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On the Fast and Precise Evaluation of the Outage Probability of Diversity Receivers Over $\alpha-\mu$, $\kappa-\mu$, and $\eta-\mu$ Fading Channels

Chaouki Ben Issaid, Mohamed-Slim Alouini, and Raul Tempone

***Abstract*—In this paper, we are interested in determining the cumulative distribution function of the sum of $\alpha-\mu$, $\kappa-\mu$, and $\eta-\mu$ random variables in the setting of rare event simulations. To this end, we present a simple and efficient importance sampling approach. The main result of this work is the bounded relative error property of the proposed estimators. Capitalizing on this result, we accurately estimate the outage probability of multibranch maximum ratio combining and equal gain diversity receivers over $\alpha-\mu$, $\kappa-\mu$, and $\eta-\mu$ fading channels. Selected numerical simulations are discussed to show the robustness of our estimators compared to naive Monte Carlo estimators.**

***Index Terms*—$\alpha-\mu$, $\kappa-\mu$, $\eta-\mu$, importance sampling, Monte Carlo, bounded relative error, outage probability, diversity techniques.**

I. Introduction

Diversity techniques are often used to reduce the fading caused by multipath transmission channel [1, Chap. 9]. They rely on receiving multiple transmitted signal replicas affected by independent fadings. When some of these techniques are considered, one of the main challenges in evaluating the system performance is that the sum of the fading envelopes or powers is involved. Although the study of diversity receivers for many important fading channels received a great deal of attention, only few works investigated the performance of diversity receivers over $\alpha-\mu$ [2, 3], $\kappa-\mu$ [4], and $\eta-\mu$ [4] fading channels.

In fact, Stacy managed in his seminal work [5] to derive an infinite series representation of the cumulative distribution function (CDF) of the sum of generalized Gamma (GG) variates, a distribution that has the same functional form as the $\alpha-\mu$ distribution. However, the author points out that such representation presents certain computational challenges. In [6], Piboongungon *et al.* investigated the average symbol error rate (SER) performance of both maximum ratio combining (MRC) and equal gain combining (EGC) receivers over GG fading channels. The authors used the moment generation function (MGF)-based approach [1] in the case of MRC receivers and the characteristic function (CHF)-based approach [7] for the EGC receivers. In [8], Sagias *et al.* proposed an upper bound for the sum of GG random variables (RVs) for the purpose of studying the performance of EGC receivers. First, they derived the expression of the CDF of the product of GG variates. Then, by exploiting the well-known inequality between the arithmetic and geometric means, the authors were able to provide an upper bound for the problem of the sum. Later on, da Costa *et al.* presented a highly accurate closed-form approximations to the probability density function (PDF) and CDF of the sum of independent identically distributed (i.i.d.) $\alpha-\mu$ variates [9]. In this work, the authors approximated the sum of i.i.d $\alpha-\mu$ RVs by one $\alpha-\mu$ RV. To determine the parameters α and μ of the approximate distribution, a moment-based method was introduced. This result was used to approximate the outage probability as well as the average bit error probability for EGC and MRC diversity techniques. Although numerical simulations show a good agreement between the approximate and the exact solution, the proposed approach is restricted to the i.i.d case. Recently, closed-form expressions for the SER of EGC and MRC receivers over $\alpha-\mu$ fading were derived in [10]. Using the Mellin transform, El Ayadi *et al.* expressed the SER in terms of the Fox H-functions.

For the $\kappa-\mu$ fading, Peppas investigated the sum of non-identical squared $\kappa-\mu$ variates in [11]. He provided a generalized Laguerre polynomial expansion for the PDF as well as the CDF of the sum. Using these results, the outage probability, the average capacity, and the average bit error probability (ABEP) of multibranch MRC receivers over $\kappa-\mu$ fading have been studied. In [12], the authors studied the outage probability of multibranch MRC receivers over $\kappa-\mu$ fading exploiting the fact that the sum of squared i.i.d $\kappa-\mu$ variates is a $\kappa-\mu$ RV with specific parameters. Da Costa et al. presented in [13] accurate closed-form approximations of the sum of i.i.d $\eta-\mu$ and $\kappa-\mu$. In fact, the sum is approximated by a single variate of the same type for which the parameters are determined using a moment-matching method. These approximations are used to investigate the ABEP as well as the level crossing rate of EGC receivers.

Peppas et al. presented in [14] an infinite series, an integral representation and a closed-form expression for the sum of independent and not necessarily identically distributed (i.n.i.d) squared $\eta-\mu$ RVs. However, the authors encouraged the use of the infinite series and integral representation instead of the closed-form expression to obtain accurate and efficient results, especially for large number of summands. Both representations involve the numerical evaluation of integrals. The performance of 1-D and 2-D RAKE receivers, specifically the outage probability, the ABEP and the Shannon capacity, are investigated. In [15], the authors derived closed-form expressions for the

The authors are in the Computer, Electrical and Mathematical Science and Engineering (CEMSE) Division, King Abdullah University of Science and Technology (KAUST), Thuwal, Makkah Province, Saudi Arabia (e-mail: {chaouki.benissaid, slim.alouini, raul.tempone}@kaust.edu.sa).

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ASEP of different digital modulation schemes using MRC diversity technique over $\eta-\mu$ fading channels, assumed to be i.n.i.d. These expression are given in terms of the Lauricella and Appell hypergeometric functions, evaluated numerically using their integral or infinite series representations. Ansari et al. [16] derived closed-from expressions for the PDF and the CDF of the sum of squared i.n.i.d $\eta-\mu$ RVs in terms of the Fox's $\bar{H}$ function. They also provided a closed-form expression for the average bit error rate (ABER) of different binary modulations with multibranch MRC scheme in terms of the extended Fox's $\bar{H}$ function, $\hat{H}$.

In summary, most of the attempts fail to provide a closed-form expression for these cumbersome problems and only propose to tackle it by deriving approximate solutions that usually involve a truncation error. In fact, determining an exact expression for the outage probability may turn to be a challenging task in many cases, and to the best of our knowledge, no closed-form results for the outage probability of multibranch diversity receivers operating over $\alpha-\mu$, $\kappa-\mu$, or $\eta-\mu$ fading channels were derived in the literature. In this case, the only way to study the system performance is by means of numerical simulations, e.g. using Monte Carlo (MC) method. Due to the simplicity of its implementation, MC can be seen as a powerful technique when the problem is too complex and difficult to be solved analytically. However, it proves its inefficiency to estimate a rare event probability, i.e. probability lower than 10^{-8}, since too many samples are needed to guarantee a good quality estimator. To solve this problem, many accelerated simulation methods have been proposed in the literature. Recently, some efforts have been made to propose efficient estimators to evaluate the outage probability with diversity techniques for wireless communication systems. For instance, the authors in [17] have addressed this problem using two unified importance sampling (IS) schemes. However, the work was limited to the i.i.d case. Later on, the authors proposed in [18] an IS scheme that evaluates the outage probability of multibranch diversity receivers over Gamma-Gamma fading channels. The proposed approach requires much less samples comparing to conventional MC schemes for achieving a given accuracy. In this paper, we show that the proposed approach in [18] can be extended to estimate accurately the left tail probability of the sum of i.n.i.d $\alpha-\mu$, $\kappa-\mu$, and $\eta-\mu$ variates. Due to its simplicity, the shifting approach is seen as one of the desirable biasing technique when dealing with the estimation of very small probabilities. In addition to that, no extra computational effort will be made when sampling from the biased PDF. To prove the bounded relative error property for the $\eta-\mu$ scenario, we derived, to the best of our knowledge, a novel asymptotic expansion of the Yacoub's integral with respect to the second argument around zero. Part of this work, dealing only with the $\alpha-\mu$ fading scenario, was presented in [19].

The rest of this paper is organized as follows. First, we describe the problem setting in Section II. In section III, we present our approach to estimate the outage probability for diversity receivers over $\alpha-\mu$, $\kappa-\mu$, and $\eta-\mu$ fading channels as well as the theorems proving the efficiency of the proposed methods in the i.n.i.d case. Prior to concluding in Section IV, we show some selected simulation results related to the evaluation of the outage probability of multibranch diversity receivers over $\alpha-\mu$, $\kappa-\mu$, and $\eta-\mu$ fading channels. We also compare the computational efficiency of our approach compared to naive MC.

II. System Model

The instantaneous signal-to-noise ratio (SNR) expression at the diversity receiver, is given by [20]

$$\gamma_{end} = \frac{E_s}{N_0\sqrt{L^{1-p+q}}}\left(\sum_{\ell=1}^{L} X_\ell^p\right)^q, \quad (1)$$

where $(p,q)=(1,2)$ for the EGC case and $(p,q)=(2,1)$ for the MRC case. The ratio $\frac{E_s}{N_0}$ is the SNR per symbol at the transmitter, L is the number of diversity branches, and $\{X_\ell\}_{\ell=1}^L$ are the channel gains which are modeled as i.n.i.d RVs.

A. $\alpha-\mu$ fading

The $\alpha-\mu$ (or GG) distribution is a generic model that covers Weibull, Nakagami-m, Gamma and other distributions as special cases. In this case, $\{X_\ell\}_{\ell=1}^L$ in (1) are modeled as i.n.i.d $\alpha-\mu$ RVs with parameters $(\alpha_\ell,\mu_\ell,\Omega_\ell)$, $\ell=1,\ldots,L$, whose PDFs are given by [3, Eq. (1)]

$$f_{X_\ell}(x) = \frac{\alpha_\ell \mu_\ell^{\mu_\ell} x^{\alpha_\ell\mu_\ell-1}}{\Omega_\ell^{\mu_\ell}\Gamma(\mu_\ell)}\exp\left(-\frac{\mu_\ell}{\Omega_\ell}x^{\alpha_\ell}\right), x\geq 0, \\ \ell=1,\ldots,L, \quad (2)$$

where α_ℓ and μ_ℓ are two positive real numbers that represent the distribution shape parameters, $\Gamma(\cdot)$ is the Gamma function [21, Sec. (8.31)], and Ω_ℓ is linked to the mean of X_ℓ as

$$\mathbb{E}\left[X_\ell\right] = \left(\frac{\Omega_\ell}{\mu_\ell}\right)^{\frac{1}{\alpha_\ell}}\frac{\Gamma\left(\mu_\ell+\frac{1}{\alpha_\ell}\right)}{\Gamma(\mu_\ell)},\ \ell=1,\ldots,L. \quad (3)$$

B. $\kappa-\mu$ fading

The $\kappa-\mu$ distribution generalizes other fading distributions such as Rice and Nakagami-m fadings. The PDF of $\{X_\ell\}_{\ell=1}^L$ in (1) is thereby given by [4, Eq. (1)]

$$f_{X_\ell}(x) = \frac{2\mu_\ell\left(1+\kappa_\ell\right)^{\frac{\mu_\ell+1}{2}}x^{\mu_\ell}}{\kappa_\ell^{\frac{\mu_\ell-1}{2}}\exp(\mu_\ell\kappa_\ell)\Omega_\ell^{\frac{\mu_\ell+1}{2}}}\exp\left(-\frac{\mu_\ell(1+\kappa_\ell)}{\Omega_\ell}x^2\right) \\ \times I_{\mu_\ell-1}\left(2\mu_\ell\sqrt{\frac{\kappa_\ell(1+\kappa_\ell)}{\Omega_\ell}}x\right), x\geq 0,\ \ell=1,\ldots,L, \quad (4)$$

where κ_ℓ and μ_ℓ are two positive real numbers, $I_\nu(\cdot)$ is the νth-order modified Bessel function of the first kind [21, Sec. (8.431)], and the mean of X_ℓ, $\ell=1,\ldots,L$, is given by

$$\mathbb{E}\left[X_\ell\right] = \left(\frac{\Omega_\ell}{\mu_\ell(1+\kappa_\ell)}\right)^{\frac{1}{2}}\frac{\Gamma\left(\mu_\ell+\frac{1}{2}\right){}_1F_1\left[\mu_\ell+\frac{1}{2};\mu_\ell;\kappa_\ell\mu_\ell\right]}{\exp(\mu_\ell\kappa_\ell)\Gamma(\mu_\ell)}, \quad (5)$$

where ${}_1F_1\left[\cdot;\cdot;\cdot\right]$ is the confluent hypergeometric function [21, Eq. (9.210.1)].

