A Large Dimensional Analysis
of Regularized Discriminant Analysis Classifiers

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Abstract

This article carries out a large dimensional analysis of standard regularized discriminant analysis classifiers designed on the assumption that data arise from a Gaussian mixture model with different means and covariances. The analysis relies on fundamental results from random matrix theory (RMT) when both the number of features and the cardinality of the training data within each class grow large at the same pace. Under mild assumptions, we show that the asymptotic classification error approaches a deterministic quantity that depends only on the means and covariances associated with each class as well as the problem dimensions. Such a result permits a better understanding of the performance of regularized discriminant analysis, in practical large but finite dimensions, and can be used to determine and pre-estimate the optimal regularization parameter that minimizes the misclassification error probability. Despite being theoretically valid only for Gaussian data, our findings are shown to yield a high accuracy in predicting the performances achieved with real data sets drawn from the popular USPS data base, thereby making an interesting connection between theory and practice.

Keywords: Linear discriminant analysis, quadratic discriminant analysis, classification, random matrix theory, consistent estimator.

* Part of this work related to the derivation of an asymptotic equivalent for the R-QDA probability of misclassification has been accepted for publication in the IEEE MLSP workshop 2017.

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1. Introduction

1.1 Overview of Discriminant Analysis for Classification

Discriminant analysis is part of a larger class of classification methods commonly known in the machine learning community as model-based classification methods (Bishop, 2006; Friedman et al., 2009; McLachlan, 2009). These methods rely on the assumption that the input data follow a certain distribution. A classifier is then designed so as to minimize a certain classification metric (Friedman et al., 2009). Linear and Quadratic discriminant analysis (LDA and QDA) merely relying on the assumption of the data following a Gaussian mixture, are among the most popular representatives (Srivastava et al., 2007). Both methods are designed to assign for a given input data the class that presents the highest posterior probability. Their major unique difference is that LDA presumes equal covariance matrices for both classes but different means whereas QDA assumes different covariances and means across classes. By construction, they both require the knowledge of the Gaussian parameters for each class. This can be performed by estimating these parameters from the available training points using maximum likelihood estimation, a way that should be effective if the number of training samples is sufficiently high. However, when the number of training samples is small compared to their dimensions, maximum likelihood covariance estimation can be inefficient, if not ill-posed, leading to high misclassification error rates. A popular approach to solve the ill-posed estimation consists in regularizing the covariance estimation (Friedman, 1989), which has led to the emergence of regularized versions of discriminant analysis, termed regularized LDA (R-LDA) and regularized QDA (R-QDA).

1.2 Previous works

A large body of research has been conducted to analyze the performance of discriminant analysis classifiers. One approach, carried out under the assumption of exact dimensions and hinging on properties of the Wishart distribution, has been pursued in (McFarland and Richards, 2002) to derive the exact misclassification error rate of QDA. Such an analysis was limited to the case in which the training sample size for each class is greater than the number of features. Moreover, it cannot be easily generalized to handle regularized discriminant analysis. An asymptotic approach has arisen in several recent works, leading to concurrent results about the misclassification error rates associated with discriminant analysis classifiers. Particularly, based on sparsity assumptions on the mean and covariance matrices, sparse variants of LDA and QDA has been proposed in (Shao et al., 2011) and (Li and Shao, 2015) and analyzed under the asymptotic regime in which the number of features $p$ is much larger than the number of training samples $n$ ($p \gg n$). A different possible regime is the one in which $n$ and $p$ grow large with the same pace, often termed as the double asymptotic regime. The major advantage of this regime is that it lends itself to the use of results from random matrix theory (Girko, 1995; Hachem et al., 2008; Benaych-Georges and Couillet, 2016). In (Raudys and Young, 2004), an extensive review of the most important results in discriminant analysis has been conducted, where important results concerning asymptotic misclassification error rates in the double asymptotic regime under the assumption of equal covariance matrices have been provided. The double asymptotic regime has also been recently considered in the analysis of the regularized LDA (Zollanvari}
and Dougherty (2015) and the analysis of euclidean distance discriminant rule (Watanabe et al. [2015]) using tools from random matrix theory and asymptotic properties of Wishart matrices, but to the best of the authors’ knowledge has not been considered for the general case in which the covariances across classes are different. As we shall see in the course of the paper, a major difficulty for the analysis of regularized QDA resides in setting of the assumptions governing the growth rate of means and covariances to be used to avoid nontrivial classification performances.

1.3 Contributions

The present work aims to provide a comprehensive understanding of the performance of regularized discriminant analysis for its two popular instances, namely R-LDA and R-QDA. Our interest is to establish such an understanding under the asymptotic regime in which the number of training samples of each class grows large with the data dimension. Under mild assumptions controlling the distance between class covariances and means, we show that the classification error approaches a non-trivial deterministic quantity that only depends on the Gaussian distribution parameters of each class and the problem dimensions. Although real data are in general far from being Gaussian draws, our asymptotic approach is shown to yield good accuracy when applied to real data. The contributions of the present work list as follows

• Under mild assumptions, building on fundamental results from random matrix theory (Benaych-Georges and Couillet 2016) and (Hachem et al. 2008), we establish the convergence of the classification error of both R-LDA and R-QDA classifiers to a deterministic error that reveals the mathematical connection between the classification error and the statistical parameters associated with each class.

• We exploit this result to allow a more efficient design of both classifiers by selecting the regularization parameter that minimizes the asymptotic classification error.

• We validate our theoretical findings using both synthetic data and real data drawn from available data bases and illustrate the good accuracy of our results for both settings.

In the remainder, we give an overview of discriminant analysis for binary classification in Section 2. The main results are presented in Section 3 while all proofs are available in the appendix section. In Section 4, we construct a general consistent estimator of the misclassification error. We validate our analysis for real data in Section 5 and conclude the paper in Section 6.

Notations

 Scalars, vectors and matrices are respectively denoted by non-boldface, boldface lowercase and boldface uppercase characters. $0_{p \times n}$ and $1_{p \times n}$ are respectively the matrix of zeros and ones of size $p \times n$, $I_p$ denotes the $p \times p$ identity matrix. The notation $\| \cdot \|$ means the Euclidean norm for vectors and the spectral norm for matrices. $(\cdot)^T$, $\text{tr}(\cdot)$ and $|\cdot|$ stands for the transpose, the trace and the determinant of a matrix respectively. For two functionals
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If \( f = \mathcal{O}(g) \), if \( \exists 0 < M < \infty \) such that \( |f| \leq Mg \). \( \mathbb{P}(.) \), \( \overset{d}{\to} \), \( \overset{p}{\to} \) and \( \overset{a.s.}{\to} \) respectively denote the probability measure, the convergence in distribution, the convergence in probability and the almost sure convergence of random variables. \( \Phi(.) \) denotes the cumulative density function (CDF) of the standard normal distribution.

2. Discriminant Analysis for Binary Classification

This paper studies binary discriminant analysis techniques which employs a discriminant rule to assign for an input data vector the class to which it most likely belongs. The discriminant rule is designed based on \( n \) available training data with known class labels. In this paper, we consider the case in which a bayesian discriminant rule is employed. Hence, we assume that observations from class \( C_i \), \( i \in \{0, 1\} \) are independent and are sampled from a multivariate Gaussian distribution with mean \( \mu_i \in \mathbb{R}^{p \times 1} \) and non-negative covariance matrix \( \Sigma_i \in \mathbb{R}^{p \times p} \). Formally speaking, an observation vector \( x \in \mathbb{R}^{p \times 1} \) is classified to \( C_i \), \( i \in \{0, 1\} \), if

\[
x = \mu_i + \Sigma_i^{1/2} z, \quad z \sim \mathcal{N}(0, I_p).
\]

As stated in [McLachlan, 2009], for distinct covariance matrices \( \Sigma_0 \) and \( \Sigma_1 \), the discriminant rule is equivalent to assigning \( x \) to class 0 if

\[
W^{QDA}(x) = -\frac{1}{2} \log \left( \frac{\Sigma_0}{\Sigma_1} \right) - \frac{1}{2} x^T \left( \Sigma_0^{-1} - \Sigma_1^{-1} \right) x + x^T \Sigma_0^{-1} \mu_0 - x^T \Sigma_1^{-1} \mu_1
- \frac{1}{2} \mu_0^T \Sigma_0^{-1} \mu_0 + \frac{1}{2} \mu_1^T \Sigma_1^{-1} \mu_1 - \log \frac{\pi_1}{\pi_0}.
\]

is positive and class 1 otherwise, where \( \pi_i \) is the prior probability for class \( i \). In particular,

\[
\begin{cases} 
  x \in C_0 & \text{if } W^{QDA} > 0 \\
  x \in C_1 & \text{otherwise.}
\end{cases}
\]

When the considered classes have the same covariance matrix, i.e., \( \Sigma_0 = \Sigma_1 \), the discriminant function simplifies to [Friedman et al., 2009; Bishop, 2006; Zollanvari and Dougherty, 2015]

\[
W^{LDA}(x) = \left( x - \frac{\mu_0 + \mu_1}{2} \right)^T \Sigma^{-1} \left( \mu_0 - \mu_1 \right) - \log \frac{\pi_1}{\pi_0}.
\]

Classification obeys hence the following rule:

\[
\begin{cases} 
  x \in C_0 & \text{if } W^{LDA}(x) > 0 \\
  x \in C_1 & \text{otherwise.}
\end{cases}
\]

Since \( W^{LDA} \) is linear in \( x \), the corresponding classification method is referred to as linear discriminant analysis. A look at the classification rules in [3], [5] reveals that they inherently assume the knowledge of the class statistics, namely their associated covariance matrices and mean vectors. In practice, these statistics can be estimated using the available training data. As such, as in any supervised learning setting, we assume that \( n_i, i \in \{0, 1\} \)
independent training samples \( T_0 = \{ x_l \in C_0 \}_{l=1}^{n_0} \) and \( T_1 = \{ x_l \in C_1 \}_{l=n_0+1}^{n_0+n_1} \) are respectively available to estimate the mean and the covariance matrix of each class \( i \). For that, we consider the following sample estimates

\[
\bar{x}_i = \frac{1}{n_i} \sum_{l \in T_i} x_l, \quad i \in \{0, 1\}
\]

\[
\hat{\Sigma}_i = \frac{1}{n_i - 1} \sum_{l \in T_i} (x_l - \bar{x}_i) (x_l - \bar{x}_i)^T, \quad i \in \{0, 1\}
\]

\[
\hat{\Sigma} = \frac{(n_0 - 1) \hat{\Sigma}_0 + (n_1 - 1) \hat{\Sigma}_1}{n-2},
\]

where \( \hat{\Sigma} \) is the pooled sample covariance matrix for both classes. To avoid singularity issues when \( n_i < p \), we use ridge estimator for the inverse of the covariance matrix (Friedman et al., 2009)

\[
H = \left( I_p + \gamma \hat{\Sigma} \right)^{-1}, \quad (6)
\]

\[
H_i = \left( I_p + \gamma \hat{\Sigma}_i \right)^{-1}, \quad i \in \{0, 1\} \quad (7)
\]

where \( \gamma > 0 \) is a regularization parameter. In doing so, we obtain the following discriminant rules

\[
\hat{W}^{R-LDA}(x) = \left( x - \frac{\bar{x}_0 + \bar{x}_1}{2} \right)^T H \left( \bar{x}_0 - \bar{x}_1 \right) - \log \frac{\pi_1}{\pi_0} . \quad (8)
\]

\[
\hat{W}^{R-QDA}(x) = \frac{1}{2} \log \frac{|H_0|}{|H_1|} - \frac{1}{2} (x - \bar{x}_0)^T H_0 (x - \bar{x}_0) + \frac{1}{2} (x - \bar{x}_1)^T H_1 (x - \bar{x}_1) - \log \frac{\pi_1}{\pi_0} . \quad (9)
\]

The corresponding classification methods will be denoted respectively by R-LDA and R-QDA. Conditioned on the training samples \( T_i, i \in \{0, 1\} \), the classification errors associated with R-LDA and R-QDA when \( x \) belongs to class \( C_i \) are given by

\[
\epsilon_i^{R-LDA} = \mathbb{P} \left[ (-1)^i \hat{W}^{R-LDA}(x) < 0 \mid x \in C_i, T_0, T_1 \right], \quad (10)
\]

\[
\epsilon_i^{R-QDA} = \mathbb{P} \left[ (-1)^i \hat{W}^{R-QDA}(x) < 0 \mid x \in C_i, T_0, T_1 \right], \quad (11)
\]

while the total classification errors are given by

\[
\epsilon^{R-QDA} = \pi_1 \epsilon_1^{R-QDA} + \pi_2 \epsilon_2^{R-QDA}.
\]

In the following, we propose an asymptotic approach to analyze the classification errors of both R-LDA and R-QDA when \( p, n_i \) grow large at the same rate. For R-LDA, our results cover a more general setting than that studied in (Zollanvari and Dougherty, 2015), in that they apply in the case where both classes have distinct covariance matrices.
3. Main Results

In this section, we present the main contributions of the article. In a first part, we carry out asymptotic analysis of the classification error for both R-LDA and R-QDA. More precisely, under some mild assumptions, we show that the classification errors converge to some deterministic quantities that depend solely on the observations’ statistics, namely the mean and covariances within each class. The interest of such results lies in that they establish a mathematical relation between the asymptotic classification error and the statistical means and covariances, thereby allowing a better understanding of the behavior of R-LDA and R-QDA in the high dimensional regime. Moreover, the obtained results can aid in more appropriate choice of the regularization parameter $\gamma$ in terms of classification error. This can be for instance achieved by selecting the regularization parameter that minimizes the asymptotic classification error. In practice, however, the asymptotic classification error could not be directly acquired as it depends on the unknown statistics. It is thus necessary to determine for it consistent estimators that depend only on the available training data. This is the objective of the second part. For R-LDA, consistent estimators have been provided in (Zollanvari and Dougherty, 2015) but only when the classes have identical covariance matrices. In this paper, we improve on these results and derive consistent estimators for the general case that accounts for different covariance matrices. Our results should have a more practical interest, since the assumption of equal covariance matrices does not always hold when real data sets are considered.

