Finite-State Mean-Field Games, Crowd Motion Problems, and its Numerical Methods

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ABSTRACT

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In this dissertation, we present two research projects, namely finite-state mean-field games and the Hughes model for the motion of crowds.

In the first part, we describe finite-state mean-field games and some applications to socio-economic sciences. Examples include paradigm shifts in the scientific community and the consumer choice behavior in a free market. The corresponding finite-state mean-field game models are hyperbolic systems of partial differential equations, for which we propose and validate a new numerical method. Next, we consider the dual formulation to two-state mean-field games, and we discuss numerical methods for these problems. We then depict different computational experiments, exhibiting a variety of behaviors, including shock formation, lack of invertibility, and monotonicity loss. We conclude the first part of this dissertation with an investigation of the shock structure for two-state problems.

In the second part, we consider a model for the movement of crowds proposed by R. Hughes in [56] and describe a numerical approach to solve it. This model comprises a Fokker-Planck equation coupled with an Eikonal equation with Dirichlet or Neumann data. We first establish a priori estimates for the solutions. Next, we consider radial solutions, and we identify a shock formation mechanism. Subsequently, we illustrate the existence of congestion, the breakdown of the model, and the trend to the equilibrium. We also propose a new numerical method for the solution of Fokker-Planck equations and then to systems of PDEs composed by a Fokker-Planck
equation and a potential type equation. Finally, we illustrate the use of the numerical method both to the Hughes model and mean-field games. We also depict cases such as the evacuation of a room and the movement of persons around Kaaba (Saudi Arabia).
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Part I

Discrete setting: finite-state mean-field games, its applications, and a numerical method for their study
Chapter 1

Finite-state mean-field games

1.1 Introduction

In the first part of this dissertation, we present finite-state mean-field games (MFG). MFG have become a powerful mathematical tool to model the dynamics of agents in economics, finance, and social sciences. Different settings were considered in the literature such as discrete and continuous in time or finite and continuous state space. Originally, finite-state mean-field games, see [67, 40, 52, 51, 42, 36], were studied as an attempt to understand the more general continuous state problems introduced by Lasry & Lions in [62, 63, 64] and by Huang et al. in [53, 54]. For additional information, see [65, 20, 2, 45, 8].

In the discrete setting, the mathematical modeling is as follows. We consider a system of $N+1$ identical players or agents that can switch between $d \in \mathbb{N}$ different states. Each player is in a state $i \in \mathcal{I} = \{1, \ldots, d\}$ and can choose a switching strategy to any other state $j \in \mathcal{I}$. The only information available to each player, besides its own state, is the number $n_j$ of players he/she sees in the different states $j \in \mathcal{I}$. The fraction of players in each state $i \in \mathcal{I}$ is denoted by $\theta_i = \frac{n_i}{N}$, and we define the probability vector $\theta = (\theta_1, \ldots, \theta_d)$, $\theta \in \mathcal{P}(\mathcal{I}) = \{\theta \in \mathbb{R}^d : \sum_i \theta_i = 1, \theta_i \geq 0\}$, which encodes the statistical information about the ensemble of players. Each player faces an optimization problem over all possible switching strategies. Here, the fundamental question is the existence of a Nash equilibrium and to determine the limit as the number of players tends to infinity. Gomes et al. showed in [42] that the $N+1$ player
Nash equilibrium always exists. The Nash equilibrium in the limit $N \to \infty$ (called mean-field Nash equilibrium) satisfies certain ordinary differential equations, which may not have unique solutions. Because of this non-uniqueness, the development of numerical methods, which capture the appropriate equilibrium, is of high interest.

The value function $U^i = U^i(\theta, t)$ for a player in the state the $i$ in the limit $N \to \infty$, satisfies the hyperbolic system

$$
\left\{
\begin{array}{l}
-U^i_t(\theta, t) = \sum_{j \in \mathcal{I}} g_j(U, \theta) \partial_{\theta_j} U^i(\theta, t) + h(U, \theta, i), \\
U(\theta, T) = U_T(\theta),
\end{array}
\right.
$$

(1.1)

at least for a short time, see Gomes et al. [42]. Note that the distribution of players in the different states is given by $\theta$, $U^i : \mathcal{P}(\mathcal{I}) \times [0, T] \to \mathbb{R}$, $g : \mathbb{R}^d \times \mathcal{P}(\mathcal{I}) \to \mathbb{R}^d$, $h : \mathbb{R}^d \times \mathcal{P}(\mathcal{I}) \times \mathcal{I} \to \mathbb{R}$, and $U_T : \mathcal{P}(\mathcal{I}) \to \mathbb{R}^d$, where $\partial_{\theta_j}$ denotes the partial derivative with respect to the variable $\theta_j$.

In general, first-order partial differential equations do not admit smooth solutions, and one needs to consider an appropriate notion of a solution. Adequate definitions of solutions are well known for conservation laws and equations which admit a maximum principle, e.g. Hamilton-Jacobi equations. Up to now, the notion of solutions for (1.1) that encodes the mean-field limit is not clear.

The main contribution of this work is a numerical method based on the Nash equilibrium equations for $N + 1$ agents. These equations, in the case of convergence, automatically give the appropriate limit. Convergence for short times was shown in [42]. For certain finite-state mean-field games, called potential mean-field games, (1.1) can be regarded as the gradient of a Hamilton-Jacobi equation, see [67, 42] and section 1.5. We use this property to validate our numerical results. Our computational experiments show the formation of shocks. Moreover, we analyze the shock structure by introducing an auxiliary conservation law and deriving a Rankine-Hugoniot con-
dition that characterizes the qualitative behavior of these systems.

The first part of this dissertation is organized as follows: in chapter 1, we describe the problem and the main assumptions. In section 1.2, we consider the N + 1 player model, which gives rise both to (1.1) and to the numerical method presented here. Subsequently, we present the formal derivation of (1.1). The model presented here extends slightly the one developed in [42] as it allows for interactions between agents as we will describe. We also deduce the limit equation that describes the asymptotic behavior of the dynamics in section 1.3, and in section 1.4 we study the characteristics of the ODE (ordinary differential equation) related to equilibrium conditions and the shocks in the PDE (partial differential equation) counterpart.

Next, in section 1.5, we introduce potential mean-field games. Those are a class of mean-field games whose study of solutions can be reduced to the one of a Hamilton-Jacobi problem. In chapter 2, we restrict our problem to two states only, and in chapter 3, we describe the so-called dual mean-field games. In this formulation, we must solve the evolution of the density of agents in each state, instead of the value function.

Finally, we use the potential mean-field games, the reduced formulations, and the proposed numerical scheme to show that the same results are obtained by solving either the primal or the dual problem. We apply two-state mean-field games to social-economics problems, in chapter 4. The problem becomes easier for having only two states, but many issues present in mean-field games can still be seen. We also illustrate some problems, as well the phenomena of loss of monotonicity and shock formation.
1.2 The setting of the problem

We consider a system of \( N + 1 \) identical players or agents. Let us fix one of them, referred to as the reference player, and denote by \( i^t \) its state at the time \( t \). All other players can be in any state \( j \in \mathcal{I} \) at each time \( t \). We denote by \( n^t_j \) the number of players (distinct from the reference player) at state \( j \) at the time \( t \), and \( \mathbf{n}^t = (n^t_1, \ldots, n^t_d) \).

Players change their states according to two mechanisms: either by a Markovian switching rate chosen by the player or by interactions with the other players. The interactions are determined by the following simplified model. We assume that any pair of players may meet at random times which follow an exponential distribution. As an outcome of the meeting, one of the two players chooses to join the state of the other player. This is a behavior often seen in opinion dynamics, where players change their opinions by interactions with other players\(^1\). We assume for the moment that all players (except the reference player) have chosen an identical Markovian strategy \( \beta(n, i, t) \in (\mathbb{R}_0^+)^d \). The reference player is allowed to choose a possibly different strategy \( \alpha(n, i, t) \in (\mathbb{R}_0^+)^d \). Given these strategies, the joint process \((i^t, \mathbf{n}^t)\) is a time-inhomogeneous Markov chain (observe that neither \( \mathbf{n}^t \) nor \( i^t \) are Markovian when considered separately). It is sufficient to specify this Markov process, as in the classical case of Markov chains with discrete space state and discrete time, by defining the generator of the process. We split the generator into three parts, each one corresponding to different interactions, which will be discussed later on. Hence we have:

\[
A \varphi^i_n = A^0 \varphi^i_n + A^1 \varphi^i_n + A^2 \varphi^i_n.
\]

Let \( e_k \) be the \( k \)-th vector of the canonical basis of \( \mathbb{R}^d \) and \( e_{jk} := e_j - e_k \). Then

\[
A^0 \varphi^i_n = \sum_{j \in \mathcal{I}} \alpha_j \left( \varphi^j_n - \varphi^i_n \right),
\]

\(^1\)More complicated models where the interaction of players is not of pairwise type can be studied. We choose such form to simplify the exposition.
\[ A_1^\alpha \varphi_n^i = \sum_{j,k \in \mathcal{I}} \gamma^{n,i}_{\beta,jk} \left( \varphi_{n+e_{jk}}^i - \varphi_n^i \right), \]

and

\[ A_2^\alpha \varphi_n^i = \sum_{j,k \in \mathcal{I}} \frac{\omega_{jk} n_j n_k}{N^2} \left( \varphi_{n+e_{jk}}^i + \varphi_{n+e_{kj}}^i - 2\varphi_n^i \right) + \sum_{j \in \mathcal{I}} \frac{\omega_{ij} n_j}{N^2} \left( \varphi_{n+e_{ij}}^j + \varphi_{n+e_{ij}}^i - 2\varphi_n^i \right). \]

The terms \( A_1^\alpha \) and \( A_1^\beta \) correspond to the transitions due to the switching strategies \( \alpha \) and \( \beta \). The first two terms of the generator are similar to the ones described in more detail in [42], but we briefly recall their interpretation for completeness:

Select one of the players, distinct from the reference player. Denote by \( k^t \) its position at the time \( t \), and by \( m^t = n^t + e_{i^t k^t} \) the process that records the number of other players in any state from the point of view of this player. Suppose further that there are no interactions (\( A_2^\alpha = 0 \)). Then for \( j \neq k \), we have for any time increment \( \delta \) small enough, that

\[ P(k^{t+\delta} = j|m^t = m, k^t = k) = \beta_j(m, k, t) \delta + o(\delta). \]

Assuming symmetry and independence of transitions from any state \( k \) to a state \( j \), \( k \neq j \), we have

\[ P(n^{t+\delta} = n + e_{jk}|n^t = n, i^t = i) = \gamma^{n,i}_{\beta,jk}(t) \delta + o(\delta), \]

where the transition rates of the process \( n^t \) are given by

\[ \gamma^{n,i}_{\beta,jk}(t) = n_k \beta_j(n + e_{ik}, k, t). \]

Similarly, the reference player switching probabilities are

\[ P(i^{t+\delta} = j|i^t = i, n^t = n) = \alpha_j(n, i, t) \delta + o(\delta). \]
The transitions between different states due to interactions give rise to the term $A_2^\omega$.

Its particular structure comes from the assumption that any two players with distinct states $j$ and $k$ can meet with rate $\frac{\omega_{jk}}{N}$ (with $\omega_{kj} = \omega_{jk} \geq 0$). As a result of this interaction, either both end in the state $j$ or state $k$ (with probability $\frac{1}{2}$ respectively).

We assume that all players have the same running cost determined by a function $c : \mathcal{I} \times \mathcal{P}(\mathcal{I}) \times (\mathbb{R}_0^+)^d \rightarrow \mathbb{R}$ as well as an identical terminal cost $U_T(\theta)$, which is Lipschitz continuous in $\theta$. The running cost $c(i, \theta, \alpha)$ depends on the state $i$ of the player, the mean field $\theta$, that is the distribution of players among states, and on the switching rate $\alpha$. As in [42], we suppose that $c$ is Lipschitz continuous in $\theta$ with a Lipschitz constant (with respect to $\theta$) bounded independently of $\alpha$. Let the running cost $c$ be differentiable with respect to $\alpha$, and $\frac{\partial c}{\partial \alpha}(i, \theta, \alpha)$ be Lipschitz with respect to $\theta$, uniformly in $\alpha$. We assume that for each $i \in \mathcal{I}$, the running cost $c(i, \theta, \alpha)$ does not depend on the $i$-th coordinate $\alpha_i$ of $\alpha$. Furthermore, we make the following additional assumptions on the cost $c$:

(A1) For any $i \in \mathcal{I}$, $\theta \in \mathcal{P}(\mathcal{I})$, $\alpha, \alpha' \in (\mathbb{R}_0^+)^d$, with $\alpha_j \neq \alpha'_j$, for some $j \neq i$,

$$c(i, \theta, \alpha') - c(i, \theta, \alpha) \geq \nabla_\alpha c(i, \theta, \alpha) \cdot (\alpha' - \alpha) + \gamma\|\alpha' - \alpha\|^2;$$

(A2) The function $c$ is superlinear in $\alpha_j$, $j \neq i$, i.e.,

$$\lim_{\alpha_j \to \infty} \frac{c(i, \theta, \alpha)}{\|\alpha\|} \to \infty.$$ 

Hypotheses (A1) and (A2) reflect generalizations, for the case where the cost depends on $\theta$, of the typical uniform convexity and superlinear growth conditions found in optimal control problems. Their requirements are better explained in a seminal paper for continuous time finite-state mean-field games [42].
Let us fix a reference player and set the Markovian strategy $\beta$ for the remaining $N$ players. The objective of the reference player is to minimize its total cost. The minimum over all Markovian strategies $\alpha$ is given by

$$u_n^{i,\beta}(t) = \inf_{\alpha} \mathbb{E}_{(h,n_t)=(i,n)}^{\beta,\alpha} \left[ \int_t^T c \left( i_s, \frac{n_s}{N}, \alpha(s) \right) ds + U_T^{ir}(\frac{n_T}{N}) \right]. \quad (1.2)$$

Define for $\varphi \in \mathbb{R}^d$, $\Delta_i \varphi := (\varphi^1 - \varphi^i, \ldots, \varphi^d - \varphi^i)$ and the generalized Legendre transform of $c$ by

$$h(z, \theta, i) = \min_{\mu \in (\mathbb{R}_0^+)^d} c(i, \theta, \mu) + \mu \cdot \Delta_i z. \quad (1.3)$$

Note that $h$ only depends on the differences between the coordinates of the variable $z$, that is, if $\Delta_i z = \Delta_i \tilde{z}$ then $h(z, \theta, i) = h(\tilde{z}, \theta, i)$. The function $u_n^{i,\beta}$ in (1.2) is the solution to the ODE

$$-\frac{\partial u_n^{i,\beta}}{\partial t} = h \left( u_n^{i,\beta}, \frac{n}{N}, i \right) + A_1^\beta u_n^{i,\beta} + A_2^\omega u_n^{i,\beta}.$$

Next, we define, for $j \neq i$,

$$\alpha_j^*(z, \theta, i) = \arg \min_{\mu \in (\mathbb{R}_0^+)^d} c(i, \theta, \mu) + \mu \cdot \Delta_i z. \quad (1.4)$$

If $h$ is differentiable, for $j \neq i$,

$$\alpha_j^*(\Delta_i z, \theta, i) = \frac{\partial h(\Delta_i z, \theta, i)}{\partial z^j}. \quad (1.5)$$

For convenience and consistency with (1.5), we require that

$$\sum_{j \in \mathcal{I}} \alpha_j^*(z, \theta, i) = 0. \quad (1.6)$$
Then the optimal strategy for the reference player is given by

$$
\bar{\alpha}(n, i, t) = \alpha^* \left( \Delta u_n^i, \frac{n}{N}, i \right).
$$

We say that a strategy $\beta$ is a Nash equilibrium if the optimal response of the reference player is $\beta$ itself, i.e., $\beta = \bar{\alpha}$. Thus by setting $u_n^i = u_n^{i,\bar{\alpha}}$, we obtain the Nash equilibrium equation for the value function, i.e.

$$
- \frac{\partial u_n^i}{\partial t} = h \left( u_n^i, \frac{n}{N}, i \right) + \sum_{j,k \in I} \gamma_{jk}^{n,i} \left( u_n^{i+e_{jk}} - u_n^i \right)
$$

$$
+ \frac{\omega_{jk} n_j n_k}{N^2} \left( u_n^{i+e_{jk}} + u_n^{i+e_{kj}} - 2u_n^i \right) + \frac{\omega_{ij} n_j}{N^2} \left( u_n^{i} + u_n^{i+e_{ij}} - 2u_n^{i} \right),
$$

where we define

$$
\gamma_{jk}^{n,i}(t) = n_k \bar{\alpha}_j \left( n + e_{ik}, k, t \right).
$$

### 1.3 Formal asymptotic behavior

Next, we investigate the asymptotic behavior of the $N+1$ player dynamics (1.7) as the number of players tends to infinity. Hence, we suppose that there exists a smooth function $U : \mathcal{P}(I) \times [0, T] \to \mathbb{R}^d$ such that

$$
u_n^i(t) = U^i \left( \frac{n}{N}, t \right).
$$

Then we have the following expansions:

$$
A_n^i u_n^i = \sum_{j,k \in I} \theta_k \left[ \left( 1 + \frac{\partial \theta_i - \partial \theta_k}{N} \right) \alpha_j^* (\Delta_k U, \theta, k) \right] \left[ \left( \partial \theta_j - \partial \theta_k \right) + \frac{\left( \partial \theta_j - \partial \theta_k \right)^2}{2N} \right] U^i
$$

$$
+ O \left( \frac{1}{N^2} \right),
$$

where we define $n_k \bar{\alpha}_j \left( n + e_{ik}, k, t \right)$. 

\[
A_2^\omega u_n^i = \sum_{j,k \in I} \frac{\omega_{jk}}{N} \theta_j \theta_k \left( \partial_{\theta_j \theta_k}^2 U^i + \partial_{\theta_j \theta_k}^2 U^i - 2\partial_{\theta_j \theta_k}^2 U^i \right) + \sum_{j \in I} \omega_{ij} (U^j - U^i)
+ \sum_{j \in I} \omega_{ij} \theta_j (\partial_{\theta_j} U^i - \partial_{\theta_j} U^i) + O \left( \frac{1}{N^2} \right).
\]

We observe that for \( j \) and \( k \) fixed, \( \frac{\theta_k \alpha_j^*}{2} (\partial_{\theta_j} - \partial_{\theta_k})^2 \) and \( \omega_{jk} \theta_j \theta_k (\partial_{\theta_j} - \partial_{\theta_k})^2 \) are degenerate elliptic operators. The first one is degenerate because \( \theta_k, \alpha_j^* \geq 0 \); the second because \( \omega_{jk} \geq 0 \). Hence, their sum is also a degenerate operator. Therefore the combination of the second order terms in the expansion of \( A_1^\alpha \) and \( A_2^\omega \) can be written as

\[
\sum_{l,m \in I} b_{lm} \partial_{\theta_l \theta_m}^2 = \sum_{j,k \in I} \frac{\theta_k \alpha_j^* + 2\omega_{jk} \theta_j \theta_k}{2} (\partial_{\theta_j} - \partial_{\theta_k})^2,
\]

for a suitable non-negative matrix \( b \). We conclude that (1.7) can be formally approximated by the parabolic system

\[
- U^i_t(\theta, t) = \sum_{j \in I} g_j^N(U, \partial \theta U, \theta, i) \partial \theta_j U^i + h(U, \theta, i) + \frac{1}{N} \sum_{l,m \in I} b_{lm}(U, \theta) \partial_{\theta_l \theta_m}^2 U^i, \quad (1.9)
\]

with suitable \( g^N : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(I) \times I \to \mathbb{R}^d \). Note that the viscosity term in the right-hand side of (1.9) does not come from an artificial argument to solve (1.9) numerically or to obtain better regularity results. On the contrary, this term is built in the original \( N \) player model. Furthermore, \( g^N \) converges locally uniformly in compact sets to

\[
g_j(U, \theta) = \sum_{i \in I} \theta_i \alpha_j^*(U, \theta, i).
\]

Therefore the limit of (1.9) is (1.1). Note that (1.1) does not depend on the interactions \( \omega_{jk} \) between players. Furthermore

\[
\sum_{j \in I} g_j(U, \theta) = \sum_{j \in I} \sum_{i \in I} \theta_i \alpha_j^*(U, \theta, i) = \sum_{i \in I} \theta_i \sum_{j \in I} \alpha_j^*(U, \theta, i) = 0, \quad (1.11)
\]
since $\sum_{j \in I} \alpha_j^*(U, \theta, i) = 0$, from (1.6). Additionally,

$$g_j(U, \theta, i) = g_j(\Delta_i U, \theta, i),$$  \hspace{1cm} (1.12)

using (1.4) in equation (1.10).