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C. $\eta-\mu$ fading

The Hoyt (Nakagami-q), the one-sided Gaussian, the Rayleigh, and the Nakagami-m distributions are special cases of the $\eta-\mu$ distribution whose PDF is [4, Eq. (17)]

$$f_{X_\ell}(x) = \frac{4\sqrt{\pi}\mu_\ell^{\mu_\ell+\frac{1}{2}} h_\ell^{\mu_\ell} x^{2\mu_\ell}}{\Gamma(\mu_\ell)\Omega_\ell^{\mu_\ell+\frac{1}{2}} H_\ell^{\mu_\ell-\frac{1}{2}}} \exp\left(-\frac{2\mu_\ell h_\ell}{\Omega_\ell}x^2\right) \times I_{\mu_\ell-\frac{1}{2}}\left(\frac{2\mu_\ell H_\ell}{\Omega_\ell}x^2\right), x \geq 0,\ \ell = 1,\ldots,L. \quad (6)$$

Here, we consider the Format 1 of the $\eta-\mu$ distribution where $\eta_\ell > 0$ and the expressions of h_ℓ and H_ℓ are given by

$$h_\ell = \frac{2+\eta_\ell^{-1}+\eta_\ell}{4}, \quad (7)$$

$$H_\ell = \frac{\eta_\ell^{-1}-\eta_\ell}{4}. \quad (8)$$

The mean of X_ℓ, $\ell = 1,\ldots,L$, can be written as

$$\mathbb{E}[X_\ell] = \left(\frac{\Omega_\ell}{2\mu_\ell}\right)^{\frac{1}{2}} \frac{\Gamma\left(2\mu_\ell+\frac{1}{2}\right)}{h_\ell^{\mu_\ell+\frac{1}{2}}\Gamma(2\mu_\ell)} \times {}_2F_1\left[\mu_\ell+\frac{3}{4}, \mu_\ell+\frac{1}{4}; \mu_\ell+\frac{1}{2}; \left(\frac{H_\ell}{h_\ell}\right)^2\right], \quad (9)$$

where ${}_2F_1[\cdot,\cdot;\cdot;\cdot]$ is the Gauss confluent hypergeometric function [21, Eq. (9.14.2)].

The quality of a communication system can be evaluated by computing the outage probability. This metric is a function of the transmission technique used, but also the channel on which the signal is transmitted. More specifically, for a given threshold γ_{th}, the outage probability P is defined as the probability that the instantaneous SNR drops below γ_{th}, i.e.

$$P = \mathbb{P}(\gamma_{end} \leq \gamma_{th}) = \mathbb{P}\left(\sum_{\ell=1}^{L} X_\ell^p \leq \left(\frac{N_0}{E_s}\sqrt{L^{1-p+q}}\gamma_{th}\right)^{\frac{1}{q}}\right). \quad (10)$$

We can see that our goal of estimating the outage probability reduces to find the CDF of the sum of powers of $\alpha-\mu$, $\kappa-\mu$, or $\eta-\mu$ RVs. In particular, we are interested in the case where the outage probability requirements are very low, i.e. in the range of 10^{-6} to 10^{-10}. For instance, this is common in areas such as wireless back-hauling using free space optics (FSO) [22] and millimeter wave [23].

III. Proposed Approach

Rare events are events with a very small probability of occurrence but have a significant contribution to the MC estimation. The IS method [24] is one of the most used approaches in the evaluation of rare events probabilities. The basic idea behind is to modify the dynamics of the simulation so that the rare event happens more frequently. This can be accomplished by changing the underlying PDF of the RV in question. The goal is thereby to find a clever change in probability that, given a certain confidence interval, reduces the number of required simulation runs. It may happen that IS does not improve the estimation when a poor choice of the new PDF, known as the biased PDF, is introduced. In fact, a bad choice may lead to a large likelihood ratio and thus we end up with an estimator having a larger variance than the MC estimator. The effectiveness of this method relies on the good change of the underlying PDF. The reader is directed to [25] for a succinct review of the use of IS in communication systems.

The outage probability is given by $P = \mathbb{E}\left[\mathbb{1}_{(S_{L,p}\leq\gamma_{p,q})}\right]$, where $\mathbb{E}[\cdot]$ is the expectation with respect to (w.r.t) the probability measure under which the PDF of X_ℓ is $f_{X_\ell}(\cdot)$, $\ell = 1,2,\ldots,L$, $\gamma_{p,q} = \left(\frac{N_0}{E_s}\sqrt{L^{1-p+q}}\gamma_{th}\right)^{\frac{1}{q}}$, and $S_{L,p} = \sum_{\ell=1}^{L} X_\ell^p$. The naive MC estimator of (10) is thus

$$\hat{P}_{MC} = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{(S_{L,p}(\omega_i)\leq\gamma_{p,q})}, \quad (11)$$

where N is the number of MC samples, $\mathbb{1}_{(\cdot)}$ is the indicator function, and $\{S_{L,p}(\omega_i)\}_{i=1}^N$ are i.i.d. realizations of the RV $S_{L,p}$. The sequence $\{X_\ell(\omega_i)\}_{\ell=1}^L$ is sampled independently according to the PDFs (2) for each realization of $S_{L,p}$.

By introducing new biased densities $\{f^*_{X_\ell}(\cdot)\}_{\ell=1}^L$, we can re-write $P = \mathbb{E}^*\left[\mathbb{1}_{(S_{L,p}\leq\gamma_{p,q})}\mathcal{L}(X_1,\ldots,X_L)\right]$ where $\mathbb{E}^*[\cdot]$ denotes the expectation w.r.t the probability measure under which the PDF of X_ℓ is $f^*_{X_\ell}(\cdot)$, $\ell = 1,2,\ldots,L$ and the likelihood ratio $\mathcal{L}(X_1,\ldots,X_L)$ is defined as

$$\mathcal{L}(X_1,\ldots,X_L) = \prod_{\ell=1}^{L} \frac{f_{X_\ell}(X_\ell)}{f^*_{X_\ell}(X_\ell)}. \quad (12)$$

In this case, the IS estimator of (10) is

$$\hat{P}_{IS} = \frac{1}{N^*}\sum_{i=1}^{N^*} \mathbb{1}_{(S_{L,p}(\omega_i)\leq\gamma_{p,q})}\mathcal{L}(X_1(\omega_i),\ldots,X_L(\omega_i)), \quad (13)$$

where for each realization $i = 1,\ldots,N^*$ and the sequence $\{X_\ell(\omega_i)\}_{\ell=1}^L$ are sampled independently according to the biased PDFs $\{f^*_\ell(\cdot)\}_{\ell=1}^L$.

IS consists in running the simulations by exhibiting a change in the original distribution, for which the studied event is no longer rare. To correct the bias introduced by this manipulation, the simulations are weighted by the likelihood ratio which is the Radon-Nikodym density of the original distribution w.r.t the biased one. The goal of a IS simulation is therefore to provide biased PDFs $\{f^*_{X_\ell}(\cdot)\}_{\ell=1}^L$ in order to reduce the variance of the estimator. The versatility of the IS method is quite large due to the different possible choices of the biased PDFs and thereby the challenge is to make the right choice of such PDFs. An inadequate choice can produce a large likelihood ratio resulting in potentially more time-consuming computations than naive MC simulation. To assess the goodness of an IS approach, many criteria has been introduced in previous works (see, for instance, [26] and references therein) among them we find the bounded relative error, one of the desirable property in the field of rare events algorithms.

In this section, a clever choice of the biased PDF is introduced. In fact, we propose to introduce the parameter $\Omega^*_\ell = \Omega_\ell - \Omega_{0,\ell}$ in the new biased PDF where $\Omega_{0,\ell}$ satisfies

$0 \leq \Omega_{0,\ell} < \Omega_\ell$ and as $\gamma_{p,q} \to 0$, it approaches Ω_ℓ, $\ell = 1, \ldots, L$. This choice is justified by the fact that the biased PDF belongs to the same family as the original PDF so the sampling from it should be simple and no extra effort will be dedicated to this task. For the selection of the parameters $\{\Omega_{0,\ell}\}_{\ell=1}^{L}$, we require that the equation $\mathbb{E}^*[S_{L,p}] = \gamma_{p,q}$ holds. This equation has infinitely many solutions among which we choose a particular one.

A. EGC case

1) $\alpha-\mu$ fading: In this case, the biased PDF is

$$f^*_{X_\ell}(x) = \frac{\alpha_\ell \mu_\ell^{\mu_\ell} x^{\alpha_\ell \mu_\ell - 1}}{(\Omega_\ell - \Omega_{0,\ell})^{\mu_\ell}\Gamma(\mu_\ell)} \exp\left(-\frac{\mu_\ell x^{\alpha_\ell}}{\Omega_\ell - \Omega_{0,\ell}}\right), x \geq 0, \quad \ell = 1, \ldots, L. \tag{14}$$

For the choice of $\Omega_{0,\ell}$, we pick a solution of the form

$$\Omega_{0,\ell} = \Omega_\ell - \theta_\ell \gamma_{1,2}^{\alpha_\ell}, \forall \ell = 1, \ldots L, \tag{15}$$

where $\theta_\ell = \left[\frac{\mu_\ell \Gamma(\mu_\ell)}{\Gamma(\mu_\ell + \frac{1}{\alpha_\ell})L}\right]^{\alpha_\ell}$.

The following theorem characterizes the efficiency of our proposed IS estimator for the estimation of the outage probability (10) for L-branch diversity receivers over $\alpha-\mu$ fading.

Theorem 1. Let $\{X_\ell\}_{\ell=1}^{L}$ be a sequence of i.n.i.d $\alpha-\mu$ RVs and $f^*_{X_\ell}(\cdot)$ be defined as in (14) where $\Omega_{0,\ell}$ is given by (15). Then, the IS estimator (13) has a bounded relative error, i.e.

$$\limsup_{\gamma_{1,2}\to 0} \frac{\mathbb{E}^*\left[\mathbb{1}_{(S_{L,1}\leq \gamma_{1,2})}\mathcal{L}^2(X_1, \ldots, X_L)\right]}{P^2} < +\infty. \tag{16}$$

provided that $\min_{1\leq \ell \leq L} \alpha_\ell > 1$.

Proof: See Appendix A. ■

2) $\kappa-\mu$ fading: The biased PDF, in the case of the $\kappa-\mu$ fading, can be written as

$$f^*_{X_\ell}(x) = \frac{2\mu_\ell (1+\kappa_\ell)^{\frac{\mu_\ell+1}{2}} x^{\mu_\ell} \exp\left(-\frac{\mu_\ell(1+\kappa_\ell)}{\Omega_\ell - \Omega_{0,\ell}}x^2\right)}{\kappa_\ell^{\frac{\mu_\ell-1}{2}} \exp(\mu_\ell\kappa_\ell)(\Omega_\ell - \Omega_{0,\ell})^{\frac{\mu_\ell+1}{2}}} \times I_{\mu_\ell-1}\left(2\mu_\ell\sqrt{\frac{\kappa_\ell(1+\kappa_\ell)}{\Omega_\ell - \Omega_{0,\ell}}}x\right), x \geq 0, \ \ell = 1, \ldots, L. \tag{17}$$

In this case, the expression of $\Omega_{0,\ell}$ is given by

$$\Omega_{0,\ell} = \Omega_\ell - \delta_\ell \left(\frac{\gamma_{1,2}}{L}\right)^2, \tag{18}$$

where $\delta_\ell = \left(\frac{\Gamma(\mu_\ell)}{\Gamma(\mu_\ell+\frac{1}{2})}\right)^2 \frac{\mu_\ell(1+\kappa_\ell)\exp(2\mu_\ell\kappa_\ell)}{{}_1F_1\left[\mu_\ell+\frac{1}{2};\mu_\ell;\kappa_\ell\mu_\ell\right]^2}$.

With the value of $\Omega_{0,\ell}$ at hand, we characterize in the following theorem the robustness of the proposed IS approach.

Theorem 2. Let $\{X_\ell\}_{\ell=1}^{L}$ be a sequence of i.n.i.d $\kappa-\mu$ RVs and $f^*_{X_\ell}(\cdot)$ be defined as in (17) where $\Omega_{0,\ell}$ is given by (18). The IS estimator (13) possesses a bounded relative error, i.e.

$$\limsup_{\gamma_{1,2}\to 0} \frac{\mathbb{E}^*\left[\mathbb{1}_{(S_{L,1}\leq \gamma_{1,2})}\mathcal{L}^2(X_1, \ldots, X_L)\right]}{P^2} < +\infty. \tag{19}$$

provided that $\min_{1\leq \ell \leq L} \mu_\ell > 1$.