3.1 Asymptotic Performance of R-LDA with Distinct Covariance Matrices

In this section, we present an asymptotic analysis of the R-LDA classifier. Our analysis is mainly based on recent results from RMT concerning some properties of Gram matrices of mixture models (Benaych-Georges and Couillet, 2016). We recall that (Zollanvari and Dougherty, 2015) made a similar analysis of R-LDA in the double asymptotic regime when both classes have a common covariance matrix, thereby not requiring these advanced tools. As such, our results can be viewed as a generalization of (Zollanvari and Dougherty, 2015) when both classes have distinct covariance matrices. This permits to evaluate the performance of R-LDA in practical scenarios when the assumption of common covariance matrices cannot always be guaranteed. To allow derivations, we shall consider the following growth rate assumptions

Assumption. 1 (Data scaling) $\frac{p}{n} \to c \in (0, \infty)$.

Assumption. 2 (Class scaling) $\frac{n_i}{n} \to c_i \in (0, \infty)$, for $i \in \{0, 1\}$.

Assumption. 3 (Covariance scaling) $\|\Sigma_i\| = O(1)$, for $i \in \{0, 1\}$.

Assumption. 4 (Mean scaling) $\|\mu_0 - \mu_1\| = O(1)$.

These assumptions are mainly considered to achieve an asymptotically non-trivial classification error. Assumption 3 is frequently met within the framework of random matrix theory (Benaych-Georges and Couillet, 2016). Under the setting of Assumption 3, Assumption 4 ensures that a nontrivial classification rate is obtained: If $\|\mu_0 - \mu_1\|$ scales faster than...
\[ O(1), \text{ then perfect asymptotic classification is achieved; however, if } \|\mu_0 - \mu_1\| \text{ scales slower than } O(1), \text{ classification is asymptotically impossible. Assumptions 1 and 2 respectively control the growth rate in the data and the training.} \]

### 3.1.1 Deterministic Equivalent

We are in a position to derive a deterministic equivalent of the misclassification error rate of the R-LDA. Indeed, conditioned on the training data \( x_1, \ldots, x_n \), the probability of misclassification is given by: \cite{Zollanvari&Dougherty2015}

\[
\epsilon_{i}^{R-LDA} = \Phi \left( \frac{(-1)^{i+1} G(\mu_i, x_0, x_1, H) + (-1)^i \log \frac{\pi_i}{\pi_0}}{\sqrt{D(x_0, x_1, H, \Sigma_i)}} \right),
\]

where

\[
G(\mu_i, x_0, x_1, H) = \left( \mu_i - \frac{x_0 + x_1}{2} \right)^T H(x_0 - x_1).
\]

\[
D(x_0, x_1, H, \Sigma_i) = (x_0 - x_1)^T H \Sigma_i H (x_0 - x_1).
\]

The total misclassification probability is thus given by

\[
\epsilon_{\text{total}} = \pi_0 \epsilon_0^{R-LDA} + \pi_1 \epsilon_1^{R-LDA}.
\]

Prior to stating the main result concerning R-LDA, we shall introduce the following quantities, which naturally appear, as a result of applying \cite{Benaych-Georges&Coillet2016}. Let \( Q(z) \) be the resolvent matrix defined as follows

\[
Q(z) = \frac{1}{z} (I_p + c_0g_0(z)\Sigma_0 + c_1g_1(z)\Sigma_1)^{-1}, \quad z \in \mathbb{C}.
\]

where \( g_i(z), i \in \{0, 1\} \), satisfies the following fixed point equations

\[
\frac{p}{n} g_i(z) = -\frac{1}{z} \frac{1}{1 + \bar{g}_i(z)}, \quad \bar{g}_i(z) = \frac{1}{p} \text{tr} \Sigma_i Q(z).
\]

Also define \( Q_i(z) \) as

\[
Q_i(z) = Q(z) \Sigma_i Q(z) + R_0 Q(z) \Sigma_0 Q(z) + R_1 Q(z) \Sigma_1 Q(z),
\]

where

\[
R_i = \frac{z^2 c_1 g_1^2(z)}{1 - z^2 c_0 g_0^2(z) \frac{1}{n} \text{tr} \Sigma_0 Q(z) \Sigma_1 Q(z) - z^2 c_1 g_1^2(z)} \frac{1}{n} \text{tr} \Sigma_0 Q(z) \Sigma_1 Q(z), \quad i \in \{0, 1\}.
\]

The quantities in \cite{Benaych-Georges&Coillet2016} can be computed in an iterative fashion where convergence is guaranteed after few iterations (see \cite{Benaych-Georges&Coillet2016} for more details). Moreover, define

\[
\overline{Q}_i(z) = \frac{(-1)^{i+1}}{2} z \mu^T Q(z) \mu + \frac{z}{2n_0} \text{tr} \Sigma_0 Q(z) - \frac{z}{2n_1} \text{tr} \Sigma_1 Q(z).
\]
\[
\mathcal{D}_i(z) \triangleq z^2 \mu^T \tilde{Q}_i(z) \mu + \frac{z^2}{n_0} \text{tr} \Sigma_0 \tilde{Q}_i(z) + \frac{z^2}{n_1} \text{tr} \Sigma_1 \tilde{Q}_i(z),
\]
where \( \mu = \mu_0 - \mu_1 \). With these definitions at hand, we state the following theorem

**Theorem 1** Under Assumptions \([3, 4]\), we have

\[
G_i(\mu_i, \bar{x}_0, \bar{x}_1, H) - \mathcal{G}_i \left( -\frac{1}{c\gamma} \right) \xrightarrow{a.s.} 0. \tag{22}
\]

\[
D_i(\bar{x}_0, \bar{x}_1, H, \Sigma_i) - \mathcal{D}_i \left( -\frac{1}{c\gamma} \right) \xrightarrow{a.s.} 0. \tag{23}
\]

As a consequence, the conditional misclassification probability converges almost surely to a deterministic quantity \( \epsilon_i^{R-LDA} \)

\[
\epsilon_i^{R-LDA} - \epsilon_i^{R-LDA} \xrightarrow{a.s.} 0, \tag{24}
\]

where

\[
\epsilon_i^{R-LDA} = \Phi \left( \frac{\left(-1\right)^{i+1} \mathcal{G}_i \left( -\frac{1}{c\gamma} \right) + \left(-1\right)^i \log \frac{\pi_0}{\pi_1}}{\sqrt{\mathcal{D}_i \left( -\frac{1}{c\gamma} \right)}} \right). \tag{25}
\]

**Proof** See Appendix \([A]\). \(\blacksquare\)

**Remark 2** As stated earlier, if \( \| \mu \| \) scales faster than \( O(1) \), perfect asymptotic classification is achieved. This can be seen by noticing that \( \frac{\mathcal{G}_i \left( -\frac{1}{c\gamma} \right)}{\sqrt{\mathcal{D}_i \left( -\frac{1}{c\gamma} \right)}} \) would grow indefinitely large with \( p \), thereby making the conditional error rates vanish.

When \( \| \Sigma_0 - \Sigma_1 \| \) converges to zero, the asymptotic misclassification error rate of each class coincides with the one derived in Zollanvari and Dougherty (2015) obtained when \( \Sigma_0 = \Sigma_1 \).

**Corollary 3** In the case where \( \| \Sigma_0 - \Sigma_1 \| = o(1) \) (including the common covariance case where \( \Sigma_0 = \Sigma_1 = \Sigma \)), the conditional misclassification error rate converges almost surely to \( \epsilon_i^{R-LDA} \)

\[
\epsilon_i^{R-LDA} = \Phi \left( \frac{\left(-1\right)^{i+1} \overline{\mathcal{G}_i} \left( -\frac{1}{c\gamma} \right) + \left(-1\right)^i \log \frac{\pi_0}{\pi_1}}{\sqrt{\overline{\mathcal{D}_i} \left( -\frac{1}{c\gamma} \right)}} \right),
\]

where

\[
\overline{\mathcal{G}_i} \left( -\frac{1}{c\gamma} \right) = \frac{\left(-1\right)^i}{2} \mu^T \left( I_p + \frac{\gamma}{1 + \gamma \delta} \Sigma \right)^{-1} \mu - \frac{n\delta}{2} \left( \frac{1}{n_0} - \frac{1}{n_1} \right)
\]

\[
\overline{\mathcal{D}_i} \left( -\frac{1}{c\gamma} \right) = \left[ \mu^T \left( I_p + \frac{\gamma}{1 + \gamma \delta} \Sigma \right)^{-2} \mu + \left( \frac{1}{n_0} + \frac{1}{n_1} \right) \text{tr} \Sigma^2 \left( I_p + \frac{\gamma}{1 + \gamma \delta} \Sigma \right)^{-2} \right]^2 - \frac{\gamma^2}{n(1 + \gamma \delta)} \text{tr} \Sigma^2 \left( I_p + \frac{\gamma}{1 + \gamma \delta} \Sigma \right)^{-2}.
\]
where \( \Sigma = \Sigma_0 \) or \( \Sigma = \Sigma_1 \), and \( \delta \) is the unique positive solution to the following equation:

\[
\delta = \frac{1}{n} \text{tr} \Sigma \left( I_p + \frac{\gamma}{1 + \gamma \delta} \Sigma \right)^{-1}.
\]

**Proof** When \( ||\Sigma_0 - \Sigma_1|| = o(1) \), we first prove that up to an error \( o(1) \), the key deterministic equivalents can be simplified to depend only on \( \Sigma_0 \) (or \( \Sigma_1 \)). Indeed, as \( ||\Sigma_0 - \Sigma_1|| = o(1) \), we have

\[
g_i(z) = \frac{1}{p} \text{tr} \Sigma \bar{Q}(z) + o(1), \quad \forall i \in \{0, 1\}.
\]

It follows that \( g_i(z) = g(z) + o(1) \) where \( g(z) = -\frac{1}{\gamma 1 + g(z)} \).

The above relations allow to simplify functionals involving matrix \( \bar{Q} \). To see that, we decompose \( \bar{Q} \) as

\[
\bar{Q}(z) = -z^{-1} (I + g(z) \Sigma_0 + c_1 g(z) (\Sigma_1 - \Sigma_0))^{-1}
\]

\[
= -z^{-1} (I + g(z) \Sigma_0)^{-1} - z^{-1} (I + g(z) \Sigma_0 + c_1 g(z) (\Sigma_1 - \Sigma_0))^{-1} - (I + g(z) \Sigma_0)^{-1}
\]

\[
\overset{(a)}{=} -z^{-1} (I + g(z) \Sigma_0)^{-1} - z^{-1} (I + g(z) \Sigma_0 + c_1 g(z) (\Sigma_1 - \Sigma_0))^{-1} c_1 g(z) (\Sigma_1 - \Sigma_0) (I + g(z) \Sigma_0)^{-1}.
\]

where (a) follows from the resolvent identity. Thus,

\[
\mu^T \Psi \mu \overset{(b)}{\leq} z^{-1} g(z) ||\Sigma_0 - \Sigma_1|| \mu^T (I + g(z) \Sigma_0 + c_1 g(z) (\Sigma_1 - \Sigma_0))^{-1} (I + g(z) \Sigma_0)^{-1} \mu = o(1).
\]

where (b) follows from the inequality \( \mu^T A B \mu \leq ||A B|| \mu ||^2 \) for \( A, B \) two \( p \times p \) matrices. Hence, for \( a, b \in \mathbb{R}^p \),

\[
a^T \bar{Q}(z) b = -z^{-1} a^T (I + g(z) \Sigma_0)^{-1} b + o(1),
\]

and \( \frac{1}{p} \text{tr} A Q = -z^{-1} \frac{1}{p} \text{tr} A (I + g(z) \Sigma_0)^{-1} + o(1) \). Using the same notations as in Zollanvari and Dougherty (2015) we have in particular for \( z = -\frac{1}{\gamma 1} \), \( g(z) = \delta + o(1) \) and \( g(z) = \frac{1}{1 + g(z)} \).

