $\mathcal{S}$ is an element of a collection of sets satisfying hypothesis (H2) then it holds that $F + \operatorname{div}\varphi$ is non degenerate.

Second, we assume F non-degenerate and $\mathcal{S}$ is a finite dimensional vector space, as in [3]. It holds again that $F + \operatorname{div}\varphi$ is non degenerate. However, unlike the hypotheses in [3], we have allowed the map f to be as singular as the map $\mathbb{R}^{d\times d} \ni \xi \mapsto |\xi|$.

We have also studied (1.10) when H is replaced by $H_0 : (0, \infty) \to \mathbb{R} \cup \{\infty\}$ defined by $H_0(1) = 0$ and $H_0(t) = \infty$ if $t \neq 1$. This case corresponds to the case of measure preserving maps. Note that H_0 is not even continuous. However, it may be obtained as a limit of functions H_n which are $C^1(0, \infty)$ convex functions and satisfy (1.2). We show that for such singular H_0, the corresponding problem

$$\inf_{u \in \mathscr{U}^1_{\mathcal{S}}} \left\{ V^f_{\mathcal{S}}[u] - \int_\Omega F \cdot u dx \right\} \tag{1.14}$$

with

$$\mathscr{U}^1_{\mathcal{S}} = \left\{ u \in \mathscr{U}_{\mathcal{S}} : \int_\Omega l(u(x))dx = \int_\Omega l(y)dy \;\forall l \in C_c(\mathbb{R}^d) \right\} \tag{1.15}$$

admits a unique minimizer. (See Theorem 4.3).

To obtain existence and uniqueness results in problem (1.14), we exploit a dual formulation and maximize $-J$ over the set that consists of (k, l, φ) such that $\varphi \in \mathcal{S}$ and $k,\ l : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ are Borel measurable, finite at least at one point, and satisfy the relations $l \equiv \infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$ and

$$k(v) + l(u) \geq u \cdot v \quad \forall u, v \in \mathbb{R}^d.$$

One shows that $-J$ admits a maximizer (k_0, l_0, φ_0) with k_0 convex and Lipschitz and the unique minimizer of problem (1.14) is u_0 given by

$$u_0 = \nabla k_0(F + \operatorname{div}\varphi_0).$$

Finally, we show convergence of a sequence of problems of the form (1.10) to (1.14). More precisely, we show that the minimizer of problem (1.14) may be obtained as limit of minimizers of problems of the form (1.10) provided that the dual problems admit regular enough maximizers. In fact, suppose the map F and the set $\mathcal{S}$ are such that for all $\varphi \in \mathcal{S}$, the map $F + \operatorname{div}\varphi$ is non-degenerate. For $(u, \beta) \in \mathscr{U}_{\mathcal{S}}$, define

$$I_n(u,\beta) = V^f_{\mathcal{S}}[u] + \int_\Omega H_n(\beta) - u \cdot F$$

and set

$$I_0(u) = V^f_{\mathcal{S}}[u] - \int_\Omega u \cdot F.$$

Thanks to Theorem 3.5, the problem

$$\inf_{(u,\beta)\in \mathscr{U}^*_{\mathcal{S}}} I_n(u,\beta) \tag{1.16}$$

admits a unique minimizer that we denote (u_n, β_n) with $u_n = \nabla k_n(F + \operatorname{div}\varphi_n)$ for some $k_n : \mathbb{R}^d \to \mathbb{R}$ convex and $\varphi_n \in \mathcal{S}$. Denote u_0 the unique minimizer of (1.14). If for all $n \in \mathbb{N}^*$ the map k_n is differentiable then the sequence $\{u_n\}_{n\in\mathbb{N}^*}$ converges almost everywhere to u_0 and in addition, the minima $\{I_n(u_n, \beta_n)\}_{n\in\mathbb{N}^*}$ converge to $I_0(u_0)$ (Cf. Theorem 4.7).

2. Preliminaries

Notation and definitions.

- Throughout this manuscript, Ω and $\Lambda \subset \mathbb{R}^d$ are two bounded convex sets; $r^* > 1$ is such that $B(0, 1/r^*) \subset \Lambda \subset B(0, r^*/2)$; $p \in (1, \infty)$ and q is its conjugate, that is, $p^{-1} + q^{-1} = 1$.
- Given $A \subset \mathbb{R}^d$, the indicator function of A is defined as

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \in A, \\ \infty & \text{otherwise.} \end{cases}$$

- For any subset $\mathcal{S}$ of $W^{1,q}_0(\Omega, \mathbb{R}^{d\times d})$, we denote by $\operatorname{span}(\mathcal{S})$ the linear subspace of $W^{1,q}_0(\Omega, \mathbb{R}^{d\times d})$ generated by $\mathcal{S}$.
- We denote by f^* the Legendre transform of the map $f : \mathbb{R}^{d\times d} \longrightarrow \mathbb{R}$ so that

$$f^*(\xi^*) = \sup_{\xi\in\mathbb{R}^{d\times d}} \{\xi \cdot \xi^* - f(\xi)\}.$$

- If $h : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{\infty\}$ is convex then the subdifferential $\partial h(x)$ of h at $x \in \operatorname{Dom}(h)$ is closed and convex. If $\partial h(x)$ non-empty we denote by $grad[h](x)$ the element of $\partial h(x)$ with minimum norm :

$$|grad[h](x)|^2 = \min\left\{|y|^2 : y \in \partial h(x)\right\}; \qquad x \in \operatorname{Dom}(h).$$

- Let $\mathcal{S} \subset W^{1,q}_0(\Omega, \mathbb{R}^{d\times d})$. We denote by $\mathscr{S}_f$ the set

$$\mathscr{S}_f := \left\{\varphi \in \mathcal{S} : \int_\Omega f^*(\varphi) \text{ is finite}\right\}. \tag{2.1}$$

- Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be measurable. We say that F is non-degenerate if for any $N \subset \mathbb{R}^d$ such that $\mathcal{L}^d(N) = 0$ we have $\mathcal{L}^d(F^{-1}(N)) = 0$.

Assumptions.

(A0) We additionally assume that the boundary of Ω is smooth and coincides the set of its extreme points.

(A1) The map $f : \mathbb{R}^{d\times d} \to \mathbb{R}$ is convex and satisfies the following three properties:

(i) There exist $a, b, c > 0$ such that for all $\xi \in \mathbb{R}^{d\times d}$,

$$c\frac{|\xi|^p}{p} + b \geq f(\xi) \geq a|\xi| - b \tag{2.2}$$

and for all $\xi^* \in \partial f(\xi)$,

$$|\xi^*|^q \leq c|\xi|^p + b. \tag{2.3}$$

(ii) The set $\mathscr{S}_f$ is non empty.

(iii) Either f is strictly convex or f is such that $\partial f^*(x^*)$ is non-empty and $grad[f^*](x^*) = 0$ for each $x^* \in \mathrm{Dom} f^*$.

(A2) The map H is $C^1(0, \infty)$, strictly convex, and such that

$$\lim_{t\to\infty} H(t) = \lim_{t\to 0^+} \frac{H(t)}{t} = +\infty.$$

(A3) The function F is measurable and belongs to $L^1(\Omega)$. Let $\mathcal{S} \subset W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. We say that F satisfies the condition $\mathbf{(ND)}_{\mathcal{S}}$ if

$$\mathrm{div}(\varphi) + F \ \text{ is non degenerate}$$

for all $\varphi \in \mathcal{S}$.

Remark 2.1. (i) As f satisfies (2.2), we have

$$-b + c^p\frac{|\xi^*|^q}{q} \leq f^*(\xi^*) \leq \chi_{\bar{B}(0,a)}(\xi^*) + b \tag{2.4}$$

for all $\xi^* \in \mathbb{R}^{d\times d}$.

(ii) If f is strictly convex then f^* is differentiable. In that case, $grad[f^*] = \nabla f^*$.

The following Lemma summarizes some elementary properties of H. We refer the reader to [3] or [2].

Lemma 2.2. *Assume* (**A3**) *holds. Then,*

(i) The map $H' : (0, \infty) \to \mathbb{R}$ is a strictly increasing bijection.

(ii) The Legendre transform H^* of H is a strictly increasing bijection from $\mathbb{R}$ to $\mathbb{R}$.

(iii) Let $g : \mathbb{R} \to \bar{\mathbb{R}}$ be defined by $g(s) = \alpha s - \beta H^*(s)$, with $\alpha, \beta > 0$. Then

$$\lim_{s\to-\infty} g(s) = \lim_{s\to\infty} g(s) = -\infty.$$

Define H_0 by

$$H_0(t) = \begin{cases} 0 & t = 1 \\ \infty & t \neq 1 \end{cases} \tag{2.5}$$

and, for $n \geq 1$,

$$H_n(t) = H(t) - H(1) + n(t-1)^2. \tag{2.6}$$

The following Lemma is straightforward.

Lemma 2.3. *Assume* (**A3**) *holds. Then,*

(i) There exists $\bar{H} \in \mathbb{R}$ *such that*

$$\bar{H} = \min_{t\in[0,\infty)} H(t).$$

(ii) The collection $\{H_n\}_{n=1}^{\infty}$ *is a non decreasing sequence of functions that converges point-wise to* H_0. *In addition, for all* $n \in \mathbb{N}^*$, *the map* H_n *is a* $C^1(0,\infty)$ *convex function that satisfies*

$$\lim_{t\to 0^+} H_n(t) = \lim_{t\to\infty} \frac{H_n(t)}{t} = +\infty.$$

(iii) Let $t > 0$. *If* $\{H_n(t)\}_{n=1}^{\infty}$ *is uniformy bounded above by a constant* c_0 *then*

$$n(t-1)^2 \le c_0 + H(1) - \bar{H}.$$

Hypothesis on the underlying sets of pseudo-gradients. We recall that in [3], the construction of $\nabla_{\mathcal{S}^\tau} u$ has relied on hypothesis on the underlying sets $\mathcal{S}^\tau$ that we summarize in *Hypothesis (H1)* below.

Hypothesis (H1).