Proof: See Appendix B. ■

3) $\eta-\mu$ fading: In this scenario, the biased PDF is given by

$$f^*_{X_\ell}(x) = \frac{4\sqrt{\pi}\mu_\ell^{\mu_\ell+\frac{1}{2}} h_\ell^{\mu_\ell} x^{2\mu_\ell} \exp\left(-\frac{2\mu_\ell h_\ell}{\Omega_\ell - \Omega_{0,\ell}}x^2\right)}{\Gamma(\mu_\ell)(\Omega_\ell - \Omega_{0,\ell})^{\mu_\ell+\frac{1}{2}} H_\ell^{\mu_\ell-\frac{1}{2}}} \times I_{\mu_\ell-\frac{1}{2}}\left(\frac{2\mu_\ell H_\ell}{\Omega_\ell - \Omega_{0,\ell}}x^2\right), x \geq 0, \ \ell = 1, \ldots, L. \tag{20}$$

Choosing $\Omega_{0,\ell}$ of the form

$$\Omega_{0,\ell} = \Omega_\ell - \beta_\ell \left(\frac{\gamma_{1,2}}{L}\right)^2, \tag{21}$$

where $\beta_{\ell,2} = \left(\frac{\Gamma(2\mu_\ell)h_\ell^{\mu_\ell+\frac{1}{2}}\sqrt{2\mu_\ell}}{\Gamma(2\mu_\ell+\frac{1}{2})\,{}_2F_1\left[\mu_\ell+\frac{3}{4},\mu_\ell+\frac{1}{4};\mu_\ell+\frac{1}{2};\left(\frac{H_\ell}{h_\ell}\right)^2\right]}\right)^2$, leads to an IS estimator with a bounded relative error.

Theorem 3. Let $\{X_\ell\}_{\ell=1}^{L}$ be a sequence of i.n.i.d $\eta-\mu$ RVs and $f^*_{X_\ell}(\cdot)$ be defined as in (20) where $\Omega_{0,\ell}$ is given by (21). The IS estimator (13) is endowed with the bounded relative error property

$$\limsup_{\gamma_{1,2}\to 0} \frac{\mathbb{E}^*\left[\mathbb{1}_{(S_{L,1}\leq \gamma_{1,2})}\mathcal{L}^2(X_1, \ldots, X_L)\right]}{P^2} < +\infty. \tag{22}$$

provided that $\min_{1\leq \ell \leq L} \mu_\ell > \frac{1}{2}$.

Proof: See Appendix C. ■

B. MRC case

1) $\alpha-\mu$ fading: We show briefly how our approach is easily extendable to the MRC case. First, we recall the expression of the outage probability in this case

$$P = \mathbb{P}\left(\sum_{\ell=1}^{L} X_\ell^2 \leq \eta_0 = \frac{N_0}{E_s}\gamma_{th}\right). \tag{23}$$

where X_ℓ^2 is also a $\alpha-\mu$ RV with parameters $(\frac{\alpha_\ell}{2}, \mu_\ell, \Omega_\ell)$. As we can see from (23), the problem is again reduced to finding the CDF of the sum of $\alpha-\mu$ variates. Using the approach described in subsection III-A1, we can easily evaluate the outage probability given by (23) for the MRC scenario. A similar theorem to Thm. 1 holds in the MRC case and its proof can be found in Appendix A.

2) $\kappa-\mu$ fading: Inspired from the EGC case, we introduce the biased PDF of a squared $\kappa-\mu$ as

$$f^*_{X_\ell}(x) = \frac{2\mu_\ell (1+\kappa_\ell)^{\frac{\mu_\ell+1}{2}} x^{\frac{\mu_\ell-1}{2}} \exp\left(-\frac{\mu_\ell(1+\kappa_\ell)}{\Omega_\ell - \Omega_{0,\ell}}x\right)}{\kappa_\ell^{\frac{\mu_\ell-1}{2}} \exp(\mu_\ell\kappa_\ell)(\Omega_\ell - \Omega_{0,\ell})^{\frac{\mu_\ell+1}{2}}} \times I_{\mu_\ell-1}\left(2\mu_\ell\sqrt{\frac{\kappa_\ell(1+\kappa_\ell)}{\Omega_\ell - \Omega_{0,\ell}}x}\right), x \geq 0, \ \ell = 1, \ldots, L. \tag{24}$$

In this case, to ensure that the IS estimator has the bounded relative error, we choose $\Omega_{0,\ell} = \Omega_\ell - \frac{\gamma_{2,1}}{L}$. The proof of the bounded relative error property can be found in Appendix B.

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3) $\eta-\mu$ fading: In the MRC case, the biased PDF of a squared $\eta-\mu$ is given by

$$f^*_{X_\ell}(x) = \frac{4\sqrt{\pi}\mu_\ell^{\mu_\ell+\frac{1}{2}} h_\ell^{\mu_\ell} x^{\mu_\ell-\frac{1}{2}}}{\Gamma(\mu_\ell)\Omega_\ell^{\mu_\ell+\frac{1}{2}} H_\ell^{\mu_\ell-\frac{1}{2}}} \exp\left(-\frac{2\mu_\ell h_\ell}{\Omega_\ell}x\right) \times I_{\mu_\ell-\frac{1}{2}}\left(\frac{2\mu_\ell H_\ell}{\Omega_\ell}x\right), x \geq 0,\ \ell = 1,\ldots,L. \quad (25)$$

with $\Omega_{0,\ell} = \Omega_\ell - \frac{\gamma_{2,1}}{L}$. With this choice, we prove, in Appendix C, that the IS estimator satisfies the bounded relative error criterion.

Our proposed IS approach can be used in both EGC and MRC cases for the estimation of the outage probability over $\alpha-\mu$, $\kappa-\mu$, and $\eta-\mu$ fadings. All proposed IS estimators are endowed with the bounded relative error. In other terms, to accurately estimate the probability P, the naive MC simulation requires a number of samples of the order of $\mathcal{O}(P^{-1})$. However, for the same accuracy requirement and when the IS estimator is endowed with the bounded relative error property, the number of simulation runs N required remains bounded independently of how small the outage probability P is.

To compare the efficiency of IS to naive MC, we need to compare the number of simulation runs required by each method to achieve the same accuracy requirement ε. To this end, we introduce the relative error of naive MC simulation

$$\varepsilon = \frac{C}{P}\sqrt{\frac{P(1-P)}{N}}, \quad (26)$$

where $C = 1.96$ which corresponds to a 95% confidence interval.

Similarly, we define the relative error of the IS method as

$$\varepsilon^* = \frac{C}{P}\sqrt{\frac{\mathbb{V}^*[\mathbb{1}_{(S_{L,p}\leq\gamma_{p,q})}\mathcal{L}(X_1,\ldots,X_L)]}{N}}. \quad (27)$$

where $\mathbb{V}^*[\cdot]$ denotes the variance w.r.t the probability measure under which the PDF of X_ℓ is $f^*_{X_\ell}(\cdot)$.

Let ϵ_0 be a fixed accuracy requirement. Using Eqs. (26) and (27), we can determine the number of samples needed by naive MC and IS simulations respectively

$$N = P(1-P)\left(\frac{C}{P\epsilon_0}\right)^2, \quad (28)$$

$$N^* = \mathbb{V}^*[\mathbb{1}_{(S_{L,p}\leq\gamma_{p,q})}\mathcal{L}(X_1,\ldots,X_L)]\left(\frac{C}{P\epsilon_0}\right)^2. \quad (29)$$

IV. Simulation Results

This section presents the numerical simulations regarding the estimation of the outage probability using both naive MC and our proposed IS method. The accuracy, as well as the efficiency, of both methods is analyzed. Table I details the fading parameters (α_ℓ,μ_ℓ), (κ_ℓ,μ_ℓ), and (η_ℓ,μ_ℓ) used in this section for two scenarios $L=3$ and $L=4$.

TABLE I
Fading parameters used to simulate the outage probability of L-branch diversity receivers over generalized fading models

Fading		Fading parameters
$\alpha-\mu$ (α_ℓ,μ_ℓ)	$L=3$	(1.5, 2), (2, 2.5), (2.2, 3.2)
	$L=4$	(1.5, 3), (1.8, 2.5), (2.5, 2.8), (2.2, 3)
$\kappa-\mu$ $(\kappa_\ell-,\mu_\ell)$	$L=3$	(1, 2), (1.5, 1), (2, 1.5)
	$L=4$	(1, 1), (1, 1.5), (1.5, 2), (2, 2)
$\eta-\mu$ (η_ℓ,μ_ℓ)	$L=3$	(1.5, 1.5), (2, 2), (2.5, 3)
	$L=4$	(2, 2), (2.5, 2.5), (3, 3), (3.5, 3.5)

In Fig. 1, we plot the outage probability P against the SNR threshold γ_{th} using naive MC (blue curve) and our proposed IS approach (red curve). The solid line represents the EGC scenario while the dashed line is for the MRC case. Similar conclusions can be drawn for both type of diversity. In fact, we notice that for the range of probabilities between 10^{-1} and 10^{-4}, both methods match for the two cases. However, as the probability becomes smaller, naive MC fails to estimate the outage probability with the same accuracy as our method. In fact, we can see that for $L=4$, naive MC with $N=10^7$ samples is unable to estimate accurately the outage probabilities below 10^{-6} while the proposed IS scheme can estimate P even with a small number of samples $N^*=10^4$. To have a clear idea on the reduction in number of samples, we turn our attention to Fig. 2 where we plot the number of required simulation runs for 5% relative error for 4-branch diversity receivers over $\alpha-\mu$ fading channel. From instance, we can see that, for $\gamma_{th}=16$ dB (respectively $\gamma_{th}=12$ dB), the gain in terms of number of samples of IS compared to naive MC is approximately 5.5×10^2 (respectively 4×10^6) for the EGC case. A significant reduction can be seen also in the MRC scenario. To further have an idea on how both methods behave as the outage probability becomes smaller, we plot in Figs. 3 and 4 the outage probability of L-branch EGC and MRC receivers over $\alpha-\mu$ fading model along with the error bars. These latter are graphical representations of the variability of the data. They allow to indicate the error estimated in a measure, in other words, an error bar indicates the uncertainty of a given value. They are often used to show the quality of a model. In our case, the error bars are based on 95% confidence interval and can be linked to the measure of relative error. To investigate fairly the errors of both methods, we fixed the same number of simulation runs $N=N^*=10^6$. From these plots, we notice that the error bars for IS are short over the entire range of outage probabilities, an indication that the estimated value is reliable and the error is low. On the other hand, naive MC error bars tends to become larger as the outage probability becomes smaller, thereby the uncertainty of the estimated value become greater.

Regarding the $\kappa-\mu$ fading model, we plot in Fig. 5 the outage probability of L-branch EGC and MRC diversity receivers for two cases $L=3$ and $L=4$. We can see that, for both EGC and MRC cases, IS estimator is able to accurately estimate the outage probability with only $N^*=10^4$ unlike naive MC whose estimator, even with $N=10^7$ samples, fails when the probability drops below 10^{-5}. The number of samples for 5% relative error for 4-branch diversity receivers over $\kappa-\mu$ fading channel for both EGC and MRC is reported in Fig. 6. For example, using our proposed IS estimator saves about 2×10^4 samples compared to naive MC when $\gamma_{th}=14$ dB for the MRC case. In fact, as the outage probability becomes smaller, the gain in terms of simulation runs becomes greater. For fixed number of samples, we plot the error bars of

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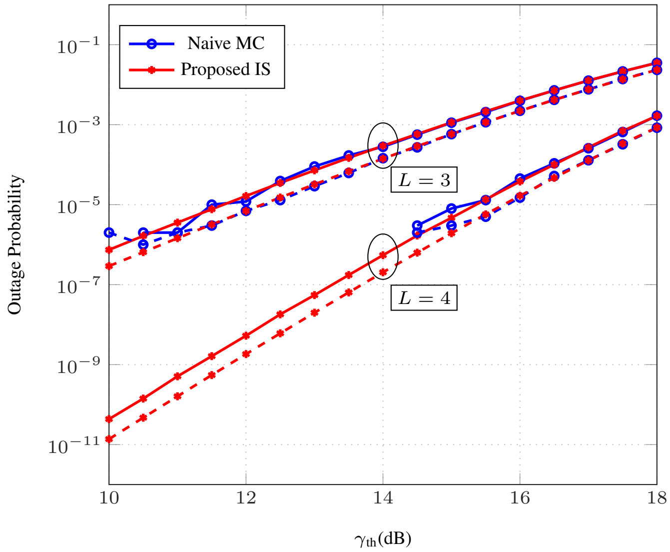

Fig. 1. Outage probability of L-branch diversity receivers over $\alpha-\mu$ fading model with $E_s/N_0 = 10$ dB and $\Omega = 5$ dB. Number of samples $N = 10^7$ and $N^* = 10^4$. EGC case: solid line and MRC case: dashed line.