Moreover,

\[
\frac{z}{2n_0} \text{tr} \Sigma_0 Q(z) - \frac{z}{2n_1} \text{tr} \Sigma_1 Q(z) = \frac{z \text{tr} \Sigma_0 Q(z)}{2} \left( \frac{1}{n_0} - \frac{1}{n_1} \right) + \frac{z}{2n_1} \text{tr} (\Sigma_0 - \Sigma_1) Q(z)
\]

\[
\leq ||\Sigma_0 - \Sigma_1|| \frac{z}{2n_1} \text{tr} Q(z)
\]

\[
= \frac{z \text{tr} \Sigma Q(z)}{2} \left( \frac{1}{n_0} - \frac{1}{n_1} \right) + o(1)
\]

\[
= -\text{tr} \Sigma (I + g(z) \Sigma)^{-1} \left( \frac{1}{n_0} - \frac{1}{n_1} \right) + o(1).
\]
It follows that

\[
\bar{G}_i \left( - \frac{1}{c \gamma} \right) = \frac{(-1)^i}{2} \mu^T \left( \mathbf{I}_p + \frac{\gamma}{1 + \gamma \delta} \Sigma \right)^{-1} \mu - \frac{n \delta}{2} \left( \frac{1}{n_0} - \frac{1}{n_1} \right) + o(1).
\]

Using the same arguments, it is easy also to show that

\[
\bar{D}_i \left( - \frac{1}{c \gamma} \right) = \left[ \frac{\mu^T \Sigma \left( \mathbf{I}_p + \frac{\gamma}{1 + \gamma \delta} \Sigma \right)^{-2} \mu + \left( \frac{1}{n_0} + \frac{1}{n_1} \right) \text{tr} \Sigma^2 \left( \mathbf{I}_p + \frac{\gamma}{1 + \gamma \delta} \Sigma \right)^{-2}}{1 - \frac{\gamma^2}{n(1 + \gamma \delta)^2} \text{tr} \Sigma^2 \left( \mathbf{I}_p + \frac{\gamma}{1 + \gamma \delta} \Sigma \right)^{-2}} \right] + o(1).
\]

Corollary 3 is useful because it allows to specify the range of applications of Theorem 1 in which the information on the covariance matrix is essential for the classification task. Also, it shows how R-LDA is robust against small perturbations in the covariance matrix. Similar observations have been made in (Wahl and Kronmal, 1977) where it was shown via a Monte Carlo study that LDA is robust against the modeling assumptions.

The observations made in Corollary 3 are illustrated in Figure 1 where

\[
\Sigma_1 - \Sigma_0 = \frac{2}{\sqrt{p}} \mathbf{I}_p
\]

which yields \(\| \Sigma_0 - \Sigma_1 \| \to 0\) as \(p \to \infty\). It is clear from the figure that the performance of R-LDA converges to the theoretical limit evinced by Theorem 1 with common covariance matrices which is exactly the same predicted by (Zollanvari and Dougherty, 2015).

3.2 Asymptotic Performance of R-QDA

In this part, we state the main results regarding the derivation of deterministic approximations of the R-QDA classification error. Such results have been obtained by considering some specific assumptions, carefully chosen such that an asymptotically non-trivial classification error (i.e., neither 0 nor 1) is achieved. We particularly highlight how the provided asymptotic approximations depend on such statistical parameters as the means and covariances within classes, thus allowing a better understanding of the performance of the R-QDA classifier. Ultimately, these results can be exploited in order to improve the performances by allowing optimal setting of the regularization parameter.

3.2.1 Technical Assumptions

We consider the following double asymptotic regime in which \(n_i, p \to \infty\) for \(i \in \{0, 1\}\) with the following assumptions met

**Assumption. 5 (Data scaling)** \(n_0 - n_1 = o(p)\) and \(\frac{p}{n} \to c \in (0, \infty)\).

**Assumption. 6 (Mean scaling)** \(\| \mu_0 - \mu_1 \|^2 = O(\sqrt{p})\).

**Assumption. 7 (Covariance scaling)** \(\| \Sigma_i \| = O(1)\).

**Assumption. 8** Matrix \(\Sigma_0 - \Sigma_1\) has exactly \(O(\sqrt{p})\) eigenvalues of order \(O(1)\). The remaining eigenvalues are of order \(O(p^{-\frac{1}{2}})\).
Assumption 5 implies also that $\pi_i \to \frac{1}{2}$ for $i \in \{0, 1\}$. As we shall see later, if this were not satisfied, the R-QDA would perform asymptotically as the classifier returning the smallest prior probability $\pi_i$. The second assumption governs the distance between the two classes in terms of the Euclidean distance of the difference between the means. This is mandatory in order to avoid asymptotic perfect classification. This is a more relaxed assumption than Assumption 2 in R-LDA since we allow larger values for $\|\mu_0 - \mu_1\|$. This can be understood as R-QDA being subject to strong noise induced when estimating $\Sigma_i$, $i \in \{0, 1\}$ which requires a large value $\|\mu_0 - \mu_1\|$ so that it can play a role in classification. A similar assumption is required to control the distance between the covariance matrices. Particularly, the spectral norm of the covariance matrices are required to be bounded as stated in Assumption 7 while their difference should satisfy Assumption 8. The latter assumption implies that for any matrix $A$ of bounded spectral norm,

$$\frac{1}{\sqrt{p}} \text{tr} A(\Sigma_0 - \Sigma_1) = O(1).$$

### 3.2.2 Central Limit Theorem (CLT)

It can be easily shown that the R-QDA conditional classification error in (11) writes as

$$\epsilon^R_{i-QDA} = P \left[ z^T B_i z + 2 z^T y_i < \xi_i | z \sim \mathcal{N}(0, I_p), T_0, T_1 \right],$$

(26)
where

\[ B_i = \Sigma_i^{1/2} (H_1 - H_0) \Sigma_i^{1/2}, \]
\[ y_i = \Sigma_i^{1/2} [H_1 (\mu_i - \bar{x}_1) - H_0 (\mu_i - \bar{x}_0)], \]
\[ \xi_i = -\log \left( \frac{|H_0|}{|H_i|} \right) + (\mu_i - \bar{x}_0)^T H_0 (\mu_i - \bar{x}_0) - (\mu_i - \bar{x}_1)^T H_1 (\mu_i - \bar{x}_1) + 2 \log \frac{\pi_1}{\pi_0}. \]

Computing \( \epsilon_i^{R-QDA} \) amounts to the cumulative distribution function (CDF) of quadratic forms of Gaussian random vectors, and hence cannot be derived in closed form in general. However, it can be still approximated by considering asymptotic regimes that allow to exploit results about central limit theorem involving quadratic forms. Under Assumptions 5-8, a central limit theorem (CLT) on the random variable \( z^T B_i z + 2 z^T y_i \) when \( z \sim N(0, I_p) \) is established.

**Proposition 4 (CLT)** Assume that assumptions 5-8 hold true. Assume also that for \( i \in \{0, 1\} \)

\[
\lim_{p \to \infty} \frac{60 \text{tr} B_i^2 + 240 \text{tr} B_i^2 \|y_i\|_2^2 + 48 \|y_i\|_2^8}{(2 \text{tr} B_i^2 + 4 \|y_i\|_2^2)^2} \to 0. \tag{27}
\]

Then,

\[
\frac{z^T B_i z + 2 z^T y_i - \text{tr} B_i}{\sqrt{2 \text{tr} B_i^2 + 4 y_i^T y_i}} \overset{d}{\to} N(0, 1). \tag{28}
\]

**Proof** The proof is mainly based on the application of the Lyapunov’s CLT for the sum of independent but non identically distributed random variables (Billingsley, 2008). The detailed proof is postponed to Appendix B.

As a by-product of the above Proposition, we obtain the following expression for the conditional classification error \( \epsilon_i \)

**Corollary 5** Under the setting of Proposition 4, the conditional classification error in (11) satisfies

\[
\epsilon_i^{R-QDA} - \Phi \left( \frac{\xi_i - \text{tr} B_i}{\sqrt{2 \text{tr} B_i^2 + 4 y_i^T y_i}} \right) \overset{\text{a.s.}}{\to} 0. \tag{29}
\]

As such an asymptotic equivalent of the conditional classification error can be derived. This is the subject of the next subsection.

### 3.2.3 Deterministic Equivalents

This part is devoted to the derivation of deterministic equivalents of some random quantities involved in the R-QDA conditional classification error. Before that, we shall introduce the following notations which basically arise as a result of applying standard results from random...
matrix theory. We define for $i \in \{0, 1\}$, $\delta_i$ as the unique positive solution to the following fixed point equation:

$$
\delta_i = \frac{1}{n_i} \text{tr} \Sigma_i \left( I_p + \frac{\gamma}{1 + \gamma \delta_i} \Sigma_i \right)^{-1}
$$

Define $T_i$ as

$$
T_i = \left( I_p + \frac{\gamma}{1 + \gamma \delta_i} \Sigma_i \right)^{-1},
$$

and the scalar $\phi_i$ and $\tilde{\phi}_i$ as

$$
\phi_i = \frac{1}{n_i} \text{tr} \Sigma_i^2 T_i^2, \quad \tilde{\phi}_i = \frac{1}{(1 + \gamma \delta_i)^2}.
$$

Define $\xi_i$, $\xi_{i-1}$ and $B_i$ as

$$
\xi_i = \frac{1}{\sqrt{p}} \left[ -\log \frac{|T_0|}{|T_1|} + \log \frac{(1 + \gamma \delta_0)^n}{(1 + \gamma \delta_1)^n} \right] + \frac{n_1}{n_1} \delta_1 \frac{n_0}{n_0} \delta_0
$$

$$
\phi_i = \frac{1}{\sqrt{p}} \text{tr} \Sigma_i (T_1 - T_0).
$$

$$
B_i = \frac{1}{\sqrt{p}} \text{tr} \Sigma_i T_1 T_0.
$$

As shall be shown in Appendix C these quantities are deterministic approximations in probability of $\xi_i$, $\xi_{i-1}$ and $B_i$. We therefore get

**Theorem 6** Under assumptions 5-8 the following convergence holds for $i \in \{0, 1\}$

$$
\epsilon_i R-QDA - \Phi \left( (1 - 1)^i \frac{\xi_i - \tilde{b}_i}{\sqrt{2B_i}} \right) \to 0.
$$

**Proof** The proof is postponed to Appendix C.

At first sight, quantity $\xi_i - \tilde{b}_i$ appears to be of order $O(\sqrt{p})$, since $\frac{1}{\sqrt{p}} \log |T_i|$ and $\frac{1}{\sqrt{p}} \text{tr} \Sigma_i T_i$ are $O(\sqrt{p})$. Following this line of thought, the asymptotic misclassification probability error is expected to converge to a trivial misclassification error. This statement is, hopefully false. Assumption 5 and 6 were carefully designed so that $\frac{1}{\sqrt{p}} \log |T_i| - \frac{1}{\sqrt{p}} \log |T_0|$ and $\frac{1}{\sqrt{p}} (n_1 \delta_1 - n_0 \delta_0)$ are of order $O(1)$. In particular, the following is proven in Appendix D.

**Proposition 7** Under Assumption 5-8 The deterministic quantities $\xi_i$ and $b_i$ are uniformly bounded when $p$ grows to infinity.

**Proof** The proof is deferred to Appendix D.

---

1. Mathematical details treating the existence and uniqueness of $\delta_i$ can be found in (Hachem et al., 2008).
Remark 8 The results of Theorem 6 along with proposition 7 show that the classification error converges to a non-trivial deterministic quantity that depends only on the statistical means and covariances within each class. The major importance of this result is that it allows to find a good choice of the regularization $\gamma$ as the value that minimizes the asymptotic classification error. While it seems to be elusive for such value to possess a closed-form expression, it can be numerically approximated by using a simple one-dimensional line search algorithm.

Remark 9 Using Assumption 8, it can be shown that $B_i$ can asymptotically simplified to

$$B_i = \frac{1}{c} \frac{\phi^2 \tilde{\phi}}{1 - \gamma^2 \phi \tilde{\phi}} + o(1).$$

(33)

where $\phi = \phi_0$ or $\phi = \phi_1$. The above relation comes from the fact that, up to an error of order $o(1)$, matrices $\Sigma_1$ or $\Sigma_0$ can be used interchangeably in $\phi_0$ or $\phi_1$ and in the terms involved in $B_i$. This, in particular, implies that $B_0$ and $B_1$ are the same up to a vanishing error. It is noteworthy to see that the same artifice could not work for the terms $\xi_i$ and $b_i$ because the normalization, being with $\frac{1}{\sqrt{p}}$, is not sufficient to provide vanishing terms.