1.4 Characteristics and shocks

We start this section by discussing the characteristic ODE for (1.1):

$$
\begin{cases}
\tilde{\theta}_i(t) = g_i(\bar{u}(t), \bar{\theta}(t)), \\
-\tilde{u}_i(t) = h(\bar{u}(t), \bar{\theta}(t), i),
\end{cases}
$$  \hspace{1cm} (1.13)

for $i \in \mathcal{I}$, $t \in [0, T]$, $\bar{\theta} : [0, T] \to \mathcal{P}(\mathcal{I})$ and $\bar{u} : [0, T] \to \mathbb{R}^d$. As presented in [42], the solutions to this ODE satisfying the initial-terminal boundary conditions

$$
\begin{cases}
\bar{\theta}(0) = \bar{\theta}_0, \\
\bar{u}(T) = U_T(\bar{\theta}(T)),
\end{cases}
$$  \hspace{1cm} (1.14)

correspond precisely to the mean-field Nash equilibria for the limit $N \to \infty$, with respect to an initial distribution of players given by $\bar{\theta}_0 \in \mathcal{P}(\mathcal{I})$. In general system (1.13)-(1.14) admits multiple solutions. Hence, the Nash equilibrium is not unique. Multiple solutions to the characteristic ODE correspond to the existence of shocks in (1.1). The importance of understanding shock formation and the corresponding relevant solutions of (1.1), which encode the mean-field limit, can not be underestimated and it is essential to select the appropriate solutions to (1.13)-(1.14).

We present in detail the numerical scheme for (1.1) in section 2.1, and the study of shock formation for two-state problems in section 2.2.
1.5 Potential mean-field games

The next topic we consider is a special class of mean-field games, in which (1.1) can be written as the gradient of a Hamilton-Jacobi equation. Suppose

\[ h(u, \theta, i) = \bar{h}(u, i) + f(i, \theta), \quad i \in \mathcal{I}, \]  

(1.15)

and \( f(i, \theta) = \partial_\theta F(\theta) \), for some potential \( F : \mathbb{R}^d \to \mathbb{R} \). This form for the Hamiltonian in (1.15) is often called separable, where the dependence on \( u \) and \( \theta \) are given by different functions. Such particular form of the Hamiltonian allows us to transform the problem of solving the hyperbolic mean-field game system in to solve a certain Hamilton-Jacobi equation. Such choice will be of paramount importance to validate our numerical scheme in chapter 4. An explicit case of Hamiltonian of this form is described in detail in section 2.1.

Now, set

\[ H(u, \theta) = \sum_{k \in \mathcal{I}} \theta_k \bar{h}(\Delta_k u, k) + F(\theta). \]  

(1.16)

Let \( \Psi_T : \mathbb{R}^d \to \mathbb{R} \) be a continuous function and consider a sufficiently smooth solution \( \Psi : \mathbb{R}^d \times [0, T] \to \mathbb{R} \) to the Hamilton-Jacobi equation

\[
\begin{aligned}
- \frac{\partial \Psi(\theta, t)}{\partial t} &= H(\partial_\theta \Psi, \theta), \\
\Psi(\theta, T) &= \Psi_T(\theta).
\end{aligned}
\]

(1.17)

In some cases, it is possible to reduce the dimensionality of the previous problem by exploring the fact that \( \theta \in \mathbb{R}^d \); more explicitly, that the number of states \( d \) is finite. This has a greater impact on problems with few states, as the two-state case later described. The reduction procedure, explained in section 3.4.1, comes with the price of introducing suitable boundary conditions described in sections 4.3 and 4.4.
Now, set
\[ U^j(\theta, t) = \partial_{\theta_j} \Psi(\theta, t). \]  
(1.18)
Differentiating (1.17) with respect to $\theta_i$, we obtain
\[ -U^i_t = \sum_{j \in I} \partial_{u_j} H(U, \theta) \partial_{\theta_i} U^j + \bar{h}(\Delta_i U, i) + \partial_{\theta_i} F. \]
The first term on the right-hand side can be written as
\[ \sum_{j \in I} \partial_{u_j} H(U, \theta) \partial_{\theta_i} U^j = \sum_{k, j \in I} \theta_k \partial_{u_j} \bar{h}(\Delta_k U, k) \partial_{\theta_i} U^j = \sum_{j \in I} g_j(U, \theta) \partial_{\theta_j} U^i, \]
taking into account the identity $\partial_{\theta_i} U^j = \partial_{\theta_j} U^i$. From this, we get that
\[ -U^i_t = \sum_{j \in I} g_j(U, \theta) \partial_{\theta_j} U^i + \bar{h}(\Delta_i U, i) + \partial_{\theta_i} F, \]
and deduce, using (1.15), that $U^i$ is indeed a solution of (1.1).

Finally, we would like to remark that potential mean-field games have similar properties and close connections to calculus of variations. A variational perspective over two-state mean-field games was explored in [48]. Also, long time convergence properties of finite-state mean-field game problems can be addressed through $\Gamma$-convergence techniques, see [36], for instance.
Chapter 2

Two-state mean-field games

In this chapter, we present simplified finite-state mean-field games where we restrict the size of the state-space to two. With minor modifications, our approach can be adapted to a larger number of states. We begin by stating the explicit equations for the case where agents can choose between two options.

2.1 The setting of the problem and the numerical scheme

Consider a two-state mean-field game, where the fraction of players in state 1 or state 2, is given by $\theta_i$, $i = 1, 2$, with $\theta_1 + \theta_2 = 1$, and $\theta_i \geq 0$. Because the limit equation (1.1) does not depend on the interactions (although the $N+1$ player model does), we set $\omega = 0$. Note that $\omega \neq 0$ would result in different numerical methods (and potentially different solutions) for (1.1). We suppose further that the running cost $c = c(i, \theta, \mu)$ in (1.2) depends quadratically on the switching rate $\mu$ (this choice allow us to compute an explicit optimal switching rate), i.e.,

$$c(i, \theta, \mu) = f(i, \theta) + c_0(i, \mu), \quad \text{with } c_0(i, \mu) = \frac{1}{2} \sum_{j \neq i} \mu_j^2. \quad (2.1)$$

Using the results from chapter 1, we have

$$h(z, \theta, 1) = f(1, \theta) - \frac{1}{2}((z^1 - z^2)^+)^2 \quad \text{and} \quad h(z, \theta, 2) = f(2, \theta) - \frac{1}{2}((z^2 - z^1)^+)^2. \quad (2.2)$$
The optimal switching rate $\alpha^*$ is given by:

$$
\alpha^*(z, \theta, 1) = \arg \min_{\mu \in \mathbb{R}^2, \mu \geq 0} \left[ f(1, \theta) + \frac{1}{2} \mu_1^2 + (\mu_1) \cdot (z_1 - z_2) \right] \Rightarrow \alpha^*_2(z, \theta, 1) = (z^1 - z^2)^+, \\
\alpha^*(z, \theta, 2) = \arg \min_{\mu \in \mathbb{R}^2, \mu \geq 0} \left[ f(2, \theta) + \frac{1}{2} \mu_1^2 + (\mu_1) \cdot (z_1 - z_2) \right] \Rightarrow \alpha^*_1(z, \theta, 2) = (z^2 - z^1)^+.
$$

Since

$$
\alpha^*_1(U, \theta, 1) = - (U^1 - U^2)^+, \quad \alpha^*_2(U, \theta, 1) = (U^1 - U^2)^+, \\
\alpha^*_1(U, \theta, 2) = (U^2 - U^1)^+, \quad \alpha^*_2(U, \theta, 2) = - (U^2 - U^1)^+,
$$

we conclude from (1.10) that

$$
g_1(U, \theta) = - \theta_1 (U^1 - U^2)^+ + \theta_2 (U^2 - U^1)^+, \quad (2.3) \\
g_2(U, \theta) = \theta_1 (U^1 - U^2)^+ - \theta_2 (U^2 - U^1)^+ = - g_1(U, \theta).
$$

Note that if the function $f$ is a gradient field, i.e., $f = \nabla F$, the two-state problem is a potential mean-field game, we will return to this result in subsection 3.4. In this case, for $p = (p_1, p_2) \in \mathbb{R}^2$, the function $H$, defined by (1.16), reads as

$$
H(p, \theta) = F(\theta) - \frac{\theta_1 ((p_1 - p_2)^+)^2 + \theta_2 ((p_2 - p_1)^+)^2}{2}. \quad (2.4)
$$

The above calculations allow us to introduce a numerical method based on the two-state mean-field model for $N+1$ players. Let $n_i, i = 1, 2$ denote the number of players in the state $i$ (as seen by the reference player excluding itself) and $N = n_1 + n_2$. The

\[\text{In case the running cost does not depend quadratically on } \mu, \text{ explicit expressions for the optimal switching rate may not be simple, reflecting on } g, \text{ and consequently on the hyperbolic PDE system of interest.}\]
vector $n$ indicates the number of players in each state, i.e. $n = (n_1, n_2) = (n_1, N-n_1)$.

As in (1.8), we have

$$\gamma_{jl}^{n,1} = n_l \alpha_j^* \left( \Delta_l u_{n+e_{jl}}, \frac{n + e_{jl}}{N}, l \right) \quad \text{and} \quad \gamma_{jl}^{n,2} = n_l \alpha_j^* \left( \Delta_l u_{n+e_{jl}}, \frac{n + e_{jl}}{N}, l \right).$$

Then equation (1.7) for the value function $u^i_n$ reads as

$$\begin{cases} 
- \frac{du^1_n}{dt} = 2 \sum_{j,l=1}^{2} \gamma_{jl}^{n,1} \left( u^1_{n+e_{jl}} - u^1_n \right) + h \left( \Delta_1 u_n, \frac{n}{N}, 1 \right), \\
- \frac{du^2_n}{dt} = 2 \sum_{j,l=1}^{2} \gamma_{jl}^{n,2} \left( u^2_{n+e_{jl}} - u^2_n \right) + h \left( \Delta_2 u_n, \frac{n}{N}, 2 \right),
\end{cases}$$

which can be rewritten as

$$\begin{cases} 
- \frac{du^1_n}{dt} = (N - n_1) \alpha_j^* \left( \Delta_2 u_{n+e_{12}}, \frac{n + e_{12}}{N}, 2 \right) \left( u^1_{n+e_{12}} - u^1_n \right) \\
\quad + n_1 \alpha_2^* \left( \Delta_1 u_n, \frac{n}{N}, 1 \right) \left( u^1_{n+e_{21}} - u^1_n \right) + h \left( \Delta_1 u_n, \frac{n}{N}, 1 \right), \\
- \frac{du^2_n}{dt} = (N - n_1) \alpha_j^* \left( \Delta_2 u_n, \frac{n}{N}, 2 \right) \left( u^2_{n+e_{12}} - u^2_n \right) \\
\quad + n_1 \alpha_2^* \left( \Delta_1 u_{n+e_{21}}, \frac{n + e_{21}}{N}, 1 \right) \left( u^2_{n+e_{21}} - u^2_n \right) + h \left( \Delta_2 u_n, \frac{n}{N}, 2 \right),
\end{cases}$$

with “$u^i_{n+e_{jl}}$” following the definition of $e_{jl}$ on page 17, where the index “$n + e_{jl}$” represents the space. Since $\theta_1 + \theta_2 = 1$, we introduce the new variable $\zeta \in [0, 1]$ and thus $\theta = (\zeta, 1 - \zeta)$. We split the domain $[0, 1]$ into $N$ equidistant subintervals and define $\zeta_k = \frac{k}{N}$, $0 \leq k \leq N$, $k \in \mathbb{N}$. The variable $\zeta_k$ corresponds to the fraction of players in the state 1. Then the fraction of players in the state 2 is given by
$1 - \zeta_k = \frac{N-k}{N}$. Consequently (2.5) takes the form

\[
\begin{align*}
-\frac{du_k^1}{dt} &= N(1 - \zeta_k)(u_{k+1}^2 - u_{k+1}^1)^+ (u_{k+1}^1 - u_k^1) + f(1, \zeta_k) \\
&\quad + N\zeta_k(u_k^1 - u_k^2)^+ (u_{k-1}^1 - u_k^1) - \frac{1}{2}((u_k^1 - u_k^2)^+)^2, \\
-\frac{du_k^2}{dt} &= N(1 - \zeta_k)(u_k^2 - u_k^1)^+ (u_{k+1}^2 - u_k^2) + f(2, 1 - \zeta_k) \\
&\quad + N\zeta_k(u_{k-1}^1 - u_{k-1}^2)^+ (u_{k-1}^2 - u_k^2) - \frac{1}{2}((u_k^2 - u_k^1)^+)^2.
\end{align*}
\]

(2.6)

Note that in (2.6), the terms $\zeta_k$ and $(1 - \zeta_k)$ vanish for $k = 0$ and $k = N$. Thus no particular care has to be taken concerning the ghost points at $\zeta_{N+1}$ and $\zeta_{-1}$. This approach is the discrete analog to not imposing boundary conditions on (1.1), since the components of the vector $\theta$ are a percentage and no agents are outside the domain (this is imposed a priori, in the construction of the model). A similar situation occurs in state constrained problems for Hamilton-Jacobi equations, see [19, 73, 74].

### 2.2 Shock structure for two-state problems

We now perform a more detailed investigation of the shock structure in the case of two-state problems. More precisely, we execute a reduction of dimension obtaining a hyperbolic scalar equation, since the key quantities depend on the difference of the utility functions only. Then, we introduce a related conservation law that yields a Rankine-Hugoniot condition for possible shocks. This new formulation allows us to study finite-state mean-field games from another numerical perspective and gives new insights into the shock structure and the qualitative behavior of solutions to (1.1).
2.2.1 Reduction of the system to a scalar problem

Let $U(\theta, t)$ be a $C^1$ solution to (1.1) with $d = 2$. We define $w(\zeta, t) = U^1(\theta, t) - U^2(\theta, t)$, where $\theta = (\zeta, 1 - \zeta)$. From (1.1) we have that

$$
(U^1 - U^2)_t = -g_1(U^1, U^2, \theta_1, \theta_2) \partial_{\theta_1} (U^1 - U^2) - g_2(U^1, U^2, \theta_1, \theta_2) \partial_{\theta_2} (U^1 - U^2)
$$

$$
- h(U^1, U^2, \theta_1, \theta_2, 1) + h(U^1, U^2, \theta_1, \theta_2, 2). \tag{2.7}
$$

Then $h$ and $g$, given by (1.3) and (1.12), can be written as

$$
h(U^1, U^2, \theta_1, \theta_2, i) = h(w(\zeta, t), 0, \zeta, 1 - \zeta, i),$$

and $g_1(U^1, U^2, \theta_1, \theta_2) = g_1(w(\zeta, t), 0, \zeta, 1 - \zeta)$. For two-state problems equation (1.11) gives $g_2 = -g_1$. Hence we obtain

$$
w_t = - g_1(w(\zeta, t), 0, \zeta, 1 - \zeta) \left[ \partial_{\theta_1} (U^1 - U^2) - \partial_{\theta_2} (U^1 - U^2) \right]
$$

$$
- h(w(\zeta, t), 0, \zeta, 1 - \zeta, 1) + h(w(\zeta, t), 0, \zeta, 1 - \zeta, 2).$$

Define $r$ and $q$ by

$$
r(w(\zeta, t), \zeta) = - g_1(w(\zeta, t), 0, \zeta, 1 - \zeta),$$

$$
q(w(\zeta, t), \zeta) = h(w(\zeta, t), 0, \zeta, 1 - \zeta, 1) - h(w(\zeta, t), 0, \zeta, 1 - \zeta, 2),$$

and denote by $\frac{\partial w}{\partial \zeta}$ the expression $\frac{\partial w(\zeta, t)}{\partial \zeta} = \left( \frac{\partial}{\partial \theta_1} - \frac{\partial}{\partial \theta_2} \right) (U^1 - U^2) |_{\zeta, 1 - \zeta}$. Then, the difference of $U^1$, equation (2.7), can be written as

$$
-w_t(\zeta, t) + r(w(\zeta, t), \zeta) \partial_{\zeta} w(\zeta, t) = q(w(\zeta, t), \zeta). \tag{2.8}
$$
Hence we obtain a single one-dimensional hyperbolic equation for \( w \), which describes the evolution of the difference \( w = U^1(\theta, t) - U^2(\theta, t) \). Equation (2.8) is fundamental for analyzing the shock structure in more detail.