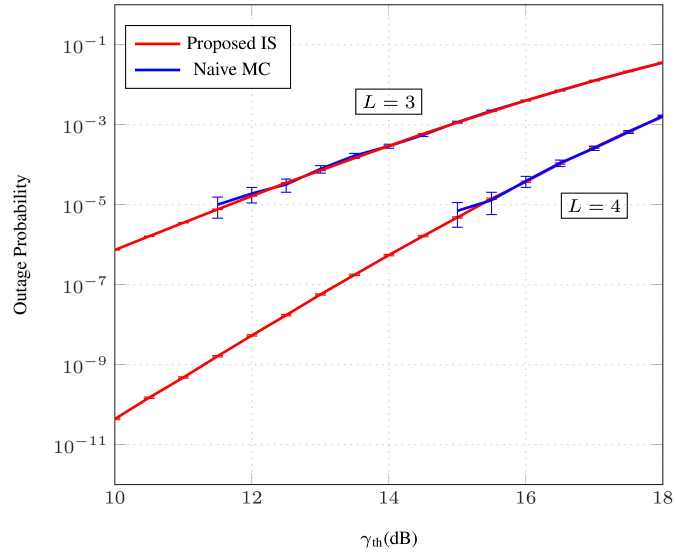

Fig. 3. Error bars of MC and IS estimators of the outage probability of L-branch EGC receivers over $\alpha-\mu$ fading model. Number of samples $N = N^* = 10^6$.

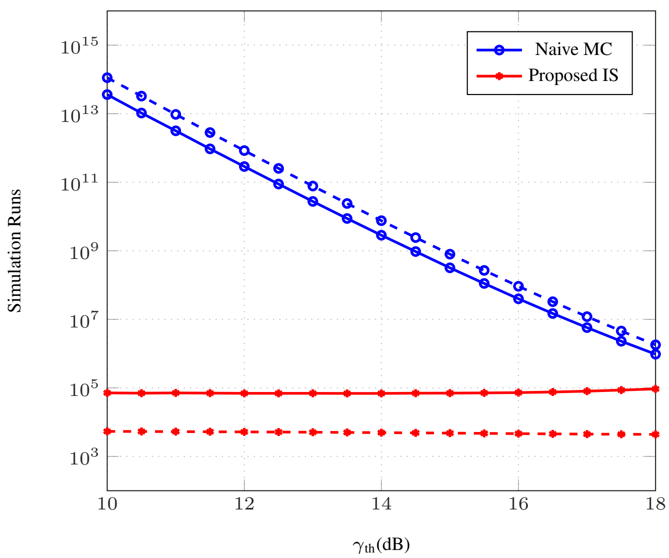

Fig. 2. Number of required simulation runs for 5% relative error for 4-branch diversity receivers over $\alpha-\mu$ fading model with $E_s/N_0 = 10$ dB and $\Omega = 5$ dB. EGC case: solid line and MRC case: dashed line.

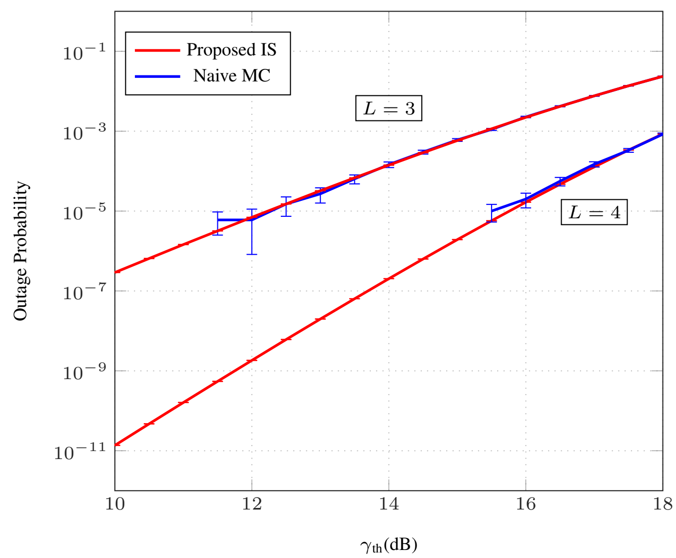

Fig. 4. Error bars of MC and IS estimators of the outage probability of L-branch MRC receivers over $\alpha-\mu$ fading model. Number of samples $N = N^* = 10^6$.

both estimators in Figs. 7 and 8. Comparing these error bars, we can see clearly that the proposed IS estimator provides better estimates than naive MC, especially when the outage probability is small.

In Fig. 9, we plot the outage probability for both diversity techniques over $\eta-\mu$ fading channel. We can observe that, for high outage probabilities, naive MC is sufficient for the estimation of P. Our method outperforms naive MC in the region of rare events, i.e $P < 10^{-5}$. The behavior of the number of required simulations runs by both methods for a fixed accuracy requirement $\varepsilon^* = \varepsilon = 5\%$ is depicted in Fig. 10 for both diversity techniques. We can notice that the number of samples N needed by naive MC to estimate P up to 95% accuracy grows rapidly whereas N^* remains almost constant. This goes hand in hand with the bounded relative error property of our IS estimator. To illustrate this idea, the number of samples N^* required by IS, for $L = 4$ is approximately 1.7×10^6 times less than the number of samples used in naive MC simulations when γ_{th}= 16 dB for the EGC case. Similar conclusions can be drawn for the MRC scenario. Error bars are reported in Figs. 11 and 12. Similar to the case of $\alpha-\mu$ and $\kappa-\mu$ fadings,our IS estimators prove their efficiency compared to naive MC. Endowed with the bounded relative error, the error bars of IS are short and almost of the same length unlike naive MC whose error bars are longer especially in the region of rare events.

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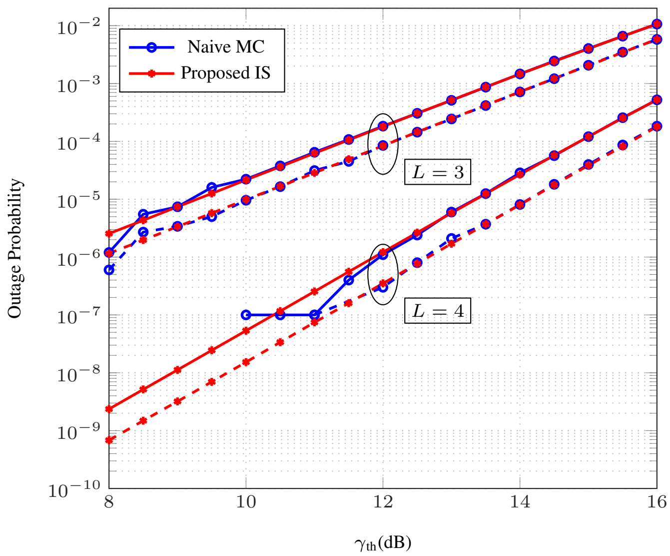

Fig. 5. Outage probability of L-branch diversity receivers over $\kappa-\mu$ fading model with $E_s/N_0 = 10$ dB and $\Omega = 5$ dB. Number of samples $N = 10^7$ and $N^* = 10^4$. EGC case: solid line and MRC case: dashed line.

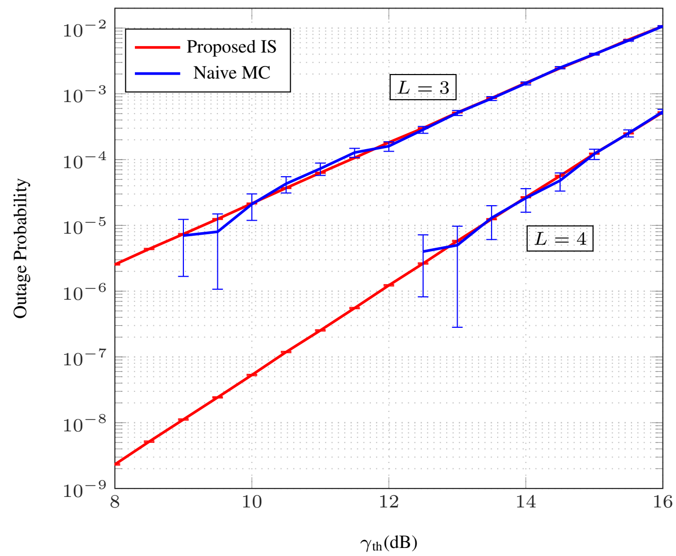

Fig. 7. Error bars of MC and IS estimators of the outage probability of L-branch EGC receivers over $\kappa-\mu$ fading model. Number of samples $N = N^* = 10^6$.

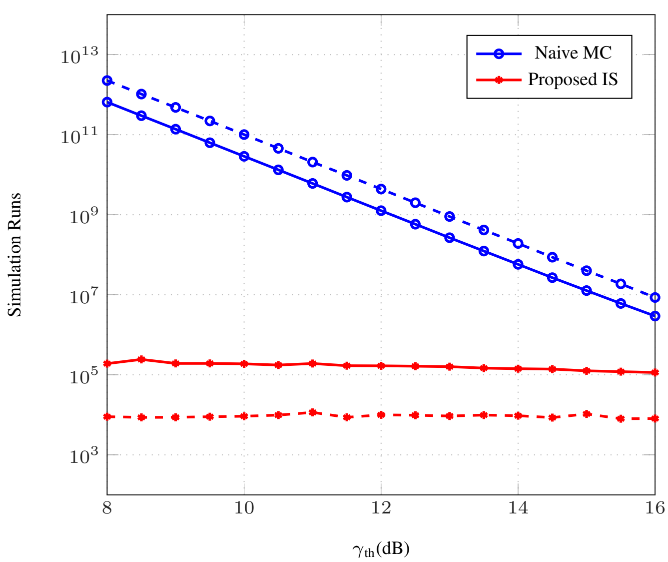

Fig. 6. Number of required simulation runs for 5% relative error for 4-branch diversity receivers over $\kappa-\mu$ fading model with $E_s/N_0 = 10$ dB and $\Omega = 5$ dB. EGC case: solid line and MRC case: dashed line.

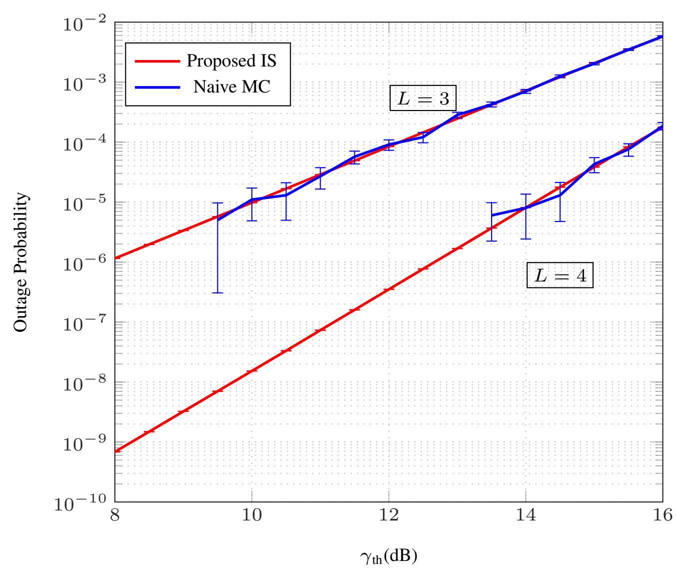

Fig. 8. Error bars of MC and IS estimators of the outage probability of L-branch MRC receivers over $\kappa-\mu$ fading model. Number of samples $N = N^* = 10^6$.

V. Conclusion

In this paper, we presented a novel approach for the efficient estimation of the left tail of the sum of $\alpha-\mu$, $\kappa-\mu$, and $\eta-\mu$ variates. We showed that the proposed estimators are endowed with the bounded relative error criterion. We also derived a novel asymptotic expansion of the Yacoub's integral with respect to the second argument around zero. Capitalizing on this result, we were able to efficiently evaluate the outage probability of L-branch diversity receivers over $\alpha-\mu$, $\kappa-\mu$, and $\eta-\mu$ fading channels. Simulation results show the accuracy, as well as the efficiency, of our proposed IS estimators compared to the naive MC estimators.