We should also mention that, although (33) takes a simpler form, we chose to work in the simulations and when computing the consistent estimates of $B_i$ with the expression (32) since we found that it provides the highest accuracy.

3.2.4 Some Special cases

1. It is important to note that we could have considered $\|\mu_0 - \mu_1\| = O(1)$. In this case, the classification error rate would still converge to a non trivial limit but would not asymptotically depend on the difference $\|\mu_0 - \mu_1\|$. This is because in this case, the difference in covariance matrices dominate that of the means and as such represent the discriminant metric that asymptotically matters. We show this behavior in Figure 2 where it is clear that the statistical means asymptotically do not play any role in the classification task.

2. Another interesting case to highlight is the one in which $\|\Sigma_0 - \Sigma_1\| = o \left( p^{-\frac{1}{2}} \right)$. From Theorem 6 and using (33), it is easy to show that the total classification error converges as

$$\epsilon_{R-QDA} - \Phi \left( -\frac{\mu^T T \mu}{2 \sqrt{p}} \sqrt{c \left( 1 - \gamma^2 \phi \tilde{\phi} \right)} \right) \xrightarrow{p} 0,$$

(34)

where $\phi$, $\tilde{\phi}$ and $T$ have respectively the same definitions as $\phi_i$, $\tilde{\phi}_i$ and $T_i$ upon dropping the class index $i$, since quantities associated with class 0 or class 1 can be used interchangeably in the asymptotic regime. It is easy to see that in this case if $\|\mu_0 - \mu_1\|^2$ scales slower than $O \left( \sqrt{p} \right)$, classification is asymptotically impossible. This must be contrasted with the results of R-LDA, which provides non-vanishing misclassification rates for $\|\mu_0 - \mu_1\| = O(1)$. This means that in this particular setting, R-QDA is asymptotically beaten by R-LDA which achieves perfect classification. In Figure 3 we...
Figure 2: Performance in terms of the testing classification error of the regularized QDA classifier with equal training ($n_0 = n_1$), $\gamma = 1$ and $[\Sigma_0]_{i,j} = 0.6^{|i-j|}$ and $\Sigma_1 = \Sigma_0 + 2S_p$. $\mu_0 = [1, 0_{1 \times (p-1)}]^T$ and $\mu_1 = \mu_0 + \frac{0.8}{\sqrt{p}}1_{p \times 1}$. The testing error is evaluated over a testing set of size 1000 samples for both classes and averaged over 500 realizations.
show the performance of R-QDA which converges to the limit predicted by equation (34).

3. When \( \| \Sigma_0 - \Sigma_1 \|_F = O(1) \) occurring for instance when \( \| \Sigma_0 - \Sigma_1 \|_1 = O(p^{-\frac{1}{2}}) \) or \( \Sigma_0 - \Sigma_1 \) is of finite rank, and \( \| \mu_0 - \mu_1 \| = O(1) \) is nearly zero and as such the misclassification error probability associated with both classes converge respectively to \( 1 - \eta \) and \( \eta \) with \( \eta \) some probability depending solely on the statistics. The total misclassification error associated with R-QDA converges to 0.5.

The above remarks should help to draw some hints on when R-LDA or R-QDA should be used. Particularly, if the Frobenius norm of \( \Sigma_0 - \Sigma_1 \) is \( O(1) \), using the information on the difference between the class covariance matrices is not recommended. We should rather rely on using the information on the difference between the classes’ means, or in other words favoring the use of R-LDA against R-QDA.
4. General Consistent Estimator of the Testing Error

In the machine learning field, evaluating the performances of algorithms is a crucial step that not only serves to ensure their efficacy but also leads to properly set the parameters involved in the design thereof, a process known in the machine learning parlance as model selection. The traditional way to evaluate performances basically consists in devoting a part of the training data to the design of the underlying method whereas performances are tested on the remaining data called testing data, treated as unseen data since they do not intervene in the design step. Among the many existing computational methods that are built on these ideas are the cross-validation (Geisser, 1977; Lachenbruch, 1975) and the bootstrap (Efron, 1983; Efron and Tibshirani, 1994) techniques. Despite being widely used in the machine learning community, these methods have the drawbacks of being computationally expensive and most importantly of relying on mere computations, which does not lead to gain a better understanding of the performances of the underlying algorithm. As far as LDA and QDA classifiers are considered, the results of the previous section allow to gain a deeper understanding of the classification performances with respect to the covariances of means of the classes. However, as these results are expressed in terms of the unknown covariances and means, they could not be relied upon to assess the classification performances. In this section, we address this question and provide consistent estimators of the classification performances for both R-LDA and R-QDA classifiers that approximate in probability their deterministic counterparts.

4.1 R-LDA

The following theorem provides the expression of the class-conditional true error estimator $\widehat{\epsilon}_i^{R-LDA}$, for $i \in \{0,1\}$.

**Theorem 10** Under Assumptions 1-4, denote

$$
\widehat{\epsilon}_i^{R-LDA} = \Phi \left( \frac{(-1)^i G(\bar{x}_i, \bar{x}_0, \bar{x}_1, H) + \hat{\theta}_i + (-1)^i \log \frac{\pi_1}{\pi_0}}{\hat{\psi}_i \sqrt{D(\bar{x}_0, \bar{x}_1, H, \hat{\Sigma}_i)}} \right),
$$

(35)

where

$$
\hat{\theta}_i = \frac{1}{n_i} \text{tr} \hat{\Sigma}_i H
$$

(36)

$$
\hat{\psi}_i = \frac{1}{1 - \frac{2}{n_i} \text{tr} \hat{\Sigma}_i H}
$$

(37)

Then,

$$
\epsilon_i^{R-LDA} - \widehat{\epsilon}_i^{R-LDA} \xrightarrow{a.s.} 0.
$$

**Proof** The proof is postponed to Appendix E.
Remark 11 From Theorem 10, it is easy to recover the general consistent estimator of the conditional classification error constructed in (Zollanvari and Dougherty, 2015). In particular, in the case where $\Sigma_0 = \Sigma_1 = \Sigma$, we have the following

$$\frac{1}{n_i} \text{tr} \hat{\Sigma} H = \frac{1}{\gamma} \left( \frac{p}{n_i} - \frac{1}{n_i} \text{tr} H \right).$$

Thus, upon dropping the class index $i$, $\hat{\theta}$ is equivalent to $\hat{\delta}$ used in (Zollanvari and Dougherty, 2015).

4.2 R-QDA

Based on the deterministic equivalent of the conditional classification error derived in Theorem 6, we construct a general consistent estimator of $\epsilon_{i}^{R-QDA}$ denoted by $\hat{\epsilon}_{i}^{R-QDA}$. The general consistent estimator of the R-QDA misclassification error is given by the following Theorem.

**Theorem 12** Under Assumptions 5-8, define

$$\hat{\epsilon}_{i}^{R-QDA} = \Phi \left( \frac{-1}{\sqrt{2\hat{B}_i}} \left( \hat{\xi}_i - \frac{\hat{b}_i}{\hat{\delta}_i} \right) \right),$$  (38)

Then,

$$\hat{\epsilon}_{i}^{R-QDA} - \epsilon_{i}^{R-QDA} \xrightarrow{p} 0.$$  (39)

where

$$\hat{\xi}_i = -\frac{1}{\sqrt{p}} \log \frac{|H_0|}{|H_1|} + \frac{(1+i+1)}{\sqrt{p}} (\bar{x}_0 - \bar{x}_1)^T H_{1-i} (\bar{x}_0 - \bar{x}_1).$$

$$\hat{\delta}_i = \frac{1}{\gamma} \left( \frac{p}{n_i} - \frac{1}{n_i} \text{tr} H_i \right).$$

$$\hat{b}_i = \frac{(-1)^i}{\sqrt{p}} \text{tr} \hat{\Sigma}_i H_{1-i} H_i + \frac{(1+i+1)}{n_i \hat{\delta}_i}.$$  (40)

$$\hat{B}_i = \left( 1 + \gamma \hat{\delta}_i \right)^{-1} \frac{1}{p} \text{tr} \hat{\Sigma}_i H_i \hat{\Sigma}_i H_i - \frac{n_i \hat{\delta}_i^2}{p} \left( 1 + \gamma \hat{\delta}_i \right)^2 + \frac{1}{p} \text{tr} \hat{\Sigma}_i H_{1-i} \hat{\Sigma}_i H_{1-i} - \frac{n_i}{p} \left( \frac{1}{n_i} \text{tr} \hat{\Sigma}_i H_{1-i} \right)^2$$

$$- 2 \left( 1 + \gamma \hat{\delta}_i \right)^{-1} \frac{1}{p} \text{tr} \hat{\Sigma}_i H_i \hat{\Sigma}_i H_{1-i} + \hat{\delta}_i \left( 1 + \gamma \hat{\delta}_i \right)^{-1} \frac{2}{p} \text{tr} \hat{\Sigma}_i H_{1-i}.$$  (41)

**Proof** See Appendix F.

4.3 Validation with synthetic data

We validate the results of Theorems 10 and 12 by examining the accuracy of the proposed general consistent estimators in terms of the RMS defined as follows

$$\text{RMS} (\hat{\epsilon}) = \sqrt{\text{Bias} (\hat{\epsilon})^2 + \text{var} (\hat{\epsilon} - \epsilon)}.$$  (42)
where
\[
\text{Bias} (\hat{\epsilon}) = \mathbb{E} [\hat{\epsilon} - \epsilon].
\] (40)

We also compare the proposed general consistent estimator (that we denote by the G-estimator) for both R-LDA and R-QDA with the following benchmark estimation techniques fully described in [Dougherty et al., 2010]

• 5-fold cross-validation with 5 repetitions (5-CV).
• 0.632 bootstrap (B632).
• 0.632+ bootstrap (B632+).
• Plugin estimator consisting of replacing the statistics in the deterministic equivalents by their corresponding sample estimates.

The first observation is that the naive plugin estimator has the worst RMS performance for both classifiers in most cases. This is simply explained by the fact that when \(p\) and \(n_i\) have the same order of magnitude, the sample estimates are inaccurate which leads to a mediocre RMS performance. On another front, it is clear for both settings \((n_i = p/2\) and \(n_i = p\)) that the proposed G-estimator achieves a suitable RMS performance beating 5-fold cross validation and the bootstrap. In Figure 5, we examine the performance of the different error estimators against the regularization parameter. As shown in the Figure 5, R-LDA is less vulnerable to the choice of \(\gamma\) as compared to R-QDA where the choice of \(\gamma\) tends to have a higher influence on the performance. Also, for both classifiers, the proposed G-estimator is able to track the empirical error and thus permits to predict the optimal regularizer with high accuracy.

5. Experiments with real data

In this section, we examine the performance of the proposed G estimator on the public USPS dataset of handwritten digits [LeCun et al., 1998] (see Figure 6). The dataset consists of 7291 training samples of 16 \(\times\) 16 grayscale images \((p = 256\) features) and 2007 testing images [http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/multiclass.html#usps]2 First, we examine the RMS performance of the different error estimators on the data for different values of the training size and for different class labels. The RMS is determined by averaging the error over a number of training sets randomly selected from the total training dataset. As shown in Figure 7, the proposed G-estimator gives a good RMS performance especially for R-QDA where it can actually outperform state-of-the-art estimators such as cross validation and Bootstrap. Moreover, it is clear that the plugin estimator has a higher RMS performance for most of the considered scenarios.

Now, we turn our attention to finding the optimal \(\gamma\) that results in the minimum testing error. Since the construction of the G-estimator is heavily based on the Gaussian assumption of the data, picking the regularizer that minimizes the estimated error using the G-estimator will not necessarily minimize the error computed on the testing data for

2. All the results of this paper can be reproduced using our Julia codes available in [https://github.com/KhalilElkhalil/Large-Dimensional-Discriminant-Analysis-Classifiers-with-Random-Matrix-Theory]
Figure 4: RMS performance of the proposed general consistent estimators (RLDA G and RQDA G) compared with the benchmark estimation techniques. We consider equal training size ($n_0 = n_1$), $\gamma = 1$ and $[\Sigma_0]_{i,j} = 0.6|i-j|$, $\Sigma_1 = \Sigma_0 + 3S_p$, $\mu_0 = [1, 0_1 \times (p-1)]^T$ and $\mu_1 = \mu_0 + \frac{0.8}{\sqrt{p}}1_{p \times 1}$. The first row treats the case where $n_0 = p/2$ whereas the second row treats the case $n_0 = p$. The testing error is evaluated over a testing set of size 1000 samples for both classes and averaged over 1000 realizations.
A Large Dimensional Analysis of Regularized Discriminant Analysis Classifiers

Figure 5: Average misclassification rate versus the regularization parameter $\gamma$. We consider $p = 100$ features with equal training size ($n_0 = n_1 = p$), $[\Sigma_0]_{i,j} = 0.6^{|i-j|}$, $\Sigma_1 = \Sigma_0 + 3S_p$, $\mu_0 = [1, 0_{1 \times (p-1)}]^T$, and $\mu_1 = \mu_0 + \frac{0.8}{\sqrt{p}}1_{p \times 1}$. The testing error is evaluated over a testing set of size 1000 samples for both classes and averaged over 1000 realizations.