### 2.2.2 A conservation law and the Rankine-Hugoniot condition

Let us consider the following conservation law associated to (2.8) on the interval \([0, 1]\),

\[
-P_t(\zeta, t) + \partial_\zeta \left( r(w(\zeta, t), \zeta) \cdot P(\zeta, t) \right) = 0, \tag{2.9}
\]

supplemented with the boundary condition \( P(0, t) = P(1, t) = 0 \) for all times \( t \in [0, T] \).

If \( P \) is a sufficiently smooth solution to (2.9) and \( P(\zeta, 0) \geq 0 \), then the maximum principle implies that \( P(\zeta, t) \geq 0 \). Furthermore, if \( \int P(\zeta, 0)d\zeta = 1 \), we have that \( \int P(\zeta, t)d\zeta = 1 \). Assuming that \( P(\zeta, 0) \) is a probability distribution, we can regard \( P(\zeta, t) \) as a probability distribution on the set \( \mathcal{P}(\mathcal{I}) \), \( \mathcal{I} = \{1, 2\} \), as we have a natural identification for two-state problems: \( \mathcal{P}(\mathcal{I}) \simeq [0, 1] \). Hence, (2.9) describes the evolution of a probability distribution on \( \mathcal{P}(\mathcal{I}) \). Uncertainty in the initial distribution of the mean field \( \zeta \) can be encoded in the initial condition \( P(\zeta, 0) \) and propagated through (2.9). Since (2.9) may not have globally smooth solutions, we use the Rankine-Hugoniot condition to characterize certain possibly discontinuous solutions. Let

\[
s : [0, T] \to [0, 1]
\]

be a \( C^1 \) curve and suppose that \( P \) is a \( C^1 \) function\(^2\) on both \( 0 < \zeta < s(t) \) and \( s(t) < \zeta < 1 \), for \( t \in [0, T] \). Assume further that (2.9) holds in this set. Let \( B : [0, 1] \times [0, T] \to \mathbb{R} \). We denote by \([B]\) the jump of \( B \) across the curve \( s(t) \), that

---

\(^2\)Since the regularity of \( P \) is unknown, the hypothesis of \( C^1 \) regularity is used to study the Rankine-Hugoniot condition and its expression, see [66].
is, \([B] = B(s^+(t), t) - B(s^-(t), t)\). Equation (2.9) leads to the Rankine-Hugoniot condition of the form

\[ [P] \dot{s} = -[r \ P]. \]

If we start with the initial condition \(P(\zeta, 0) = 1\), then the support of \(P\) is the closure of the set of all mean-field states which can be reached from some initial mean-field state (all possible choices of \(\zeta\) at time 0). Suppose \(\mathcal{B} = \{(\zeta, t) = (s(t), t), \ 0 \leq t \leq T\}\) and that there is a discontinuity in \(P\) at the boundary \(\mathcal{B}\) of the set \(P = 0\). Then we conclude from the Rankine-Hugoniot condition that

\[ \dot{s} = -r. \]

Hence \(\mathcal{B}\) is a characteristic for (2.8). From the discussion above, it is clear that the solution to (2.9) with the initial condition \(P(\zeta, 0) = 1\) will track the transport of a uniform measure by the characteristics to (2.8). Therefore numerical simulations of this quantity provide an indicator whether characteristics run into shocks or not.

By using (2.9), we can also derive local Lipschitz bounds for the solution to (2.8):

**Proposition 1.** There exists a time \(t_0 < T\) and a constant \(C\), depending only on the \(L^\infty\) norm and on the Lipschitz constant of \(w(x, T)\), such that for any \(C^1\) solution \(w : [0, 1] \times (t_0, T] \to \mathbb{R}\) to (2.8), we have for \(t_0 < t \leq T\) the following estimate

\[ \|\partial_\zeta w(\cdot, t)\|_\infty \leq \frac{C}{(t - t_0)}. \]

**Proof 1.** Let \(P\) be a solution to (2.9) with \(P(0, t) = P(1, t) = 0, \ t \leq T, P(\zeta, t) \geq 0\), with \(\int P(\zeta, t) d\zeta = 1\). We differentiate equation (2.8) with respect to \(\zeta\), multiply by \(2\partial_\zeta w\) and obtain

\[ -\partial_{\zeta} S + r \partial_{\zeta} S = -2\partial_{\zeta} r \partial_{\zeta} w - 2\partial_{w} r S \partial_{\zeta} w + 2\partial_{w} q S + 2\partial_{\zeta} q \partial_{\zeta} w, \]
using that $S(\zeta, t) = (\partial_\zeta w(\zeta, t))^2$. Then, we deduce the following estimate:

$$-\frac{d}{dt} \int S(\zeta, t) P(\zeta, t) \, d\zeta \leq 2 \int (-\partial_\zeta r r \partial_\zeta w - \partial_w r S \partial_\zeta w + \partial_w q S + \partial_\zeta q \partial_\zeta w) P(\zeta, t) \, d\zeta$$

$$\leq C_1 \int S^{3/2} P(\zeta, t) \, d\zeta + C_2,$$

where we use the fact that $w$ is bounded, see [42], in the last step. Then

$$\int S(\zeta, t) P(\zeta, t) \, d\zeta \leq C + \int_t^T \|S(\cdot, s)\|_\infty^{3/2} \, ds + \|S(\cdot, T)\|_\infty.$$

Taking $P(\zeta, t)$ to be an arbitrary probability measure on $[0, 1]$ this yields that

$$\|S(\cdot, t)\|_\infty \leq C + \int_t^T \|S(\cdot, s)\|_\infty^{3/2} \, ds + \|S(\cdot, T)\|_\infty.$$

This estimate does not give global bounds, but a nonlinear version of Gronwall’s inequality yields the existence of a time $t_0 < T$, such that the estimate holds up to $t_0$.

In chapter 4, we will investigate the numerical behavior of the conservation law obtained here. Methods used to examine two-state mean-field games are used in the next chapter to describe and study dual two-state mean-field games.
Chapter 3

Dual two-state mean-field games

Now, we discuss the dual version of finite-state mean-field games. We restrict ourselves to the two-state case, simplifying the presentation and helping to compare both solutions numerically, via the primal and dual formulations. In the dual problem, we solve the PDE associated with the evolution of the density of agents in each state as a function of the value function.

Let us start by considering (1.1) reduced to the two-state case:

\[
\begin{align*}
- U^i_t(\theta, t) &= \sum_{j=1}^{2} g_j(U, \theta) \frac{\partial U^i(\theta, t)}{\partial \theta_j} + h(U, \theta, i), \\
U(\theta, T) &= U_T(\theta).
\end{align*}
\]

Here \( U^i : \mathcal{P}(\mathcal{I}) \times [0, T] \to \mathbb{R} \), \( g : \mathbb{R}^2 \times \mathcal{P}(\mathcal{I}) \to \mathbb{R}^2 \), \( h : \mathbb{R}^2 \times \mathcal{P}(\mathcal{I}) \times \mathcal{I} \to \mathbb{R} \), \( U^i_T : \mathcal{P}(\mathcal{I}) \to \mathbb{R} \) and \( i \in \mathcal{I} = \{1, 2\} \). The characteristics for (3.1) are a system of a finite-state Hamilton-Jacobi equation, coupled with a transport equation for a probability measure, see [42].

Motivated by the discussion in [67], we consider the dual equation to (3.1), which is

\[
\begin{align*}
\Theta^i_t(v, t) &= g_i(v, \Theta) + \sum_{j=1}^{2} h(v, \Theta, j) \frac{\partial \Theta^i(v, t)}{\partial v^j}, \\
\Theta(v, T) &= \Theta_T(v),
\end{align*}
\]

where \( \Theta^i : \mathbb{R}^2 \times [0, T] \to \mathbb{R} \), \( \Theta^i_T : \mathbb{R}^2 \to \mathbb{R} \) and \( i \in \mathcal{I} = \{1, 2\} \). For potential mean-field games, both (3.1) and (3.2) can be regarded as gradients of a Hamilton-
Jacobi equation [67, 42]. Thanks to a special reduction, which we discuss in the following section, this gradient structure can be used to study a wide range of two-state problems.

3.1 Reduced primal problem

We explore the particular structure of (3.1) for the two-state problems by transforming it into a scalar problem. We repeat some of the steps of the reduction performed in chapter 2, to contrast it with the dual case.

Observe that \( \theta \) is a probability vector. Thus, we rewrite \( \theta \) as \( \theta = (\theta_1, \theta_2) = (\zeta, 1 - \zeta), \zeta \in [0, 1] \). Let \( U \) be a \( C^1 \) solution to (3.1). Because \( h(z, \theta, i) \) depends only on the differences of the coordinates of \( z \), we define

\[
w(\zeta, t) = U^1(\zeta, 1 - \zeta, t) - U^2(\zeta, 1 - \zeta, t),
\]

and set \( w_T(\zeta) = U^1_T(\zeta, 1 - \zeta) - U^2_T(\zeta, 1 - \zeta) \). Thus, the hyperbolic system (3.1) is reduced to a scalar equation called the reduced primal equation (see [43]):

\[
-w_t(\zeta, t) + r(w, \zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) = q(w, \zeta), \tag{3.3}
\]

where

\[
r(w, \zeta) = -g(1, \zeta, 1 - \zeta),
q(w, \zeta) = h(w, 0, \zeta, 1 - \zeta, 1) - h(w, 0, \zeta, 1 - \zeta, 2),
\]

and \( \frac{\partial w}{\partial \zeta} \) denotes \( \frac{\partial w}{\partial \zeta} = \left( \frac{\partial}{\partial \theta_1} - \frac{\partial}{\partial \theta_2} \right) (U^1 - U^2) |_{(\zeta,1-\zeta)}. \) Note that (3.3) is supplemented with the terminal condition \( w(\zeta, T) = w_T(\zeta) \), and since \( r(w, 0) \leq 0 \) and \( r(w, 1) \geq 0 \) no further boundary conditions are required.
3.2 Dual problem

Now, we present a transformation introduced by Lions [67] to convert system (1.1) or (3.1) into an equivalent system of linear PDEs. This procedure is related to the hodograph transformation (see, for instance, [33]), a technique often used to convert certain nonlinear PDEs into a linear PDE by interchanging the dependent and independent variables. Note that this transformation is similar to the generalized coordinate techniques in classical Hamiltonian dynamics.

For fixed time $t$, we consider the function $U(\theta, t)$, solution to (3.1), mapping from an open set of $\mathbb{R}^2$ into $\mathbb{R}^2$ and its inverse $\Theta(v, t) = (\Theta^1(v, t), \Theta^2(v, t))$, defined by

$$\Theta(U(\theta, t), t) = \theta.$$ 

Using (3.1), we obtain that $\Theta(v, t)$ satisfies the dual system (3.2).

3.3 Reduced dual problem

We now apply a similar reduction procedure to the dual problem. This reduction transforms the dual system (3.2) into a scalar equation. Let $\tilde{v} = v^1 - v^2$. We consider the set where $\Theta^1 + \Theta^2 = 1$ and deduce that

$$\Theta_t^1 = g_1(\tilde{v}, 0, \Theta^1, 1 - \Theta^1) + h(\tilde{v}, 0, \Theta^1, 1 - \Theta^1, 1) \frac{\partial \Theta^1}{\partial v^1} + h(\tilde{v}, 0, \Theta^1, 1 - \Theta^1, 2) \frac{\partial \Theta^1}{\partial v^2}.$$ 

Next, we look for solutions depending only on $\tilde{v}$, that is, $\Theta^1(v^1, v^2, t) = Z(\tilde{v}, t)$. The equation for $Z$ is given by

$$Z_t = [h(\tilde{v}, 0, Z, 1 - Z, 1) - h(\tilde{v}, 0, Z, 1 - Z, 2)] \frac{\partial Z}{\partial \tilde{v}} + g_1(\tilde{v}, 0, Z, 1 - Z),$$
which can be rewritten as

\[-Z_t(\tilde{v}, t) + q(\tilde{v}, Z) \frac{\partial Z}{\partial \tilde{v}} = r(\tilde{v}, Z).\]  

Equation (3.4) is supplemented with the boundary conditions: \(\lim_{\tilde{v} \to -\infty} Z(\tilde{v}, t) = 1\) and \(\lim_{\tilde{v} \to +\infty} Z(\tilde{v}, t) = 0\). These conditions are motivated by the following considerations: if \(\tilde{v}\) is very negative, the best state regarding the utility function is state 1. Hence, all players would switch to it. Similarly, if \(\tilde{v}\) is very large, then all players will switch to state 2.

### 3.4 Potential two-state mean-field games

We now consider the class of potential mean-field games restricted to the case of two state problems. Suppose that

\[h(z, \theta, i) = \tilde{h}(z, i) + f(i, \theta), \quad i \in \{1, 2\}.\]  

The functions \(h\) that admit the decomposition as in (3.5) are called *separable* throughout this text. We show in this section that separable mean-field games are potential. We are not aware of other classes of potential mean-field games. Separable mean-field games occur naturally in many problems. Various examples in the realm of socio-economic sciences were discussed in [43].

In this section, we suppose also that \(f(i, \theta) = \frac{\partial F(\theta)}{\partial \theta_i}\) for some potential \(F : \mathbb{R}^2 \to \mathbb{R}\).

Define \(H : \mathbb{R}^2 \times \mathcal{P}(\mathcal{I}) \to \mathbb{R}\) by

\[H(z, \theta) = \theta_1 \tilde{h}(\Delta_1 z, 1) + \theta_2 \tilde{h}(\Delta_2 z, 2) + F(\theta).\]  

Let \(\Psi_T : \mathbb{R}^2 \to \mathbb{R}\) be a continuous function and consider a smooth enough solution
\(\Psi : \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R}\) of the Hamilton-Jacobi equation

\[
\begin{dcases}
    - \frac{\partial \Psi(\theta, t)}{\partial t} = H(\partial_\theta \Psi, \theta), \\
    \Psi(\theta, T) = \Psi_T(\theta).
\end{dcases}
\]  \tag{3.7}

By setting \(U^i(\theta, t) = \frac{\partial \Psi(\theta, t)}{\partial \theta_i}\), we obtain that

\[-U^i = g_1(U; \theta) \frac{\partial U^i}{\partial \theta_1} + g_2(U; \theta) \frac{\partial U^i}{\partial \theta_2} + \tilde{h}(\Delta_i U, i) + \frac{\partial F(\theta)}{\partial \theta_i},\]

and deduce that \(U^i\) solves the PDE in (3.1). In section 4.4, we describe the numerical solution of (3.7) and how to handle boundary conditions.

### 3.4.1 Reduced potential mean-field games - I

The reduction to the scalar case from section 3.1 can be performed in the potential case. Once again, we set \(\theta = (\zeta, 1 - \zeta), \zeta \in [0, 1]\), and define

\[\Upsilon_T(\zeta) = \Psi_T(\zeta, 1 - \zeta).\]

Let \(\Upsilon : [0, 1] \times [0, T] \rightarrow \mathbb{R}\) solve the Hamilton-Jacobi equation

\[
\begin{dcases}
    - \frac{\partial \Upsilon(\zeta, t)}{\partial t} = \tilde{H}(\partial_\zeta \Upsilon, \zeta), \\
    \Upsilon(\zeta, T) = \Upsilon_T(\zeta),
\end{dcases}
\]  \tag{3.8}

where \(\tilde{H} : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}\) is

\[\tilde{H}(\partial_\zeta \Upsilon, \zeta) = \zeta \tilde{h}(\partial_\zeta \Upsilon, 0, 1) + (1 - \zeta) \tilde{h}(\partial_\zeta \Upsilon, 0, 2) + F(\zeta, 1 - \zeta).\]  \tag{3.9}

Then, \(\Psi(\zeta, 1 - \zeta, t) = \Upsilon(\zeta, t)\) solves (3.7).
The natural boundary conditions for (3.8), taking into account that \( \zeta \in [0, 1] \), are the state constrained boundary conditions, as discussed in [43]. These can be implemented in practice by taking large Dirichlet data for the boundary values of \( \Upsilon \) at \( \zeta = 0, 1 \). A solution to the reduced primal system can be addressed via the reduced potential system by setting \( w_p = \frac{\partial \Upsilon(\zeta, t)}{\partial \zeta} \) and observing that \( w_p \) is a solution to (3.3).

### 3.4.2 Reduced potential mean-field games - II

In this subsection, we prove the proposition connecting separable and potential mean-field games.

**Proposition 2.** A separable two-state mean-field game has an associated reduced equation that admits a potential.

**Proof 2.** Taking into account the expression of \( q(w, \zeta) \) in the reduced primal formulation and using that \( h \) is separable, we obtain:

\[
q(w, \zeta) = \left[ \tilde{h}(w, 0, 1) - \tilde{h}(w, 0, 2) \right] + \left[ f(1, \zeta) - f(2, \zeta) \right].
\]

Because we are dealing with a problem in one dimension, \( f(1, \zeta) - f(2, \zeta) \) is the derivative of some potential \( \tilde{F} : \mathbb{R} \to \mathbb{R} \). So, without any additional assumptions, we conclude that the reduced equation admits a potential.

### 3.5 Potential formulation for dual systems

Suppose (3.5) holds and let \( H \) be given by (3.6). Fix \( V_T : \mathbb{R}^2 \to \mathbb{R} \) of class \( C^1 \) and take \( V : \mathbb{R}^2 \times [0, T] \to \mathbb{R} \) as a smooth solution to the dual Hamilton-Jacobi equation

\[
\begin{aligned}
\frac{\partial V(v, t)}{\partial t} &= H(v, \partial_v V), \\
V(v, T) &= V_T(v).
\end{aligned}
\] (3.10)
Note that analogously to the primal case, the function $\Theta(v, t) = D_v V(v, t)$ solves the PDE in (3.2).

### 3.5.1 Reduced potential for dual systems

As in the previously reduced cases, suppose

$$V_T(v^1, v^2) = \Phi_T(v^1 - v^2).$$

Let $\Phi(\tilde{v}, t)$ solve

$$\begin{cases}
\frac{\partial \Phi(\tilde{v}, t)}{\partial t} = \tilde{H}(\tilde{v}, \partial_{\tilde{v}} \Phi), \\
\Phi(\tilde{v}, T) = \Phi_T(\tilde{v}),
\end{cases} \tag{3.11}$$

where $\tilde{H} : \mathbb{R} \times [0, 1] \to \mathbb{R}$ is

$$\tilde{H}(\tilde{v}, \partial_{\tilde{v}} \Phi) = \partial_{\tilde{v}} \Phi \tilde{h}(\tilde{v}, 0, 1) + (1 - \partial_{\tilde{v}} \Phi) \tilde{h}(\tilde{v}, 0, 2) + F(\partial_{\tilde{v}} \Phi, 1 - \partial_{\tilde{v}} \Phi).$$

Then, it follows that $V(v^1, v^2, t) = \Phi(v^1 - v^2, t)$ solves the PDE in (3.10).