Appendix A
Proof of Bounded Relative Error for $\alpha-\mu$ Fading

Proof: The likelihood ratio is given by

$$\mathcal{L}(X_1,\dots,X_L) = \prod_{\ell=1}^{L} \frac{f_{X_\ell}(X_\ell)}{f^*_{X_\ell}(X_\ell)}$$
$$= \prod_{\ell=1}^{L} \left(\frac{\Omega_\ell - \Omega_{0,\ell}}{\Omega_\ell}\right)^{\mu_\ell} \prod_{\ell=1}^{L} \exp\left(\mu_\ell \left[\frac{1}{\Omega_\ell - \Omega_{0,\ell}} - \frac{1}{\Omega_\ell}\right] x_\ell^{\alpha_\ell}\right). \tag{A.1}$$

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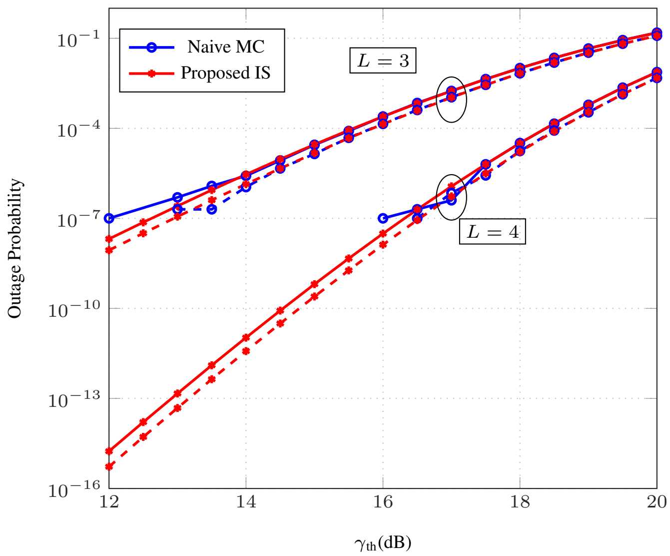

Fig. 9. Outage probability of L-branch diversity receivers over $\eta-\mu$ fading model with $E_s/N_0 = 10$ dB and $\Omega = 5$ dB. Number of samples $N = 10^7$ and $N^* = 10^4$. EGC case: solid line and MRC case: dashed line.

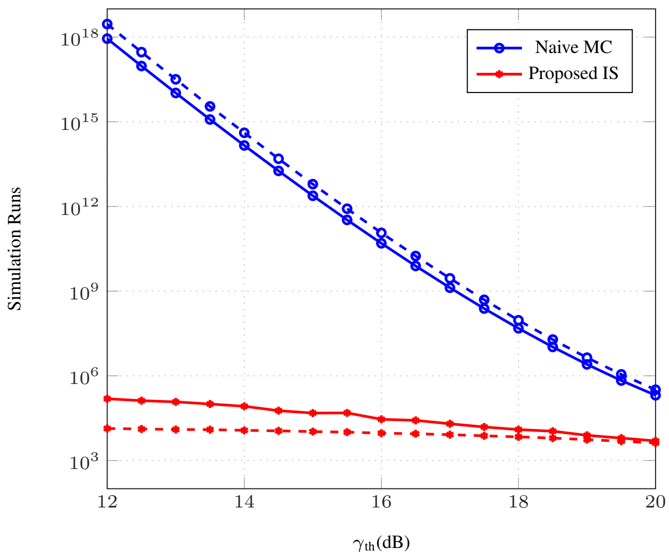

Fig. 10. Number of required simulation runs for 5% relative error for 4-branch diversity receivers over $\eta-\mu$ fading model with $E_s/N_0 = 10$ dB and $\Omega = 5$ dB. EGC case: solid line and MRC case: dashed line.

Replacing the expression of $\Omega_{0,\ell}$ in (14), we get

$$\mathcal{L}(X_1,\ldots,X_L) = \prod_{\ell=1}^{L}\left(\frac{\theta_\ell \gamma_{p,q}^{\alpha_\ell}}{\Omega_\ell}\right)^{\mu_\ell} \times \exp\left(\sum_{\ell=1}^{L}\mu_\ell\left[\frac{1}{\theta_\ell\gamma_{p,q}^{\alpha_\ell}} - \frac{1}{\Omega_\ell}\right]x_\ell^{\alpha_\ell}\right). \tag{A.2}$$

Thereby, the likelihood can be bounded by

$$\mathcal{L}(X_1,\ldots,X_L) \leq \prod_{\ell=1}^{L}\left(\frac{\theta_\ell}{\Omega_\ell}\right)^{\mu_\ell}\gamma_{p,q}^{\sum_{\ell=1}^{L}\mu_\ell\alpha_\ell} \times \exp\left(\sum_{\ell=1}^{L}\frac{\mu_\ell}{\theta_\ell}\left(\frac{x_\ell}{\gamma_{p,q}}\right)^{\alpha_\ell}\right). \tag{A.3}$$

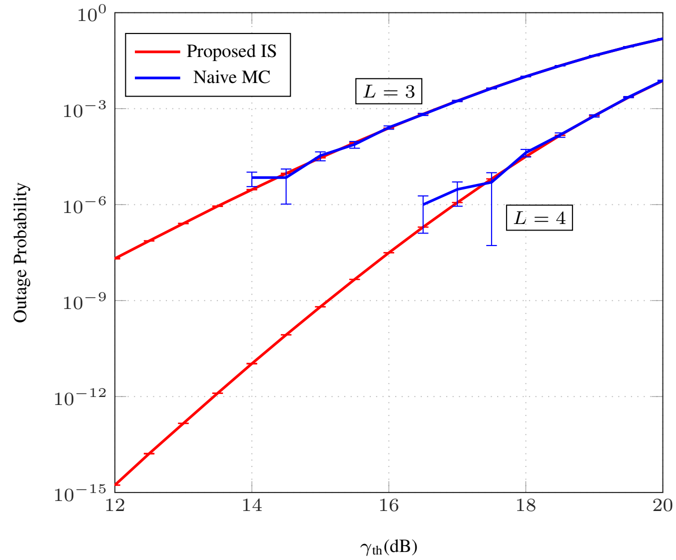

Fig. 11. Error bars of MC and IS estimators of the outage probability of L-branch EGC receivers over $\eta-\mu$ fading model. Number of samples $N = N^* = 10^6$

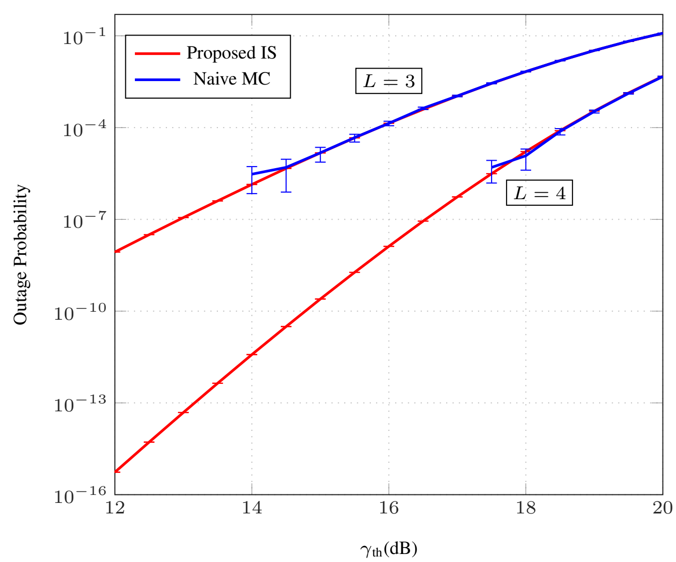

Fig. 12. Error bars of MC and IS estimators of the outage probability of L-branch MRC receivers over $\eta-\mu$ fading model. Number of samples $N = N^* = 10^6$.

Let $\eta_0 = \max_{1\leq\ell\leq L}\frac{\mu_\ell}{\theta_\ell}$, $\alpha_{max} = \max_{1\leq\ell\leq L}\alpha_\ell$, and $\alpha_{min} = \min_{1\leq\ell\leq L}\alpha_\ell$. We define the two sets

$$I \triangleq \left\{\frac{x_\ell}{\gamma_{p,q}} : \frac{x_\ell}{\gamma_{p,q}} < 1\right\}, \tag{A.4}$$

$$J \triangleq \left\{\frac{x_\ell}{\gamma_{p,q}} : \frac{x_\ell}{\gamma_{p,q}} \geq 1\right\}. \tag{A.5}$$

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We can write

$$\sum_{\ell=1}^{L}\left(\frac{x_\ell}{\gamma_{p,q}}\right)^{\alpha_\ell}=\sum_{\ell\in I}\left(\frac{x_\ell}{\gamma_{p,q}}\right)^{\alpha_\ell}+\sum_{\ell\in J}\left(\frac{x_\ell}{\gamma_{p,q}}\right)^{\alpha_\ell} \leq\sum_{\ell\in I}\left(\frac{x_\ell}{\gamma_{p,q}}\right)^{\alpha_{min}}+\sum_{\ell\in J}\left(\frac{x_\ell}{\gamma_{p,q}}\right)^{\alpha_{max}}. \tag{A.6}$$

Using the embedding inequality of L^1 into L^α for $\alpha \geq 1$ [27], we can write, for any positive real numbers $\{v_\ell\}_{\ell=1}^L$

$$\sum_{\ell=1}^{L} v_\ell^\alpha \leq \left(\sum_{\ell=1}^{L} v_\ell\right)^\alpha. \tag{A.7}$$

Since $\min_{1\leq\ell\leq L}\alpha_\ell > 1$, we can use this inequality for $v_\ell=\frac{x_\ell}{\gamma_{p,q}}$, $\ell=1,\dots,L$ to get

$$\sum_{\ell=1}^{L}\left(\frac{x_\ell}{\gamma_{p,q}}\right)^{\alpha_\ell}\leq\left(\sum_{\ell\in I}\frac{x_\ell}{\gamma_{p,q}}\right)^{\alpha_{min}}+\left(\sum_{\ell\in J}\frac{x_\ell}{\gamma_{p,q}}\right)^{\alpha_{max}} \leq\left(\sum_{\ell=1}^{L}\frac{x_\ell}{\gamma_{p,q}}\right)^{\alpha_{min}}+\left(\sum_{\ell=1}^{L}\frac{x_\ell}{\gamma_{p,q}}\right)^{\alpha_{max}}. \tag{A.8}$$

Thus, we can write

$$\mathcal{L}(X_1,\dots,X_L)\leq\prod_{\ell=1}^{L}\left(\frac{\theta_\ell}{\Omega_\ell}\right)^{\mu_\ell}\gamma_{p,q}^{\sum_{\ell=1}^{L}\mu_\ell\alpha_\ell}\times \exp\left(\eta_0\left[\frac{1}{\gamma_{p,q}^{\alpha_{min}}}\left(\sum_{\ell=1}^{L}x_\ell\right)^{\alpha_{min}}+\frac{1}{\gamma_{p,q}^{\alpha_{max}}}\left(\sum_{\ell=1}^{L}x_\ell\right)^{\alpha_{max}}\right]\right). \tag{A.9}$$

Therefore, we obtain the following upper bound

$$\mathbb{E}^*\left[\mathbb{1}_{\{S_{L,p}\leq\gamma_{p,q}\}}\mathcal{L}^2(X_1,\dots,X_L)\right]\leq\prod_{\ell=1}^{L}\left(\frac{\theta_\ell}{\Omega_\ell}\right)^{2\mu_\ell}\gamma_{p,q}^{2\sum_{\ell=1}^{L}\mu_\ell\alpha_\ell} \times\exp(4\eta_0). \tag{A.10}$$

On the other hand, we have that

$$\bigcap_{\ell=1}^{L}\{X_\ell\leq\frac{\gamma_{p,q}}{L}\}\subset\{\sum_{\ell=1}^{L}X_\ell\leq\gamma_{p,q}\}. \tag{A.11}$$

In the i.n.i.d scenario, this leads to

$$P\geq\prod_{\ell=1}^{L}\mathbb{P}\left(X_\ell\leq\frac{\gamma_{p,q}}{L}\right). \tag{A.12}$$