Figure 6: Samples of digits from the USPS dataset.
Figure 7: RMS performance of the proposed general consistent estimators (R-LDA G and R-QDA G) compared with the benchmark estimation techniques. We consider equal training size ($n_0 = n_1$) and $\gamma = 1$. The first gives the performance for the USPS data with digits (5, 2) whereas the second row considers the digits (5, 6).
Figure 8: Average misclassification rate versus the regularization parameter $\gamma$ of the USPS dataset for different instances of digits and assuming equal training size ($n_0 = n_1$). The first two columns are associated with the performance of the R-LDA classifier whereas the last two are associated with R-QDA. The solid red line refers to the performance of the proposed G-estimator whereas the dotted black line refers to the empirical error computed using the testing data.
Table 1: Estimates of the optimal regularizer using the two-step optimization method with their corresponding testing error.

<table>
<thead>
<tr>
<th>Dataset</th>
<th>$\hat{\gamma}$ (R-LDA)</th>
<th>$\epsilon_{\text{testing}}$ (R-LDA)</th>
<th>$\hat{\gamma}$ (R-QDA)</th>
<th>$\epsilon_{\text{testing}}$ (R-QDA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>USPS(5, 2), $n_0 = 100$</td>
<td>3.54</td>
<td>0.0307</td>
<td>0.896</td>
<td>0.0111</td>
</tr>
<tr>
<td>USPS(5, 2), $n_0 = 400$</td>
<td>20.98</td>
<td>0.0251</td>
<td>1.049</td>
<td>0.0223</td>
</tr>
<tr>
<td>USPS(5, 6), $n_0 = 100$</td>
<td>4.753</td>
<td>0.0272</td>
<td>0.567</td>
<td>0.0272</td>
</tr>
<tr>
<td>USPS(5, 6), $n_0 = 400$</td>
<td>12.1572</td>
<td>0.01818</td>
<td>0.562</td>
<td>0.009</td>
</tr>
</tbody>
</table>

USPS. One straightforward approach is to compute the testing error for all possible value of $\gamma$ in the range $(0, \infty)$, then pick the regularizer resulting in the minimum error. Obviously, this approach is far from being practical and is simply unfeasible. Motivated by this issue, we propose a two-stage optimization explained as follows.

5.1 Two-stage optimization

Although real data are far from being Gaussian, the proposed G-estimator can be used to have a glimpse on the optimal regularizer. More specifically, we can use the G-estimator to determine the interval in which the optimal regularizer is likely to belong, then we perform cross validation (or testing if we have enough testing data) for multiple values of $\gamma$ inside that interval and finally pick the value that results on the minimum cross-validation error (or testing error). As seen in Figure 8, both the R-LDA and the R-QDA G-estimators are able to mimic the real behavior of the testing error when $\gamma$ varies for both situations when $n_0 < p$ and $n_0 > p$. Similarly to synthetic data, Figure 8 also shows how R-QDA is vulnerable to the choice of $\gamma$ which justifies the need to find a good regularization parameter $\gamma$. In Table 1 we provide numerical values for the output of the two-step optimization using a confidence interval $\left(\hat{\gamma}_G - \frac{2}{\sqrt{p}}\right)^+, \hat{\gamma}_G + \frac{2}{\sqrt{p}}\right)$ with a uniform grid of 50 points where $\hat{\gamma}_G$ is a minimizer of the G-estimator built based on the Gaussian assumption.

6. Conclusion

This work permits to derive closed form expressions of the asymptotic misclassification rate for both classifiers R-LDA and R-QDA in the regime where the data dimension and the training size tend to infinite at the same pace. The obtained results allow a better fundamental understanding of the underlying classification algorithms and shed light on the way we should optimize the regularization parameter.

One way to extend this work is to consider the general framework of regularized discriminant analysis where we can consider the general class of discriminant analysis classifiers in (Friedman, 1989) that encompasses as a special case the R-LDA and R-QDA.

3. Usually, we perform cross validation or Bootstrap to have an estimate of the error from the training set, but since we have enough testing data we rely on the testing error for the USPS dataset.
4. $x^+ = \max(x, 0)$, for $x \in \mathbb{R}$.
Appendices

A. Proof of Theorem 1

A.1 Notations

Through this appendix, the following notations are used. For $i \in \{0, 1\}$, we let $X_i \in \mathbb{R}^{p \times n_i}$ the matrix of $n_i$ observations associated with class $i$. Thus, there exists $Y_i = \Sigma_i^{1/2} Z_i$ such that $X_i = \Sigma_i Z_i + \mu_i 1_{n_i}$, where $1_{n_i} \in \mathbb{R}^{p \times n_i}$ is the vector of all ones, and $Z_i = [z_{i,1}, \cdots, z_{i,n_i}] \in \mathbb{R}^{p \times n_i}$ where $z_{i,j}$ are independent random vectors with standard multivariate Gaussian distribution. We define the following resolvent matrix:

$$Q(z) = \left( \frac{Y_0 Y_0^T}{p} + \frac{Y_1 Y_1^T}{p} - zI_p \right)^{-1}$$

(41)

the behavior of which has extensively been studied in [Benaych-Georges and Couillet, 2016, Proposition 5]. Particularly, it was shown that under assumptions 1, 2 and 3, $Q(z)$ is equivalent to a deterministic matrix $\tilde{Q}(z)$ in the sense that

$$\frac{1}{p} \text{tr} M (Q(z) - \tilde{Q}(z)) \overset{p}{\to} 0,$$

$$u^T (Q(z) - \tilde{Q}(z)) v \overset{p}{\to} 0,$$

(42)

for all deterministic matrices $M$ of bounded spectral norms and all deterministic vectors $u$ and $v$ of bounded euclidean norms. Moreover, it has been shown in [Benaych-Georges and Couillet, 2016, Proposition 6] that

$$Q(z) \Sigma_i Q(z) \leftrightarrow \tilde{Q}_i(z), \text{ for } i \in \{0, 1\}.$$

(43)

where $\tilde{Q}_i$ is given in [18]. Based on the results of [42] and [43], we successively prove [22] and [23].

A.2 Proof of (22)

With the aforementioned notations at hand, it is easy to show that $\hat{\Sigma}_i$ can be expressed as

$$\hat{\Sigma}_i = \frac{1}{n_i - 1} \left( Y_i Y_i^T - Y_i \frac{1_{n_i} 1_{n_i}^T}{n_i} Y_i^T \right).$$

Let $\frac{1_{n_i} 1_{n_i}^T}{n_i} = O_i E_i O_i^T$, be the eigenvalue decomposition of $\frac{1_{n_i} 1_{n_i}^T}{n_i}$ where $E_i = \text{diag} \left( [1, 0_{(n_i-1) \times 1}] \right)$ and $O_i$ is a $n_i \times n_i$ orthogonal matrix with first column $\frac{1}{\sqrt{n_i}} 1_{n_i}$. Let $\tilde{Y}_i = Y_i O_i$. Hence

$$\hat{\Sigma}_i = \frac{1}{n_i - 1} Y_i O_i O_i^T \tilde{Y}_i^T - \frac{1}{n_i - 1} Y_i O_i E_i O_i^T Y_i^T$$

$$= \frac{1}{n_i - 1} \tilde{Y}_i \tilde{Y}_i^T - \frac{1}{n_i - 1} \tilde{Y}_i, 1 \tilde{Y}_i, 1^T,$$

(44)
where \( \bar{y}_{i,1} \) being the first column of \( \bar{Y}_i \). Since the Gaussian distribution is invariant to multiplication by a unitary matrix, \( \bar{Y}_i \) has the same distribution as \( Y_i \), and as such matrix \( H \) can be expressed as:

\[
H = \left( \mathbf{I}_n + \frac{\gamma}{n-2} \bar{Y}_0 \bar{Y}_0^T + \frac{\gamma}{n-2} \bar{Y}_1 \bar{Y}_1^T - \frac{\gamma}{n-2} \bar{Y}_{0,1} \bar{Y}_{0,1}^T - \frac{\gamma}{n-2} \bar{Y}_{1,1} \bar{Y}_{1,1}^T \right)^{-1}. \tag{45}
\]

Then, the following relation holds for \( z = -\frac{n}{p\gamma} \)

\[
H = -z \mathbf{Q}(z) + O_{\|\|}(p^{-1}),
\]

where \( O_{\|\|}(p^{-1}) \) refers to a matrix whose spectral norm is \( O(p^{-1}) \). We are now ready to handle the term \( G(\mu_i, \bar{x}_0, \bar{x}_1, H) \). To start with, we first express it as

\[
G(\mu_i, \bar{x}_0, \bar{x}_1, H) = \left( \frac{(-1)^i}{2} \mu^T - \frac{1}{2n_0} 1^T \bar{y}_0 - \frac{1}{2n_1} 1^T \bar{y}_1 \right) H \left( \frac{1}{n_0} \bar{y}_{0,1} - \frac{1}{n_1} \bar{y}_{1,1} + \mu \right).
\]

and thus can be expanded as

\[
G(\mu_i, \bar{x}_0, \bar{x}_1, H) = \left( \frac{(-1)^i}{2} \mu^T H \mu + \frac{(-1)^i}{2n_0} \mu^T \bar{Y}_0 1_{n_0} + \frac{(-1)^i+1}{2n_1} \mu^T \bar{Y}_1 1_{n_1} \right) - \frac{1}{2n_0} \mu^T \bar{Y}_0 1_{n_0} \\
- \frac{1}{2n_1} \mu^T \bar{Y}_1 1_{n_1} + \frac{1}{2n_0} \mu^T \bar{Y}_0 1_{n_0} - \frac{1}{2n_1} \mu^T \bar{Y}_1 1_{n_1} + \frac{1}{2n_1} \bar{Y}_1^T \bar{Y}_1 1_{n_1}.
\]

It follows from [45] that \( \bar{y}_{0,1} = \frac{1}{\sqrt{n_0}} Y_0 \bar{1} \) and \( \bar{y}_{1,1} = \frac{1}{\sqrt{n_1}} Y_1 \bar{1} \) are independent of \( H \). The following convergence holds thus true

\[
\frac{1}{n_0} \mu^T \bar{Y}_0 1_{n_0} \xrightarrow{a.s.} 0.
\]

\[
\frac{1}{n_1} \mu^T \bar{Y}_1 1_{n_1} \xrightarrow{a.s.} 0.
\]

\[
\frac{1}{n_1} \bar{Y}_1^T \bar{Y}_1 1_{n_1} \xrightarrow{a.s.} 0.
\]

On the other hand, we have

\[
\mu^T H \mu = -z \mu^T \mathbf{Q}(z) \mu.
\tag{46}
\]

and thus from [42]

\[
\mu^T H \mu + z \mu^T \mathbf{Q}(z) \mu \xrightarrow{p} 0.
\tag{47}
\]

Moreover,

\[
\frac{1}{n_i} 1^T \bar{y}_i^T \bar{Y}_i 1_{n_i} = \frac{1}{n_i} \bar{y}_i^T \bar{Y}_i
\]
Again, from the independence of $\tilde{y}_i$ and $H$ and the application of the trace Lemma (Couillet and Debbah [2011, Theorem 3.7]) it follows that:

$$\frac{1}{n_i^2} Y_i^T H Y_i 1_{n_i} - \frac{1}{n_i} \text{tr} \Sigma_i H \xrightarrow{a.s.} 0,$$

which gives using (42):

$$\frac{1}{n_i^2} Y_i^T H Y_i 1_{n_i} + \frac{z}{n_i} \text{tr} \Sigma_i \tilde{Q}(z) \xrightarrow{P} 0.$$

This completes the proof of (22).

A.3 Proof of (23)

Using the notations employed in the proof of (22), $D(x_0, x_1, H, \Sigma_i)$ can be expressed as:

$$D(x_0, x_1, H, \Sigma_i) = \left( \mu^T + 1_{n_0}^T Y_0 \right) H \Sigma_i H \left( \mu + 1_{n_1}^T Y_1 \right) .$$

As in the proof (22), from the independence of $\frac{1}{n_1} Y_1 1$ and $\frac{1}{n_0} Y_0 1$ of $H$, it is easy to see that the cross-products in (48) will converge to zero almost surely. We thus have:

$$D(x_0, x_1, H, \Sigma_i) = \mu^T H \Sigma_i H \mu + \frac{1}{n_0^2} Y_0^T H \Sigma_i H Y_0 1_{n_0} + \frac{1}{n_1^2} Y_1^T H \Sigma_i H Y_1 1_{n_1} .$$

Finally, we use (43) to obtain

$$\mu^T H \Sigma_i H \mu - z^2 \mu^T \tilde{Q}(z) \mu \xrightarrow{P} 0$$

$$\frac{1}{n_j^2} Y_j^T H \Sigma_i H Y_j 1_{n_j} - \frac{z^2}{n_j} \text{tr} \Sigma_j \tilde{Q}(z) \xrightarrow{P} 0, \quad j = 1 - i.$$

which completes the proof of (23).