The boundary conditions associated with the dual problem suggest asymptotically linear boundary conditions for $\Phi$. More precisely,

$$\lim_{\tilde{v} \to -\infty} \frac{\partial \Phi(\tilde{v}, t)}{\partial \tilde{v}} = 1 \quad \text{and} \quad \lim_{\tilde{v} \to +\infty} \frac{\partial \Phi(\tilde{v}, t)}{\partial \tilde{v}} = 0.$$

Furthermore, a solution to the reduced dual equation can be constructed via the reduced potential dual equation (3.11) by taking $Z_{\rho} = \frac{\partial \Phi(\tilde{v}, t)}{\partial \tilde{v}}$, and observing that $Z_{\rho}$ solves the reduced dual equation (3.4).
3.6 Legendre transform

Using the Legendre transform as in [67], we can relate the various terminal conditions for (3.1), (3.2), (3.7), and (3.10).

To do so, we fix a convex function $\Psi_T(\theta)$ and the corresponding solution $\psi(\theta, t)$ of (3.7). Then $U(\theta, t) = \frac{\partial \Psi(\theta, t)}{\partial \theta}$ solves (3.1) with terminal data $U_T(\theta) = \frac{\partial \Psi_T(\theta)}{\partial \theta}$.

To define the corresponding solutions to (3.2) and (3.10) we consider the Legendre transform $V_T$ of $\Psi_T$:

$$V_T(v) = \sup_{\theta} v \cdot \theta - \Psi_T(\theta).$$

Then, by the usual properties of the Legendre transform, and under sufficient regularity and convexity assumptions, the inverse of the map $\theta \mapsto \frac{\partial \Psi_T(\theta)}{\partial \theta}$ is $v \mapsto \frac{\partial V_T(v)}{\partial v}$.

Furthermore, as observed before, let $V$ solve (3.10) with terminal data $V_T$. Then $\Theta(v, t) = \frac{\partial V(v, t)}{\partial v}$ solves (3.2). Hence, the terminal data $\Theta_T$ is the inverse of $U_T$ and, at least for $t$ close enough to $T$, $\Theta(v, t)$ is the inverse of $U(\theta, t)$, by the properties discussed previously. Besides, at least for $t$ close enough to $T$, $\Psi(\theta, t)$ is the Legendre transform of $V(v, t)$.

3.7 Some reduced systems

Finally, we examine (3.3) in the case where $h$ is given by (2.2) and $g$ by (2.3). The reduced primal system becomes

$$-w_t(\zeta, t) - \frac{(1 - 2\zeta)|w| - w \partial w(\zeta, t)}{2} = \frac{1}{2} |w| w - [f(1, \zeta, 1 - \zeta) - f(2, \zeta, 1 - \zeta)].$$

Moreover, the reduced dual system reads as

$$-Z_t(\bar{v}, t) + \left(f(1, Z) - f(2, Z) - \frac{1}{2} |\bar{v}| \bar{v}\right) \frac{\partial Z(\bar{v}, t)}{\partial \bar{v}} = \frac{(1 - 2Z)|\bar{v}| - \bar{v}}{2}.$$
The function $\tilde{H}$ in (3.9) is

$$\tilde{H}(\tilde{v}, \zeta) = -\frac{(\tilde{v}^-)^2 + \zeta|\tilde{v}|\tilde{v}}{2} + F(\zeta, 1 - \zeta).$$

And the potential versions are given respectively by:

$$-\frac{\partial Y(\zeta, t)}{\partial t} = -\frac{1}{2} \left\{ \left( \frac{\partial Y}{\partial \zeta} \right)^2 + \zeta \frac{\partial Y}{\partial \zeta} \right\} + F(\zeta, 1 - \zeta),$$

$$Y(\zeta, T) = Y_T(\zeta),$$

the reduced potential formulation for the primal problem, and

$$\frac{\partial \Phi(\tilde{v}, t)}{\partial t} = -\frac{1}{2} \left[ (\tilde{v}^-)^2 + \frac{\partial \Phi}{\partial \tilde{v}} |\tilde{v}| \tilde{v} \right] + F \left( \frac{\partial \Phi}{\partial \tilde{v}}, 1 - \frac{\partial \Phi}{\partial \tilde{v}} \right),$$

$$\Phi(\tilde{v}, T) = \Phi_T(\tilde{v}),$$

the reduced potential formulation for the dual problem.
Chapter 4

Socioeconomic applications of two-state mean-field games and numerical investigations

In this chapter, we use the tools developed for two-state problems and present some applications to socioeconomics. We consider two examples: the paradigm shift problem and the consumer choice problem. In section 4.3, we exhibit their numerical simulations. Next, we exemplify the use of the numerical scheme presented in chapter 2. We now make use of it in a two-fold. First, we illustrate the examples of sections 4.1, 4.2. Then, we investigate characteristics of mean-field games, as the shock formation, the loss of monotonicity and the lack of invertibility.

4.1 Paradigm shift

According to Kuhn [59], a paradigm shift corresponds to a change in a basic assumption within the ruling theory of science. Classical cases of paradigm shifts are the transition from Ptolemaic cosmology to Copernican one, the development of quantum mechanics which replaced classical mechanics on the microscopic scale or the acceptance of Mendelian inheritance as opposed to Pangenesis.

A common assumption in theoretical models for research dynamics is the fact that scientists are rewarded from recognition from others, see for example [12, 11]. This recognition may take the form of citations, prizes or other financial incentives and is further amplified by scientific activity, like conferences and collaborations, in this field. Bensancenot and Dogguy modeled a paradigm shift in a scientific community
by a two-state mean-field game approach and analyzed the competition between two different scientific hypotheses [9]. In our example, we consider a simpler model but follow their general ideas and assumptions.

Let us consider a scientific community with \( N \) researchers working on two different hypotheses. Each researcher working on the paradigm \( i, i = 1, 2 \), wants to maximize its productivity measured by a cost function of the form (2.1). Here the function \( f = f(i, \theta) \) corresponds to the productivity of a researcher working on paradigm \( i \), and \( c_0(i, \mu) = -\frac{1}{2} \sum_{j \neq i}^{2} \mu_j^2 \) to the cost of switching to the other objective. Observe the negative sign of the switching costs, since agents want to maximize their productivity. We assume that the productivity is directly related to the number of researchers working on the paradigm since, for example, more researchers imply more scientific activities like conference and collaborations taking place. In the case of two different fields, \( \theta_1 \) gives the fraction of researchers working on paradigm 1 and \( \theta_2 = 1 - \theta_1 \) on paradigm 2. We choose the functions \( f(i, \theta), i = 1, 2 \), of the form

\[
\begin{align*}
  f(1, \theta) &= [a_1 \theta_1^r + (1 - a_1)(1 - \theta_1)^r]^{\frac{1}{r}}, \\
  f(2, \theta) &= [a_2 (1 - \theta_2)^r + (1 - a_2) \theta_2^r]^{\frac{1}{r}}. 
\end{align*}
\]

These functions are called \textit{productivity functions with constant elasticity of substitution} and are commonly used in economics to combine two or more productive inputs (in our case scientific activities in the different fields) to an output quantity. The constant \( r \in \mathbb{R}, r \neq 0 \), denotes the \textit{elasticity of substitution}, and it measures how easily one can substitute one input for the other. The constants \( a_i \in [0, 1] \) measure the dependence of paradigm \( i \) with respect to the other. If \( a_i \) is close to one, the field is more autonomous and little influenced by the activity in the other field. The choice of \( f \) is also motivated by the assumption that scientists place greater weight on theories which are accepted by a larger community than otherwise.
4.2 Consumer choice

Consumer choice models relate preferences to consumption expenditure. Research has shown that consumer behavior is a very complex process, which involves elements of psychology, sociology, marketing, and economics. We propose a very stylized model, in which consumers preferences are solely based on the price of the goods available and the fraction of consumers using them. This model may apply to study the dynamics in the mobile phone sector, where consumer choices are strongly influenced by the number of people using the same provider and the costs of the contract. We also refer to [50] for other economic applications of two-state mean-field games.

Let us consider two choices of consumption goods and denote by \( \theta_1 \) the fraction of agents consuming the good 1 and by \( \theta_2 = 1 - \theta_1 \) the fraction that consumes the good 2. We assume that the price of a good is strongly determined by the consumption rate, in particular, we choose, for \( i = 1, 2 \),

\[
f(i, \theta) = \begin{cases} 
\frac{\theta_i^{1-\eta} - 1}{1 - \eta} + s_i, & \eta > 0, \eta \neq 1, \\
\ln(\theta_i) + s_i, & \eta = 1,
\end{cases}
\]

where \( s_i \in \mathbb{R}^+ \) corresponds to the minimum price of the good. In economic literature, the function \( f \) is called the isoeelastic utility function. It expresses the utility in terms of the consumption \( \theta \) and exhibits a constant risk aversion \( \eta \). Risk aversion measures the reluctance of agents to accept a bargain with an uncertain payoff in place of another bargain with a more certain, but possibly lower, expected payoff.

4.3 Numerical examples

In the following, we illustrate the behavior of the proposed numerical method for different applications. Due to the hyperbolic nature of (1.1), shocks may arise even
in the case of smooth terminal conditions. As discussed in section 1.2, shocks in the utility function are caused by the non-uniqueness of the Nash mean-field equilibrium. When shocks are present, a small variation in the initial distribution of players may give rise to a substantially different behavior of the mean field through the ODEs (1.13)-(1.14). Hence, shock formation in the utility function indicates the existence of a threshold for the preference of a state depending on the distribution of players.

Let $N = 100$, i.e. the interval $[0, 1]$ is discretized into 100 equidistant intervals. Each grid point corresponds to the percentage of players being in state 1. System (2.6) is solved using an explicit in time discretization with time steps of size $\Delta t = 10^{-4}$. In all examples in this section, the terminal time $T$ is set to $T = 10$ if not stated otherwise. The simulations were performed on a 2.3GHz i7 computer with 16GB of RAM using the software Matlab. For the next three examples, the running time is around three seconds.

**Example 1 (Shock formation):** In this first example, we would like to illustrate the formation of shocks, a phenomenon often observed in the derivatives of solutions of Hamilton-Jacobi equations. We choose a terminal cost of the form

$$U^1(\theta, 10) = \theta_1 - \frac{1}{2} \text{ and } U^2(\theta, 10) = \theta_2 - \frac{1}{2},$$

a running cost as in (2.1) with $f(1, \theta) = 1 - \theta_1$ and $f(2, \theta) = 1 - \theta_2 = \theta_1$. Figure 4.1 illustrates the formation of a shock for smooth terminal data. This shock is also present when we consider the difference $U^1 - U^2$ of the utilities. This difference is a relevant variable in this problem, since both $g$ and $h$, given by (1.12) and (1.3) respectively, depend only on the difference between the utilities.

**Example 2 (Paradigm shift):** In this example, we illustrate the outcome of a two-state mean-field game modeling a paradigm shift within a scientific community. 
Note that we use the negative cost functional since we always consider minimization problems. The terminal utilities are given by

\[ U^1(\theta, T = 10) = 1 - \theta_1 \text{ and } U^2(\theta, T = 10) = \theta_2, \]

and the parameters in (4.1) are set to \( a_1 = \frac{1}{2}, \ a_2 = \frac{9}{10}, \ r = \frac{3}{4}. \) In figure 4.2 we observe
the paradigm shift within the scientific community. At $T = 10$ the optimal states are $\theta_1 = 1$ and $\theta_2 = 1$ since the functions $U^1$ and $U^2$ take their minimum value at these points respectively. In figure 4.2 we observe that this is not the case at $t = 0$. Here $U^1$ takes its minimum value at $\theta_1 = 0$, i.e. paradigm 1 is not popular anymore.

**Example 3 (Consumer choice):** In our third example, we consider the consumer choice behavior, see section (4.2). We set the final utility function to

$$U^1(\theta, 10) = 1 - \theta_1$$
$$U^2(\theta, 10) = \theta_2.$$

At the time $T = 10$ the utility functions take their minimum value at $\theta_1 = 1$ and $\theta_2 = 0$, i.e. their minimum corresponds to the case that either all of them choose product 1 or product 2, respectively. Figure 4.3 illustrates the utility functions for
two sets of parameters, namely

\[ \eta = 0.5, \ s_1 = 0.075, \ s_2 = 0.1 \quad \text{and} \quad \eta = 1, \ s_1 = 0.1, \ s_2 = 0.075. \]

We observe for the second set of parameters that \( U^1 \) takes its minimum value on the interval \( \theta_1 \in [0.65, 1] \). The jump at \( \theta_1 \approx 0.65 \) in both utilities indicates the existence of a critical acceptance rate. If less than 65% of the consumers buy product 1, the price is increasing, and the product will not be competitive.

### 4.4 Potential mean-field games

To validate our methods, we consider again Example 1, that can be written as a potential mean-field game. We remark that the numerical method proposed in this work can be applied to mean-field games which do not admit a potential formulation. In Example 1, we have that \( F \) and \( H \) in (2.4) are given by

\[ F(\theta_1, \theta_2) = \theta_1 \theta_2, \]
\[ H(p_1, p_2, \theta_1, \theta_2) = -\frac{1}{2} \theta_1((p_1 - p_2)^+)^2 - \frac{1}{2} \theta_2((p_2 - p_1)^+)^2 + F(\theta_1, \theta_2). \]

Then, we compare the numerical simulations of (2.6) with the ones for the corresponding Hamilton-Jacobi equation as we explain in what follows.

For \( i = 1, 2 \), set

\[ f(i, \theta) = \frac{\partial}{\partial \theta_i} F(\theta_1, \theta_2) \quad \text{and} \quad \Psi_T(\theta_1, \theta_2) = \frac{1}{2} (\theta_1 - \frac{1}{2})^2 + \frac{1}{2} (\theta_2 - \frac{1}{2})^2. \]

To simplify the numerical implementation, we perform a dimension reduction. Define \((\theta_1, \theta_2) = (\zeta, 1 - \zeta)\), with \( \zeta \in [0, 1] \). Let \( \Psi \) be the solution to (3.7) and define

\[ \Upsilon(\zeta, t) = \Psi(\zeta, 1 - \zeta, t). \quad (4.2) \]

We observe that

\[ -\frac{\partial \Upsilon}{\partial t} = \tilde{H}(\partial_\zeta \Upsilon, \zeta), \]

where \( \tilde{H} : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \) is

\[ \tilde{H}(\partial_\zeta \Upsilon, \zeta) = -\frac{1}{2} \zeta \left[ (\partial_\zeta \Upsilon)^+ \right]^2 - \frac{1}{2} (1 - \zeta) \left[ (-\partial_\zeta \Upsilon)^+ \right]^2 + F(\zeta, 1 - \zeta). \]

We use Godunov's method and an explicit Runge-Kutta method to discretize (3.7). Particular care has to be taken at the boundary. Since \( \zeta \in [0, 1] \) represents the first component of a probability vector, the natural boundary conditions for this problem are state constraint. A way to implement this is by supplementing (3.7) with large Dirichlet boundary values, i.e.,

\[ \Upsilon(0, t) = \Upsilon(1, t) = c_D \quad \text{with} \quad c_D \in \mathbb{R}^+. \]

To implement the Dirichlet boundary conditions, we follow the works of Abgrall and
Waagan, see [1, 77]. Again, we consider an equidistant discretization of the interval $[0,1]$ into $N$ subintervals of size $\Delta \zeta$, and we approximate the solution $\Upsilon(\zeta, \tau)$ to (3.7) by $\Upsilon^\tau(\zeta)$ for $\tau = T - l\Delta t$, and $\zeta = k \Delta \zeta$, $0 \leq k \leq N$; $l, k \in \mathbb{N}_0$. We set $\Upsilon^T(\zeta) = \Psi_0(\zeta,1-\zeta)$. Then, the Godunov scheme can be written as

$$\Upsilon^{\tau-\Delta t} = \Upsilon^\tau - \Delta t \hat{H}(\delta^-_\zeta \Upsilon^\tau, \delta^+_\zeta \Upsilon^\tau, \zeta),$$

(4.3)

where $\delta^+_\zeta$ and $\delta^-_\zeta$ are the difference operators

$$\delta^-_\zeta \Phi(\zeta) = \frac{\Phi(\zeta) - \Phi(\zeta - \Delta \zeta)}{\Delta \zeta} \text{ and } \delta^+_\zeta \Phi(\zeta) = \frac{\Phi(\zeta + \Delta \zeta) - \Phi(\zeta)}{\Delta \zeta},$$

and $\hat{H}$ in (4.3) is given by

$$\hat{H}(\alpha, \beta, \zeta) = \begin{cases} 
\min_{\alpha \leq q \leq \beta} \hat{H}(q, \zeta), & \text{if } \alpha \leq \beta; \\
\max_{\beta \leq q \leq \alpha} \hat{H}(q, \zeta), & \text{if } \beta \leq \alpha.
\end{cases}$$

At the boundary $\zeta = 0, 1$, we set

$$\Upsilon^{\tau-\Delta t}(0) = \min [\Upsilon^\tau(0) - \Delta t H^- (\delta^+_\zeta \Upsilon^\tau(0), 0), c_D],$$

$$\Upsilon^{\tau-\Delta t}(1) = \min [\Upsilon^\tau(1) - \Delta t H^+ (\delta^-_\zeta \Upsilon^\tau(1), 1), c_D],$$

where $H^-(p, \zeta) = \hat{H}(0, p, \zeta)$ and $H^+(p, \zeta) = \hat{H}(p, 0, \zeta)$.

We denote by $\frac{\partial \Upsilon}{\partial \zeta}$ the expression $\frac{\partial \Upsilon(\zeta, t)}{\partial \zeta} = \left( \frac{\partial}{\partial \theta_1} - \frac{\partial}{\partial \theta_2} \right) \Psi|_{(\zeta,1-\zeta)}$. Accordingly, we have that

$$\frac{\partial \Upsilon(\zeta, t)}{\partial \zeta} = U^1(\theta, t) - U^2(\theta, t),$$

by using (4.2) and (1.18), providing us a method of evaluating the difference $U^1 - U^2$ via the potential version of the mean-field game. Figure 4.4 shows the derivative of $\Upsilon$ with respect to $\zeta$, i.e. the difference $U^1 - U^2$ evaluated via the potential version,
as well as the difference $U^1 - U^2$ calculated via our numerical method from (2.6) at the time $t = 0$. The same spatial and temporal discretization (i.e. $N = 100$ and $\Delta t = 10^{-4}$) were used in both simulations.