We recall that the CDF of a GG RV is given by [5]

$$F_{X_\ell}(x)=\frac{\gamma\left(\mu_\ell,\frac{\mu_\ell}{\Omega_\ell}x^{\alpha_\ell}\right)}{\Gamma(\mu_\ell)}, \tag{A.13}$$

where $\gamma(\cdot,\cdot)$ is the lower incomplete Gamma defined in [21, Eq. (8.350.1)]. Since we have $\gamma(s,z)\underset{z\to 0}{\sim}\frac{z^s}{s}$ [28, Eq. (2.272)], then we can write

$$\gamma\left(\mu_\ell,\frac{\mu_\ell}{\Omega_\ell}\left(\frac{\gamma_{p,q}}{L}\right)^{\alpha_\ell}\right)\underset{\gamma_{p,q}\to 0}{\sim}\frac{\mu_\ell^{\mu_\ell-1}\gamma_{p,q}^{\alpha_\ell\mu_\ell}}{\Omega_\ell^{\mu_\ell}L^{\alpha_\ell\mu_\ell}} \tag{A.14}$$

Thereby, the CDF has the following asymptotic expansion

$$F_{X_\ell}\left(\frac{\gamma_{p,q}}{L}\right)\underset{\gamma_{p,q}\to 0}{\sim}\frac{\mu_\ell^{\mu_\ell-1}\gamma_{p,q}^{\alpha_\ell\mu_\ell}}{\Omega_\ell^{\mu_\ell}L^{\alpha_\ell\mu_\ell}\Gamma(\mu_\ell)} \tag{A.15}$$

A lower bound on P is given by

$$P\geq\prod_{\ell=1}^{L}F_{X_\ell}\left(\frac{\gamma_{p,q}}{L}\right)\underset{\gamma_{p,q}\to 0}{\sim}\prod_{\ell=1}^{L}\frac{\mu_\ell^{\mu_\ell-1}\gamma_{p,q}^{\alpha_\ell\mu_\ell}}{\Omega_\ell^{\mu_\ell}L^{\alpha_\ell\mu_\ell}\Gamma(\mu_\ell)}. \tag{A.16}$$

Thus, we get as $\gamma_{p,q}\to 0$

$$\frac{1}{P^2}\leq\prod_{\ell=1}^{L}\left[\frac{\Omega_\ell^{\mu_\ell}L^{\alpha_\ell\mu_\ell}\Gamma(\mu_\ell)}{\mu_\ell^{\mu_\ell-1}}\right]^2\gamma_{p,q}^{-2\sum_{\ell=1}^{L}\alpha_\ell\mu_\ell}. \tag{A.17}$$

Combining (A.10) and (A.17), we obtain

$$\limsup_{\gamma_0\to 0}\frac{\mathbb{E}^*\left[\mathbb{1}_{\{S_{L,p}\leq\gamma_{p,q}\}}\mathcal{L}^2(X_1,\dots,X_L)\right]}{P^2} \leq\prod_{\ell=1}^{L}\left[\frac{[\Gamma(\mu_\ell)]^{1+\mu_\ell\alpha_\ell}\mu_\ell^{\alpha_\ell\mu_\ell-\mu_\ell+1}}{[\Gamma(\mu_\ell+\frac{1}{\alpha_\ell})]^{\mu_\ell\alpha_\ell}}\right]^2\exp(4\eta_0). \tag{A.18}$$

and hence the proof is concluded. ■

APPENDIX B
PROOF OF BOUNDED RELATIVE ERROR FOR $\kappa-\mu$ FADING

Proof: Due to the similarity between the proofs of the bounded relative error for the case of the sum of $\kappa-\mu$ and squared $\kappa-\mu$ and to avoid redundancy, we define, for $i\in\{1,2\}$, the following PDFs

$$\psi_{Y_{\ell,i}}(x)=\frac{B_{\ell,i}x^{\xi_{\ell,i}}}{\Omega_\ell^{\frac{\mu_\ell+1}{2}}}\exp\left(-\frac{\mu_\ell(1+\kappa_\ell)}{\Omega_\ell}x^i\right) \times I_{\mu_\ell-1}\left(2\mu_\ell\sqrt{\frac{\kappa_\ell(1+\kappa_\ell)}{\Omega_\ell}}x^i\right), \tag{B.1}$$

where $\{Y_{\ell,1}\}_{\ell=1}^L$ and $\{Y_{\ell,2}\}_{\ell=1}^L$ are respectively a set of squared $\kappa-\mu$ and $\kappa-\mu$ RVs with respective PDFs $\{\psi_{X_{\ell,1}}(\cdot)\}_{\ell=1}^L$ and $\{\psi_{X_{\ell,2}}(\cdot)\}_{\ell=1}^L$ and the constants $\{B_{\ell,i}\}_{i=1}^2$ and $\{\xi_{\ell,i}\}_{i=1}^2$ are given by

$$B_{\ell,1}=\frac{\mu_\ell\left(1+\kappa_\ell\right)^{\frac{\mu_\ell+1}{2}}}{\kappa_\ell^{\frac{\mu_\ell-1}{2}}\exp(\mu_\ell\kappa_\ell)},\ B_{\ell,2}=2B_{\ell,1}, \tag{B.2}$$

$$\xi_{\ell,1}=\frac{\mu_\ell-1}{2},\ \xi_{\ell,2}=\mu_\ell. \tag{B.3}$$

For $i\in\{1,2\}$, we define the biased PDFs as

$$\psi_{Y_{\ell,i}}(x)=\frac{B_{\ell,i}x^{\xi_{\ell,i}}}{(\Omega_\ell-\Omega_{0,\ell,i})^{\frac{\mu_\ell+1}{2}}}\exp\left(-\frac{\mu_\ell(1+\kappa_\ell)}{\Omega_\ell-\Omega_{0,\ell,i}}x^i\right) \times I_{\mu_\ell-1}\left(2\mu_\ell\sqrt{\frac{\kappa_\ell(1+\kappa_\ell)}{\Omega_\ell-\Omega_{0,\ell,i}}}x^i\right). \tag{B.4}$$

The likelihood ratios can be written as

$$\mathcal{L}_i=\prod_{\ell=1}^{L}\left(\frac{\Omega_\ell-\Omega_{0,\ell,i}}{\Omega_\ell}\right)^{\frac{\mu_\ell+1}{2}}\frac{I_{\mu_\ell-1}\left(2\mu_\ell\sqrt{\frac{\kappa_\ell(1+\kappa_\ell)}{\Omega_\ell}}Y_{\ell,i}^i\right)}{I_{\mu_\ell-1}\left(2\mu_\ell\sqrt{\frac{\kappa_\ell(1+\kappa_\ell)}{\Omega_\ell-\Omega_{0,\ell,i}}}Y_{\ell,i}^i\right)} \times\exp\left(\sum_{\ell=1}^{L}\mu_\ell(1+\kappa_\ell)\left[\frac{1}{\Omega_\ell-\Omega_{0,\ell,i}}-\frac{1}{\Omega_\ell}\right]Y_{\ell,i}^i\right). \tag{B.5}$$

To bound the ratio of the modified Bessel function, we assume that $\min_{1\leq\ell\leq L}\mu_\ell > 1$ to have [29, Eq.(1.1)]

$$\frac{I_{\mu_\ell-1}\left(2\mu_\ell\sqrt{\frac{\kappa_\ell(1+\kappa_\ell)}{\Omega_\ell}Y_{\ell,i}^i}\right)}{I_{\mu_\ell-1}\left(2\mu_\ell\sqrt{\frac{\kappa_\ell(1+\kappa_\ell)}{\Omega_\ell-\Omega_{0,\ell,i}}Y_{\ell,i}^i}\right)} \leq \left(\frac{\Omega_\ell-\Omega_{0,\ell,i}}{\Omega_\ell}\right)^{\frac{\mu_\ell-1}{2}} \times \exp\left(2\mu_\ell\left[\frac{1}{\Omega_\ell-\Omega_{0,\ell,i}}-\frac{1}{\Omega_\ell}\right]\sqrt{\kappa_\ell(1+\kappa_\ell)Y_{\ell,i}^i}\right). \quad \text{(B.6)}$$

Therefore, we can write

$$\mathcal{L}_i \leq \prod_{\ell=1}^{L}\left(\frac{\Omega_\ell-\Omega_{0,\ell,i}}{\Omega_\ell}\right)^{\mu_\ell} \times \exp\left(\sum_{\ell=1}^{L}\mu_\ell(1+\kappa_\ell)\left[\frac{1}{\Omega_\ell-\Omega_{0,\ell,i}}-\frac{1}{\Omega_\ell}\right]Y_{\ell,i}^i\right) \times \exp\left(2\sum_{\ell=1}^{L}\mu_\ell\left[\frac{1}{\Omega_\ell-\Omega_{0,\ell,i}}-\frac{1}{\Omega_\ell}\right]\sqrt{\kappa_\ell(1+\kappa_\ell)Y_{\ell,i}^i}\right). \quad \text{(B.7)}$$

The expression of $\Omega_{0,\ell,i}$ as function of $\gamma_{p,q}$ is given by

$$\Omega_{0,\ell,i} = \Omega_\ell - \delta_{\ell,i}\left(\frac{\gamma_{p,q}}{L}\right)^i, \quad \text{(B.8)}$$

where

$$\delta_{\ell,1} = 1, \quad \text{(B.9)}$$

$$\delta_{\ell,2} = \left(\frac{\Gamma(\mu_\ell)}{\Gamma(\mu_\ell+\frac{1}{2})}\right)^2\mu_\ell(1+\kappa_\ell)\frac{\exp(2\mu_\ell\kappa_\ell)}{{}_1F_1\left[\mu_\ell+\frac{1}{2};\mu_\ell;\kappa_\ell\mu_\ell\right]^2}. \quad \text{(B.10)}$$

Thus, the upper bound of the likelihood ratio becomes

$$\mathcal{L}_i \leq \prod_{\ell=1}^{L}\left(\frac{\delta_{\ell,i}}{\Omega_\ell}\right)^{\mu_\ell}\left(\frac{\gamma_{p,q}}{L}\right)^{i\sum_{\ell=1}^{L}\mu_\ell} \times \exp\left(\left(\frac{L}{\gamma_{p,q}}\right)^i\sum_{\ell=1}^{L}\frac{\mu_\ell(1+\kappa_\ell)}{\delta_{\ell,i}}Y_{\ell,i}^i\right) \times \exp\left(2\left(\frac{L}{\gamma_{p,q}}\right)^{\frac{i}{2}}\sum_{\ell=1}^{L}\mu_\ell\sqrt{\frac{\kappa_\ell(1+\kappa_\ell)}{\delta_{\ell,i}}Y_{\ell,i}^i}\right). \quad \text{(B.11)}$$

By defining $\mu_0 = \max_{1\leq\ell\leq L}\mu_\ell$, $\kappa_0 = \max_{1\leq\ell\leq L}\kappa_\ell$, and $\delta_{0,i} = \min_{1\leq\ell\leq L}\delta_{\ell,i}$, we get

$$\mathcal{L}_i \leq \prod_{\ell=1}^{L}\left(\frac{\delta_{\ell,i}}{\Omega_\ell}\right)^{\mu_\ell}\left(\frac{\gamma_{p,q}}{L}\right)^{i\sum_{\ell=1}^{L}\mu_\ell} \times \exp\left(\left(\frac{L}{\gamma_{p,q}}\right)^i\frac{\mu_0(1+\kappa_0)}{\delta_{0,i}}\sum_{\ell=1}^{L}Y_{\ell,i}^i\right) \times \exp\left(2\left(\frac{L}{\gamma_{p,q}}\right)^{\frac{i}{2}}\mu_0\sqrt{\frac{\kappa_0(1+\kappa_0)}{\delta_{0,i}}}\sum_{\ell=1}^{L}\sqrt{Y_{\ell,i}^i}\right). \quad \text{(B.12)}$$