A collection $\{\mathfrak{A}_n\}_{n=1}^{\infty}$ of subsets of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ satisfies *Hypothesis (H1)* if

(i) $\mathfrak{A}_n$ of a finite dimensional subspace of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ for each $n \in \mathbb{N}^*$.
(ii) The map $\nabla\varphi$ has a countable range whenever $\varphi \in \mathfrak{A}_n$, for any $n \in \mathbb{N}^*$.
(iii) The set $\cup_{n\in\mathbb{N}^*}\mathfrak{A}_n$ is dense in $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$.
(iv) For $i \le j$, we have the inclusion $\mathfrak{A}_i \subset \mathfrak{A}_j$.

An explicit construction of sets satisfying *Hypothesis (H1)* is provided in [3]. Here, we build on the conditions of *Hypothesis (H1)* and we relax conditions on the underlying sets:

Hypothesis (H2).

A collection $\{\mathfrak{Q}_n\}_{n=1}^{\infty}$ of subsets of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ satisfies *Hypothesis (H2)* if

(i) Span($\mathfrak{Q}_n$) is of finite dimension and $\mathfrak{Q}_n$ is a non-empty closed and convex subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$.
(ii) The map $\operatorname{div}\varphi$ is non-degenerate whenever $\varphi \in \mathfrak{Q}_n$, for any $n \in \mathbb{N}^*$.
(iii) The set $\cup_{n\in\mathbb{N}^*}\mathfrak{Q}_n$ is dense in $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$.
(iv) For $i \le j$, the inclusion $\mathfrak{Q}_i \subset \mathfrak{Q}_j$ holds.

The following result is found in Theorem 2.57 in [5]. We reproduce it here for the reader's convenience.

Theorem 2.4. *Let* $E \subset \mathbb{R}^d$ *be a non-empty compact set and* E_{ext} *be the set of its extreme points. Then, there exists* $\varphi_E : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{\infty\}$ *(called the Choquet function) a convex function, strictly convex on* E, *so that*

$$E_{ext} = \{x \in E : \varphi_E(x) = 0\}.$$

$$\varphi_E(x) \le 0 \Leftrightarrow x \in E.$$

The next lemma asserts that a collection of sets can be constructed to satisfy *Hypothesis (H2)*. The construction uses the previous theorem.

Lemma 2.5. *Assume* (**A0**) *holds. Then, there exists a collection of sets* $\{\mathfrak{Q}_n\}_{n=1}^{\infty}$ *satisfying the requirements of Hypothesis (H2).*

Remark 2.6. *The condition* (**A0**) *in Lemma 2.5 is only needed for requirement (ii) of Hypothesis (H2).*

Proof. Suppose ψ is a Choquet function finite on $\bar{\Omega}$. Then, as the boundary of Ω coincides with the set of extreme points of Ω, ψ vanishes on $\partial\Omega$. Let $\varphi_0 : \Omega \to \mathbb{R}^{d\times d}$ be defined by

$$\varphi_0 = \begin{pmatrix} \psi & 0 & \cdots & 0 \\ 0 & \psi & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi \end{pmatrix}.$$

As ψ is convex on $\bar{\Omega}$, $\varphi_0 \in W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and it follows that $\operatorname{div} \varphi_0 = \nabla\psi$. Thus, for almost every x in Ω, we have

$$\det(\nabla(\operatorname{div} \varphi_0)(x)) = \det(\nabla^2\psi(x)) > 0.$$

Thanks to Lemma 5.5.3 in [1], the map $\operatorname{div} \varphi_0$ is non-degenerate. Let $\{\mathfrak{A}_n\}_{n=1}^{\infty}$ be a collection of sets satisfying Hypothesis (H1). One readily checks that the family of sets defined by

$$\mathfrak{Q}_n = \left\{ \varphi + \epsilon\varphi_0 : \varphi \in \mathfrak{A}_n ; \epsilon \geq \frac{1}{n} \right\}$$

for $n \in \mathbb{N}^*$, satisfies hypothesis (H2).

□

Special displacements. To $\mathcal{S} \subset W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ we associate $\mathscr{U}_{\mathcal{S}}$, the set of all $u : \Omega \to \bar{\Lambda}$ measurable such that there exists $\bar{c} = \bar{c}(u, \Omega, \Lambda) > 0$ satisfying :

$$\left| \int_\Omega u \cdot \operatorname{div} \varphi \right| \leq \bar{c} \|\varphi\|_{L^q(\Omega,\mathbb{R}^{d\times d})} \quad \forall \varphi \in \mathcal{S}. \tag{2.7}$$

If $\operatorname{span}(\mathcal{S})$ is of finite dimension then $\mathscr{U}_{\mathcal{S}}$ is the set of all measurable maps. At any rate, $\mathscr{U}_{\mathcal{S}}$ contains $W^{1,p}(\Omega, \mathbb{R}^d)$. We introduce the following set

$$\mathscr{U}_{\mathcal{S}}^1 = \left\{ u \in \mathscr{U}_{\mathcal{S}} : \int_\Omega l(u(x))dx = \int_\Omega l(y)dy \ \forall l \in C_c(\mathbb{R}^d) \right\}$$

and

$$\mathscr{U}_{\mathcal{S}}^* = \left\{ (u, \beta) : u \in \mathscr{U}_{\mathcal{S}};\ \beta : \Omega \to [0, \infty);\ \int_\Omega l(u(x))\beta(x)dx = \int_\Omega l(y)dy \ \forall l \in C_c(\mathbb{R}^d) \right\}.$$

Notice that $\mathscr{U}_{\mathcal{S}}^1 = \{u \in \mathscr{U}_{\mathcal{S}} : (u, 1) \in \mathscr{U}_{\mathcal{S}}^*\}$. This corresponds to measure preserving displacements.

Extended pseudo-projected gradient. Let $\mathcal{S} \subset W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and $u \in \mathscr{U}_{\mathcal{S}}$. Define

$$\mathcal{G}_{\mathcal{S}}(u) := \left\{ G \in L^p(\Omega, \mathbb{R}^{d\times d}) : \int_\Omega u \operatorname{div} \varphi = -\int_\Omega \langle G, \varphi \rangle \ \forall \varphi \in \mathcal{S} \right\}.$$

Consider the operator

$$V_{\mathcal{S}}^f(u) := \sup_{\varphi\in\mathcal{S}} \int_\Omega \left(-u \operatorname{div} \varphi - f^*(\varphi)\right) = \sup_{\varphi\in\mathscr{S}_f} \int_\Omega \left(-u \operatorname{div} \varphi - f^*(\varphi)\right). \tag{2.8}$$

We denote by $\Phi_S(u)$ the set of maximizers of Problem (2.8). The following results are essentially found in Proposition 3.1 in [3].

Proposition 2.7. *Suppose that $\mathcal{S}$ is a finite dimensional subspace of $W_0^{1,q}(\Omega,\mathbb{R}^{d\times d})$ and f is C^1 and strictly convex. Suppose, in addition that there exist constants $c_1, c_2, c_3 > 0$ such that*

$$\begin{aligned} -c_3 + c_2|\xi|^p \le f(\xi) &\le c_3 + c_1|\xi|^p \\ |Df(\xi)| &\le c_3 + c_1|\xi|^{p-1} \\ |Df^*(\xi)| &\le c_3 + c_1|\xi|^{q-1} \end{aligned}$$

for all $\xi \in \mathbb{R}^{d\times d}$. Then, there exists a unique map denoted $\nabla_{\mathcal{S}} u$ that minimizes

$$\inf_{G\in\mathcal{G}_{\mathcal{S}}(u)} \int_\Omega f(G)dx.$$

Moreover, $\nabla_{\mathcal{S}} u$ uniquely satisfies $G \in \mathcal{G}_{\mathcal{S}}(u)$ and $Df(G) \in \mathcal{S}$.

In the next Proposition, we establish similar results as in Proposition 2.7 but under weaker assumptions on $\mathcal{S}$ and f.

Proposition 2.8. *Assume $(\mathbf{A_1})$ holds. Assume $\mathcal{S}$ is a finite dimensional non-empty closed and convex subset of $W_0^{1,q}(\Omega,\mathbb{R}^{d\times d})$ and let $u \in \mathscr{U}_{\mathcal{S}}$.*

(1) For all $G \in \mathcal{G}_{\mathcal{S}}(u)$, $\varphi \in \mathcal{S}$, we have

$$\int_\Omega f(G) \ge \int_\Omega \left(-u \operatorname{div}\varphi - f^*(\varphi)\right).$$

(2) The supremum in problem (2.8) is attained.

(3) A map $\bar\varphi$ belongs to $\Phi_{\mathcal{S}}(u)$ if and only if $\bar\varphi$ belongs to $\mathscr{S}_f$ and

$$\int_\Omega \left(grad[f^*](\bar\varphi)\cdot(\varphi-\bar\varphi) + u\cdot(\operatorname{div}\varphi - \operatorname{div}\bar\varphi)\right)dx \ge 0$$

for all $\varphi \in \mathscr{S}_f$.

(4) Suppose the hypotheses of Lemma 2.7 are satisfied. Then we have

$$\int_\Omega f(\nabla_{\mathcal{S}} u) = V_{\mathcal{S}}^f(u)$$

and $\Phi_{\mathcal{S}}(u) = \{Df(\nabla_{\mathcal{S}} u)\}$.

Proof. 1.) Let $\varphi \in \mathcal{S}$ and $G \in \mathcal{G}_{\mathcal{S}}(u)$, By using the Legendre transformation,

$$\int_\Omega f(G) \ge \int_\Omega G\cdot\varphi - f^*(\varphi)dx = \int_\Omega u\cdot\operatorname{div}\varphi - f^*(\varphi)dx.$$

2.) Let $\varphi \in \mathcal{S}$. We use (2.7) and (2.4) to get

$$\begin{aligned} \int_\Omega \left(u\operatorname{div}\varphi + f^*(\varphi)\right) &\ge -\bar c\|\varphi\|_{L^q(\Omega,\mathbb{R}^{d\times d})} + \int_\Omega f^*(\varphi) \\ &\ge -\bar c\|\varphi\|_{L^q(\Omega,\mathbb{R}^{d\times d})} + q^{-1}c^{-q}\|\varphi\|^q_{L^q(\Omega,\mathbb{R}^{d\times d})}. \end{aligned} \tag{2.9}$$

In light of (2.9), $q > 1$ implies that the map $\mathscr{S}_f \ni \varphi \mapsto T(\varphi) := \int_\Omega (u\operatorname{div}\varphi + f^*(\varphi))$ is L^q-coercive. Moreover, the convexity of f^* guarantees that T is weakly lower semi-continuous. The direct methods of the calculus of variations thus yield the existence of a maximizer in problem (2.8).