![Figure 4.4: Derivative of $\Upsilon$ with respect to $\zeta$ versus the difference $U^1 - U^2$ calculated through our numerical method for Example 1.](image)

### 4.5 Shock structure for two-state problems

Finally, we discuss equations (2.8) and (2.9), as well as the numerical results for Example 1 presented in section 4.3. Equation (2.8) can be written as

$$-w_t(\zeta, t) = \frac{(1 - 2\zeta)|w| - w}{2} \partial_\zeta w(\zeta, t) + \frac{1}{2} |w| w - [f(1, \zeta, 1 - \zeta) - f(2, \zeta, 1 - \zeta)],$$

using that $h$ and $g$ are given by (2.2) and (2.3) respectively. Similarly, (2.9) takes the form

$$-P_t(\zeta, t) + \partial_\zeta \left[ \frac{(1 - 2\zeta)|w| - w}{2} P(\zeta, t) \right] = 0. \quad (4.4)$$

Now, we observe that the difference $w = U^1 - U^2$ satisfies

$$w(0, t) \geq 0 \text{ and } w(1, t) \leq 0. \quad (4.5)$$
The case in which (4.5) is not true, corresponds to no agents being in the state 1, i.e., $\zeta = 0$ in a situation where state 1 would be preferred to state 2 (and analogously for $\zeta = 1$). Therefore, (4.4) does not require any boundary conditions since for $\zeta = 0$ and $\zeta = 1$ the advection term vanishes, i.e., $\frac{(1 - 2\zeta)|w| - w}{2} = 0$.

We solve numerically (4.4) using an upwind finite difference scheme with the same parameters as in Section 4.3. The initial datum $P(\zeta, 0)$ is set to

$$P(\zeta, 0) = 1 \text{ for } \zeta \in (0, 1) \text{ and } P(\zeta, 0) = 0 \text{ for } \zeta \in \{0, 1\}.$$  

Due to the vanishing advection terms at $\zeta = 0, 1$, we set homogeneous Dirichlet boundary conditions for $P$, that is $P(\zeta, t) = 0$ if $\zeta \in \{0, 1\}$. We choose a uniform grid with $N = 125$ points and times steps $\Delta t = 10^{-4}$. This particular value comes from the Courant-Friedrichs-Lewy (CFL) condition, see the already mentioned [1, 77].

The evolution of $P$ for Example 1 is illustrated in figure 4.5. We observe that the function $P$ vanishes on a neighborhood of the shock in $w$ (see Figure 4.5), which is located at $\zeta = 0.5$. This suggests that the appropriate notion of a weak solution to
(2.8) has the property that characteristics can only cross once at the original point. This is similar to what happens for entropy solutions for conservation laws or viscosity solutions of Hamilton-Jacobi equations.

### 4.6 Dual problems

Now, we compare the numerical simulations of the primal, dual and potential mean-field game for different examples. Let $\zeta \in \mathcal{I}$ denote the fraction of players being in the state 1. We discretize the domain $[0, 1]$ into $N = 200$ equidistant intervals. The time steps are set to $\Delta t = 10^{-5}$, if not stated otherwise.

We solve the primal problem using the numerical discretization introduced in section 2.1. The corresponding potential mean-field game, the Hamilton-Jacobi equation (3.7), is solved using Godunov’s method. The simulations of its dual formulation, equation (3.4), are based on a finite-differences scheme using an upwind discretization for the convection term.

The numerical illustration of dual solutions is only a first step in the understanding of the dual formulation for finite-state mean-field games. We do not claim that the dual formulation brings some advantage in computational time. This kind of statement for general Hamiltonians is a difficult problem to address. Concerning accuracy, through our numerical experiments, solutions computed via the primal and dual formulations coincide to the order of the discretization size.

**Example 4 (Shock formation - Dual):** In this example involving the dual problem, we solve (3.3) with terminal data

$$w(\zeta, T = 5) = 2\zeta - 1,$$
and running costs as in (2.1):

\[ f(1, \theta) = 1 - \theta_1 \text{ and } f(2, \theta) = 1 - \theta_2. \]

Hence, \( F(\theta) = \theta_1 \theta_2 \). Figure 4.6 depicts the solution, and we observe the formation of a shock in the primal version and on the corresponding potential formulation. A boundary layer can also be seen in the dual variable \( Z \). This results from the discontinuities of \( Z(\bar{v}, T) \) at the boundary due to the limiting boundary conditions.

Figure 4.6: Simulations for Example 4.

Now, in Figure 4.7, we compare the primal and dual solutions computed via the
reduced version and using the potential formulation.

(a) Reduced Primal ($w_p$) via the Reduced Potential Primal ($\Upsilon$) at time $t = 0$.

(b) Reduced Potential ($\Phi$) for the Dual Problem at time $t = 0$.

(c) Comparison of the solutions $w$, $w_p$ to the primal problem at time $t = 0$.

(d) Comparison of the solutions $z$ and $z_p$ to the dual problem at time $t = 0$.

Figure 4.7: Simulations for Example 4 via the reduced versions.

Example 5 (Monotonicity loss): In our fifth example, we illustrate the behavior of solutions when $w$ loses its monotone behavior. In this case, the function $w$ is not invertible anymore. Hence, we expect different shocks in the dual variable.

We choose

$$F(\theta) = \kappa \theta_1^2 \theta_2^2, \quad \kappa \in \mathbb{R}^+.$$
Then $f(1, \theta) = 2 \kappa \theta_1^2 \theta_2$ and $f(2, \theta) = 2 \kappa \theta_1 \theta_2^2$. The terminal condition is set to

$$w(\zeta, T = 0.25) = 2\zeta - 1.$$ 

In Figure 4.8, we observe the loss of monotonicity of $w$ at time $t = 0$. In this case, it is not possible to invert $w$ anymore. The formation of a discontinuity is also visible in the evolution of $Z$.

![Graphs showing the evolution of $w$, $\tilde{\upsilon}$, and $Z$ over time](image)

Figure 4.8: Simulations for Example 5.
Chapter 5

Conclusions

In the first part of this dissertation, we have presented an effective numerical method for two-state mean-field games and discussed examples in socio-economic sciences. We compared our method with the numerical results obtained from classical and well-established schemes for Hamilton-Jacobi equations. If the original equation, (1.1), is as potential mean-field game, our numerical scheme gives identical results to the ones obtained from discretizing the Hamilton-Jacobi equation itself. This is an interesting observation because we know of no reason which implies that the solutions to (1.1) for potential mean-field games agree with the gradients of viscosity solutions to (1.17). Indeed, it is conceivable that a different notion of a weak solution rather than viscosity solutions could be more appropriate to describe mean-field problems. In the case where the original equation does not admit a potential, our method provides a new way to approximate the solution to (1.1). Furthermore, this method is an effective way to select among all possible solutions to the mean-field equilibrium equations (1.13) - (1.14) that arise in the limit $N \to \infty$.

As presented in the examples, our method captures shocks effectively. We analyzed the shock structure using an associated conservation law and proved a local Lipschitz estimate for the solutions of (1.1). We have shown that our numerical method can be used in the case of non-uniqueness of mean-field equilibrium because it is automatically compatible with the limit $N \to \infty$. To the best of our knowledge, our method is the only one available in the literature that can address problems where there is no uniqueness of mean-field Nash equilibrium, and that provides a selection
criterion for (1.13)-(1.14). Shock formation is not only a challenging issue in mathematical terms, but it also gives interesting insights into the dynamic behavior of interacting agents. The shocks correspond to rapid changes in the utility function and indicate the existence of thresholds in the distribution of agents, where the market behavior changes significantly, as seen in our socio-economic examples. Also, its presence indicates the lack of uniqueness of a mean-field Nash equilibrium.

\[1\] By the time of conclusion of this dissertation, a preprint addressing a new numerical method for finite-state mean-field games based on a monotonicity hypothesis appeared in the literature, see [39]. This new method is not suitable to treat problems where uniqueness of solutions is not known, as one of the studied examples in this dissertation.
Part II

Continuous setting: crowd motion problems, the Hughes model, and a numerical method for Fokker-Planck equations and derived systems
Chapter 6

Introduction to crowd motion problems and the Hughes model

Understanding the dynamics of pedestrian crowds is of great significance to prevent catastrophic emergency evacuations. In this dissertation, we consider an extension of the partial differential equation (PDE) model due to R. Hughes in [56] that describes the evolution of a pedestrian crowd. Our system of PDEs comprises a continuity equation or Fokker-Planck equation with viscosity $\varepsilon \geq 0$ coupled to an Eikonal equation

$$
\left\{
\begin{array}{l}
\rho_t(x,t) - \text{div}(\rho(1 - \rho)^2 Du) = \varepsilon \Delta \rho, \\
|Du(x)|^2 = \frac{1}{(1 - \rho)^2},
\end{array}
\right.
$$

(6.1)

where $\rho : \Omega \times \mathbb{R}_+ \to \mathbb{R}$, the density of agents, and $u : \Omega \to \mathbb{R}$, the time to exit $\Omega$ through its boundary $\partial \Omega$, are the unknowns and the given initial data is $\rho(x,0) = \rho_0(x)$ and $u(x) = u_0$, with $x$ representing the spatial variable and $t$ the time. Also $\rho_t$ stands for the time derivative of $\rho$; $Du$ and $\Delta u$ the spatial gradient and the Laplacian of $u$ and $u_\nu$, $\rho_\nu$ are the directional derivatives of $u$ and $\rho$ with respect to the outer unit normal vector to $\partial \Omega$. We also suppose $\Omega$ is an open domain of $\mathbb{R}^d$, with $\partial \Omega = \Gamma_d \cup \Gamma_n$, the Dirichlet and the Neumann parts of the boundary respectively which are the relevant boundary conditions in applications and $d \leq 3$. The Fokker-Planck equation describes the evolution of the crowd density $\rho$ ($0 \leq \rho \leq 1$). The solution to the Eikonal equation represents the time an agent/person/pedestrian takes to exit the domain $\Omega$ taking into account congestion effects; its gradient, $Du$, determines the optimal
direction of movement for each agent (in the steepest descent way) assuming the
rest of the population is frozen (due to the stationary character of the Eikonal). The
constraint $\rho \leq 1$ corresponds to the maximal density of the population. The Dirichlet
condition ($\rho$ or $u$ prescribed on $\partial \Omega$) corresponds to the areas on the boundary where
agents/people/pedestrians can leave. The Neumann condition ($\rho_\nu$ and $u_\nu$ prescribed
on $\partial \Omega$) is a no-flow constraint at the boundary. The correct interpretation of these
boundary conditions is essential in the design of numerical schemes. The case $\varepsilon = 0$
is the model in [56].

Significant progress has been achieved in the understanding of pedestrian and
crowd models ([5], [29], [6], [32]). However, even some aspects of one-dimensional
models are not completely understood. Microscopic ([27, 30]) and mean-field game
([13, 72]) interpretations were used to study the macroscopic dynamics that the
Hughes model describes. Numerical approaches to these problems were developed
in [25, 37].

In section 7.1, we establish new a priori estimates for solutions of (6.1) that give a
partial regularity for the solutions. Then, in section 7.2, we consider a radial problem
to examine the behavior of the model in two and three dimensions under a simplified
setting. We show that with zero viscosity ($\varepsilon = 0$), the model admits shocks, which
we also illustrate numerically. The shocks represent situations where nearby agents
follow a substantially different direction or where there are discontinuities in the
density $\rho$. The presence of shocks makes the study of numerical methods and the
proof of the existence of solutions into non-trivial problems. In subsection 7.2.4, we
present some conserved quantities for the radial problem in two and three-dimensions.
Next, we study the one-dimensional flow problem, in section 7.3. This formulation
postulates that agents arrive at a prescribed rate, that we call current, on one side of
an interval and leave through the other side. We show that the Hughes model may
not be well-posed as the density $\rho$ can exceed the maximal congestion threshold. In
these examples, we identify two mechanisms of loss of regularity: shocks in the zero-viscosity problem and congestion in the flow problem. We also illustrate the trend to equilibrium in section 7.3.2.

As an approach to better understand properties of the Hughes model, we introduce variations on it. This is done in Chapter 8, where we present three variations of the original model. Those include the introduction of viscosity in the Eikonal equation, a time-dependent version of the Eikonal equation - analogously to mean-field games, and a model for pilgrims around Kaaba (Saudi Arabia).

To study the Hughes model and its variation, we need to make use of a numerical method. We need then to pass through the study of a numerical method for Fokker-Planck and Hamilton-Jacobi equations. In Chapter 9, we introduce a numerical approach to study Fokker-Planck equations via the adjoint structure of its Hamilton-Jacobi counterpart. In this way, we can use the same type of discretization in space for both the Fokker-Planck and the Eikonal/Hamilton-Jacobi equation present in the system. We examine the adjoint structure in Section 9.2. Next, in Section 9.3, we proof key features of the method: positivity and mass-conservation. Then, in Section 9.4, we describe the numerical method and its properties, including consistency. Some sample schemes are then studied in detail.

Next, we devote Chapter 10 to extend our numerical scheme to address systems of partial differential equations (PDEs). Those systems arise in multiple applications such as mean-field games (MFG), population models, traffic flow problems, and modeling in chemotaxis. In Section 10.1, we exemplify via the Hughes model how to regard those systems of PDEs under the setting of the numerical approach introduced in Chapter 9. Then, in the subsequent sections, we present six examples of measure-potential systems. Those include the Hughes model in one- and two-dimensions, forward-forward mean-field games, and the Kaaba model. Next, in Section 10.6, we depict the non-optimality characteristic of the Hughes model.
Finally, in Chapter 11, we finish the second part of the dissertation with the conclusions and remarks on future works.
Chapter 7

Main results to the Hughes model

7.1 Estimates

We begin our study of the Hughes model by investigating a priori estimates; that is, estimates that are valid for smooth enough solutions. These estimates extend the ones obtained in [45] in the periodic setting. First, for the Neumann or Dirichlet boundary conditions and \( \varepsilon > 0 \), we prove that smooth solutions satisfy \( 0 \leq \rho \leq 1 \) for all times whenever the initial condition does too. This result should be contrasted with the flow problem considered in Section 7.3. Next, we prove that \( Du \in L^p \) (Lebesgue space) for any \( 1 < p < \infty \), and then \( \frac{1}{1-\rho} \in L^p \). This last estimate gives a quantitative control on the congestion. Here, we consider the Hughes model in an open domain, \( \Omega \subset \mathbb{R}^d \), with Dirichlet-Neumann conditions: \( \partial \Omega = \Gamma_d \cup \Gamma_n \), \( \nu \) the outer unit normal to \( \partial \Omega \), \( \rho = 0 \) in \( \Gamma_d \times [0,T] \), \( \rho_\nu = 0 \) in \( \Gamma_n \times [0,T] \) and viscosity \( \varepsilon = 1 \).

Lemma 1. Let \( \rho : \Omega \times [0,T] \rightarrow \mathbb{R} \) solve

\[
\rho_t - \text{div}(\rho(1-\rho)g(x,t)) = \Delta \rho,
\]

where \( \Omega \) is an open subset of \( \mathbb{R}^d \) with \( \partial \Omega = \Gamma_d \cup \Gamma_n \), \( \rho = 0 \) in \( \Gamma_d \times [0,T] \), and \( \rho_\nu = 0 \) in \( \Gamma_n \times [0,T] \). Assume also \( 0 \leq \rho(x,0) \leq 1 \) and \( g \in C^\infty \). Then \( 0 \leq \rho(x,t) \leq 1 \).

Proof. Note that \( \tilde{\rho} = 1 - \rho \) satisfies

\[
\tilde{\rho}_t - \text{div}(\rho\tilde{\rho}g(x,t)) = \Delta \tilde{\rho}.
\]
Because $\tilde{\rho}(x, 0) \geq 0$ (and, with Dirichlet boundary data, $\tilde{\rho} \geq 0$ in $\partial \Omega$), we have $\tilde{\rho} \geq 0$.

**Proposition 3.** Let $(u, \rho)$ solve (6.1) with $\varepsilon = 1$. Suppose $u = 0$ in $\Gamma_d \times [0, T]$, $u_\nu = 0$ on $\Gamma_n \times [0, T]$ and $0 < \rho < 1$ at $t = 0$. Then, for any $\alpha < -1$,

$$\frac{d}{dt} \int_{\Omega} (1 - \rho)^{\alpha+1} \leq C \int_{\Omega} (1 - \rho)^{\alpha+1}. \quad (7.1)$$

Furthermore,

$$\int_0^T \int_{\Omega} |D(1 - \rho)^{\frac{\alpha+1}{2}}|^2 \leq C.$$

**Proof.** Multiply the first equation in (6.1) by $-(\alpha + 1)(1 - \rho)^\alpha$. Then,

$$\frac{d}{dt} \int_{\Omega} (1 - \rho)^{\alpha+1} \leq c \int_{\Omega} (1 - \rho)^{\alpha+1}\rho D\rho Du - \alpha(\alpha + 1) \int_{\Omega} (1 - \rho)^{\alpha-1}|D\rho|^2$$

$$- (\alpha + 1) \int_{\partial\Omega} (1 - \rho)^{\alpha+2}\rho u_\nu - (\alpha + 1) \int_{\partial\Omega} (1 - \rho)^\alpha \rho_\nu$$

(using Cauchy’s inequality)

$$\leq - \frac{\alpha(\alpha + 1)}{2} \int_{\Omega} (1 - \rho)^{\alpha-1}|D\rho|^2 + \int_{\Omega} (1 - \rho)^{\alpha+3}\rho^2|Du|^2$$

$$- (\alpha + 1) \int_{\partial\Omega} (1 - \rho)^{\alpha+2}\rho u_\nu - (\alpha + 1) \int_{\partial\Omega} (1 - \rho)^\alpha \rho_\nu$$

(using the Eikonal eq. and the Lemma 1)

$$\leq - \frac{\alpha(\alpha + 1)}{2} \int_{\Omega} (1 - \rho)^{\alpha-1}|D\rho|^2 + C \int_{\Omega} (1 - \rho)^{\alpha+1}$$

$$- (\alpha + 1) \int_{\partial\Omega} (1 - \rho)^{\alpha+2}\rho u_\nu - (\alpha + 1) \int_{\partial\Omega} (1 - \rho)^\alpha \rho_\nu.$$

Now, we observe that, on $\Gamma_n$, $u_\nu = 0$ and, on $\Gamma_d$, we have $\rho = 0$. Hence,

$$(1 - \rho)^{\alpha+2}\rho u_\nu = 0 \text{ in } \partial\Omega.$$
Similarly, \((1 - \rho)\alpha \rho \leq 0\) in \(\Gamma_d\) and vanishes in \(\Gamma_n\). Hence, it is also non-positive in \(\partial \Omega\). Thus, taking into account that \(\alpha + 1 \leq 0\), integrating in time, observing that the only positive term on the right-hand side is \(\int (1 - \rho)^{\alpha + 1}\), and using Gronwall’s inequality, we get the desired estimates. \(\square\)

**Proposition 4.** Under the same hypotheses of Proposition 3, we have that, for any \(1 < p < \infty\),

\[
\sup_{0 \leq t \leq T} \int_{\Omega} |Du|^{2p} < C_p.
\]

**Proof.** By using (7.1) in the Eikonal equation, we have

\[
|Du|^2 = \frac{1}{(1 - \rho)^2} \in L^p, \forall p.
\]

\(\square\)

### 7.2 Shocks in radial solutions

To understand the behavior of the Hughes model, we begin by considering radial solutions. In this case, (6.1) becomes a scalar PDE. Thanks to this simplification, we prove the existence of shocks in the zero viscosity problem. Thus, shocks exist in general two or three-dimensional problems because radial solutions are a particular case.