B. Proof of Proposition 4

To reduce the amount of notations, we drop the class subscript $i$. In all the proof B plays the role of $B_i$ and $y$ plays the role of $y_i$, for $i \in \{0, 1\}$. To begin with, let $B = U_B \Sigma_B U_B^T$ be the eigenvalue decomposition of $B$, so that $z^T B z + 2z^T y$ has the same distribution as:

$$g(z) \triangleq \sum_{j=1}^p (\alpha_j z_j^2 + 2z_j \tilde{y}_j),$$

where $\tilde{y} = U_B y$, $\alpha_i$ diagonal elements of $\Sigma_B$ and $z_j$ and $\tilde{y}_j$ are respectively the $j$th entries of $\omega$ and $\tilde{y}$. Let $\Psi = [X_0, X_1]$ be the observations associated with class 0 and 1. Then, conditioning on $\Psi$, $g(z)$ is the sum of independent but not identically distributed r.v.s. $r_j = \alpha_j z_j^2 + 2z_j \tilde{y}_j$. To prove the CLT, we resort to the Lyapunov CLT Theorem, (Billingsley 2008, Theorem 27.3). We first calculate the mean and the variance of $r_j$ conditioned on $\Psi$:

$$E[r_j | \Psi] = \alpha_j$$

$$\text{var}[r_j | \Psi] = \sigma_j^2 = 2\alpha_j^2 + 4\tilde{y}_j^2.$$
Define the total variance $s_p^2$ as

$$s_p^2 = \sum_{j=1}^{p} \sigma_j^2 = 2 \text{tr} B^2 + 4 \tilde{y}^T \tilde{y}. \quad (49)$$

To prove the CLT, it suffices to check the Lyapunov's condition. Under the setting of Proposition 4,

$$\lim_{p \to \infty} \frac{1}{s_p^4} \sum_{j=1}^{p} \mathbb{E} \left[ |r_j - \alpha_j|^4 \right] = \lim_{p \to \infty} \frac{\sum_{j=1}^{p} 60 \alpha_j^4 + 240 \alpha_j^2 \tilde{y}_j^2 + 48 \tilde{y}_j^4}{2 \text{tr} B^2 + 4 \tilde{y}^T \tilde{y}} \leq \lim_{p \to \infty} \frac{60/p^2 \text{tr} B^2 + 240/p^2 \text{tr} B^2 \|y\|^2_2 + 48/p^2 \|y\|^4_2}{(2/p \text{tr} B^2 + 4/p\|y\|^2_2)^2}. \quad (50)$$

C. Proof of Theorem 6

The proof consists in showing the following convergences

$$\frac{1}{\sqrt{p}} \xi_i - \bar{\xi}_i \stackrel{a.s.}{\to} 0. \quad (50)$$

$$\frac{1}{\sqrt{p}} \text{tr} B_i - \bar{b}_i \stackrel{a.s.}{\to} 0. \quad (51)$$

$$\frac{1}{p} \text{tr} B_i^2 - \bar{B}_i \stackrel{a.s.}{\to} 0. \quad (52)$$

$$\frac{1}{p} y_i^T y_i \stackrel{a.s.}{\to} 0. \quad (53)$$

and establishing that the condition in 4 holds with probability 1. We will prove sequentially equations (50)-(53).

C.1 Proof of (50)

Using the simplified expression of $\hat{\Sigma}_i$ in (44), we can write

$$H_i = \left( I_p + \frac{\gamma}{n_i - 1} Y_i Y_i^T - \frac{\gamma}{n_i - 1} y_i y_i^T \right)^{-1}. \quad (54)$$

Recall that $\frac{1}{\sqrt{p}} \xi_i$ writes as

$$\frac{1}{\sqrt{p}} \xi_i = -\frac{1}{\sqrt{p}} \log \left| H_0 \right| + \frac{1}{\sqrt{p}} (\mu_i - \bar{x}_0)^T H_0 (\mu_i - \bar{x}_0) - \frac{1}{\sqrt{p}} (\mu_i - \bar{x}_1)^T H_1 (\mu_i - \bar{x}_1) + \frac{2}{\sqrt{p}} \log \frac{\pi_1}{\pi_0}. \quad (54)$$

Under Assumption 7, Matrix $H_i$ follows the model in Hachem et al. (2008). According to Hachem et al. (2008, Theorem 1),

$$\frac{1}{p} \log |H_i| - \frac{1}{p} \left( \log |T_i| - n_i \log (1 + \gamma \delta_i) + \gamma \frac{n_i \delta_i}{1 + \gamma \delta_i} \right) \stackrel{a.s.}{\to} 0. \quad (54)$$
The convergence holds with rate $O(p^{-1})$ hence,

$$\frac{1}{\sqrt{p}} \log |H_i| - \frac{1}{\sqrt{p}} \left( \log |T_i| - n_i \log (1 + \gamma \delta_i) + \gamma \frac{n_i \delta_i}{1 + \gamma \delta_i} \right) \xrightarrow{p} 0.$$  

and

$$\frac{1}{\sqrt{p}} \left( (\mu_i - \bar{x}_0)^T H_0 (\mu_i - \bar{x}_0) - (\mu_i - \bar{x}_1)^T H_1 (\mu_i - \bar{x}_1) \right) - \frac{(-1)^{i+1}}{\sqrt{p}} \mu_i^T T_{1-i} \mu \xrightarrow{p} 0.$$  

C.2 Proof of (51)

We start by writing $\frac{1}{\sqrt{p}} \text{tr} B_i$ as

$$\frac{1}{\sqrt{p}} \text{tr} B_i = \frac{1}{\sqrt{p}} \text{tr} \Sigma_i^{1/2} (H_1 - H_0) \Sigma_i^{1/2}$$

$$= \frac{1}{\sqrt{p}} \text{tr} \Sigma_i (H_1 - H_0)$$

$$= \frac{1}{\sqrt{p}} \text{tr} \Sigma_i H_1 - \frac{1}{\sqrt{p}} \text{tr} \Sigma_i H_0$$

From (Hachem et al., 2008), we know that

$$\frac{1}{p} \text{tr} \Sigma_i H_i - \frac{1}{p} \text{tr} \Sigma_i T_i \xrightarrow{a.s.} 0$$  (55)

where the above convergence holds with rate $O(p^{-1})$.

Thus,

$$\frac{1}{\sqrt{p}} \text{tr} B_i - \frac{1}{\sqrt{p}} \text{tr} \Sigma_i (T_1 - T_0) \xrightarrow{p} 0.$$  

which yields the convergence in probability of $\frac{1}{\sqrt{p}} \text{tr} B_i$ to $\bar{b}_i$.

C.3 Proof of (52)

To prove (52), we need the following lemma, the proof of which is omitted since it follows from the techniques established in (Hachem et al., 2008).

Lemma 13 For all $A$ and $B$ of finite spectral norms, we have the following convergence

for $i \in \{0, 1\}$

$$\frac{1}{p} \text{tr} AH_i BH_i - \left[ \frac{1}{p} \text{tr} T_i^2 AB + \frac{\gamma^2 \phi_i}{1 - \gamma^2 \phi_i \delta_i} \frac{1}{n_i} \text{tr} A \Sigma_i T_i^2 \frac{1}{n_i} \text{tr} B \Sigma_i T_i^2 \right] \xrightarrow{a.s.} 0.$$  (56)

With the above Lemma at hand, we are now ready to handle $\frac{1}{p} \text{tr} B_i^2$.

$$\frac{1}{p} \text{tr} B_i^2 = \frac{1}{p} \text{tr} \Sigma_i (H_1 - H_0) \Sigma_i (H_1 - H_0)$$

$$= \frac{1}{p} \text{tr} \Sigma_i H_1 \Sigma_i H_1 + \frac{1}{p} \text{tr} \Sigma_i H_0 \Sigma_i H_0 - \frac{2}{p} \text{tr} \Sigma_i H_0 \Sigma_i H_1.$$  

29
By product of Lemma 13, we can easily get
\[
\frac{1}{p} \text{tr} \left( \Sigma_i H_i \Sigma_i \right) - \frac{c \phi_i}{1 - \gamma^2 \phi_i} \xrightarrow{a.s.} 0. \tag{57}
\]
\[
\frac{1}{p} \text{tr} \left( \Sigma_i H_{1-i} \Sigma_i H_{1-i} - \frac{1}{p} \text{tr} \Sigma_i^2 T^2_{1-i} \right) + \frac{\gamma^2 c \phi_{1-i} \left( \frac{1}{n_i} \text{tr} \Sigma_i \Sigma_i T^2_{1-i} \right)^2}{1 - \gamma^2 \phi_{1-i} \phi_{1-i}} \xrightarrow{a.s.} 0. \tag{58}
\]
Finally, it is straightforward to obtain
\[
\frac{1}{p} \text{tr} \Sigma_i H_i \Sigma_i H_0 - \frac{1}{p} \text{tr} \Sigma_i T_i \Sigma_i T_0 \xrightarrow{a.s.} 0. \tag{59}
\]
This completes the proof of (52).

C.4 Proof of (53)
By simple manipulations, we can show that
\[
\frac{1}{p} \text{tr} \Sigma_i y_i^T y_i - \frac{1}{p} \mu^T H_j \Sigma_i \mu = 0, \tag{60}
\]
where by Assumptions 6 and 7
\[
\frac{1}{p} \mu^T H_j \Sigma_i \mu = O \left( \frac{1}{\sqrt{p}} \right).
\]
Finally, by applying the continuous mapping theorem (Serfling, 2002), we complete the proof of Theorem 6.

Now, to conclude we need to check that the condition in Proposition 4 holds with probability 1. This can be easily seen by replacing in (27), \( \frac{1}{p} \text{tr} B^2 \) by its deterministic equivalent and noting that it has order \( O(1) \).

D. Proof of Proposition 7
To prove Proposition 7, it suffices to show:
\[
\frac{1}{\sqrt{p}} \left( \frac{n_0 \delta_0}{1 + \gamma \delta_0} - \frac{n_1 \delta_1}{1 + \gamma \delta_1} \right) = O(1) \tag{61}
\]
\[
\frac{1}{\sqrt{p}} (n_1 \log (1 + \gamma \delta_1) - n_0 \log (1 + \gamma \delta_0)) = O(1) \tag{62}
\]
\[
\frac{1}{\sqrt{p}} (\log |T_0| - \log |T_1|) = O(1) \tag{63}
\]
Without loss of generality and relying on Assumption 5, we assume \( n_0 = n_1 = \frac{n}{2} \). To begin with, we first note that:
\[
\frac{\delta_0}{1 + \gamma \delta_0} - \frac{\delta_1}{1 + \gamma \delta_1} = \frac{1}{(1 + \gamma \delta_0) (1 + \gamma \delta_1)} (\delta_0 - \delta_1).
\]
On the other hand,
\[
\delta_0 - \delta_1 = \frac{2}{n} \text{tr} \Sigma_0 T_0 - \frac{2}{n} \text{tr} \Sigma_1 T_1
\]
\[
= \frac{2}{n} \text{tr} (\Sigma_0 (T_0 - T_1)) + \frac{2}{n} \text{tr} (\Sigma_0 - \Sigma_1) T_1.
\]

Recall that for invertible square matrices \( A \) and \( B \), we have the \textit{resolvent identity} given by
\[
A^{-1} - B^{-1} = A^{-1} (B - A) B^{-1}.
\] (64)

Thus,
\[
\delta_0 - \delta_1 = \frac{2}{n} \text{tr} \Sigma_0 T_0 (T_1^{-1} - T_0^{-1}) T_1 + \frac{2}{n} \text{tr} (\Sigma_0 - \Sigma_1) T_1
\]
\[
= \frac{2}{n} \text{tr} \Sigma_0 T_0 \left( \frac{\gamma}{1 + \gamma \delta_1} \Sigma_1 - \frac{\gamma}{1 + \gamma \delta_0} \Sigma_0 \right) T_1 + \frac{2}{n} \text{tr} (\Sigma_0 - \Sigma_1) T_1
\]
\[
= \frac{2 \gamma}{n} \text{tr} \Sigma_0 T_0 \left( \frac{\Sigma_1}{1 + \gamma \delta_1} - \frac{\Sigma_1}{1 + \gamma \delta_0} + \frac{\Sigma_1}{1 + \gamma \delta_0} - \frac{\Sigma_0}{1 + \gamma \delta_0} \right) T_1 + \frac{2}{n} \text{tr} (\Sigma_0 - \Sigma_1) T_1
\]
\[
= \frac{\gamma^2 (\delta_0 - \delta_1)}{(1 + \gamma \delta_0)(1 + \gamma \delta_1)} \frac{2}{n} \text{tr} \Sigma_0 T_0 \Sigma_1 T_1 - \frac{\gamma}{1 + \gamma \delta_0} \frac{2}{n} \text{tr} \Sigma_0 T_0 (\Sigma_0 - \Sigma_1) T_1 + \frac{2}{n} \text{tr} (\Sigma_0 - \Sigma_1) T_1.
\]