3.) Let $\bar{\varphi} \in \Phi_{\mathcal{S}}(u)$ so that $\bar{\varphi} \in \mathscr{S}_f$. Let $\varphi \in \mathscr{S}_f$ and $\epsilon \in (0,1)$. The convexity of f^* ensures that $\bar{\varphi} + \epsilon(\varphi - \bar{\varphi}) \in \mathscr{S}_f$ and the maximality property of $\bar{\varphi}$ implies that

$$\int_\Omega u \cdot \operatorname{div} \bar{\varphi} + f^*(\bar{\varphi}) \leq \int_\Omega u \cdot (\operatorname{div} \bar{\varphi} + \epsilon \operatorname{div}(\varphi - \bar{\varphi})) + f^*(\bar{\varphi} + \epsilon(\varphi - \bar{\varphi})). \tag{2.10}$$

We rewrite (2.10), in turn, as

$$\int_\Omega \frac{f^*(\bar{\varphi} + \epsilon(\varphi - \bar{\varphi})) - f^*(\bar{\varphi})}{\epsilon} + u \cdot \operatorname{div}(\varphi - \bar{\varphi}) dx \geq 0. \tag{2.11}$$

Thus, as $\epsilon \to 0$, relation (2.11) yields

$$\int_\Omega \big(grad[f^*](\bar{\varphi}) \cdot (\varphi - \bar{\varphi}) + u \cdot \big(\operatorname{div} \varphi - \bar{\operatorname{div}} \varphi \big) \big) \, dx \geq 0.$$

One shows the converse implication by first noticing that as f^* is convex, the range of the map $grad[f^*](\bar{\varphi})$ lies in the sub-differential of f^* so that $f^*(\varphi) - f^*(\bar{\varphi}) \geq grad[f^*](\bar{\varphi})\,(\varphi - \bar{\varphi})$. The rest of the argument is straightforward.

4.) Thanks to lemma 2.7, $Df(\nabla_{\mathcal{S}} u) \in \mathcal{S}$. Next, we set $\varphi_0 := Df(\nabla_{\mathcal{S}} u)$. By definition of f^*,

$$f(\nabla_{\mathcal{S}} u) + f^*(\varphi) \geq \varphi \cdot \nabla_{\mathcal{S}} u$$

for all $\varphi \in \mathcal{S}$. As f is convex and $\varphi_0 = Df(\nabla_{\mathcal{S}} u)$, we have

$$f(\nabla_{\mathcal{S}} u) + f^*(\varphi_0) = \varphi_0 \cdot \nabla_{\mathcal{S}} u,$$

Thus,

$$\int_\Omega f(\nabla_{\mathcal{S}} u) \geq \int_\Omega \varphi \cdot \nabla_{\mathcal{S}} u - \int_\Omega f^*(\varphi) = \int_\Omega -u \operatorname{div} \varphi - \int_\Omega f^*(\varphi)$$

and

$$\int_\Omega f(\nabla_{\mathcal{S}} u) = \int_\Omega \varphi_0 \cdot \nabla_{\mathcal{S}} u - \int_\Omega f^*(\varphi_0) = \int_\Omega -u \operatorname{div} \varphi_0 - \int_\Omega f^*(\varphi_0).$$

We deduce that $\varphi_0 \in \Phi_{\mathcal{S}}(u)$. Since f is strictly convex, we conclude $\Phi_{\mathcal{S}}(u) = \{Df(\nabla^{\mathcal{S}} u)\}$ and moreover, $\displaystyle\int_\Omega f(\nabla_{\mathcal{S}} u) = V_{\mathcal{S}}^f(u)$.

□

In the next Proposition, we establish a convergence result in the spirit of (1.7). We also connect the operator $V_{\mathcal{S}}^f$ with the usual notions of gradient and total variation.

Proposition 2.9. *Assume* $(\mathbf{A_1})$ *holds. Assume that* $\mathcal{S}_n$ *is a finite dimensional non-empty closed and convex subset of* $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ *for each* $n \geq 1$. *The following holds.*

(1) If $\{\mathcal{S}_n\}_{n=1}^\infty$ *is a monotonically increasing family of subsets of some set* $\mathcal{S}_0$ *and* $\cup_{n\in\mathbb{N}^*}\mathcal{S}_n$ *is dense in* $\mathcal{S}_0$ *with respect to the* $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ *norm then*

$$\lim_{n\to\infty} V_{\mathcal{S}_n}^f[u] = V_{\mathcal{S}_0}^f[u]$$

for any $u \in \mathscr{U}_{\mathcal{S}}$.

(2) If $\mathcal{S} = W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ *and* $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ *then* $V_{\mathcal{S}}^f[u] = \displaystyle\int_\Omega f(\nabla u) dx$.

(3) Assume $u \in BV(\Omega, \mathbb{R}^{d\times d})$ *and* $f(\xi) = |\xi|$ *for all* $\xi \in \mathbb{R}^{d\times d}$. *If* $\mathcal{S} = W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ *then* $V_{\mathcal{S}}^f[u]$ *is the total variation of* u.

Remark 2.10. *A consequence of Proposition 2.9 is the following : If the sequence of sets* $\{\mathcal{S}_n\}_{n\in\mathbb{N}^*}$ *is monotonically increasing to* $W_0^{1,q}(\Omega,\mathbb{R}^{d\times d})$ *and* $u\in W^{1,p}(\Omega,\mathbb{R}^d)$ *we have*

$$\lim_{n\to\infty} V^f_{\mathcal{S}_n}[u] = \int_\Omega f(\nabla u)dx.$$

Proof. 1.) Recall that

$$V^f_{\mathcal{S}_n}[u] = \sup_{\varphi\in\mathcal{S}_n}\left\{\int_\Omega (-u\cdot\operatorname{div}\varphi - f^*(\varphi))\,dx\right\}.$$

As $\{\mathcal{S}_n\}_{n=1}^\infty$ is a monotonically increasing, $\lim\limits_{n\to\infty} V^f_{\mathcal{S}_n}[u]$ exists. Moreover, since $\mathcal{S}_n\subset\mathcal{S}_0$ for all $n\geq 1$,

$$\lim_{n\to\infty} V^f_{\mathcal{S}_n}[u] \leq V^f_{\mathcal{S}_0}[u]. \tag{2.12}$$

Let $\epsilon>0$ and choose $\varphi^\epsilon\in\mathcal{S}_0$ such that

$$V^f_{\mathcal{S}_0}[u] \leq \epsilon + \int_\Omega (-u\cdot\operatorname{div}\varphi^\epsilon - f^*(\varphi^\epsilon))\,dx.$$

Let $\{\varphi^\epsilon_n\}_{n\in\mathbb{N}^*}$ be a sequence converging to φ^ϵ in $W_0^{1,q}(\Omega,\mathbb{R}^{d\times d})$ and such that $\varphi^\epsilon_n\in\mathcal{S}_n$ for all $n\in\mathbb{N}^*$. Then

$$\begin{aligned}
V^f_{\mathcal{S}_0}[u] \leq& \epsilon + \int_\Omega (-u\cdot\operatorname{div}\varphi^\epsilon - f^*(\varphi^\epsilon))\,dx\\
=&\epsilon + \lim_{n\to\infty}\int_\Omega (-u\cdot\operatorname{div}\varphi^\epsilon_n - f^*(\varphi^\epsilon_n))\,dx\\
\leq&\epsilon + \limsup_{n\to\infty} V^f_{\mathcal{S}_n}[u]\\
=&\epsilon + \lim_{n\to\infty} V^f_{\mathcal{S}_n}[u].
\end{aligned}$$

As ϵ is arbitrary, we have

$$\lim_{n\to\infty} V^f_{\mathcal{S}_n}[u] \geq V^f_{\mathcal{S}_0}[u]. \tag{2.13}$$

From (2.12) and (2.13), we conclude that $\lim\limits_{n\to\infty} V^f_{\mathcal{S}_n}[u] = V^f_{\mathcal{S}_0}[u]$.
2.) One has

$$\begin{aligned}
V^f_{\mathcal{S}}[u] =& \sup_{\varphi\in W_0^{1,q}(\Omega,\mathbb{R}^{d\times d})}\left\{\int_\Omega (-u\cdot\operatorname{div}\varphi - f^*(\varphi))\,dx\right\}\\
=& \sup_{\varphi\in W_0^{1,q}(\Omega,\mathbb{R}^{d\times d})}\left\{\int_\Omega (\nabla u\cdot\varphi - f^*(\varphi))\,dx\right\}\\
\leq& \int_\Omega f(\nabla u)dx.
\end{aligned}$$

The inequality above is obtained by using the definition of the Legendre transform f^* of f. Let $\bar\varphi\in\partial f(\nabla u)$. Then $f^*(\bar\varphi)+f(\nabla u)=\nabla u\cdot\bar\varphi$. Thanks to the growth conditions (2.2) and

(2.3) on f, it holds that $\bar{\varphi} \in L^q(\Omega, \mathbb{R}^{d\times d})$. Since $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ is dense in $L^q(\Omega, \mathbb{R}^{d\times d})$ for the $L^q(\Omega, \mathbb{R}^{d\times d})$ norm, we get

$$\begin{aligned}\int_\Omega f(\nabla u) &= \int_\Omega (\nabla u \cdot \bar{\varphi} - f^*(\bar{\varphi})) \\ &\leq \sup_{\varphi \in W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})} \left\{ \int_\Omega (\nabla u \cdot \varphi - f^*(\varphi))\, dx \right\} \\ &= V_{\mathcal{S}}^f[u].\end{aligned}$$

We conclude that $V_{\mathcal{S}}^f[u] = \displaystyle\int_\Omega f(\nabla u)$.