Now, we assume radial symmetry, what corresponds to a model where agents want to get away from the origin. In dimension \(d > 1\), assume \(u = u(r, t)\), where \(r\) is the radius. The Eikonal equation in (6.1) gives that

\[
u_r = \pm \frac{1}{1 - \rho}.
\]

We select the negative root because it corresponds to agents leaving the origin. Since
\( \rho \) is radial, \( \rho = \rho(r, t) \), we rewrite the Fokker-Planck equation in (6.1) in polar coordinates, and we use the preceding equation to get

\[
\rho_t + \frac{d-1}{r} \rho(1 - \rho) + \rho_r (1 - 2\rho) = \varepsilon \left( \rho_{rr} + \frac{d-1}{r} \rho_r \right).
\]

When \( \varepsilon = 0 \), we obtain the first-order PDE

\[
\rho_t + \frac{d-1}{r} \rho(1 - \rho) + \rho_r (1 - 2\rho) = 0.
\]

To study shocks, we use the method of characteristics. For that, we first solve the characteristic system

\[
\begin{cases}
\frac{dr}{dt} = 1 - 2\rho, \\
\frac{d\rho}{dt} = -\frac{d-1}{r} \rho(1 - \rho),
\end{cases}
\quad (7.2)
\]

with \( r_0, \rho_0(r_0) \) as initial conditions at \( t = 0 \) for \( r(t) \) and \( \rho(t) \), respectively. We define the auxiliary function \( V \):

\[
V(\rho) = \rho(1 - \rho),
\quad (7.3)
\]

and, analogously, \( V_0(\rho_0) = \rho_0(1 - \rho_0) \). Next, using implicit differentiation, we rewrite (7.2) as

\[
\begin{cases}
\frac{dr}{dt} = \frac{\partial V}{\partial \rho}, \\
\frac{d\rho}{dt} = -\frac{d-1}{r} V.
\end{cases}
\quad (7.4)
\]

Then, we compute \( \frac{\partial V}{\partial r} = -\frac{d-1}{r} V \), and conclude that

\[
V = \left( \frac{r_0}{r} \right)^{d-1} V_0.
\quad (7.5)
\]

Now, solving for \( \rho \) in (7.3) gives two roots. If \( 0 < \rho_0(r_0) \leq \frac{1}{2} \), here called
regime 1,
\[ \rho_1 = \frac{1 - \sqrt{1 - 4V}}{2} , \]
while, if \( 1/2 < \rho_0(r_0) < 1 \), here called regime 2,
\[ \rho_2 = \frac{1 + \sqrt{1 - 4V}}{2} . \]

Using these expressions on the right-hand side of the ordinary differential equation (ODE) (7.2) for \( r(t) \) and (7.5), we get
\[ \dot{r}_1(t) = 1 - 2\rho_1 = \sqrt{1 - 4V_0} \left[ \frac{r_0}{r_1(t)} \right]^{d-1} , \]
and
\[ \dot{r}_2(t) = 1 - 2\rho_2 = -\sqrt{1 - 4V_0} \left[ \frac{r_0}{r_2(t)} \right]^{d-1} . \]

Now, using (7.4) and (7.5), we obtain the ODE describing the time evolution of \( \rho \):
\[ \dot{\rho}(t) = -\frac{d - 1}{r(t)} \left[ \frac{r_0}{r(t)} \right]^{d-1} V_0 . \quad (7.6) \]

In the next two subsections, we present the particular cases of the radial solutions in dimension 2 and 3. We also make use of the following remark.

**Remark 1.** The function defined by \( x \mapsto \sqrt{1 - 4x(1 - x)} \), is identical to \(-2x + 1\) in \([0, 1/2]\) and identical to \(2x - 1\) in \([1/2, 1]\).

### 7.2.1 Dimension 2

For the Hughes model in dimension 2, we solve:
\[ \dot{r}_1(t) = \sqrt{1 - 4V_0} \frac{r_0}{r_1(t)} , \quad \text{with} \ 0 < \rho_0(r_0) \leq 1/2, \]
\[ r_2(t) = -\sqrt{1 - 4V_0 \frac{r_0}{r_2(t)}}, \text{ with } 1/2 < \rho_0(r_0) < 1, \]

with the initial conditions \( r_1(0) = r_2(0) = r_0 \). We have the following implicit representation for \( r_1(t) \) and \( r_2(t) \):

**Regime 1:** \( r_1(t) \) satisfies

\[
2r_0\rho_0(1 - \rho_0) \log \left[ 2r_1(t) \left( 1 + \sqrt{1 - \frac{4r_0\rho_0(1 - \rho_0)}{r_1(t)}} \right) - 4r_0\rho_0(1 - \rho_0) \right] \\
+ r_1(t) \sqrt{1 - \frac{4r_0\rho_0(1 - \rho_0)}{r_1(t)}} \\
= t + r_0 \left\{ 2\rho_0(1 - \rho_0) \log \left[ 4r_0(1 - \rho_0)^2 \right] - 2\rho_0 + 1 \right\}. 
\]

**Regime 2:** \( r_2(t) \) satisfies

\[
2r_0\rho_0(1 - \rho_0) \log \left[ 2r_2(t) \left( 1 + \sqrt{1 - \frac{4r_0\rho_0(1 - \rho_0)}{r_2(t)}} \right) - 4r_0\rho_0(1 - \rho_0) \right] \\
+ r_2(t) \sqrt{1 - \frac{4r_0\rho_0(1 - \rho_0)}{r_2(t)}} \\
= -t + r_0 \left\{ 2\rho_0(1 - \rho_0) \log \left[ 4r_0\rho_0^2 \right] + 2\rho_0 - 1 \right\}. 
\]

In dimension 2, the equation for the time evolution of \( \rho \), equation (7.6), becomes

\[ \dot{\rho}(t) = -\frac{r_0}{r(t)^2} V_0. \]

The above equation gives an implicit representation for \( \rho \). We solve it numerically and present the parametric plot of the radius \( r(t) \) versus the density \( \rho(t) \) in subsection 7.2.3.
7.2.2 Dimension 3

For the Hughes model in dimension 3, we solve:

\[
\dot{r}_1(t) = \sqrt{1 - \frac{4V_0r_0^2}{r_1(t)^2}}, \text{ with } 0 < \rho_0(r_0) \leq 1/2,
\]

and

\[
\dot{r}_2(t) = -\sqrt{1 - \frac{4V_0r_0^2}{r_2(t)^2}}, \text{ with } 1/2 < \rho_0(r_0) < 1,
\]

with the initial conditions \( r_1(0) = r_2(0) = r_0 \) and obtain the explicit formulas

\[
r_1(t) = \sqrt{4r_0^2V_0 + t^2 + 2tC_1 + C_1^2},
\]

and

\[
r_2(t) = \sqrt{4r_0^2V_0 + t^2 - 2tC_2 + C_2^2},
\]

where \( C_1 = C_2 = \pm r_0\sqrt{1 - 4V_0} \) because \( r(0) = r_0 \).

Remark 1 then implies that \( C_1 = r_0(-2\rho_0 + 1) \) and \( C_2 = r_0(2\rho_0 - 1) \). Plugging them back in the expressions for \( r_1(t) \), \( r_2(t) \) we obtain:

\[
r_1(t) = \sqrt{t^2 + 2r_0(1 - 2\rho_0)t + r_0^2},
\]

\[
r_2(t) = \sqrt{t^2 - 2r_0(2\rho_0 - 1)t + r_0^2}.
\]

Because the expressions for \( r_1(t) \) and \( r_2(t) \) agree, there is no need to consider two separate regimes. Thus,

\[
r(t) = \sqrt{t^2 + 2r_0(1 - 2\rho_0)t + r_0^2}.
\]
Now, according to (7.6), the density $\rho(t)$ solves

$$\dot{\rho}(t) = -2 \frac{r_0^2}{r(t)^3} V_0.$$ 

The solution with the initial condition $\rho(0) = \rho_0$ is

$$\rho(t) = \frac{1}{2} \left[ 1 - \frac{t + r_0(1 - 2\rho_0)}{r(t)} \right] = \frac{1}{2} \left[ 1 - \frac{t + r_0(1 - 2\rho_0)}{\sqrt{t^2 + 2r_0(1 - 2\rho_0)t + r_0^2}} \right].$$

7.2.3 Numerical experiments

Now, we numerically investigate the formation of shocks for the radial Hughes model without viscosity. We construct three density profiles, $\rho_0(r_0)$, with support in the interval $[0, 1]$. For each profile, we plot the graphs of $r(t)$ versus $\rho(t)$. The different colors correspond to the solution at the different times. The blue curve in each plot is the initial density profile $\rho_0$; see Figure 7.2 for the two-dimensional case and Figure 7.3 for the three-dimensional one.

We observe the formation of shocks in the solutions when the curve is not a graph anymore, i.e., a non-single-valued function. Once a shock happens, the characteristic’s method is not valid anymore and the above expressions for $r(t)$ and $\rho(t)$ lack meaning. We also point out that the mass density, $r^{d-1}\rho$, is conserved up to the shock.

![Figure 7.1: Initial densities for radial case.](image-url)
(a) Case 1: $\rho_0$ concentrated around 0.2 with support on $0 \leq r_0 \leq 0.6$ and maximum value of 0.35.

(b) Case 2: $\rho_0$ concentrated around 0.75 with support on $0.5 \leq r_0 \leq 1.0$ and maximum value of 0.8.

(c) Case 3: $\rho_0$ with support on $0 \leq r_0 \leq 1.0$ and maximum value of 0.4.

Figure 7.2: Shocks along the time evolution of three given profiles in 2-D.

Cases 1 and 2 have initial profiles with supports of similar size, but case 2 presents shocks in a shorter time (both in 2-D and 3-D cases). This shows the dependence on the intensity of $\rho_0$; in case 2, $\rho_0$ reaches 0.8, while in case 1, it is not greater than 0.35. Case 1 and 2 illustrate the dependence on the value of $\rho_0$ for the time the shock appears. Both cases have initial profiles with supports of similar size, but case 2 presents shocks in a shorter time (in both in 2-D and 3-D cases). Also in case 2, $\rho_0$ reaches 0.8, while in case 1, it is not greater than 0.35.

Finally, the presence of shocks in our examples implies the possibility of the existence of shocks for the Hughes model in any dimension. This implication can be seen by considering solutions that depend only on two or three dimensions in a higher-dimensional space. We are not aware of any proof of this fact in the literature. We also remark that the study of radial symmetry essentially reduces the Hughes PDE
7.2.4 Conserved quantities

We present some conserved quantities for the models previously described. Despite the theoretical point of view, we could use those conserved quantities to validate our numerical methods.

Table 7.1 lists those quantities. We have looked for conserved quantities that are polynomials on $r$, $\rho$, and some of their powers. The column ”Pol” describes for which polynomials we searched conserved quantities. The column ”Deg” lists up the polynomial degree we looked for those quantities. The $b_i's$ stand for any constant, including zero. Then, for example, in the first equation, we have that the integral of all the powers of $\rho$ up to degree 6 are conserved, as well their sums.
7.3 Flow problem - stationary case

The flow problem in dimension one consists of people entering a domain from one side at a fixed rate and exiting through the other side. If the flow is large enough, the maximal density \( \rho = 1 \) may be achieved or exceeded, and the model may break.

In the one-dimensional flow problem in \([0, 1]\), agents arrive at \( x = 0 \) and are only allowed to leave through \( x = 1 \). By computing \( u_x \) in the Eikonal equation and substituting in the Fokker-Planck equation, we obtain

\[
\rho_t + \rho_x (1 - 2\rho) = \varepsilon \rho_{xx}. \tag{7.7}
\]

The corresponding stationary equation is

\[
\varepsilon \rho_{xx} + 2\rho \rho_x - \rho_x = \frac{d}{dx} \left[ \varepsilon \rho_x + \rho^2 - \rho \right] = 0.
\]

We can then formulate the stationary flow problem as:

\[
\begin{align*}
\varepsilon \rho_x + \rho^2 - \rho &= j, \quad x \in [0, 1], \\
\rho(1) &= 0,
\end{align*} \tag{7.8}
\]

where \( j \) is a prescribed net current of agents entering \([0, 1]\).

Our interest is to understand the behavior of the solutions of (7.8) as the current \( j \) becomes large; that is, a large flow of agents entering the domain. By solving numerically the ODE (7.8), for \( \varepsilon = 1 \) and \( j \) between 0 and 1.5, with increments of 0.1, we observe the different solutions for \( \rho \). For \( j > 1.2 \), the density \( \rho \) is larger than 1, see Figure 7.4(a). This behavior is remarkable because solutions of the reduced equation (7.8) are \( C^\infty \), but in the original model, the equations are singular when \( \rho = 1 \).

Finally, in Figure 7.4(b), we depict the effect of a small viscosity (\( \varepsilon = 0.01 \)) for
<table>
<thead>
<tr>
<th>Eq. Type</th>
<th>Equation</th>
<th>Pol</th>
<th>Deg</th>
<th>Conserved Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D w/o visc.</td>
<td>$\rho_t + \rho_x(1 - 2\rho) = 0$</td>
<td>$\rho, \rho', \rho''$</td>
<td>6</td>
<td>$\int_R b_1 \rho + b_2 \rho^2 + b_3 \rho^3 + \cdots + b_6 \rho^6$</td>
</tr>
<tr>
<td>1D w/ visc.</td>
<td>$\rho_t + \rho_x(1 - 2\rho) = \varepsilon \rho_{xx}$</td>
<td>$\rho, \rho', \rho''$</td>
<td>15</td>
<td>$\int_R b_1 \rho$</td>
</tr>
<tr>
<td>Rad.2D no visc.</td>
<td>$\rho_t + \frac{1}{r} \rho(1 - \rho) + \rho_r(1 - 2\rho) = 0$</td>
<td>$\rho, r, 1/r$</td>
<td>6</td>
<td>$\int b_1 r \rho + \frac{1}{2} b_2 r^2 \rho^2(-3 + 2\rho)$</td>
</tr>
<tr>
<td>Rad.2D w. visc.</td>
<td>$\rho_t + \frac{1}{r} \rho(1 - \rho) + \rho_r(1 - 2\rho) = \varepsilon [\rho_{rr} + \frac{1}{r} \rho_r]$</td>
<td>$\rho, r, 1/r$</td>
<td>30</td>
<td>$\int b_1 r \rho$</td>
</tr>
<tr>
<td>Rad.3D no visc.</td>
<td>$\rho_t + \frac{2}{r} \rho(1 - \rho) + \rho_r(1 - 2\rho) = 0$</td>
<td>$\rho, r, 1/r$</td>
<td>15</td>
<td>$\int b_1 r^2 \rho - \frac{1}{4} b_2 r^4 \rho^2(-3 + 2\rho) + \frac{1}{15} b_3 r^6 \rho^3(-10 + 15\rho + 6\rho^2)$</td>
</tr>
<tr>
<td>Rad.3D w. visc.</td>
<td>$\rho_t + \frac{2}{r} \rho(1 - \rho) + \rho_r(1 - 2\rho) = \varepsilon [\rho_{rr} + \frac{2}{r} \rho_r]$</td>
<td>$\rho, r, 1/r, \frac{1}{r^2}$</td>
<td>15</td>
<td>$\int b_1 r^2 \rho - \frac{1}{15} b_3 r^6 \rho^3(-35 + 84\rho - 70\rho^2 + 20\rho^3)$</td>
</tr>
</tbody>
</table>

Table 7.1: Conserved Quantities - radial formulation for the Hughes model.
a range of admissible currents (the ones avoiding $\rho > 1$). The current for which the model reaches the maximal density 1 is called the critical current.

### 7.3.1 Dependence on viscosity

We now investigate the dependence of the solutions on the viscosity. In Figure 7.5(a), we see that, for small viscosity, the model breaks down. However, large viscosity seems to have a stabilizing effect. In Figure 7.5(a), we used a current with a fixed value $j = 0.5$ and viscosities from 0.3 to 1.5 with increments of 0.15. In Figure 7.5(b), which also corresponds to the supercritical case, we depict the behavior with the same current $j = 0.5$, but with viscosities in the interval (0.22, 0.36), with increments of 0.02.

Now, in the subcritical case, Figure 7.5(c), for $j = 0.2$, the different solutions for the density $\rho$ show an upper bound when using different viscosities (from 0.02 to 0.1 with increments of 0.02).

To better understand the relation between viscosity and the critical current, we solve (7.8) with different viscosities and compute the critical current for which the density reaches one. At this density, the model breaks down as shown in Figure 7.6.
Figure 7.5: Dependence of congestion on viscosity.

Figure 7.6: Critical current \( j \) as a function of viscosity.

### 7.3.2 The trend to equilibrium

We investigate the relation between the solution to the stationary problem, (7.8), and the time-dependent one, (7.7). Both numerical solutions to the stationary and the time-dependent versions are calculated using the function NDSolve included in the software Mathematica.

First, as an example, we solve the problem with the following initial/boundary
conditions, and \( \varepsilon = 0.05 \):

\[
\begin{aligned}
 j &= \varepsilon \rho_x(0, t) + \rho^2(0, t) - \rho(0, t) = -0.2(1 - e^{-10t}), \\
\rho(1, t) &= 0, \\
\rho(x, 0) &= 10 \ x^2(1 - x)^2.
\end{aligned}
\]

For large times, the stationary solution coincides with the time-dependent one. The transient period of time in which this fails is due to the form of initial condition. This behavior is depicted in Figure 7.7(a). The time-dependent solution is shown via the filled graph, while the plot of the stationary one uses a wire mesh.