Therefore,
\[
(\delta_0 - \delta_1) \left[ 1 - \frac{\gamma^2}{(1 + \gamma \delta_0)(1 + \gamma \delta_1)} \frac{1}{n} \text{tr} \Sigma_0 T_0 \Sigma_1 T_1 \right] = \frac{\gamma}{1 + \gamma \delta_0} \frac{2}{n} \text{tr} \Sigma_0 T_0 (\Sigma_0 - \Sigma_1) T_1 + \frac{2}{n} \text{tr} (\Sigma_0 - \Sigma_1) T_1.
\]

or equivalently
\[
\delta_0 - \delta_1 = \left[ 1 - \frac{\gamma^2}{(1 + \gamma \delta_0)(1 + \gamma \delta_1)} \frac{1}{n} \text{tr} \Sigma_0 T_0 \Sigma_1 T_1 \right]^{-1}
\]
\[
\times \left[ -\frac{\gamma}{1 + \gamma \delta_0} \frac{2}{n} \text{tr} \Sigma_0 T_0 (\Sigma_0 - \Sigma_1) T_1 + \frac{2}{n} \text{tr} (\Sigma_0 - \Sigma_1) T_1 \right].
\]

All in all, we have
\[
\delta_0 - \delta_1 = \frac{1}{(1 + \gamma \delta_0)(1 + \gamma \delta_1)} \left[ 1 - \frac{\gamma^2}{(1 + \gamma \delta_0)(1 + \gamma \delta_1)} \frac{1}{n} \text{tr} \Sigma_0 T_0 \Sigma_1 T_1 \right]^{-1}
\]
\[
\times \left[ -\frac{\gamma}{1 + \gamma \delta_0} \frac{2}{n} \text{tr} \Sigma_0 T_0 (\Sigma_0 - \Sigma_1) T_1 + \frac{1}{n} \text{tr} (\Sigma_0 - \Sigma_1) T_1 \right].
\] (65)

To guarantee that the left hand side of (65) does not blow up, we shall prove that
\[
\liminf_p \left( 1 - \frac{\gamma^2}{(1 + \gamma \delta_0)(1 + \gamma \delta_1)} \frac{1}{n} \text{tr} \Sigma_0 T_0 \Sigma_1 T_1 \right) > 0.
\] (66)

or equivalently
\[
\limsup_p \frac{\gamma^2}{(1 + \gamma \delta_0)(1 + \gamma \delta_1)} \frac{2}{n} \text{tr} \Sigma_0 T_0 \Sigma_1 T_1 < 1.
\] (67)
For that, recall that for a symmetric matrix $A$ and non-negative definite matrix $B$ (Lee 2008), we have

$$\operatorname{tr} AB \leq \|A\| \operatorname{tr} B.$$  \hspace{1cm} (68)

Thus,

$$\frac{\gamma^2}{(1 + \gamma \delta_0)(1 + \gamma \delta_1)} \frac{1}{n} \operatorname{tr} \Sigma_0 T_0 \Sigma_1 T_1 \leq \frac{\gamma^2 \|\Sigma_0 T_0\|}{(1 + \gamma \delta_0)(1 + \gamma \delta_1)} \frac{1}{n} \operatorname{tr} \Sigma_1 T_1$$

$$= \frac{\gamma \delta_1}{1 + \gamma \delta_1} \frac{\|\Sigma_0 T_0\|}{1 + \gamma \delta_0} < 1$$

$$< \frac{\gamma \|\Sigma_0 T_0\|}{1 + \gamma \delta_0}.$$

Since $\Sigma_0$ and $T_0$ share the same eigenvectors, there exists a $\lambda$ an eigenvalue of $\Sigma_0$ such that

$$\|\Sigma_0 T_0\| = \frac{\lambda}{1 + \gamma \delta_0}.$$

Thus,

$$\frac{2 \gamma^2}{(1 + \gamma \delta_0)(1 + \gamma \delta_1)} \frac{1}{n} \operatorname{tr} \Sigma_0 T_0 \Sigma_1 T_1 < \frac{\gamma \|\Sigma_0 T_0\|}{1 + \gamma \delta_0} = \frac{\frac{\gamma \lambda}{1 + \gamma \delta_0}}{1 + \gamma \delta_0} = \frac{\frac{\gamma \lambda}{1 + \gamma \delta_0}}{1 + \gamma \delta_0} < 1.$$

Thus, (67) holds. Using Assumptions 5 and 7 and by (67)

$$\frac{1}{(1 + \gamma \delta_0)(1 + \gamma \delta_1)} \left(1 - \frac{2 \gamma^2}{(1 + \gamma \delta_0)(1 + \gamma \delta_1)} \frac{1}{n} \operatorname{tr} \Sigma_0 T_0 \Sigma_1 T_1\right)^{-1} = \mathcal{O}(1),$$

which implies using Assumption 8 that

$$\delta_0 - \delta_1 = \mathcal{O} \left(\frac{1}{\sqrt{p}}\right).$$  \hspace{1cm} (69)

This gives the claim of (61). For (62), and using the inequality $\log(x) \leq x - 1$, for $x > 0$ we can show that

$$|\log (1 + \gamma \delta_1) - \log (1 + \gamma \delta_0)|$$

$$= |\log \frac{1 + \gamma \delta_1}{1 + \gamma \delta_0}|$$

$$= |\log \left(1 + \gamma \frac{\delta_1 - \delta_0}{1 + \gamma \delta_0}\right)|$$

$$\leq \gamma |\delta_0 - \delta_1| \frac{\delta_0 - \delta_1}{1 + \gamma \min(\delta_0, \delta_1)}.$$
Following the result of (61), (62) also holds. As for (63), it suffices to notice that:

\[
\frac{1}{\sqrt{p}} \log \det T_0 T_0^{-1} = \frac{1}{\sqrt{p}} \log \det \left( I_p + T_0^\frac{1}{2} \left( \gamma \delta_1 \Sigma_1 - \gamma \delta_0 \Sigma_0 \right) T_0^\frac{1}{2} \right) \\
= \frac{1}{\sqrt{p}} \log \det \left[ I_p + \gamma \delta_1 T_0 (\Sigma_1 - \Sigma_0) + \frac{\delta_1 - \delta_0}{\delta_0} (I_p - T_0) \right]
\]

Define \( \Phi \) the matrix that has the same eigenvectors as \( \Sigma_1 - \Sigma_0 \) with eigenvalues \( \phi_i = |\lambda_i (\Sigma_1 - \Sigma_0)|, i = 1, \ldots, p \). Then, since \( T_0 \preceq I_p \), we have the following:

\[
\left| \frac{1}{\sqrt{p}} \log \det T_0 T_0^{-1} \right| \leq \frac{1}{\sqrt{p}} \log \det \left[ I_p + \gamma \delta_1 T_0^\frac{1}{2} \Phi T_0^\frac{1}{2} + \frac{\delta_1 - \delta_0}{\delta_0} I_p \right]
\]

Given that \( 0 \preceq \Phi \), we have

\[
\frac{1}{\sqrt{p}} \log \det \left[ I_p + \gamma \delta_1 T_0^\frac{1}{2} \Phi T_0^\frac{1}{2} + \frac{\delta_1 - \delta_0}{\delta_0} I_p \right] \leq \frac{1}{\sqrt{p}} \log \det \left[ I_p + \gamma \delta_1 T_0^\frac{1}{2} \Phi T_0^\frac{1}{2} \right] \\
\leq \frac{1}{\sqrt{p}} \log \det \left[ I_p + \frac{\delta_1 - \delta_0}{\delta_0} I_p \right] \leq \frac{1}{\sqrt{p}} \log \det \left[ I_p + \frac{\delta_1 - \delta_0}{\delta_0} I_p \right] = O(1)
\]

This completes the proof.

E. Proof of Theorem 10

We start the proof by showing the following:

\[
G (\bar{x}_i, \bar{x}_0, \bar{x}_1, H) + (-1)^{i+1} \hat{\theta}_i - G (\mu_i, \bar{x}_0, \bar{x}_1, H) \overset{a.s.}{\longrightarrow} 0.
\]

To this end, note that

\[
G (\bar{x}_i, \bar{x}_0, \bar{x}_1, H) = G (\mu_i, \bar{x}_0, \bar{x}_1, H) + \frac{1}{n_i} \sum_{j=1}^{n_i} \tilde{y}_{ij} H (\bar{x}_0 - \bar{x}_1).
\]

Using the same arguments used to prove (22), we can easily show that

\[
G (\bar{x}_i, \bar{x}_0, \bar{x}_1, H) = G (\mu_i, \bar{x}_0, \bar{x}_1, H) + \frac{(-1)^i}{n_i} \log \Sigma_i H + o(1)
\]

It remains now to show that

\[
\hat{\theta}_i - \frac{1}{n_i} \log \Sigma_i H \overset{a.s.}{\longrightarrow} 0. \quad (70)
\]

To this end, we examine the convergence of the quantity \( \frac{1}{n_i} \log \hat{\Sigma}_i H \). Recall from (44) that

\[
\hat{\Sigma}_i = \frac{1}{n_i - 1} \tilde{Y}_i \tilde{Y}_i^T - \frac{1}{n_i - 1} \tilde{Y}_{i,1} \tilde{Y}_{i,1}^T \\
= \frac{1}{n_i - 1} \sum_{j=1}^{n_i} \tilde{y}_{ij} \tilde{y}_{ij}^T - \frac{1}{n_i - 1} \tilde{y}_{i,1} \tilde{y}_{i,1}^T
\]
Let $H_{[j]} = \left( \gamma \hat{\Sigma} - \frac{\gamma}{n-2} \tilde{y}_{i,j} \tilde{y}_{i,j}^T + I_p \right)^{-1}$ Thus,
\[
\frac{1}{n_i} \text{tr} \hat{\Sigma}_i H = \frac{1}{n_i} \sum_{j=2}^{n_i} \frac{1}{1 + \frac{\gamma}{n-2} \tilde{y}_{i,j}^T H_{[j]} \tilde{y}_{i,j}} \frac{1}{n_i} \sum_{j=2}^{n_i} \frac{1}{1 + \frac{\gamma}{n-2} \tilde{y}_{i,j}^T H_{[j]} \tilde{y}_{i,j}}
\]
Thanks to the independence between $H_{[j]}$ and $\tilde{y}_{i,j}$ and by simple application of the trace Lemma (Couillet and Debbah, 2011), we have
\[
\frac{1}{n_i} \text{tr} \hat{\Sigma}_i H - \frac{1}{n_i} \text{tr} \Sigma_i H \frac{1 + \frac{\gamma}{n-2} \text{tr} \Sigma_i H}{1 + \frac{\gamma}{n-2} \text{tr} \Sigma_i H} \overset{a.s.}{\rightarrow} 0.
\]
By simple manipulations, we have the convergence in (70).

Now, using the same tricks consisting in using the inversion Lemma along with the trace Lemma, we obtain
\[
\hat{\psi}_i^2 D \left( x_0, \bar{x}_1, H, \hat{\Sigma}_i \right) - D \left( x_0, \bar{x}_1, H, \Sigma_i \right) \overset{a.s.}{\rightarrow} 0.
\]

F. Proof of Theorem 12

The proof consists in proving the following convergences:

\[
\hat{\xi}_i - \xi_i \overset{p}{\rightarrow} 0. \tag{71}
\]

\[
\hat{b}_i - b_i \overset{p}{\rightarrow} 0. \tag{72}
\]

\[
\hat{B}_i - B_i \overset{p}{\rightarrow} 0. \tag{73}
\]

The proof of (71) is straightforward and relies on the following facts
\[
\frac{1}{\sqrt{p}} (\mu_i - \bar{x}_i)^T H_i (\mu_i - \bar{x}_i) \overset{p}{\rightarrow} 0.
\]
\[
\frac{1}{\sqrt{p}} (\bar{x}_0 - \bar{x}_1)^T H_{1-i} (\bar{x}_0 - \bar{x}_1) - \frac{1}{\sqrt{p}} (\mu_i - \bar{x}_{1-i})^T H_{1-i} (\mu_i - \bar{x}_{1-i}) \overset{p}{\rightarrow} 0.
\]

The proof of (72) relies on the fact that $\hat{\delta}_i$ is a consistent estimator of $\delta_i$ as shown in (Zollanvari and Dougherty, 2015), the variance of which can be shown of order $O(p^{-2})$. Thus,
\[
\frac{1}{\sqrt{p}} \text{tr} \Sigma_i U_i - \frac{n_i}{\sqrt{p}} \hat{\delta}_i \overset{p}{\rightarrow} 0.
\]

Also, we have, for $i \in \{0, 1\}$,
\[
\frac{1}{\sqrt{p}} \text{tr} \hat{\Sigma}_i H_{1-i} - \frac{1}{\sqrt{p}} \text{tr} \Sigma_i H_{1-i} \overset{p}{\rightarrow} 0.
\]

which gives the convergence in (72).
F.1 Proof of (73)

The proof of (73) is a bit more involved than those of (71) and (72), as we will show in the following. The proof mainly relies on the application of the inversion lemma followed by the trace lemma [Couillet and Debbah (2011)].