4.) The total variation of $u \in BV(\Omega, \mathbb{R}^{d\times d})$ is

$$\|Du\|(\Omega) = \sup \left\{ \int_\Omega u \cdot \operatorname{div} \varphi : \varphi \in C_c^1(\Omega, \mathbb{R}^{d\times d}); |\varphi| \leq 1 \right\} \tag{2.14}$$

while, using the the Legendre transform of $f(\xi) = |\xi|$, we obtain

$$V_{\mathcal{S}}^f(u) = \sup \left\{ \int_\Omega u \cdot \operatorname{div} \varphi : \varphi \in W_0^{1,q}(\Omega, \mathbb{R}^{d\times d}); |\varphi| \leq 1 \right\}. \tag{2.15}$$

It follows directly from (2.14) and (2.15) that $\|Du\|(\Omega) \leq V_{\mathcal{S}}^f[u]$. The converse inequality $\|Du\|(\Omega) \geq V_{\mathcal{S}}^f[u]$ follows from the density of $C_c^1(\Omega, \mathbb{R}^{d\times d})$ in $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and an argument similar to the one made in the proof of (1) in the proposition.

□

3. Minimization with general displacements.

We consider the following :

$$\inf_{(u,\beta)\in \mathscr{U}_{\mathcal{S}}^*} \left\{ I(u,\beta) = V_{\mathcal{S}}^f(u) + \int_\Omega (H(\beta) - F \cdot u)\, dx \right\}. \tag{3.1}$$

This problem will be studied via a dual problem that we will formulate next.

3.1. **An auxiliary problem.** For $l, k : \mathbb{R}^d \to (-\infty, \infty]$, define for $u, v \in \mathbb{R}^d$

$$l^\#(v) := \sup_{u\in\mathbb{R}^d, t>0} \{u \cdot v - l(u)t - H(t)\} \tag{3.2}$$

and

$$k_\#(u) := \sup_{v\in\mathbb{R}^d, t>0} \{(1/t)\,(u \cdot v - k(v) - H(t))\}. \tag{3.3}$$

It is known that $((l^\#)_\#)^\# = l^\#$ and $((k_\#)^\#)_\# = k_\#$. Call $\mathcal{C}$ the set of all functions (k,l) with $k,\ l : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ Borel measurable, finite at least at one point, and satisfying $l \equiv \infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$ and such that

$$k(v) + tl(u) + H(t) \geq u \cdot v \quad \forall u, v \in \mathbb{R}^d, t > 0. \tag{3.4}$$

Call $\mathcal{C}'$ the set of all functions $(k,l) \in \mathcal{C}$ such that $l = k_\#$ and $k = l^\#$. Let $\mathscr{A}$ be the set of (k,l,φ) such that $(k,l) \in \mathcal{C}$ and $\varphi \in S$. Consider the following functional defined on $\mathscr{A}$:

$$J(k,l,\varphi) := \int_\Omega k(F + \operatorname{div} \varphi) + \int_\Lambda l + \int_\Omega f^*(\varphi).$$

If $\mathscr{A}'$ denotes the subset of $\mathscr{A}$ consisting of all $(k,l,\varphi)\in\mathscr{A}$ that satisfy $(k,l)\in\mathcal{C}'$. It holds that

$$\inf\{J(k,l,\varphi):(k,l,\varphi)\in\mathscr{A}\}=\inf\{J(k,l,\varphi):(k,l,\varphi)\in\mathscr{A}'\}\,. \tag{3.5}$$

Indeed, the key observation to this end is that for $(k,l,\varphi)\in\mathscr{A}$, one has $l\geq k_{\#}$ and $k\geq (k_{\#})^{\#}$ so that

$$J(k,l,\varphi)\geq J((k_{\#})^{\#},k_{\#},\varphi)\quad\text{and}\quad ((k_{\#})^{\#},k_{\#},\varphi)\in\mathscr{A}'.$$

For $R>0$, we set

$$\mathscr{A}_R=\{(k,l,\varphi)\in\mathscr{A}':J(k,l,\varphi)\leq R\}\,.$$

Lemma 3.1. *Assume* (**A1**), (**A2**) *and* (**A3**) *holds. Let* $(k,l,\varphi)\in\mathscr{A}_R$. *Set* $s_l:=-\inf_{u\in\bar\Lambda} l(u)$. *Then,*

$$\int_\Omega k(F+\operatorname{div}\varphi)\geq \mathcal{L}^d(\Omega)H^*(s_l)-r^*\|F\|_{L^1(\Omega)}.$$

Moreover, there exists $M:=M(R,F,f,\Omega,\Lambda)>0$ *such that*

$$|s_l|\leq M. \tag{3.6}$$

Proof. As Λ is bounded and l is convex, we choose $u_l\in\overline{\Lambda}$ such that $-l(u_l)=s_l$. Since $k:=l^{\#}$, in view of (3.2), we have

$$-tl(u_l)-H(t)+u_l\cdot v=ts_l-H(t)+u_l\cdot v\leq H^*(s_l)+u_l\cdot v\leq k(v). \tag{3.7}$$

Using the last inequality in (3.7), one gets

$$\int_\Omega k(F+\operatorname{div}\varphi)\geq\int_\Omega (H^*(s_l)+u_l\cdot(F+\operatorname{div}\varphi)) \tag{3.8}$$

$$=H^*(s_l)\mathcal{L}^d(\Omega)+\int_\Omega u_l\cdot F. \tag{3.9}$$

We have used the fact that u_l is a constant vector and $\varphi\in W_0^{1,q}(\Omega,\mathbb{R}^{d\times d})$ to obtain the equality in 3.9. Hence,

$$\int_\Omega k(F+\operatorname{div}\varphi)\geq \mathcal{L}^d(\Omega)H^*(s_l)-r^*\|F\|_{L^1(\Omega)}.$$

Thus,

$$R\geq J(k,l,\varphi)\geq -s_l\mathcal{L}^d(\Lambda)+\mathcal{L}^d(\Omega)H^*(s_l)-r^*\|F\|_{L^1(\Omega)}+\inf f^*.$$

Thanks to Lemma 2.2 (iii), s_l is bounded uniformly in l.

□

Lemma 3.2. *Assume* (**A1**), (**A2**) *and* (**A3**) *hold.*

(1) There exists $M>0$ *such that for all* $(k,l,\varphi)\in\mathscr{A}_R$ *one has*

$$\int_\Lambda |l(y)|dy\leq M. \tag{3.10}$$

(2) There exist $a_0,b_0,c_0>0$ *such that for all* $(k,l,\varphi)\in\mathscr{A}_R$, *the map* k *is* r^**-Lipschitz, and one has for all* $v\in\mathbb{R}^d$

$$-c_0+a_0|v|\leq k(v)\leq b_0+r^*|v|. \tag{3.11}$$

Proof. 1.) Recall that for $(k, l, \varphi) \in \mathcal{A}_R$, one has

$$J(k, l, \varphi) = \int_\Omega k(F + \operatorname{div} F) + \int_\Lambda l + \int_\Omega f^*(\varphi).$$

By Lemma 3.1, for all $(k, l, \varphi) \in \mathcal{A}_R$, if we define $s_l := -\inf_{u \in \bar{\Lambda}} l(u)$, we get

$$R \geq J(k, l, \varphi) \geq \mathcal{L}^d(\Omega) H^*(s_l) - r^* \|F\|_{L^1(\Omega)} + \int_\Lambda l(y) dy + \mathcal{L}^d(\Omega) \inf f^*.$$

Rearranging the terms, we get:

$$\int_\Lambda l(y) dy \leq R - \mathcal{L}^d(\Omega) H^*(s_l) + r^* \|F\|_{L^1(\Omega)} - \inf f^* \mathcal{L}^d(\Omega).$$

By definition of s_l we also have $-s_l \mathcal{L}^d(\Omega) \leq \int_\Lambda l(y) dy$ and thus

$$-s_l \mathcal{L}^d(\Omega) \leq \int_\Lambda l(y) dy \leq R - \mathcal{L}^d(\Omega) H^*(s_l) + r^* \|F\|_{L^1(\Omega)} - \inf f^* \mathcal{L}^d(\Omega).$$

We next exploit Lemma 3.1 to deduce (3.10).
2.) Let $(k, l, \varphi) \in \mathscr{A}_R$. Since $k = l^\#$, by equation (3.2), k is a r^*-Lipschitz as Λ has diameter less or equal to r^*. Next

$$\begin{aligned} k(0) &= \sup_{u \in \bar{\Lambda}, t > 0} \{-t l(u) - H(t)\} \\ &= \sup_{t > 0} \{-t s_l - H(t)\}. \end{aligned}$$

As s_l is uniformly bounded, the growth condition on H ensures that $|k(0)|$ is uniformly bounded say by some $b_0 > 0$. We get then the inequality $k(v) \leq b_0 + r^* |v|$ for all $v \in \mathbb{R}^d$. Because of the hypothesis on the domain Λ, we take $a_0 > 0$ such that $B(0, a_0) \subset \Lambda$. As $(k, l, \varphi) \in \mathcal{A}_R$, we use relation (3.4) to obtain for $v \neq 0$

$$k(v) \geq v \cdot \left(a_0 \frac{v}{|v|} \right) - l \left(a_0 \frac{v}{|v|} \right) - H(1). \tag{3.12}$$

Since l is convex and $\int_\Lambda |l| dy$ is uniformly bounded (thanks to equation (3.10)), we deduce that $\sup_{y \in \bar{B}(0, a_0)} |l|(y)$ is bounded by a constant independent of l (see for instance Theorem 1, p. 236 in [8]). Thus equation (3.12) implies that there exists $c_0 > 0$ such that $k(v) \geq a_0 |v| - c_0$ for all $v \in \mathbb{R}^d$.