Now, in the subcritical case, the second aspect we observe is the trend to the equilibrium of the time-dependent solution. We solve now the system, with \( \varepsilon = 0.05 \),

\[
\begin{aligned}
 j &= \varepsilon \rho_x(0, t) + \rho^2(0, t) - \rho(0, t) = -0.2(1 - e^{-10t}), \\
\rho(1, t) &= 0, \\
\rho(x, 0) &= 10 \ x^2(1 - x)^2.
\end{aligned}
\]

In Figure 7.7(b), we can see that, for large times, the asymptotic solution approaches the stationary solution and then coincides with it. We are not aware of any
proof or theoretical result concerning this property.
Chapter 8

Variations of the Hughes model

In chapter 6, we presented the Hughes model for crowd motion. Now, we consider variations of it, and we explain their interest. We explore these new models, and we aim to understand its characteristics and possible issues. A second reason to consider these generalizations is that different formulations may provide a better description of real world problems.

There are few articles on the rigorous analysis of the Hughes model, despite a good understanding of its solutions being of great importance. Some authors have studied variations of the model, and their primary interest was the existence of solutions. Unfortunately, they were not able to prove the convergence of these solutions to the Hughes model [29].

In what follows, we present three variations of the model. First, Model I adds viscosity to the Eikonal equation, an attempt to increase regularity and provide better a priori estimates. We then describe Model II, which extends the stationary Eikonal equation to a time-dependent Hamilton-Jacobi equation. This new system is based on a variational problem, while one of the main difficulties of the Hughes models is the lack of a variational formulation. This difference on the Hughes model is what makes the search for the existence of solutions, uniqueness, and numerical schemes a hard problem. Last, in Model III, we propose a modification of the Hughes model to study the movement of crowds around Kaaba (Saudi Arabia).
8.1 Model I - Hughes model with diffusion

In the first model we address, we add a small viscosity term $\varepsilon_2$ to the second equation in (6.1). We assume that $\rho$ has an initial condition $0 < \rho < 1$ at $t = 0$. We consider the system

$$
\begin{cases}
\rho_t - \text{div} (\rho(1-\rho)^2 \nabla u) = \varepsilon_1 \Delta \rho, \\
-\varepsilon_2 \Delta u + |\nabla u|^2 = \frac{1}{(1-\rho)^2},
\end{cases}
$$

with Dirichlet and Neumann data, where we take $u = 0$ in $\partial \Omega \times [0, T]$, and we assume $\partial \Omega = \Gamma_d \cup \Gamma_n$, with $\rho = 0$ in $\Gamma_d \times [0, T]$, and $\rho = 0$ in $\Gamma_n \times [0, T]$. We depict an example for this model in Chapter 10, Figure 10.7, while simulating the evacuation of a domain without boundaries. We can then compare with Figure 10.5, where no kind of viscosity is present; and with Figure 10.6, where viscosity is present only in the Fokker-Planck equation.

8.2 Model II - A time-dependent Hughes Model

The second model we study concerns particular population models related to the Hughes one, see [29, 14], that can be written as a system of equations of the form

$$
\begin{cases}
\rho_t - \text{div}(h(\rho) \nabla u) = \Delta \rho, \\
-u_t + f(\rho)|\nabla u|^2 = \Delta u + g(\rho),
\end{cases}
$$

for suitable functions $f, g, h : \mathbb{R}^+ \to \mathbb{R}$, together with initial-terminal conditions

$$
\begin{cases}
u(x, T) = u_0(x), \\
\rho(x, 0) = \rho_0(x).
\end{cases}
$$

Here, $\rho : \Omega \times [0, T] \to \mathbb{R}$, with the assumption, as before, that $\partial \Omega = \Gamma_d \cup \Gamma_n$, with $\rho = 0$ in $\Gamma_d \times [0, T]$, and $\rho = 0$ in $\Gamma_n \times [0, T]$. We assume $u = 0$ in $\Gamma_d \times [0, T]$ and
By applying ideas similar to the ones in [41], we can obtain various estimates for these equations. One key point is that, although these equations do not have the standard mean-field structure, a Hamilton-Jacobi equation coupled with the adjoint of its linearization, many of the techniques for mean-field games can still be applied. An important case is the PDE:

\[
\begin{aligned}
\rho_t - \text{div}(\rho(1 - \rho)^2 Du) &= \Delta \rho, \\
-u_t + \frac{|Du|^2}{2} (1 - \rho)(1 - 3\rho) &= \Delta u,
\end{aligned}
\]

where \( f(\rho) = \frac{1}{2}(1 - \rho)(1 - 3\rho), g(\rho) = 0, h(\rho) = \rho(1 - \rho)^2 \). This example is motivated by variational problems in crowd dynamics [14], and it could be solved numerically with the method we propose in Chapter 9.

8.3 Model III - Modeling crowds around the Kaaba

In Chapter 6, we introduced the Hughes model. Now, we show how we can modify it to address a real-world problem, namely the movement of pilgrims around the Kaaba, in Saudi Arabia. The Kaaba, Arabic for “The Cube,” is a building in Saudi Arabia. Pilgrims visiting it must turn around it seven times. During the annual Hajj pilgrimage, the number of persons visiting the site can reach one million. Such large gathering poses significant crowd motion problem.

The new physical domain is a 2-D strip whose length is 14\( \pi \), and it corresponds to the number of turns a person has already performed. The height of the strip describes the radius from the Kaaba to the outer part of the ring where pilgrims can move. We normalize the height and set it equal one. If we write \( \bar{\rho} = \sum_{j=0}^{6} \rho(r, \theta + 2j\pi) \), then the
Hughes model for the Kaaba then reads as

\[
\begin{align*}
\rho_t(r, \theta, t) - \text{div}(\rho(1 - \bar{\rho})^2 Du) &= \varepsilon \Delta \rho, \\
|Du(r, \theta, t)|^2 &= \frac{1}{(1 - \bar{\rho})^2}.
\end{align*}
\]

Concerning boundary conditions, we impose \( u = +\infty \) on the set \([0, 14\pi] \times \{1\}\), i.e., on the top of the strip and on the bottom part, we have a no flow condition. The exception is in part of the sets \([0, 2\pi] \times \{1\}\) and \((12\pi, 14\pi] \times \{1\}\), where we define \( u = 0 \). In this way, we have Neumann boundary conditions for \( \theta \), representing the inflow and outflow of persons.

Our model can also include potentials in specific regions of the domain, which would represent places where people would like to be close or far from it. Situations like this in the Kaaba are, for instance, the black stone present in one of the Kaaba’s corner, in which people tend to agglomerate in an attempt of touching it, or the fact people give preference to turn around the building with a short radius.

No public data about the number of people around Kaaba is available, and the few articles related to the movement around Kaaba are in the field of Physics and Computer Science simulation, and they lack mathematical rigor. Many of the existing simulations use models based on fluid flow, ignoring the “thinking” component of people, who tend to move to lower-density regions, see [31], and the review paper [58].

Another important question in crowd motion problems is the design of routes. Empirical studies show that adding certain obstacles on the path of crowds helps in reducing the formation of high-density regions. Our proposed model can encode obstacles and may help to understand their impact on the flow.

Finally, we present simulations of the Kaaba model in Section 10.8, after explaining how to adapt the numerical approach presented in chapter 9.
Chapter 9

Numerical methods for Fokker-Planck equations

9.1 Introduction

Fokker-Plank (FP) equations model the time evolution of a probability density. The general set up is as follows. Given an open subset of $\mathbb{R}^d$, $\Omega$, a terminal time, $T > 0$, and a (drift) vector field, $b(x, t) : \Omega \times [0, T] \to \Omega$, we seek to find a time-dependent probability distribution, $\rho : \Omega \times [0, T] \to \mathbb{R}$, solving

$$
\begin{align*}
\partial_t \rho - \varepsilon \Delta \rho + \text{div}(b(x, t)\rho) &= 0, \quad \text{in } \Omega \times [0, T], \\
\rho(\cdot, 0) &= \rho_0(\cdot), \quad \text{in } \Omega.
\end{align*}
$$

Besides, we supplement the above problem with boundary conditions on $\partial \Omega \times [0, T]$, where $\partial \Omega$ is the boundary of $\Omega$. We are particularly interested in problems where $b$ may depend on $\rho$ either directly or through an unknown function determined by an additional partial differential equation. Two examples discussed in this dissertation are the forward-forward mean-field game (MFG) and the Hughes model.

The Fokker-Planck equation was introduced in statistical mechanics. This equation has multiple applications in economics [50, 61], crowd motion models [55, 60], and biological models [28, 49]. Due to the complex structure of those equations, the computation of explicit solutions is not possible in general. Hence, effective numerical methods for the approximation of solutions of FP equations have a broad interest.

Here, we propose a technique to obtain approximation schemes for FP equations using their representation as the adjoint of the linearization of Hamilton-Jacobi (HJ)
equations. In this way, all monotone numerical schemes proposed in the context of HJ equations give rise to consistent schemes for FP equations. In particular, as required by the nature of the problem, these schemes preserve positivity and are conservative, i.e., under suitable boundary conditions mass is preserved.

Previously, the adjoint structure of the FP equation was used by several authors, for example, in [3] and in [4]. In those references, the authors propose a finite-difference scheme which is the adjoint of the linearization of the upwind scheme used to approximate a convex Hamiltonian. In [21, 22, 23, 24, 26], the authors propose a semi-Lagrangian numerical method using a slightly different procedure, but based on a similar principle.

The main contribution described in this chapter is to show how to use the adjoint structure with a broad class of numerical solvers, and without limitations on the problem dimension. Here, in contrast to the above references, we develop semi-discrete schemes, where the spatial variable is discretized. To construct the semi-discretization, we apply symbolic calculus to assemble the schemes by exact formula manipulation. The evolution in time then corresponds to a system of ordinary differential equations (ODE). These can be solved with different methods, depending on the smoothness of the solution and desired accuracy. The semi-discrete schemes obtained symbolically are compiled before the numerical simulation of the resulting system of ODEs, reducing substantially both the computational time and the work of writing the code of such schemes.

9.2 Adjoint structure

The relation between a FP equation and its adjoint equation is well known. In recent works, [15, 16, 17, 18, 34, 44, 76], this relation was used to study regularity properties, vanishing viscosity limits, and rates of convergence of numerical methods. Those results are based on the observation that a FP equation is the adjoint of the
linearization of a certain HJ equation.

9.2.1 Linearization and duality

Here, we discuss the relation between FP and HJ equations. First, we consider the HJ operator

\[ HJ(u) := -u_t(x, t) + H(x, Du(x, t)) - \varepsilon \Delta u(x, t), \]

(9.2)

with the Hamiltonian \( H = H(x, p) : \Omega \times \mathbb{R}^d \to \mathbb{R}, \Omega \subset \mathbb{R}^d \). Further, we define the nonlinear generator

\[ A^{HJ} u := H(x, Du(x, t)) - \varepsilon \Delta u(x, t). \]

Here, we write \( Du = D_x u \) for the gradient in the variable \( x = (x_1, \cdots, x_d) \). The parameter \( \varepsilon \) is called the viscosity.

To linearize (9.2) around a generic point \( u_0 \), we expand \( u = u_0 + \lambda w \), then take the derivative in \( \lambda \), and, finally, consider the limit \( \lambda \to 0 \). Now, we compute this linearization. Boundary conditions are discussed in the next subsection.

The expansion \( HJ(u_0 + \lambda w) \) gives

\[ - \partial_t (u_0 + \lambda w) + H(x, D(u_0 + \lambda w)) - \varepsilon \Delta (u_0 + \lambda w) \]

\[ = -(u_0)_t - \lambda w_t + H(x, Du_0 + \lambda Dw) - \varepsilon \Delta u_0 - \lambda \varepsilon \Delta w. \]

We suppose \( H \) has enough regularity so that we can take the derivative of the preceding expression with respect to \( \lambda \). Next, we let \( \lambda \to 0 \), obtaining the operator

\[ L(w) := -w_t + D_p H(x, Du) \cdot Dw - \varepsilon \Delta w, \quad (9.3) \]
the linearization of the HJ operator. The (linear) generator of $L$ is

$$A^L w := D_p H(x, Du) \cdot Dw - \varepsilon \Delta w.$$ 

Finally, we compute the adjoint of $L$ by integration by parts. We fix smooth functions, $w$ and $\rho$, and derive the identity

$$\iint_{[0,T] \times \Omega} (-w_t + D_p H(x, Du) \cdot Dw - \varepsilon \Delta w) \rho$$

$$= \iint_{[0,T] \times \Omega} (\rho_t - \text{div}_x(D_p H(x, Du) \rho) - \varepsilon \Delta \rho) w$$

$$+ \iint_{[0,T] \times \partial \Omega} (D_p H(x, Du)) \cdot n \rho w + \varepsilon \frac{\partial \rho}{\partial n} w - \varepsilon \rho \frac{\partial w}{\partial n}$$

$$- \int_{\Omega} \rho(x, T) w(x, T) - \rho(x, 0) w(x, 0),$$

where $n$ is the normal vector to the boundary, $\partial \Omega$. The last calculation shows that the adjoint of $L$ is the following FP operator

$$L^* \rho := \rho_t - \text{div}_x(D_p H(x, Du) \rho) - \varepsilon \Delta \rho,$$ 

whose generator is $A^{FP} \rho := -\text{div}_x(D_p H(x, Du) \rho) - \varepsilon \Delta \rho$.

### 9.2.2 Boundary conditions

Now, we address the boundary conditions for (9.1) on $\partial \Omega \times [0, T]$. The discussion of initial conditions is straightforward. Two usually used boundary conditions for FP equations are Dirichlet data and a prescribed flow via Neumann conditions. Typically, the Dirichlet one vanishes on the boundary, corresponding to the case where particles exit once they reach the boundary. In the Dirichlet case, the data vanishes on the boundary. This corresponds to the case where particles exit once they reach the
boundary. The prescribed flow case represents a current of particles or agents crossing the boundary. Thus, with a zero flow, the mass is conserved.

Both the Dirichlet condition and the zero flow Neumann condition determine cancellations in the boundary integrals in (9.4). This suggests different conditions for the HJ operator, its linearized version, and its adjoint, the FP operator.

The first case corresponds to a FP equation with Dirichlet boundary conditions:

\[
\begin{align*}
\rho_t(x,t) - \text{div}(D_p H(x,Du) \rho) &= \varepsilon \Delta \rho, \quad \text{in } \Omega \times [0,T], \\
\rho(\cdot,t) &= 0, \quad \text{on } \partial \Omega \times [0,T].
\end{align*}
\]

We consider the HJ operator with the boundary conditions

\[
\begin{align*}
-u_t(x,t) + H(x,Du(x,t)) - \varepsilon \Delta u(x,t) &= 0, \quad \text{in } \Omega \times [0,T], \\
u(\cdot,t) &= g_1(\cdot,t), \quad \text{for any } g_1, \quad \text{on } \partial \Omega \times [0,T],
\end{align*}
\]

and the linearized operator as

\[
\begin{align*}
-w_t + D_p H(x,Du) \cdot Dw - \varepsilon \Delta w &= 0, \quad \text{in } \Omega \times [0,T], \\
w(\cdot,t) &= 0, \quad \text{on } \partial \Omega \times [0,T].
\end{align*}
\]

The second case corresponds to a FP equation with a flux through the boundary

\[
\begin{align*}
\rho_t(x,t) - \text{div}(D_p H(x,Du) \rho) - \varepsilon \Delta \rho(x,t) &= 0 \quad \text{in } \Omega \times [0,T], \\
D_p H(x,Du) \rho + \varepsilon \frac{\partial \rho}{\partial n}(x,t) &= g_2(x,t), \quad \text{on } \partial \Omega \times [0,T],
\end{align*}
\]

where \( g_2 \) is the desired in/out-flow through \( \partial \Omega \). We can consider diverse boundary conditions for the HJ operator: Dirichlet type, state-constraint, reflection at the boundary, and Neumann type. In the following example, we use Neumann conditions
with zero flow. The Hamilton-Jacobi operator is

\[
\begin{cases}
- u_t(x,t) + H(x, Du(x,t)) - \varepsilon \Delta u(x,t), & \text{in } \Omega \times [0,T], \\
\frac{\partial u}{\partial n}(x,t) = 0, & \text{on } \partial \Omega \times [0,T],
\end{cases}
\]

with the corresponding linearization

\[
\begin{cases}
- w_t + D_p H(x, Du) \cdot Dw - \varepsilon \Delta w, & \text{in } \Omega \times [0,T], \\
\frac{\partial w}{\partial n} (\cdot, t) = 0, & \text{on } \partial \Omega \times [0,T].
\end{cases}
\]

In the schemes we describe in section 9.4, we focus on the spatial discretization of the previous operators. The time discretization can be chosen separately, depending on the application. This is the reason we simplified the discussion above, and we avoided initial conditions, which are straightforward.

We now discuss the connection between stochastic differential equations with their density formulation, the Fokker-Planck equation, and then the associated Hamiltonian, via the adjoint structure. This pathwise interpretation given by the solution of the SDE provides understanding on how to treat boundary conditions.

A nonlinear FP equation is related to the solution of a stochastic differential equation of McKean-Vlasov type (or mean-field type), see [68, 69, 70, 75]. More precisely, we consider the stochastic differential equation (SDE)

\[
\begin{cases}
\frac{dX(t)}{dt} = b(X(t), \rho(X(t), t), t) dt + \sqrt{2\varepsilon} dW(t), \\
X(0) = X^0,
\end{cases}
\]

(9.6)

where \( b : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^d \) is a regular vector-valued function, \( X^0 \) is a random vector in \( \mathbb{R}^d \), independent of the Brownian motion \( W(\cdot) \), with density \( \rho_0 \), and \( \rho(\cdot, t) \) is the density of \( X(t) \). It can be shown (see [57]) that under suitable growth and regularity conditions for \( b \) (9.6) admits a unique solution and \( \rho \) is the unique classical
solution of the nonlinear FP equation

\[ \partial_t \rho - \varepsilon \Delta \rho + \text{div}(b(x, \rho, t) \rho) = 0. \]

With Dirichlet conditions, those trajectories end at the boundary; for zero-flux conditions, they are reflected, see [10, 38]. Moreover, the Hamilton-Jacobi operator from which (9.2.2) can be derived is

\[ -u_t - \varepsilon \Delta u - b(x, \rho, t) \cdot Du. \]

**Remark 2.** Our methods can be extended to study stationary FP equations. In this case, the associated Hamilton-Jacobi operator is stationary.

### 9.3 Properties

In this section, we show how to use a duality argument to obtain properties of FP equations from corresponding properties of the evolution semigroup associated with a HJ equation. The arguments detailed here are valid without any substantial changes for the discretized problems using the semigroups associated with the discretized equation. This discretization can be performed in multiple ways, including finite-differences and semi-Lagrangian schemes.