Since the RV are positive, we have

$$\sum_{\ell=1}^{L}Y_{\ell,i}^i \leq \left(\sum_{\ell=1}^{L}Y_{\ell,i}\right)^i, \quad \text{(B.13)}$$

and using Cauchy-Schwarz-Buniakowsky inequality [21, Sec 11.311],

$$\sum_{\ell=1}^{L}\sqrt{Y_{\ell,i}^i} \leq \sqrt{L}\sqrt{\sum_{\ell=1}^{L}Y_{\ell,i}^i}, \quad \text{(B.14)}$$

we obtain the following upper bound

$$\mathbb{E}^*\left[\mathbb{1}_{(S_{L,p}\leq\gamma_{p,q})}\mathcal{L}_i^2\right] \leq \prod_{\ell=1}^{L}\left(\frac{\delta_{\ell,i}}{\Omega_\ell}\right)^{2\mu_\ell}\left(\frac{\gamma_{p,q}}{L}\right)^{2i\sum_{\ell=1}^{L}\mu_\ell} \times \exp\left(2\frac{L^i\mu_0(1+\kappa_0)}{\delta_{0,i}}\right)\exp\left(4L^{\frac{i+1}{2}}\mu_0\sqrt{\frac{\kappa_0(1+\kappa_0)}{\delta_{0,i}}}\right). \quad \text{(B.15)}$$

On the other hand, we have

$$P \geq \prod_{\ell=1}^{L}\mathbb{P}\left(Y_{\ell,i}\leq\frac{\gamma_{p,q}}{L}\right) = \prod_{\ell=1}^{L}F_{Y_{\ell,i}}\left(\frac{\gamma_{p,q}}{L}\right). \quad \text{(B.16)}$$

The CDF of the RV $Y_{\ell,i}$ is given by [4, Eq.(3)]

$$F_{Y_{\ell,i}}(x) = 1 - Q_{\mu_\ell}\left(2\sqrt{2\kappa_\ell\mu_\ell}, \sqrt{\frac{2\mu_\ell(1+\kappa_\ell)}{\Omega_\ell}x^i}\right), \quad \text{(B.17)}$$

where $Q\mu(\cdot,\cdot)$ is the generalized Marcum-Q function defined as [4, Eq.(4)]

$$Q\mu(a,b) = \frac{1}{a^{\mu-1}}\int_b^\infty x^\mu\exp\left(-\frac{x^2+a^2}{2}\right)I_{\mu-1}(ax) \quad \text{(B.18)}$$

Using [30, Eq.(8)], we obtain the asymptotic expansion around $x=0$ of CDF of $Y_{\ell,i}$

$$F_{Y_{\ell,i}}(x) \underset{x\to 0}{\sim} \frac{\exp(-\kappa_\ell\mu_\ell)}{\Gamma(\mu_\ell+1)}\left(\frac{\mu_\ell(1+\kappa_\ell)}{\Omega_\ell}x^i\right)^{\mu_\ell}. \quad \text{(B.19)}$$

Therefore, we get the following upper bound

$$\frac{1}{P^2} \leq \prod_{\ell=1}^{L}\left[\frac{\Gamma(\mu_\ell+1)\exp(\kappa_\ell\mu_\ell)\Omega_\ell^{\mu_\ell}}{(1+\kappa_\ell)^{\mu_\ell}\mu^{\mu_\ell}}\right]^2\left(\frac{L}{\gamma_{p,q}}\right)^{2i\sum_{\ell=1}^{L}\mu_\ell}. \quad \text{(B.20)}$$

From equations (B.15) and (B.20), we end up having

$$\limsup_{\gamma_{p,q}\to 0}\frac{\mathbb{E}^*\left[\mathbb{1}_{(S_{L,p}\leq\gamma_{p,q})}\mathcal{L}_i^2\right]}{P^2} \leq \prod_{\ell=1}^{L}\frac{\left(\Gamma(\mu_\ell+1)\right)^2\exp(2\kappa_\ell\mu_\ell)\delta_\ell^{4\mu_\ell}}{(1+\kappa_\ell)^{2\mu_\ell}\mu^{2\mu_\ell}} \times \exp\left(2L^i\mu_0\left(2\sqrt{\frac{L\kappa_0(1+\kappa_0)}{\delta_{0,i}}}+\frac{1+\kappa_0}{\delta_{0,i}}\right)\right). \quad \text{(B.21)}$$

■

Appendix C
Proof of Bounded Relative Error for $\eta-\mu$ Fading

Proof: To prove the bounded relative error property for the $\eta-\mu$ case, we define, for $i\in\{1,2\}$, the PDFs

$$\phi_{X_{\ell,i}}(x) = \frac{C_{\ell,i}\,x^{\alpha_{\ell,i}}}{\Omega_\ell^{\mu_\ell+\frac{1}{2}}}\exp\left(-\frac{2\mu_\ell h_\ell}{\Omega_\ell}x^i\right)I_{\mu_\ell-\frac{1}{2}}\left(\frac{2\mu_\ell H_\ell}{\Omega_\ell}x^i\right), \quad \text{(C.1)}$$

where $\{\phi_{X_{\ell,1}}(\cdot)\}_{\ell=1}^{L}$ and $\{\phi_{X_{\ell,2}}(\cdot)\}_{\ell=1}^{L}$ are respectively the PDFs of the squared $\eta-\mu$ RVs $\{X_{\ell,1}\}_{\ell=1}^{L}$ and of the $\eta-\mu$ RVs $\{X_{\ell,2}\}_{\ell=1}^{L}$, respectively.
For $\ell=1,\dots L$, the constants $\{C_{\ell,i}\}_{i=1}^{2}$ and $\{\alpha_{\ell,i}\}_{i=1}^{2}$ are defined as

$$C_{\ell,1}=\frac{2\sqrt{\pi}\mu_{\mu_\ell+\frac{1}{2}}h_\ell^{\mu_\ell}}{\Gamma(\mu_\ell)H_\ell^{\mu_\ell-\frac{1}{2}}},\ C_{\ell,2}=2C_{\ell,1}, \tag{C.2}$$

$$\alpha_{\ell,1}=\mu_\ell-\frac{1}{2},\ \alpha_{\ell,2}=2\mu_\ell. \tag{C.3}$$

The biased PDFs, for $i\in\{1,2\}$, are given by

$$\phi^*_{X_{\ell,i}}(x)=\frac{C_{\ell,i}}{(\Omega_\ell-\Omega_{0,\ell,i})^{\mu_\ell+\frac{1}{2}}}x^{\alpha_{\ell,i}}\exp\left(-\frac{2\mu_\ell h_\ell}{\Omega_\ell-\Omega_{0,\ell,i}}x^i\right)\times I_{\mu_\ell-\frac{1}{2}}\left(\frac{2\mu_\ell H_\ell}{\Omega_\ell-\Omega_{0,\ell,i}}x^i\right). \tag{C.4}$$

We define the likelihood ratios, for $i\in\{1,2\}$ as

$$\mathcal{L}_i=\prod_{\ell=1}^{L}\left(\frac{\Omega_\ell-\Omega_{0,\ell,i}}{\Omega_\ell}\right)^{\mu_\ell+\frac{1}{2}}\frac{I_{\mu_\ell-\frac{1}{2}}\left(\frac{2\mu_\ell H_\ell}{\Omega_\ell}X_\ell^i\right)}{I_{\mu_\ell-\frac{1}{2}}\left(\frac{2\mu_\ell H_\ell}{\Omega_\ell-\Omega_{0,\ell,i}}X_\ell^i\right)}\times\exp\left(2\sum_{\ell=1}^{L}\mu_\ell h_\ell\left[\frac{1}{\Omega_\ell-\Omega_{0,\ell,i}}-\frac{1}{\Omega_\ell}\right]X_\ell^i\right). \tag{C.5}$$

Since $\Omega_\ell-\Omega_{0,\ell,i}<\Omega_\ell$, then using [29, Eq.(1.1)] and provided $\min\limits_{1\le\ell\le L}\mu_\ell>\frac{1}{2}$, we can write

$$\frac{I_{\mu_\ell-\frac{1}{2}}\left(\frac{2\mu_\ell H_\ell}{\Omega_\ell}X_\ell^i\right)}{I_{\mu_\ell-\frac{1}{2}}\left(\frac{2\mu_\ell H_\ell}{\Omega_\ell-\Omega_{0,\ell,i}}X_\ell^i\right)}\le\left(\frac{\Omega_\ell-\Omega_{0,\ell,i}}{\Omega_\ell}\right)^{\mu_\ell-\frac{1}{2}}\times\exp\left(2\mu_\ell H_\ell\left[\frac{1}{\Omega_\ell-\Omega_{0,\ell,i}}-\frac{1}{\Omega_\ell}\right]X_\ell^i\right). \tag{C.6}$$

Therefore, the likelihood ratio can be bounded by

$$\mathcal{L}_i\le\prod_{\ell=1}^{L}\left(\frac{\Omega_\ell-\Omega_{0,\ell,i}}{\Omega_\ell}\right)^{2\mu_\ell}\times\exp\left(2\sum_{\ell=1}^{L}\mu_\ell(h_\ell+H_\ell)\left[\frac{1}{\Omega_\ell-\Omega_{0,\ell,i}}-\frac{1}{\Omega_\ell}\right]X_\ell^i\right). \tag{C.7}$$

Regarding the choice of $\Omega_{0,\ell,i}$, we recall that its expression is given by

$$\Omega_{0,\ell,i}=\Omega_\ell-\beta_{\ell,i}\left(\frac{\gamma_{p,q}}{L}\right)^i, \tag{C.8}$$

where

$$\beta_{\ell,1}=1, \tag{C.9}$$

$$\beta_{\ell,2}=\left(\frac{\Gamma(2\mu_\ell)h_\ell^{\mu_\ell+\frac{1}{2}}\sqrt{2\mu_\ell}}{\Gamma(2\mu_\ell+\frac{1}{2})\,{}_2F_1\left[\mu_\ell+\frac{3}{4},\mu_\ell+\frac{1}{4};\mu_\ell+\frac{1}{2};\left(\frac{H_\ell}{h_\ell}\right)^2\right]}\right)^2. \tag{C.10}$$

Replacing (C.8) in (C.7), we get

$$\mathcal{L}_i\le\prod_{\ell=1}^{L}\left(\frac{\beta_{\ell,i}}{\Omega_\ell}\right)^{2\mu_\ell}\left(\frac{\gamma_{p,q}}{L}\right)^{2i\sum_{\ell=1}^{L}\mu_\ell}\times\exp\left(2\left(\frac{L}{\gamma_{p,q}}\right)^i\sum_{\ell=1}^{L}\frac{\mu_\ell(h_\ell+H_\ell)}{\beta_{\ell,i}}X_\ell^i\right). \tag{C.11}$$

Let $\mu_0=\max\limits_{1\le\ell\le L}\mu_\ell$, $h_0=\max\limits_{1\le\ell\le L}h_\ell$, $H_0=\max\limits_{1\le\ell\le L}H_\ell$, and $\beta_{0,i}=\min\limits_{1\le\ell\le L}\beta_{\ell,i}$, then we can write

$$\mathcal{L}_i\le\prod_{\ell=1}^{L}\left(\frac{\beta_{\ell,i}}{\Omega_\ell}\right)^{2\mu_\ell}\left(\frac{\gamma_{p,q}}{L}\right)^{2i\sum_{\ell=1}^{L}\mu_\ell}\times\exp\left(2\left(\frac{L}{\gamma_{p,q}}\right)^i\frac{\mu_0(h_0+H_0)}{\beta_{0,i}}\sum_{\ell=1}^{L}X_\ell^i\right). \tag{C.12}$$

Exploiting the fact that $\sum_{\ell=1}^{L}X_{\ell,i}^i\le\left(\sum_{\ell=1}^{L}X_{\ell,i}\right)^i$, we get

$$\mathbb{E}^*\left[\mathbb{1}_{(S_{L,p}\le\gamma_{p,q})}\mathcal{L}_i^2\right]\le\prod_{\ell=1}^{L}\left(\frac{\beta_{\ell,i}}{\Omega_\ell}\right)^{4\mu_\ell}\left(\frac{\gamma_{p,q}}{L}\right)^{4i\sum_{\ell=1}^{L}\mu_\ell}\times\exp\left(4\frac{L^i\mu_0(h_0+H_0)}{\beta_{0,i}}\right). \tag{C.13}$$

On the other hand, to lower bound the probability P, we can write

$$P\ge\prod_{\ell=1}^{L}F_{X_{\ell,i}}\left(\frac{\gamma_{p,q}}{L}\right). \tag{C.14}$$