□

Proposition 3.3. *Assume* (**A1**), (**A2**), *and* (**A3**) *hold. Assume $\mathcal{S}$ is a finite dimensional non-empty closed and convex subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d \times d})$. Then, the functional J admits a minimizer (k_0, l_0, φ_0) in $\mathscr{A}'$.*

Proof. Let $(\bar{k}, \bar{l}, \bar{\varphi}) \in \mathscr{A}$. Set $R = J(\bar{k}, \bar{l}, \bar{\varphi})$. Take a minimizing sequence $\{(k_n, l_n, \varphi_n)\}_{n\in\mathbb{N}^*}$ of Problem (3.5) that is in $\mathscr{A}_R$. By Lemma 3.1 and the growth condition on f^* we may assume without lost of generality that $\{\varphi_n\}_{n=1}^{\infty}$ converges to some $\varphi_0 \in \mathcal{S}$ weakly in $L^q(\Omega, \mathbb{R}^{d\times d})$. Since Span$(\mathcal{S})$ is finite dimensional, $\{\varphi_n\}_{n=1}^{\infty}$ converges to some $\varphi_0 \in \mathcal{S}$ strongly in the $L^q(\Omega, \mathbb{R}^{d\times d})$ norm. We deduce

$$\int_\Omega f^*(\varphi_0) \le \liminf_{n\to\infty} \int_\Omega f^*(\varphi_n). \tag{3.13}$$

From Lemma 3.2, as l_n is convex, passing to a subsequence if necessary, we may assume that (k_n, l_n) converges locally uniformly to $(k_0, l_0) \in \mathcal{C}'$. The Lebesgue dominated convergence together with Inequality (3.11) yield

$$\int_\Omega k(F + \operatorname{div}\varphi_0) \le \liminf_{n\to\infty} \int_\Omega k_n(F + \operatorname{div}\varphi_n). \tag{3.14}$$

Since $\{l_n\}_{n=1}^{\infty}$ is uniformly bounded below (thanks to Lemma 3.1), by Fatou's Lemma we get

$$\int_\Lambda l_0 \le \liminf_{n\to\infty} \int_\Lambda l_n. \tag{3.15}$$

By inequalities (3.13), (3.14) and (3.15), we get

$$J(k_0, l_0, \varphi_0) \le \liminf_{n\to\infty} J(k_n, l_n, \varphi_n)$$

and (k_0, l_0, φ_0) is a minimizer of J over $\mathscr{A}'$.

□

3.2. **A uniqueness result.** We prove next the main result of this subsection. We will need the following Lemma whose proof can be found in [3] or in Lemma 4.1.10 of [2].

Lemma 3.4. *Consider a lower semicontinuous function $l_0 : \mathbb{R}^d \to \bar{\mathbb{R}}$ such that $\inf_{\bar{\Lambda}} l_0 > -\infty$; l_0 is finite on Λ and $l_0 \equiv +\infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$. Set $k = (l_0)_\#$. Suppose that $v \in \mathbb{R}^d$ and k is differentiable at v.*

1. *There exist unique $u_0 \in \bar{\Lambda}$ and $t_0 > 0$ such that $k(v) = -t_0 l(u_0) - H(t_0) - u_0 \cdot v$. In addition, u_0 and t_0 are characterized by $u_0 = \nabla k(v)$ and $H'(t_0) + l(u_0) = 0$.*
2. *Let $l \in C_b(\mathbb{R}^d)$ and let $1 \ge \epsilon > 0$. Define $l_\epsilon = l_0 + \epsilon l$ and $k_\epsilon(v) = (l_\epsilon)_\#$.*
   - *(a) There exists a constant M independent of v and ϵ such that ,*
   $$\left|\frac{k_\epsilon(v) - k(v)}{\epsilon}\right| \le M.$$
   - *(b) We have*
   $$\lim_{\epsilon\to 0} \frac{k_\epsilon(v) - k(v)}{\epsilon} = -t_0 l(u_0).$$

Next, we give the main result of this subsection.

Theorem 3.5. *Assume* (**A1**), (**A2**), *and* (**A3**) *hold. Assume $\mathcal{S}$ is a finite dimensional non-empty closed and convex subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. Assume F satisfies the condition **(ND)**$_{\mathcal{S}}$. Then, Problems* (3.1) *and* (3.5) *are dual. Problem* (3.5) *admits a maximizer* (k_0, l_0, φ_0) *with*

$k_0 = l_0^{\#}$ and $l_0 = (k_0)_{\#}$. Problem (3.1) admits a unique minimizer (u_0, β_0). Moreover u_0 satisfies

$$\begin{cases} u_0 = & \nabla k_0(F + \operatorname{div}\varphi_0) \\ \varphi_0 \in & \Phi_S(u_0). \end{cases}$$

Proof. **Step 1**. For $(u,\beta) \in \mathscr{U}_{\mathcal{S}}^*$ and $(k,l,\varphi) \in \mathscr{A}$, one has

$$\begin{aligned} I(u,\beta) =& V_{\mathcal{S}}^f(u) + \int_\Omega (H(\beta) - F\cdot u)\,dx \\ &\geq \int_\Omega (-u\cdot(div\,\varphi + F))\,dx - \int_\Omega f^*(\varphi)dx + \int_\Omega H(\beta)dx + \int_\Omega \beta l(u)dx - \int_\Lambda l(y)dy \\ &\geq \int_\Omega -k(div\,\varphi + F) - \int_\Omega f^*(\varphi) - \int_\Lambda l(y)dy. \end{aligned}$$

Thus $I(u,\beta) \geq -J(k,l,\varphi)$ with equality if and only if $\varphi \in \Phi_{\mathcal{S}}(u)$ and

$$k(F + \operatorname{div}\varphi) + \beta l(u) + H(\beta) = u\cdot(F + \operatorname{div}\varphi).$$

Note that if k is convex, the map $\nabla k(F + \operatorname{div}\varphi)$ is well defined as the map $F + \operatorname{div}\varphi$ is non-degenerate. Using Lemma 3.4(i), it follows that if k is convex, then $I(u,\beta) = -J(k,l,\varphi)$ if and only if

$$\begin{cases} \varphi & \in \Phi_{\mathcal{S}}(u) \\ u & = \nabla k(F + \operatorname{div}\varphi) \\ \beta & = (H')^{-1}(-l(u)) \end{cases}. \tag{3.16}$$

Step 2. Let (k_0, l_0, φ_0) be a maximizer of Problem (3.5) with $k_0 = l_0^{\#}$ and $l_0 = (k_0)_{\#}$. The $u_0 = \nabla k_0(F + \operatorname{div}\varphi_0)$ is well defined as k_0 is convex and we set $\beta_0 = (H')^{-1}(-l(u_0))$. We are to show that $(u_0, \beta_0) \in \mathscr{U}_{\mathcal{S}}^*$ and $\varphi_0 \in \Phi_{\mathcal{S}}(u_0)$.
Step 3. Let $\bar{l} \in C_c(\mathbb{R}^d)$. For $\epsilon \in (0,1)$, define $l_\epsilon = l_0 + \epsilon\bar{l}$ and $k_\epsilon = (l_\epsilon)^{\#}$. Using Lemma 3.4, one has

$$\lim_{\epsilon\to 0^+} \int_\Omega (1/\epsilon)\left(k_0(F + \operatorname{div}\varphi_0) - k_\epsilon(F + \operatorname{div}\varphi_0)\right)dx = \int_\Omega \beta_0\bar{l}(\nabla k_0(F + \operatorname{div}\varphi_0)) = \int_\Omega \beta_0\bar{l}(u_0). \tag{3.17}$$

Since $J(k_0, l_0, \varphi_0) \leq J(k_\epsilon, l_\epsilon, \varphi_0)$, we deduce that $-\int_\Lambda \bar{l} + \int_\Omega \beta_0\bar{l}(u_0) \leq 0$. As we can replace $\bar{l}$ by $-\bar{l}$, one deduces that $\int_\Lambda \bar{l} = \int_\Omega \beta_0\bar{l}(u_0)$. Therefore $(u_0, \beta_0) \in \mathscr{U}_S^*$.
Step 4. Let $\varphi \in \mathcal{S}$. For $\epsilon \in (0,1)$, set $\varphi_\epsilon = \epsilon\varphi + (1-\epsilon)\varphi_0$. By the convexity of $\mathcal{S}$, the map φ_ϵ belongs to $\mathcal{S}$. As $J(k_0, l_0, \varphi_0) \leq J(k_0, l_0, \varphi_\epsilon)$, we have

$$\int_\Omega (1/\epsilon)\left(k_0(F + \operatorname{div}\varphi_0) - k_0(F + \operatorname{div}\varphi_0 + \epsilon\operatorname{div}(\varphi - \varphi_0))\right) + f^*(\varphi_0) - f^*(\varphi_0 + \epsilon(\varphi - \varphi_0)) \leq 0. \tag{3.18}$$

Thanks to Lemma 3.4 the latter inequality implies

$$\begin{aligned}&-\int_\Omega u_0\cdot \operatorname{div}(\varphi-\varphi_0)-grad[f^*](\varphi_0)\cdot(\varphi-\varphi_0)\\ =&-\int_\Omega \nabla k_0(F+\operatorname{div}\varphi_0)\cdot \operatorname{div}(\varphi-\varphi_0)-grad[f^*](\varphi_0)\cdot(\varphi-\varphi_0)\\ \le& 0.\end{aligned}$$

It follows from Proposition 2.8 that $\varphi_0\in\Phi_{\mathcal{S}}(u_0)$.