Recall that

\[
\frac{1}{p} \text{tr} \mathbf{B}^2 = \frac{1}{p} \text{tr} \Sigma \mathbf{H}_1 \Sigma \mathbf{H}_1 - \frac{2}{p} \text{tr} \Sigma \mathbf{H}_0 \Sigma \mathbf{H}_1 + \frac{1}{p} \text{tr} \Sigma \mathbf{H}_0 \Sigma \mathbf{H}_0
\]

Without loss of generality, we can assume \( i = 1 \), the other case follows naturally. We start by handling the term \( \frac{1}{p} \text{tr} \Sigma_1 \mathbf{H}_1 \Sigma_1 \mathbf{H}_1 \). The common method here is to replace \( \Sigma_1 \) by its sample estimate \( \hat{\Sigma}_1 \), then compute the limit of the obtained expression and perform the necessary corrections to obtain the estimate of interest. In fact, we have

\[
\frac{1}{p} \text{tr} \hat{\Sigma}_1 \mathbf{H}_1 \hat{\Sigma}_1 \mathbf{H}_1 = \frac{1}{p} \sum_{j=1}^{n_1-2} \sum_{k=1}^{n_1-2} \frac{1}{p} \text{tr} \frac{\tilde{y}_{1,j} \tilde{y}_{1,j}^T}{n_1-1} \mathbf{H}_1 \frac{\tilde{y}_{1,k} \tilde{y}_{1,k}^T}{n_1-1} \mathbf{H}_1
\]

\[
= \frac{1}{p} \sum_{j=1}^{n_1-2} \sum_{k=1}^{n_1-2} \text{tr} \frac{\tilde{y}_{1,j} \tilde{y}_{1,j}^T}{n_1-1} \mathbf{H}_1 \frac{\tilde{y}_{1,k} \tilde{y}_{1,k}^T}{n_1-1} \mathbf{H}_1 + \frac{1}{p} \sum_{j=1}^{n_1-2} \text{tr} \frac{\tilde{y}_{1,j} \tilde{y}_{1,j}^T}{n_1-1} \mathbf{H}_1 \frac{\tilde{y}_{1,j} \tilde{y}_{1,j}^T}{n_1-1} \mathbf{H}_1
\]

Using the inversion lemma, we handle the first term in the previous equation as follows

\[
\frac{1}{p} \sum_{j=1}^{n_1-2} \sum_{k=1}^{n_1-2} \text{tr} \frac{\tilde{y}_{1,j} \tilde{y}_{1,j}^T}{n_1-1} \mathbf{H}_1 \frac{\tilde{y}_{1,k} \tilde{y}_{1,k}^T}{n_1-1} \mathbf{H}_1 = \frac{1}{p} \sum_{j=1}^{n_1-2} \sum_{k=1}^{n_1-2} \left( \frac{1}{n_1-1} \tilde{y}_{1,j} \tilde{y}_{1,j}^T \mathbf{H}_1 \frac{\tilde{y}_{1,j} \tilde{y}_{1,j}^T}{n_1-1} \mathbf{H}_1 \right)^2
\]

where

\[
\mathbf{H}_{1,j} = \left( \mathbf{I}_p + \gamma \hat{\Sigma}_i - \frac{\gamma}{n_1-1} \tilde{y}_{1,j} \tilde{y}_{1,j}^T \right)^{-1}
\]

and

\[
\mathbf{H}_{1,j,k} = \left( \mathbf{I}_p + \gamma \hat{\Sigma}_i - \frac{\gamma}{n_1-1} \tilde{y}_{1,j} \tilde{y}_{1,j}^T - \frac{\gamma}{n_1-1} \tilde{y}_{1,k} \tilde{y}_{1,k}^T \right)^{-1}
\]

We now refer to the use of the trace lemma to replace the denominator by its deterministic equivalents, thus we ultimately get

\[
\frac{1}{p} \sum_{j=1}^{n_1-2} \sum_{k=1}^{n_1-2} \text{tr} \frac{\tilde{y}_{1,j} \tilde{y}_{1,j}^T}{n_1-1} \mathbf{H}_1 \frac{\tilde{y}_{1,k} \tilde{y}_{1,k}^T}{n_1-1} \mathbf{H}_1 = \frac{n_i}{p} \left( \frac{1}{n_1-1} \text{tr} \mathbf{H}_1 \mathbf{H}_1 \right) + o(1).
\]

Using similar steps, the second term can be approximated as follows

\[
\frac{1}{p} \sum_{j=1}^{n_1-2} \sum_{k=1}^{n_1-2} \text{tr} \frac{\tilde{y}_{1,j} \tilde{y}_{1,j}^T}{n_1-1} \mathbf{H}_1 \frac{\tilde{y}_{1,j} \tilde{y}_{1,j}^T}{n_1-1} \mathbf{H}_1 = \frac{n_i}{p} \left( \frac{1}{n_1-1} \text{tr} \mathbf{H}_1 \mathbf{H}_1 \right)^2 + o(1).
\]

We thus obtain

\[
\frac{1}{p} \text{tr} \hat{\Sigma}_1 \mathbf{H}_1 \hat{\Sigma}_1 \mathbf{H}_1 = \frac{n_i}{p} \left( \frac{1}{n_1-1} \text{tr} \mathbf{H}_1 \mathbf{H}_1 \right) + \frac{n_i}{p} \left( \frac{1}{n_1-1} \text{tr} \mathbf{H}_1 \mathbf{H}_1 \right)^2 + o(1).
\]
We will now handle the term \( \frac{1}{p} \text{tr} \Sigma_1 H_0 \Sigma_1 H_0 \). Again, we start by replacing \( \Sigma_1 \) by \( \hat{\Sigma}_1 \). In doing so, we obtain:

\[
\frac{1}{p} \text{tr} \hat{\Sigma}_1 H_0 \hat{\Sigma}_1 H_0 = \frac{1}{p} \sum_{j=2}^{n-1} \sum_{k=2}^{n-1} \frac{\tilde{y}_{1,j} \tilde{y}^T_{1,j} H_0 \tilde{y}_{1,k} \tilde{y}^T_{1,k} H_0}{n_1 - 1}
\]

\[
= \frac{1}{p} \sum_{j=2}^{n-1} \left( \frac{\tilde{y}_{1,j} \tilde{y}^T_{1,j} H_0 \tilde{y}_{1,j} \tilde{y}^T_{1,j}}{n_1 - 1} \right)^2 + \frac{1}{p} \sum_{j=2}^{n-1} \sum_{k \neq j} \frac{\tilde{y}_{1,j} \tilde{y}^T_{1,j} H_0 \tilde{y}_{1,k} \tilde{y}^T_{1,k} H_0}{n_1 - 1}
\]

\[
= \frac{n_1}{p} \left( \frac{1}{n_1 - 1} \text{tr} \Sigma_1 H_0 \right)^2 + \frac{1}{p} \text{tr} \Sigma_1 H_0 \Sigma_1 H_0 + o(1)
\]

It remains now to handle the term \( \frac{1}{p} \text{tr} \Sigma_1 H_1 \Sigma_1 H_0 \). Using the same reasoning, we have:

\[
\frac{1}{p} \text{tr} \hat{\Sigma}_1 H_1 \hat{\Sigma}_1 H_0 = \frac{1}{p} \sum_{j=2}^{n-1} \sum_{k=2}^{n-1} \frac{\tilde{y}_{1,j} \tilde{y}^T_{1,j} H_1 \tilde{y}_{1,k} \tilde{y}^T_{1,k} H_0}{n_1 - 1}
\]

\[
= \frac{1}{p} \sum_{j=2}^{n-1} \frac{1}{(n_1 - 1)^2} \tilde{y}_{1,j}^T H_1 \tilde{y}_{1,j} \tilde{y}^T_{1,j} H_0 \tilde{y}_{1,j} + \frac{1}{p} \sum_{j=2}^{n-1} \sum_{k \neq j} \frac{1}{(n_1 - 1)^2} \tilde{y}_{1,j} \tilde{y}^T_{1,j} H_1 \tilde{y}_{1,k} \tilde{y}^T_{1,k} H_0
\]

Using the inversion Lemma along with the trace Lemma, we ultimately find:

\[
\frac{1}{p} \text{tr} \hat{\Sigma}_1 H_1 \hat{\Sigma}_1 H_0 = \frac{n_1}{p} \frac{1}{n_1 - 1} \text{tr} \Sigma_1 H_1 - \frac{n_1}{p} \frac{1}{n_1 - 1} \text{tr} \Sigma_1 H_0 + \frac{1}{p} \text{tr} \Sigma_1 H_1 \Sigma_1 H_0 + \frac{1}{p} \text{tr} \Sigma_1 H_1 \Sigma_1 H_0 + o(1).
\]

Now, we will put things together. We have the following

\[
\frac{1}{p} \text{tr} \Sigma_1 H_1 \Sigma_1 H_1 = \left( 1 + \frac{\gamma}{n_1 - 1} \right) \text{tr} \Sigma_1 H_1 \frac{1}{p} \text{tr} \hat{\Sigma}_1 H_1 \hat{\Sigma}_1 H_1
\]

\[
- \frac{n_1}{p} \left( \frac{1}{n_1} \text{tr} \Sigma_1 H_1 \right)^2 \left( 1 + \frac{\gamma}{n_1 - 1} \text{tr} \Sigma_1 H_1 \right)^2 + o(1).
\]

\[
\frac{1}{p} \text{tr} \Sigma_1 H_0 \Sigma_1 H_0 = \frac{n_1}{p} \frac{1}{n_1 - 1} \text{tr} \Sigma_1 H_0 - \frac{n_1}{p} \left( \frac{1}{n_1} \text{tr} \Sigma_1 H_0 \right)^2 + o(1).
\]

and

\[
\frac{1}{p} \text{tr} \Sigma_1 H_1 \Sigma_1 H_0 = \left( 1 + \frac{\gamma}{n_1 - 1} \text{tr} \Sigma_1 H_1 \right) \frac{1}{p} \text{tr} \hat{\Sigma}_1 H_0 \hat{\Sigma}_1 H_0
\]

\[
- \frac{n_1}{p} \frac{1}{n_1} \text{tr} \Sigma_1 H_0 \frac{1}{n_1} \Sigma_1 H_1 \left( 1 + \frac{\gamma}{n_1 - 1} \text{tr} \Sigma_1 H_1 \right) + o(1).
\]

A consistent estimator of \( \frac{1}{p} \text{tr} B^2 \) is thus given by

\[
\frac{1}{p} \text{tr} B^2 = \left( 1 + \frac{\gamma}{n_1 - 1} \text{tr} \Sigma_1 H_1 \right) \frac{1}{p} \text{tr} \hat{\Sigma}_1 H_1 \hat{\Sigma}_1 H_1 - \frac{n_1}{p} \left( \frac{1}{n_1} \text{tr} \Sigma_1 H_1 \right)^2 \left( 1 + \frac{\gamma}{n_1 - 1} \text{tr} \Sigma_1 H_1 \right)^2
\]

\[
+ \frac{1}{p} \text{tr} \hat{\Sigma}_1 H_0 \hat{\Sigma}_1 H_0 - \frac{n_1}{p} \left( \frac{1}{n_1} \text{tr} \Sigma_1 H_0 \right)^2 - 2 \left( 1 + \frac{\gamma}{n_1 - 1} \text{tr} \Sigma_1 H_1 \right)^2 \frac{1}{p} \text{tr} \hat{\Sigma}_1 H_1 \hat{\Sigma}_1 H_0
\]

\[
+ \frac{2n_1}{p} \frac{1}{n_1} \text{tr} \Sigma_1 H_0 \frac{1}{n_1} \text{tr} \Sigma_1 H_1 \left( 1 + \frac{\gamma}{n_1 - 1} \text{tr} \Sigma_1 H_1 \right).
\]
We replace $\frac{1}{n_1} \text{tr} \Sigma_1 H_1$ and $\frac{1}{n_1} \text{tr} \Sigma_1 H_0$ by their respective consistent estimates $\hat{\delta}_1$ and $\frac{1}{n_1} \text{tr} \hat{\Sigma}_1 H_0$ to get the consistent estimate for $\frac{1}{p} \text{tr}\mathbf{B}_1^2$. By this, we achieve the proof of the theorem.

References