We recall the expression of the CDF $F_{X_{\ell,i}}(\cdot)$ [4, Eq.(19)]

$$F_{X_{\ell,i}}(x)=1-Y_{\mu_\ell}\left(\frac{H_\ell}{h_\ell},\sqrt{\frac{2h_\ell\mu_\ell}{\Omega_\ell}x^i}\right),\ i\in\{1,2\}, \tag{C.15}$$

where the Yacoub's integral $Y_\mu(\cdot,\cdot)$ is definite as [4, Eq.(20)]

$$Y_\mu(a,b)=\frac{2^{\frac{3}{2}-\mu}\sqrt{\pi}(1-a^2)^\mu}{a^{\mu-\frac{1}{2}}\Gamma(\mu)}\int_b^\infty x^{2\mu}\exp(-x^2)I_{\mu-\frac{1}{2}}(ax^2)dx, \tag{C.16}$$

with $-1<a<1$ and $b\ge0$.
From Appendix D, we have that

$$F_{X_{\ell,i}}(x)\underset{x\to0}{\sim}\frac{\sqrt{\pi}\left(h_\ell^2-H_\ell^2\right)^{\mu_\ell}\mu_\ell^{2\mu_\ell-1}}{\Gamma(\mu_\ell)\Gamma(\mu_\ell+\frac{1}{2})\Omega_\ell^{2\mu_\ell}}x^{2i\mu_\ell}. \tag{C.17}$$

Going back to (C.14), we can write around $\gamma_{p,q}=0$

$$P\ge\pi^{\frac{L}{2}}\prod_{\ell=1}^{L}\frac{(h_\ell^2-H_\ell^2)^{\mu_\ell}\mu_\ell^{2\mu_\ell-1}}{\Gamma(\mu_\ell)\Gamma(\mu_\ell+\frac{1}{2})\Omega_\ell^{2\mu_\ell}}\left(\frac{\gamma_{p,q}}{L}\right)^{2i\sum_{\ell=1}^{L}\mu_\ell}. \tag{C.18}$$

Thus, we get

$$\frac{1}{P^2}\le\pi^{-L}\prod_{\ell=1}^{L}\frac{\left[\Gamma(\mu_\ell)\Gamma(\mu_\ell+\frac{1}{2})\right]^2\Omega_\ell^{4\mu_\ell}}{(h_\ell^2-H_\ell^2)^{2\mu_\ell}\mu_\ell^{4\mu_\ell-2}}\left(\frac{L}{\gamma_{p,q}}\right)^{4i\sum_{\ell=1}^{L}\mu_\ell}. \tag{C.19}$$

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Combining both equation (C.13) and (C.19), we obtain the following upper bound

$$\limsup_{\gamma_{p,q}\to 0}\frac{\mathbb{E}^*\left[\mathbb{1}_{(S_{L,p}\leq\gamma_{p,q})}\mathcal{L}_i^2\right]}{P^2}\leq\prod_{\ell=1}^{L}\frac{\left[\Gamma(\mu_\ell)\Gamma(\mu_\ell+\frac{1}{2})\right]^2\beta_{\ell,i}^{4\mu_\ell}}{\pi(h_\ell^2-H_\ell^2)^{2\mu_\ell}\mu_\ell^{4\mu_\ell-2}}$$
$$\times\exp\left(4\frac{L^i\mu_0(h_0+H_0)}{\beta_{0,i}}\right). \quad \text{(C.20)}$$

and hence the proof is concluded. ∎

APPENDIX D
EXPANSION OF $Y_\mu(a,b)$ AROUND $b=0$

In this appendix, we derive an asymptotic expansion of the Yacoub's integral $Y_\mu(a,b)$ around $b=0$. In fact, we have

$$Y_\mu(a,b)=\frac{2^{\frac{3}{2}-\mu}\sqrt{\pi}(1-a^2)^\mu}{a^{\mu-\frac{1}{2}}\Gamma(\mu)}I_\mu(a,b), \quad \text{(D.1)}$$

with $-1<a<1$ and $b\geq 0$. In the rest of derivation, we assume $\mu>\frac{1}{2}$ to ensure the convergence of the integrals and

$$I_\mu(a,b)=\int_b^\infty x^{2\mu}\exp(-x^2)I_{\mu-\frac{1}{2}}(ax^2)dx$$
$$=I_{\mu,1}(a,b)-I_{\mu,2}(a,b) \quad \text{(D.2)}$$

where

$$I_{\mu,1}(a,b)=\int_0^\infty x^{2\mu}\exp(-x^2)I_{\mu-\frac{1}{2}}(ax^2)dx, \quad \text{(D.3)}$$
$$I_{\mu,2}(a,b)=\int_0^b x^{2\mu}\exp(-x^2)I_{\mu-\frac{1}{2}}(ax^2)dx. \quad \text{(D.4)}$$

Let us first focus on computing I_1. To do that, we have the following equality for $|a|<1$ [31, Eq.(2.6)]

$$\int_0^\infty t^sK_\mu(2t)I_\nu(2at)dt=\frac{a^\nu}{4\Gamma(\nu+1)}\Gamma\left(\frac{\mu+\nu+s+1}{2}\right)\times$$
$$\Gamma\left(\frac{\nu-\mu+s+1}{2}\right)\times$$
$${}_2F_1\left[\frac{\mu+\nu+s+1}{2},\frac{\nu-\mu+s+1}{2};\nu+1;a^2\right] \quad \text{(D.5)}$$

For $\mu=-\frac{1}{2}$, we have $K_{-\frac{1}{2}}(2t)=\frac{\sqrt{\pi}}{2}\frac{e^{-2t}}{\sqrt{t}}$ [21, Eq.(8.469.3)]. Thus, we get

$$\int_0^\infty t^sK_\mu(2t)I_\nu(2at)dt=\frac{\sqrt{\pi}}{2}\int_0^\infty t^{s-\frac{1}{2}}\exp(-2t)I_\nu(2at)dt$$
$$=\frac{\sqrt{\pi}}{2^{s+\frac{1}{2}}}\int_0^\infty x^{2s}\exp(-x^2)I_\nu(ax^2)dx. \quad \text{(D.6)}$$

From (D.5) and (D.6), we obtain

$$\int_0^\infty x^{2s}\exp(-x^2)I_\nu(ax^2)dx=\frac{2^{s+\frac{1}{2}}a^\nu}{4\sqrt{\pi}\Gamma(\nu+1)}\Gamma\left(\frac{1}{4}+\frac{\nu+s}{2}\right)$$
$$\times\Gamma\left(\frac{3}{4}+\frac{\nu+s}{2}\right){}_2F_1\left[\frac{1}{4}+\frac{\nu+s}{2},\frac{3}{4}+\frac{\nu+s}{2};\nu+1;a^2\right]. \quad \text{(D.7)}$$

For the special case when $s=\mu$ and $\nu=\mu-\frac{1}{2}$, we have

$$\int_0^\infty x^{2\mu}\exp(-x^2)I_{\mu-\frac{1}{2}}(ax^2)dx=\frac{2^{\mu-\frac{3}{2}}}{\sqrt{\pi}}a^{\mu-\frac{1}{2}}\Gamma(\mu)$$
$$\times{}_2F_1\left[\mu,\mu+\frac{1}{2};\mu+\frac{1}{2};a^2\right]. \quad \text{(D.8)}$$

Since ${}_2F_1\left[\mu,\mu+\frac{1}{2};\mu+\frac{1}{2};a^2\right]=(1-a^2)^{-\mu}$ [32, Eq.(21)], we can write

$$I_{\mu,1}(a,b)=\frac{2^{\mu-\frac{3}{2}}a^{\mu-\frac{1}{2}}\Gamma(\mu)}{\sqrt{\pi}(1-a^2)^\mu} \quad \text{(D.9)}$$

Now, we turn our attention to find an expansion of $I_{\mu,2}(a,b)$ around $b=0$. We know that we can write $\exp(-x^2)=1+\mathcal{O}(x^2)$ and that [30, Eq. (1.1)]

$$I_\nu(x)=\frac{x^\nu}{2^\nu\Gamma(\nu+1)}+\mathcal{O}(x^{\nu+2}). \quad \text{(D.10)}$$

An equivalent to $I_{\mu,2}(a,b)$ around $b=0$ is

$$\int_0^{2\mu}\left(1+\mathcal{O}(x^2)\right)\left(\frac{a^{\mu-\frac{1}{2}}x^{4\mu-1}}{2^{\mu-\frac{1}{2}}\Gamma(\mu+\frac{1}{2})}+\mathcal{O}(x^{2\mu+3})\right)dx$$
$$=\frac{a^{\mu-\frac{1}{2}}b^{4\mu}}{2^{\mu+\frac{3}{2}}\mu\Gamma(\mu+\frac{1}{2})}+\mathcal{O}(b^{4\mu+2}). \quad \text{(D.11)}$$

Replacing (D.9) and (D.11) in (D.2), we get

$$I_\mu(a,b)\underset{b\to 0}{\sim}\frac{2^{\mu-\frac{3}{2}}a^{\mu-\frac{1}{2}}\Gamma(\mu)}{\sqrt{\pi}(1-a^2)^\mu}-\frac{a^{\mu-\frac{1}{2}}b^{4\mu}}{2^{\mu+\frac{3}{2}}\mu\Gamma(\mu+\frac{1}{2})}+\mathcal{O}(b^{4\mu+2}). \quad \text{(D.12)}$$

Therefore, the expansion of $Y_\mu(a,b)$ around $b=0$ is

$$Y_\mu(a,b)\underset{b\to 0}{\sim}1-\frac{\sqrt{\pi}(1-a^2)^\mu b^{4\mu}}{\mu\Gamma(\mu)\Gamma(\mu+\frac{1}{2})2^{2\mu}}+\mathcal{O}(b^{4\mu+2}). \quad \text{(D.13)}$$

which concludes the proof.

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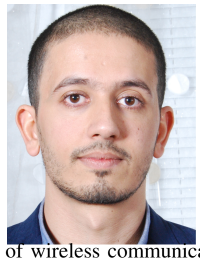
Mohamed-Slim Alouini (S'94, M'98, SM'03, F'09) was born in Tunis, Tunisia. He received the Ph.D. degree in Electrical Engineering from the California Institute of Technology (Caltech), Pasadena, CA, USA, in 1998. He served as a faculty member in the University of Minnesota, Minneapolis, MN, USA, then in the Texas A&M University at Qatar, Education City, Doha, Qatar before joining King Abdullah University of Science and Technology (KAUST), Thuwal, Makkah Province, Saudi Arabia as a Professor of Electrical Engineering in 2009. His current research interests include the modeling, design, and performance analysis of wireless communication systems.

Raul Tempone was born in Montevideo, Uruguay, in 1969. He received the B.E. degree in industrial engineering at the University of the Republic, Montevideo, Uruguay in 1995. After his graduation he worked on the optimal dispatch of electricity for the Uruguayan system using techniques from nonlinear stochastic programming and visited the Royal Institute of Technology (KTH) in Stockholm, Sweden, to study further numerical analysis. He obtained a MSc in Engineering Mathematics in 1999 (inverse problems for incompressible flows, supervised by Jesper Oppelstrup, KTH) and a PhD in Numerical Analysis in 2002 (a posteriori error estimation and control for stochastic differential equations, supervised by Anders Szepessy, KTH). He later moved to ICES, UT Austin, to work as a postdoc from 2003 until 2005 in the area of numerical methods for PDEs with random coefficients (supervised by Ivo Babuska). In 2005 he became an assistant professor with the School of Computational Sciences and the Department of Mathematics at Florida State University, Tallahassee. In 2007 he was awarded the first Dahlquist fellowship by KTH and COMSOL for his contributions to the field of numerical approximation of deterministic and stochastic differential equations. In 2009 he joined KAUST as an Associate Professor in Applied Mathematics (founding faculty). He later became there the Director of the KAUST Center for Uncertainty Quantification and was promoted to Full Professor in 2015.

Dr. Tempone's research interests are in the mathematical foundation of computational science and engineering. More specifically, he has focused on a posteriori error approximation and related adaptive algorithms for numerical solutions of various differential equations, including ordinary differential equations, partial differential equations, and stochastic differential equations. He is also interested in the development and analysis of efficient numerical methods for uncertainty quantification and Bayesian model validation. The areas of application he considers include, among others, engineering, chemistry, biology, physics as well as social science and computational finance.



Chaouki Ben Issaid was born in Sfax, Tunisia. He received the Diplôme d'Ingénieur degree from l'École Polytechnique de Tunisie, La Marsa, Tunisia, in 2013. He also holds the Master degree in applied mathematics and computational science from King Abdullah University of Science and Technology (KAUST), Thuwal, Saudi Arabia. Currently, he is working toward the Ph.D degree in statistics at KAUST. His current research interests include efficient Monte Carlo simulations for the performance of wireless communication systems.