Step 5. Since $(u_0,\beta_0)\in\mathscr{U}^*_{\mathcal{S}}$, $\varphi_0\in\Phi_{\mathcal{S}}(u_0)$, $u_0=\nabla k_0(F+\operatorname{div}\varphi_0)$, and $\beta_0=(H')^{-1}(-l(u_0))$, we deduce that $I(u_0,\beta_0)=J(k_0,l_0,\varphi_0)$ and u_0 is a minimizer of Problem (3.1) thanks to relation (3.16). Suppose $(u_1,\beta_1)\in\mathscr{U}^*_{\mathcal{S}}$ is another minimizer of Problem (3.1). Then we have $I(u_1,\beta_1)=J(k_0,l_0,\varphi_0)$ and by relation (3.16), we get $u_1=\nabla k_0(F+\operatorname{div}\varphi_0)$ which implies $u_1=u_0$. Next the strict convexity of H yields that $\beta_0=\beta_1$. We conclude that (u_0,β_0) is the unique minimizer of Problem (3.1) and u_0 is characterized by

$$\begin{cases}u_0= & \nabla k_0(F+\operatorname{div}\varphi_0)\\ \varphi_0\in & \Phi_{\mathcal{S}}(u_0).\end{cases}$$

□

Corollary 3.6. *Assume* (**A0**), (**A1**), (**A2**), *and* (**A3**) *hold. Assume $\mathcal{S}$ is a finite dimensional non-empty closed and convex subset of $W^{1,q}_0(\Omega,\mathbb{R}^{d\times d})$ and $\nabla\varphi$ is non-degenerate whenever $\varphi\in\mathcal{S}$. Suppose F has a countable range (thus degenerate) Then, F satisfies the condition **(ND)**$_{\mathcal{S}}$ and problem* (3.1) *admits a unique solution.*

Corollary 3.7. *Assume* (**A1**), (**A2**), *and* (**A3**) *hold. Assume $\mathcal{S}$ is a finite dimensional subspace of $W^{1,q}_0(\Omega,\mathbb{R}^{d\times d})$ and $\nabla\varphi$ has a countable range whenever $\varphi\in\mathcal{S}$. Suppose F is non-degenerate. Then, F satisfies the condition **(ND)**$_{\mathcal{S}}$ and problem* (3.1) *admits a unique solution.*

4. The incompressible case

Throughout this section, we assume that $\mathcal{S}$ is a subset of $W^{1,q}_0(\Omega,\mathbb{R}^{d\times d})$ We consider the following problem:

$$\inf_{u\in\mathcal{U}^1_{\mathcal{S}}}\left\{I_0(u):=V^f_{\mathcal{S}}(u)-\int_\Omega F\cdot u dx\right\} \tag{4.1}$$

and we recall that the set $\mathscr{U}^1_{\mathcal{S}}$ is defined as

$$\mathscr{U}^1_{\mathcal{S}}=\left\{u\in\mathscr{U}_{\mathcal{S}}:\int_\Omega l(u(x))dx=\int_\Lambda l(y)dy\ \forall l\in C_c(\mathbb{R}^d)\right\}.$$

We assume $\mathcal{L}^d(\Omega)=\mathcal{L}^d(\Lambda)$ so that $\mathscr{U}^1_{\mathcal{S}}$ is non-empty.

4.1. Existence and uniqueness via duality.

We study Problem (4.1) via duality. Let $u\in\mathscr{U}^1_{\mathcal{S}}$, $\varphi\in\mathcal{S}$, $l\in C(\bar\Lambda)$ and $k:\mathbb{R}^d\to\mathbb{R}$ satisfy $k(v)+l(u)\ge u\cdot v$ for all $u\in\Lambda$ and all

$v \in \mathbb{R}^d$. One has

$$V_{\mathcal{S}}^f(u) - \int_\Omega F \cdot u dx = - \int_\Omega u \cdot (F + \operatorname{div} \varphi) + \int_\Omega l(u) - \int_\Lambda l(y) dy - \int_\Omega f^*(\varphi) \tag{4.2}$$

$$\geq - \int_\Omega k(F + \operatorname{div} \varphi) - \int_\Lambda l(y) dy - \int_\Omega f^*(\varphi). \tag{4.3}$$

This suggests that we consider the dual problem

$$M_0 := \inf_{(k,l,\varphi) \in A_0} \left\{ J(k,l,\varphi) := \int_\Omega k(F + \operatorname{div} \varphi) + \int_\Lambda l(y) dy + \int_\Omega f^*(\varphi) \right\} \tag{4.4}$$

with A_0 being the set of all (k, l, φ) such that $\varphi \in \mathcal{S}$, $l \in C(\bar{\Lambda})$, $\inf_\Lambda l = 0$ and $k : \mathbb{R}^d \to \mathbb{R}$ satisfies $k(v) + l(u) \geq u \cdot v$ for all $u \in \Omega$ and all $v \in \mathbb{R}^d$.

4.1.1. *Existence and regularity of minimizers of Problem* (4.4). Denote by C_0, the set of (k, l) such that $l \in C(\bar{\Lambda})$, $\inf_\Lambda l = 0$ and $k : \mathbb{R}^d \to \mathbb{R}$ satisfies $k(v) + l(u) \geq u \cdot v$ for all $u \in \Omega$ and all $v \in \mathbb{R}^d$. For $(k, l) \in C_0$, we define

$$l^\&(v) = \max \left\{ u \cdot v - l(u); u \in \bar{\Lambda} \right\}$$

for all $v \in \mathbb{R}^d$, and

$$k_\&(u) = \sup \left\{ u \cdot v - k(v); v \in \mathbb{R}^d \right\}$$

for all $u \in \bar{\Lambda}$. We have the following Lemma:

Lemma 4.1. *Suppose that $(k, l) \in C_0$. It holds that $(l^\&, (l^\&)_\&) \in C_0$, $l^\& \leq k$, $0 \leq (l^\&)_\& \leq l$, $((l^\&)_\&)^\& = l^\&$ and $l^\&(0) = 0$.*

Proof. Consider the set $\mathcal{C}$ of all (k, l) such that $k : \mathbb{R}^d \to \mathbb{R}$ and $l : \bar{\Lambda} \to \mathbb{R}$ satisfy

$$k(v) + l(u) \geq u \cdot v; \quad \forall u \in \bar{\Lambda}; \quad \forall v \in \mathbb{R}^d. \tag{4.5}$$

Claim 1: If $(k, l) \in \mathcal{C}$ then $(k, k_\&) \in \mathcal{C}$ and $(l^\&, l) \in \mathcal{C}$.
Proof of claim 1. Let $u \in \bar{\Lambda}$. By definition of $k_\&$, for all $v \in \mathbb{R}^d$, one has $k_\&(u) \geq u \cdot v - k(v)$. Thus $(k, k_\&) \in \mathcal{C}$.
Similarly, let $v \in \mathbb{R}^d$. By definition of $l^\&$, for all $u \in \bar{\Lambda}$, one has $l^\&(v) \geq u \cdot v - l(u)$. Thus $(l^\&, l) \in \mathcal{C}$.
Claim 2: If $(k, l) \in \mathcal{C}$ then $k \geq l^\&$ and $l \geq k_\&$.
Proof of claim 2. Since for all $v \in \mathbb{R}^d$ and all $u \in \bar{\Lambda}$ we have $k(v) \geq uv - l(u)$, we deduce that $k \geq l^\&$. Similarly for all $v \in \mathbb{R}^d$ and all $u \in \bar{\Lambda}$ we have $l(u) \geq uv - k(v)$, we deduce that $l \geq k_\&$.
Claim 3: If $(k, l) \in \mathcal{C}$ then $(l^\&, (l^\&)_\&) \in \mathcal{C}$, $k \geq l^\&$ and $l \geq (l^\&)_\&$.
Proof of claim 3. Suppose $(k, l) \in \mathcal{C}$. By claim 2, $k \geq l^\&$. By claim 1, $(l^\&, l) \in \mathcal{C}$. It follows that, on the one hand, using claim 1 one more time we get $(l^\&, (l^\&)_\&) \in \mathcal{C}$ and claim 2, on the other hand, we get $l \geq (l^\&)_\&$.
Claim 4: If $(k, l) \in \mathcal{C}_0$ then $l^\&(0) = 0$ and $\inf_{u \in \bar{\Omega}} (l^\&)_\&(u) = 0$.

Proof of claim 4. We have

$$l^\&(0) = \sup_{u \in \bar{\Lambda}} \{u \cdot 0 - l(u)\} = - \inf_{u \in \bar{\Lambda}} l(u). \tag{4.6}$$

Since $(k,l) \in \mathcal{C}_0$, we have $\inf_{u \in \bar{\Lambda}} l(u) = 0$ which, in light of (4.6), yields $l^{\&}(0) = 0$. Next, for $u \in \bar{\Lambda}$,

$$(l^{\&})_{\&}(u) \geq u \cdot 0 - l^{\&}(0) = 0.$$

Hence $\inf_{u \in \bar{\Lambda}} (l^{\&})_{\&}(u) \geq 0$. By claim 3, $l \geq (l^{\&})_{\&}$. Thus

$$\inf_{u \in \bar{\Lambda}} (l^{\&})_{\&}(u) \leq \inf_{u \in \bar{\Lambda}} l(u) = 0.$$

We deduce that $\inf_{u \in \bar{\Lambda}} (l^{\&})_{\&}(u) = 0$.
Claim 5: If $(k,l) \in \mathcal{C}$, then $l^{\&} = ((l^{\&})_{\&})^{\&}$.
Proof of claim 5. Since $l \geq (l^{\&})_{\&}$, we use the monotonicity of the operator $(\cdot)^{\&}$ to deduce that $l^{\&} \leq ((l^{\&})_{\&})^{\&}$. By claim 3, $(l^{\&}, (l^{\&})_{\&}) \in \mathcal{C}$. We exploit then claim 2 to deduce that $l^{\&} \geq ((l^{\&})_{\&})^{\&}$.

□

Let us denote by C_0' the set of all $(k,l) \in C_0$ such that $l^{\&} = k$, $k_{\&} = l$, $k(0) = 0$, and $l \geq 0$, and by A_0' the set of all $(k,l,\varphi) \in A_0$ with $(k,l) \in C_0'$. One readily checks that, in light of Lemma 4.1, Problem (4.4) has the same infimum value as

$$\inf_{(k,l,\varphi) \in A_0'} \left\{ J(k,l,\varphi) := \int_{\Omega} k(F + \operatorname{div} \varphi) dx + \int_{\Omega} f^*(\varphi) dx + \int_{\Lambda} l(y) dy \right\}. \tag{4.7}$$

Lemma 4.2. *Assume* (**A1**) *and* (**A3**) *hold. Assume that the set $\mathcal{S}$ is a finite dimensional non-empty closed and convex subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. Then, problem* (4.7) *admits a minimizer $(k_0, l_0, \varphi_0) \in A_0'$ with k_0 convex and r^*-Lipschitz and $k_0(0) = 0$.*

Proof. Consider a minimizing sequence $(k_n, l_n, \varphi_n)_n$ of Problem (4.7). Since $k_n = l_n^{\&}$ and $l_n = (k_n)^{\&}$, k_n is r^*-Lipschitz. As $k_n(0) = 0$, we use Ascoli-Arzela theorem to deduce that a subsequence of $\{k_n\}_{n=1}^{\infty}$ converges locally uniformly to some k_0. Next, using the growth condition (2.4) on f^* as well as the facts that k_n is r^*-Lipschitz, $k_n(0) = 0$, we establish the following estimate :

$$J(k_n, l_n, \varphi_n) \geq \int_{\Omega} \left(-r^* |F + \operatorname{div} \varphi_n| + c^p \frac{|\varphi_n|^q}{q} - b \right) + \int_{\Lambda} l_n(y) dy \tag{4.8}$$

As the left hand side of (4.8) is bounded, $l_n \geq 0$ and $\mathcal{S}$ is finite dimensional, we deduce from 4.8 that a subsequence of $\{\varphi_n\}_{n=1}^{\infty}$ converges strongly to some φ_0 in $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. Invoking (4.8) again, we show that $\left\{ \int_{\Lambda} l_n(y) dy \right\}_{n=1}^{\infty}$ is bounded. This, combined with the fact that l_n is non negative and convex, yield the existence of a subsequence of $\{l_n\}_{n=1}^{\infty}$ that converges locally uniformly to some l_0 (see for instance Theorem 1, p. 236 in [8]). One readily checks that $(k_0, l_0, \varphi_0) \in A_0'$. We next exploit lower semi-continuity properties of the functional J to conclude that (k_0, l_0, φ_0) is a minimizer of J over A_0'.

□

4.1.2. *A duality result.* We have the following theorem.

Theorem 4.3. *Assume* (**A1**) *and* (**A3**) *hold. Assume $\mathcal{S}$ is a finite dimensional non-empty closed and convex subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. Suppose the map F satisfies the condition $\boldsymbol{(ND)}_{\mathcal{S}}$. Then Problems* (4.1) *and* (4.4) *are dual. Problem* (4.4) *admits a maximizer (k_0, l_0, φ_0) with $k_0 = l_0^{\&}$ and $l_0 = (k_0)_{\&}$. Problem* (4.1) *admits a unique minimizer u_0. Moreover u_0 satisfies*

$$\begin{cases} u_0 = & \nabla k_0(F + \operatorname{div}\varphi) \\ \varphi_0 \in & \Phi_S(u_0) \end{cases}.$$

Proof. Suppose $u \in \mathscr{U}_S^1$ and $(k, l, \varphi) \in A_0$. Using 4.2 and 4.3, we see that $I_0(u) \geq -J(k, l, \varphi)$ with equality if and only if $\varphi \in \Phi_S(u)$ and for almost every $x \in \Omega$, we have $l(u) + k(F + \operatorname{div}\varphi) = u \cdot (F + \operatorname{div}\varphi)$. The latter condition reduces to $u(x) = \nabla k(F(x) + \operatorname{div}\varphi(x))$ if k is convex, under the assumption $F + \operatorname{div}\varphi$ is non-degenerate. Now, let $(k_0, l_0, \varphi_0) \in A_0'$ be a minimizer of J over A_0. Since $F + \operatorname{div}\varphi_0$ is non-degenerate and k_0 is convex, the map $u_0 = \nabla k_0(F + \operatorname{div}\varphi_0)$ is well defined.

Variation around l_0**.** Let $\bar{l} \in C_c(\mathbb{R}^d)$. For $\epsilon \in (0, 1)$, set $l_\epsilon = l_0 + \epsilon\bar{l}$ and $k_\epsilon = (l_\epsilon)^{\&}$. Let $v \in \mathbb{R}^d$ be a point where k_0 is differentiable. Using the measurable selection theorem, one deduces that there exists $T_\epsilon : \mathbb{R}^d \to \mathbb{R}^d$ measurable such that for all $\epsilon \in [0, 1)$

$$k_\epsilon(v) = T_\epsilon(v) \cdot v - l_\epsilon(T_\epsilon(v)).$$

Then, for $\epsilon \in (0, 1)$, we have

$$\bar{l}(T_\epsilon(v)) \leq -(1/\epsilon)\,(k_\epsilon(v) - k_0(v)) \leq \bar{l}(T_0(v)) \tag{4.9}$$

and

$$|(1/\epsilon)\,(k_\epsilon(v) - k_0(v))| \leq \|\bar{l}\|_{L^\infty(\mathbb{R}^d)}. \tag{4.10}$$

Moreover,

$$\lim_{\epsilon\to 0^+} -(1/\epsilon)\,(k_\epsilon(v) - k_0(v)) = \bar{l}(T_0(v)). \tag{4.11}$$

Confer p. 95 of [2] for (4.9)- (4.11). Hence, as

$$T_0(F + \operatorname{div}\psi_0) = \nabla k_0(F + \operatorname{div}\psi_0) = u_0 \quad a.e.$$

one has

$$\lim_{\epsilon\to 0^+} \int_\Omega (1/\varepsilon)\,(k_0(F + \operatorname{div}\psi_0) - k_\epsilon(F + \operatorname{div}\psi_0))\,dx = \int_\Omega \bar{l}(T_0(F + \operatorname{div}\psi_0)) = \int_\Omega \bar{l}(u_0). \tag{4.12}$$

Since $J(k_0, l_0, \varphi_0) \leq J(k_\epsilon, l_\epsilon, \varphi_0)$, we deduce from (4.12) that $-\int_\Lambda \bar{l} + \int_\Omega \bar{l}(u_0) \leq 0$. As we can replace l by $-l$, one deduces that $\int_\Lambda \bar{l} = \int_\Omega \bar{l}(u_0)$. As a result, $u_0 \in \mathscr{U}_S^1$.

Variation around φ_0. Let $\varphi \in \mathcal{S}$. For $\epsilon \in (0,1)$, by convexity of $\mathcal{S}$, we have $\varphi_\epsilon := \epsilon\varphi_1 + (1-\epsilon)\varphi_0 \in \mathcal{S}$. Then $J(k_0, l_0, \varphi_0) \leq J(k_0, l_0, \varphi_\epsilon)$. This implies that

$$\int_\Omega (1/\epsilon)\left(k_0(F + \operatorname{div}\varphi_0) - k_0(F + \operatorname{div}\varphi_0 + \epsilon \operatorname{div}(\varphi - \varphi_0))\right) + f^*(\varphi_0) - f^*(\varphi_0 + \epsilon(\varphi - \varphi_0) \leq 0.$$

As ϵ tends to 0^+, the above equation yields

$$\begin{aligned}
&-\int_\Omega u_0 \cdot \operatorname{div}(\varphi - \varphi_0) - grad[f^*](\varphi_0) \cdot (\varphi - \varphi_0) \\
=&-\int_\Omega \nabla k_0(F + \operatorname{div}\varphi_0) \cdot \operatorname{div}(\varphi - \varphi_0) - grad[f^*](\varphi_0) \cdot (\varphi - \varphi_0) \\
\leq& 0.
\end{aligned}$$

It follows from Proposition 2.8 that $\varphi_0 \in \Phi_{\mathcal{S}}(u_0)$.

□

Corollary 4.4. *Assume* (**A0**), (**A1**), *and* (**A3**) *hold. Assume $\mathcal{S}$ is a finite dimensional non-empty closed and convex subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ and $\nabla\varphi$ is non-degenerate whenever $\varphi \in \mathcal{S}$. Suppose F has a countable range (thus degenerate). Then, F satisfies the condition* ***(ND)***$_{\mathcal{S}}$ *and problem* (4.1) *admits a unique solution.*

Corollary 4.5. *Assume* (**A1**), *and* (**A3**) *hold. Assume $\mathcal{S}$ is a finite dimensional subspace of and $\nabla\varphi$ has a countable range whenever $\varphi \in \mathcal{S}$. Suppose F is non-degenerate. Then, F satisfies the condition* ***(ND)***$_{\mathcal{S}}$ *and problem* (4.1) *admits a unique solution.*

4.2. **A Link between Problem** (3.1) **and Problem** (4.1)**.** Here, we explore the relationships between problem (3.1) and problem (4.1). For this purpose, we make a further assumption of the domains Ω and Λ by requiring that $\Omega = \Lambda$. Assume (**A1**) holds and recall $\{H_n\}_{n=0}^\infty$ as defined in (2.5) and (2.6). Then, Lemma 2.3 ensures that (**A1**) holds for H_n for $n \geq 1$.

Define

$$I_n(u, \beta) := V_{\mathcal{S}}^f(u) + \int_\Omega H_n(\beta) - u \cdot F$$

and

$$I_0(u) := V_{\mathcal{S}}^f(u) - \int_\Omega u \cdot F.$$

Let (u_n, β_n) be the unique minimizer of the problem

$$\inf_{(u,\beta)\in\mathscr{U}_{\mathcal{S}}^*} I_n(u, \beta),$$

as given by Theorem 3.5. Recall that C_0 is the set of all (k, l) such that $l \in C(\bar{\Lambda})$ and $k : \mathbb{R}^d \to \mathbb{R}$ satisfies for all $u \in \Lambda$ and all $v \in \mathbb{R}^d$:

$$k(v) + l(u) \geq u \cdot v. \tag{4.13}$$

Let C_n be the set of all (k, l) such that $l \in C(\bar{\Lambda})$ and $k : \mathbb{R}^d \to \mathbb{R}$ satisfying:

$$k(v) + tl(u) + H_n(t) \geq u \cdot v; \quad \forall u \in \Lambda;\ \forall \in \mathbb{R}^d. \tag{4.14}$$

We denote $\mathcal{A}_0$ the set of all (k,l,φ) satisfying $(k,l)\in C_0$ and $\varphi\in S$. Similarly $\mathcal{A}_n$ denotes he set of all (k,l,φ) satisfying $(k,l)\in C_n$ and $\varphi\in S$. If $(k,l,\varphi)\in A_0\cup A_n$, we still set

$$J(k,l,\varphi)=\int_\Omega k(F+\operatorname{div}\varphi)+\int_\Lambda l(y)dy+\int_\Omega f^*(\varphi).$$

For $n\in\mathbb{N}$, (k_n,l_n,φ_n) is a minimizer of J over A_n as given by Lemma 3.3 and Lemma 4.2. We suppose in addition that k_n is convex and r^*-Lipschitz.

Lemma 4.6. *Assume* (**A1**), (**A2**), *and* (**A3**) *hold. Assume $\mathcal{S}$ is a finite dimensional non-empty closed and convex subset of $W_0^{1,q}(\Omega,\mathbb{R}^{d\times d})$.*

(1) The sequence $\{I_n(u_n,\beta_n)\}_{n\in\mathbb{N}^*}$ is bounded.
(2) The sequence $\{\beta_n\}_{n\in\mathbb{N}^*}$ converges to 1 in $L^2(\Omega)$.
(3) The sequence $\{\varphi_n\}_{n\in\mathbb{N}^*}$ admits a subsequence that converges to some $\bar\varphi$ in S with respect to the $W_0^{1,q}\left(\Omega,\mathbb{R}^{d\times d}\right)-$ norm.

Proof. **Step 1.** Let $\bar u\in\mathscr{U}^1_{\mathcal{S}}$. We have $(u,1)\in\mathscr{U}^*_{\mathcal{S}}$ and thus $I_n(u_n,\beta_n)\le I_n(\bar u,1)$ for all $n\ge 1$. As $H_n(1)=0$, it holds that $I_n(\bar u,1)=V^f_{\mathcal{S}}(\bar u)-\int_\Omega \bar u\cdot F$ which is finite. Hence

$$R_0:=V^f_{\mathcal{S}}(\bar u)-\int_\Omega \bar u\cdot F\ge I_n(u_n,\beta_n). \tag{4.15}$$

On the other hand, we use growth condition (2.4) to get

$$I_n(u_n,\beta_n)\ge\int_\Omega(-b+u_n\cdot F)dx\ge -b\mathcal{L}^d(\Omega)-r^*\|F\|_{L^1(\Omega,\mathbb{R}^d)}:=-R_1. \tag{4.16}$$

We use (4.15) and (4.16) to prove (1).
Step 2. Let $\varphi_0\in\mathcal{S}$. As u_n has values in Λ, it holds that

$$V^f_{\mathcal{S}}(u_n)=\sup_{\varphi\in S}\int_\Omega(-u_n\operatorname{div}\varphi-f^*(\varphi))dx\ge\int_\Omega(-r^*|\operatorname{div}\varphi_0|-f^*(\varphi_0))dx=:R_2. \tag{4.17}$$

and

$$\int_\Omega -u_n\cdot F dx\ge -r^*\|F\|_{L^1(\Omega,\mathbb{R}^d)}. \tag{4.18}$$

We combine (4.15), (4.16), (4.17), (4.18) to get

$$R_2-r^*\|F\|_{L^1(\Omega,\mathbb{R}^d)}+\int_\Omega H_n(\beta_n)\le I_n(u_n,\beta_n)\le R_0. \tag{4.19}$$

Setting $c_0\mathcal{L}^d(\Omega):=R_0-R_2+r^*\|F\|_{L^1(\Omega,\mathbb{R}^d)}$, we use lemma 2.3 and (4.19) to obtain

$$\int_\Omega n(\beta_n(x)-1)^2dx\le(c_0+\bar H-H(1))\mathcal{L}^d(\Omega).$$

This establishes (2).
Step 3. As $\{H_n\}_{n=1}^\infty$ is a non decreasing sequence that converges to H_0, it holds that $C_{n+1}\subset C_n\subset C_0$ for all $n\in\mathbb{N}$. Thus, as $(k_n,l_n)\in C_n$, we have $(k_n,l_n)\in C_0$ so that

$$k_n(F+\operatorname{div}\varphi_n)+l_n(x)\ge x\cdot(F+\operatorname{div}\varphi_n). \tag{4.20}$$

Since $-J(k_n, l_n, \varphi_n) = I_n(u_n, \beta_n)$, we have $J(k_n, l_n, \varphi_n) \leq R_1$ for all $n \in \mathbb{N}^*$. This, combined with $\Omega = \Lambda$, and (4.20) yields

$$R_1 \geq \int_\Omega (k_n(F + \operatorname{div} \varphi_n) + l_n(x) + f^*(\varphi_n))\, dx \geq \int_\Omega (x \cdot (F + \operatorname{div} \varphi_n) + f^*(\varphi_n))\, dx. \tag{4.21}$$

In view of the growth condition (2.4) and boundedness of Ω, (4.21) implies

$$R_1 \geq \int_\Omega \left(r^*|F + \operatorname{div} \varphi_n| - b + c^p \frac{|\varphi_n|^q}{q} \right) dx. \tag{4.22}$$

As, $\mathcal{S}$ is of finite dimension and the div operator is continuous on $\mathcal{S}$, we conclude that $\{\varphi_n\}_{n=1}^\infty$ is convergent up to a subsequence in $W_0^{1,q}\left(\Omega, \mathbb{R}^{d\times d}\right)$ which allows us to conclude (3).

□

Theorem 4.7. *Assume* (**A1**), (**A2**), *and* (**A3**) *hold. Assume $\mathcal{S}$ is a finite dimensional non-empty closed and convex subset of $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$. Suppose that k_n is differentiable for all $n \in \mathbb{N}^*$. The sequence $\{u_n\}_{n\in\mathbb{N}^*}$ converges almost everywhere to the unique minimizer u_0 of I_0 over $\mathscr{U}_S^1$. In addition, the minima $\{I_n(u_n, \beta_n)\}_{n=1}^\infty$ converge to $I_0(u_0)$.*

Proof. **Step 1.** For $n \in \mathbb{N}^*$, set $\bar{k}_n = k_n - k(0)$. Note that $\bar{k}_n(0) = 0$. As the k_n are r^*-Lipschitz, so are $\bar{k}_n$ and we obtain that, up to a subsequence, $\left\{\bar{k}_n\right\}_{n=1}^\infty$ converges locally uniformly to a certain $\bar{k}$. Since $F + \operatorname{div} \varphi_n$ is non-degenerate, we have that $\nabla \bar{k}_n(F + \operatorname{div} \varphi_n)$ is well-defined. Furthermore, Lemma 4.6 ensures that $\{\varphi_n\}_{n=1}^\infty$ converges to some $\bar{\varphi} \in S$ with respect to $W^{1,q}\left(\Omega, \mathbb{R}^d\right)$ (up to a subsequence). As a result, $\{\operatorname{div} \varphi_n\}_{n=1}^\infty$ converges to $\operatorname{div} \bar{\varphi}$ in $L^q(\Omega, \mathbb{R}^d)$. Since $\mathcal{S}$ is of finite dimension, the L^q convergence of $\{\operatorname{div} \varphi_n\}_{n=1}^\infty$ reduces to a pointwise convergence. Next, using the convexity of the $\bar{k}_n$ and the pointwise convergence of $\{\operatorname{div} \varphi_n\}_{n=1}^\infty$ to $\operatorname{div} \varphi_n$, we deduce that up to a subsequence $\left\{\nabla \bar{k}_n(F + \operatorname{div} \varphi_n)\right\}_{n=1}^\infty$ converges a.e to $\nabla \bar{k}(F + \operatorname{div} \bar{\varphi})$ (cf. [14] Theorem 25.7). Theorem 3.5 ensures that $\nabla \bar{k}_n(F + \operatorname{div} \varphi_n) = u_n$. If we denote $\bar{u} := \nabla \bar{k}(F + \operatorname{div} \bar{\varphi})$, then, up to a subsequence, the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges a.e to $\bar{u}$.

Step 2. As for all $n \in \mathbb{N}^*$ and all $l \in C_b(\mathbb{R}^d)$ one has $\int_\Omega \beta_n l(u_n) dx = \int_\Omega l dx$. The strong convergence in $L^2(\Omega)$ of $\{\beta_n\}_{n=1}^\infty$ to 1, and the almost everywhere convergence of $\{u_n\}_{n\in\mathbb{N}}$ converges to $\bar{u}$ ensure that $\lim_{n\to\infty} \int_\Omega \beta_n l(u_n) dx = \int_\Omega l(\bar{u}) dx$. As a result, $\int_\Omega l(\bar{u}) dx = \int_\Omega l dx$ and thus $\bar{u} \in \mathscr{U}_S$.

Step 3.. We recall that

$$I(u, \beta) = V_S^f(u) + \int_\Omega (H(\beta) - u \cdot F)\, .$$

By applying the Lebesgue dominated convergence theorem, we have

$$\liminf_n I(u_n, \beta_n) \geq V_S^f(\bar{u}) + \int_\Omega -\bar{u} \cdot F = I_0(\bar{u}).$$

Let u_0 be the unique minimizer of I_0 over $\mathscr{U}_S^1$ as given by theorem 4.3. Then,

$$\liminf_n I(u_n, \beta_n) \geq I_0(\bar{u}) \geq I_0(u_0). \tag{4.23}$$

Meanwhile, as $C_n \subset C_0$ and (k_0, l_0, φ_0) is a minimizer of J over C_0, we have

$$J(k_0, l_0, \varphi_0) \leq J(k_n, l_n, \varphi_n).$$

This, along with the duality established in Theorem 3.5 imply that

$$\limsup_n I(u_n, \beta_n) \leq \limsup_n -J(k_n, l_n, \varphi_n) \leq -J(k_0, l_0, \varphi_0) = I_0(u_0). \tag{4.24}$$

We combine (4.23) and (4.24) to obtain $I_0(\bar{u}) = I_0(u_0)$. As u_0 is the unique minimizer of I_0 over $\mathscr{U}_S^1$ we have $u_0 = \bar{u}$. We note that the limit $\bar{u}$ does not depend on the subsequence of $\{u_n\}_n$ chosen. Thus, the whole sequence $\{u_n\}_n$ converges a.e. to u_0. In addition, $\{I_n(u_n, \beta_n)\}_n$ converges a.e. to $I_0(u_0)$.

□

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