We denote by \( \langle f, g \rangle = \int_\Omega f \, g \) the duality product, and by \( S_t \) the semigroup corresponding to the evolution of time given by the linearized operator (9.3). This semigroup preserves order; that is \( v \leq w \) implies \( S_t v \leq S_t w \). Moreover, we assume that \( S_t 1 = 1 \). We note that the property \( S_t 1 = 1 \) depends on the boundary conditions for (9.3). We define the adjoint \( S_t^* \) of \( S_t \) by

\[ \langle S_t^* u, v \rangle = \langle u, S_t v \rangle. \]
We have then the following results:

**Proposition 9.3.1** (Positivity). Suppose $S_t$ is monotone. Then, the evolution of the initial density $\rho_0$ through the adjoint semigroup, $S_t^*$, preserves positivity. That is, if $\rho_0 \geq 0$, we have $S_t^* \rho \geq 0$, for all $t \in [0, T]$.

*Proof:* Denote by $w_T$ the terminal condition for the linearized operator. First, note that $w_T \geq 0$ implies $S_t w_T \geq 0$. This follows from the maximum principle for HJ equations. Thus, for $w_T \geq 0$, we have

$$\langle S_t^* \rho, w_T \rangle = \langle \rho, S_t w_T \rangle \geq 0,$$

since $\rho \geq 0$, and $S_t w_T \geq 0$. Accordingly, $S_t^* \rho \geq 0$.

Now, we show that if $S_t 1 = 1$ the mass is conserved by $S_t^*$. For example, under periodic boundary conditions, the evolution of (9.3) preserves constants. This is not the case under Dirichlet boundary conditions, where mass loss through the boundary occurs.

**Proposition 9.3.2** (Conservation of Mass). Suppose $S_t 1 = 1$. Let $\rho_0$ be the initial density probability distribution, i.e. $\int_{\Omega} \rho_0 = 1$. Then, for all $t \in [0, T]$, the evolution of this probability measure through the adjoint semigroup, $S_t^* \rho_0$, is also a probability measure.

*Proof:* First, observe that $S_t 1 = 1$. Then,

$$\int_{\Omega} S_t^* \rho_0 = \langle S_t^* \rho_0, 1 \rangle = \langle \rho_0, S_t 1 \rangle = \langle \rho_0, 1 \rangle = \int_{\Omega} \rho_0 = 1.$$

*Remark 3.* The previous assumptions on $S_t$ are not restrictive for our applications. For discrete problems, the linearized semigroups $S_t$ for HJ equations are usually monotone and, in cases where mass conservation for the FP equation holds, also satisfy
$S_t 1 = 1$. Often, it is easier to check monotonicity and that $S_t 1 = 1$ than the corresponding properties for $S^*_t$.

**Remark 4.** If the viscosity vanishes ($\varepsilon = 0$), our approach is still valid (provided enough regularity is present). A first-order HJ operator gives rise to a continuity equation (CE), i.e. a FP equation without viscosity. This case is considered in chapter 10, where we extend our numerical scheme to address systems of partial differential equations (PDEs). Those systems arise in multiple applications such as MFGs, population models, traffic flow problems, and modeling in chemotaxis.

### 9.4 Numerical approach

Our numerical approach relies on the relation between the HJ framework and the corresponding adjoint FP equation. Once we choose a semi-discrete (discrete in space) numerical scheme for (9.2), we can reuse it to construct an approximation for (9.5).

Before proceeding, we define additional notation. To simplify, we consider a scheme for the case where the domain $\Omega$ is $\mathbb{T}^2$ (2-D torus). Let $\mathbb{T}^2_{\Delta x}$ be an uniform grid on $\mathbb{T}^2$, with constant discretization parameter $\Delta x > 0$. Let $x_{i,j}$ denote a generic point in $\mathbb{T}^2_{\Delta x}$. The space of grid functions defined on $\mathbb{T}^2_{\Delta x}$ is denoted by $G(\mathbb{T}^2_{\Delta x})$, and the functions $U, M \in G(\mathbb{T}^2_{\Delta x})$ (approximations of respectively $u$ and $\rho$) are called $U_{i,j}$ and $M_{i,j}$, when evaluated at $x_{i,j}$. We utilize a semi-discrete numerical scheme $N(x, p): \mathbb{T}^2_{\Delta x} \times \mathbb{R}^d \to \mathbb{R}$ monotone and consistent to approximate the operator $H(x, p)$ by the discrete operator

$$N(x, DU),$$

where $DU$ is a discretization of the gradient operator on $U$. Thanks to the adjoint structure, we use the scheme (9.7) to assemble a discrete operator $K$

$$K(x, DU, M) := (DU N(x, DU))^T M,$$
that discretizes the spatial part of the FP operator (9.5). The discrete approximation $M$ for the solution of the FP equation is then given by the ODE

$$M_t - K(x, DU, M) - \varepsilon \Delta_d M = 0,$$

or simply,

$$M_t - (D_U N(x, DU))^T M - \varepsilon \Delta_d M = 0, \quad (9.9)$$

i.e., a discrete equivalent to the adjoint structure seen in (9.5). Here, the nonlinear part of the operator corresponds to the discrete operator $D_U N(x_{i,j}, DU)$, and $\Delta_d M$ is a discretization of the Laplacian, which is added to the scheme to increase the stability, if necessary.

We note that the operators $N$ and $K$ depend on the monotone approximation scheme used to discretize the HJ equation. The operator $K$ can be computed using a symbolic differentiation as we show in section 10.1. Also, the Hamiltonian $H(x, p)$ must be sufficiently regular in the $p$ variable so that the scheme is properly defined. The properties of positivity and mass conservation are valid at the discrete level as consequence of the semigroup arguments in section 9.3.

We now prove how the consistency of schemes for HJ equations transfers to schemes for FP equations.

**Proposition 9.4.1 (Consistency).** Suppose $u$ is the solution to a linearized HJ equation and $\rho$ the solution to the associated FP equation. Also, that $u$ and $\rho$ are $C^\infty$, and consider their restriction to the grid points. Denote by $S^N_t$ the linearized semigroup corresponding to a discretization of the HJ equation with $o(1)$ error, i.e.,

$$S^N_t u = S_t u + o(1), \quad (9.10)$$

with $S_t$ as in the previous sections. Then, the adjoint semigroup $(S^N_t)^*$ operating on
the discretization of the FP equation possesses the same order of error, i.e.,

\[(S_t^N)^* \rho = (S_t)^* \rho + o(1). \tag{9.11}\]

**Proof:** By the adjoint structure between the HJ and FP equations, and the hypothesis of consistency in equation (9.10), we have

\[
\langle (S_t^N)^* \rho, u \rangle = \langle \rho, S_t^N u \rangle = \langle \rho, S_t u \rangle + \langle \rho, o(1) \rangle = \langle S_t^* \rho, u \rangle + o(1),
\]

which proves (9.11). \hfill \blacksquare

**Remark 5** (Convergence). We stress that the main novelty in the current paper is a systematic approach to build schemes for FP equations from schemes for HJ equations. Naturally, different schemes will have different convergence properties, which must be examined case by case. Furthermore, to address the convergence of a method, we must know the regularity of the vector field in the FP equation and the existence of a solution. In the numerical simulations presented in chapter 10, such regularity is not always known. Also, for the crowd motion model we consider, the existence of solutions is still an open problem.

### 9.4.1 Finite differences

Now, we consider an explicit scheme using our method. We describe an upwind discretization for the Hamiltonian, which we assume to be

\[H(x, p) = g(x) + |p|^\alpha, \text{ with } \alpha > 1, \text{ and } p = (p_1, p_2, p_3, p_4). \tag{9.12}\]

We define the standard finite-difference operators as

\[
(D_1^+ u)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x}, \quad (D_2^\pm u)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{\Delta x},
\]
\[ \Delta_d u = \frac{1}{\Delta x^2} \left(4u_{i,j} - u_{i+1,j} - u_{i,j+1} - u_{i-1,j} - u_{i,j-1}\right). \]

The approximation of the operator \( H(x,p) \) is

\[ N(x,p) = g(x) + G(p_1^+, p_2^+, p_3^-, p_4^+), \]

where for a real number \( r \), we define the operators

\[ r^+ := \max(0, r), \quad r^- := \max(0, -r), \tag{9.13} \]

and

\[ G(p) = G(p_1, p_2, p_3, p_4) := (p_1^2 + p_2^2 + p_3^2 + p_4^2)^{\frac{3}{2}}. \]

The operators \( r^+ \) and \( r^- \) are chosen to preserve the monotonicity of the scheme for the HJ operator, which is well defined backward in time.

Now, we compute the operator \( K(x, DU, M) \), and we obtain

\[
K(x_{i,j}, [DU]_{i,j}, M_{i,j}) = \\
\frac{1}{\Delta x} \left[ M_{i,j} \frac{\partial N}{\partial p_1}(x_{i,j}, [DU]_{i,j}) - M_{i-1,j} \frac{\partial N}{\partial p_1}(x_{i-1,j}, [DU]_{i-1,j}) \\
+ M_{i+1,j} \frac{\partial N}{\partial p_2}(x_{i+1,j}, [DU]_{i+1,j}) - M_{i,j} \frac{\partial N}{\partial p_2}(x_{i,j}, [DU]_{i,j}) \\
+ M_{i,j} \frac{\partial N}{\partial p_3}(x_{i,j}, [DU]_{i,j}) - M_{i-1,j} \frac{\partial N}{\partial p_3}(x_{i,j-1}, [DU]_{i,j-1}) \\
+ M_{i,j+1} \frac{\partial N}{\partial p_4}(x_{i,j+1}, [DU]_{i,j+1}) - M_{i,j} \frac{\partial N}{\partial p_4}(x_{i,j}, [DU]_{i,j}) \right]. \tag{9.14}
\]

We then use this expression for the operator in (9.8).

This scheme is similar to the one in [3]. The advantage here is the fact that the properties of positivity and mass conservation are automatically obtained from the
hypotheses on the discretization of the HJ operator; unlike typical schemes for FP equations, where such properties must be proved a posteriori. Analogously, in case we have chosen the semi-Lagrangian scheme for HJ equations previously presented, we would be creating a ”dual semi-Lagrangian” method for FP equations. This is the main point in this work: the possibility of generating ”dual methods” with desired properties for FP equations based on well-established methods for HJ equations. We explicit this possible method in the following section.

9.4.2 Semi-Lagrangian scheme

To describe a semi-Lagrangian scheme appropriate to approximate (9.12), we introduce the operator

\[ D^\gamma U_{i,j} := \max_{\gamma \in B(0,1)} \frac{I[U](x_{i,j}, \gamma) - U(x_{i,j})}{h}, \tag{9.15} \]

where \( B(0,1) \) is the unitary ball in \( \mathbb{R}^2 \), \( h \) a parameter of the same order of \( \sqrt{\Delta x} \), and

\[ I[U](x_{i,j}, \gamma) = \frac{1}{2} \sum_{i=1}^{2} \left( I(U)(x_{i,j} + \gamma h + e_i \sqrt{2 \varepsilon h}) + I(U)(x + \gamma h - e_i \sqrt{2 \varepsilon h}) \right). \tag{9.16} \]

Here, \( I[u](x) \) is an interpolation operator on the matrix \( U \), and \( e_i \) is the \( i \) unitary vector of an orthonormal basis of the space. Details on how to choose the interpolation operator are discussed in [35]. In our case, the discrete operator has the form

\[ N(x, D^\gamma U_{i,j}) := g(x) + (D^\gamma U_{i,j})^\alpha. \tag{9.17} \]

We take the adjoint of the linearized of \( N \), by using (9.15), and we apply it to (9.8); analogously as performed for the finite-difference scheme. This scheme formally differs from the one proposed in [26], since the characteristic of the system (9.16) is computed in the dual formulation of the problem. We note that the operator \( N(x, p) \) in (9.17)
is monotone by construction, see [35].

We conclude by remarking that we will use the finite-difference method in the following chapter while simulating several systems of PDEs. Examples include mean-field games and the Hughes model.
Chapter 10

Measure-Potential models and its numerical simulations

One immediate application of the numerical scheme of the previous chapter is to solve “measure-potential” systems of PDEs. These systems comprise a first equation for the evolution of a measure coupled to a second one for a potential or value function. Typically, the potential determines the drift for the convection in the first equation. Many problems have this structure: mean-field games, traffic-flow models, crowd motion, and chemotaxis.

Now we describe the structure of this chapter. In section 10.1, we use the Hughes model in 1-D to exemplify how to translate a measure-potential system of PDEs into the setting developed in the previous chapter. Then, we present solutions to this problem with both initial/Dirichlet conditions and an imposed-flow type, in sections 10.2 and 10.3, respectively. In the first case, agents/people/pedestrians can leave the domain from both sides. In the second case, we impose a flow of agents entering with reflecting boundary conditions on one side (preventing agents from leaving immediately) and Dirichlet conditions on the other one. We then move to a different problem, in Section 10.4, where we illustrate forward-forward mean-field games (FFMFG). Model II, in Section 8.2 of Chapter 8, possesses similar structure. Next, in Section 10.5, we use the numerical method to depict several properties of the Hughes model. We show that the symmetry is preserved over time in the absence of boundaries or external potentials. We also verify the positivity and mass-preservation, and we compare the influence of viscosity when it is present in the Fokker-Planck equation and the Eikonal. In Section 10.6, we show a property of the Hughes model: the non-
optimality on the evacuation time, due to the stationary character of the Eikonal. In Section 10.7, we use the Hughes model in 2-D simulating the evacuation of a room with size ratios similar to a typical office or bedroom. Finally, in Section 10.8, we present the simulation of the Kaaba model, already described in Section 8.3. All the simulations were performed on a 2.3GHz i7 computer with 16GB of RAM using the software Mathematica. The source-codes are available in Appendix A.

10.1 Example: Hughes model in 1-D

We use the Hughes model, equation (6.1), in 1-D as an example of the applicability of the numerical scheme of chapter 9 for systems of PDEs comprising a Fokker-Planck equation. In contrast with MFG problems, the Hughes model does not have the adjoint structure built-in.

We now describe how the Hughes system can be studied via our framework. Performing the same steps as in section 9.2, with the HJ operator

\[- u_t + f(\rho)H(x, Du) - \varepsilon \Delta u, \]  (10.1)

where \( f(\rho) \) is a regular function of the density, we obtain the associated FP equation

\[ \rho_t - \text{div} \left( f(\rho)D_pH(x, Du)\rho \right) = \varepsilon \Delta u. \]  (10.2)

By setting \( f(\rho) = (1 - \rho)^2 \), and \( H(x, p) = \frac{|p|^2}{2} \), (10.2) becomes the first equation of (6.1); and (10.1) is the adjoint operator we must study. More explicitly, starting by the spatial part of the HJ equation, the adjoint structure is:

\( (1 - \rho)^2 \frac{|Du|^2}{2} \xrightarrow{\text{Linearization}} (1 - \rho)^2 Du \cdot Dw \xrightarrow{\text{Adjoint}} - \text{div} \left[ \rho(1 - \rho)^2 Du \right], \)

where \( u, w, \) and \( \rho \) follow the notation of section 9.2.
We illustrate the use of finite differences to discretize the generator of the HJ operator. We choose the monotone scheme:

\[ N_n(u) \equiv (1 - \rho_n)^2 \left[ \frac{\max\{u_n - u_{n-1}, 0\}^2}{2h^2} + \frac{\max\{u_n - u_{n+1}, 0\}^2}{2h^2} \right]. \tag{10.3} \]

Once we discretize the domain \( \Omega \) (a finite interval of \( \mathbb{R} \) in this example), we calculate the previous discrete operators in each grid point, taking into account the discretized versions of the boundary conditions. We have then a matrix, with the dimensions of the grid, whose entries have the expressions of the form of (10.3). The matrix entries (a vector in the 1-D case) have the \( u_j \)'s \((j \in \{0, 1, \ldots, P\}, P \) being the size of the grid in this 1-D example) in its literal form, as a parameter. The numerical value of this matrix will only be evaluated once we have an approximation for the values of \( u \). This will be done by solving the EK equation with an initial guess. Since the EK equation is a particular case of a HJ equation, we choose to discretize it in space in the same form as the HJ operator associated to the FP equation. We use the monotone scheme

\[ \tilde{N}_n(u) \equiv \frac{\max\{u_n - u_{n-1}, 0\}^2}{h^2} + \frac{\max\{u_n - u_{n+1}, 0\}^2}{h^2} - \frac{1}{(1 - \rho_n)^2}. \tag{10.4} \]

We are then left to solve, at the discrete level, the system

\[
\begin{cases}
\rho_t + (D_u N_n(u))^T \rho = 0, \\
\tilde{N}_n(u) = 0,
\end{cases}
\tag{10.5}
\]

at each point of the grid. In this way, all the spatial part of the operators was treated. First equation of (10.5) is a time-dependent matrix ODE in \( \rho \), whose spatial part has \( u_j \)'s as parameters (updated by the solution of the second equation). This ODE is supplemented with a discretization of the initial condition for the density, defined in the original continuous Hughes system (6.1). The second equation, the discretized
version of the EK equation, is a difference equation in the \( u_j \) variable, with \( \rho_j \) as a parameter (updated by the approximation of the solution of the first equation at each time iteration).

Up to now, we have only treated the spatial part of the operators, and we have chosen to discretize with a monotone finite-difference scheme. To solve in time the first ODE of (10.5), we can use ordinary solvers, based on explicit or implicit schemes. The experience with our simulations tells us that the use of explicit Euler methods is fast unless the matrix \((D_u N_n(u))^T\) becomes stiff. In this case, a better approach is the use of Backward Differentiation Formulas (BDF). Now, for the second ODE of (10.5), we use a method described by Oberman in [71], based on fixed-point arguments. Alternatively, other methods could be applied, as policy iteration or fast marching (not suitable in case viscosity is present). The system (10.5) is then solved as follows: at each time iteration of the solver for the discrete FP equation, we solve one discrete EK equation. In sections 10.5 and 10.7, we present simulations of the Hughes model in 2-D while following this approach.

10.2 Example 1 - Hughes 1-D initial/boundary conditions

For the first numerical example, we solve the Hughes model with a low viscosity \( \varepsilon = 0.01 \), and the following initial/boundary conditions:

\[
\begin{align*}
\rho(0, t) &= 0, \\
\rho(1, t) &= 0, \\
\rho(x, 0) &= 0.9 \sin^2(3\pi x), \\
u(0, t) &= 0, \\
u(1, t) &= 0.
\end{align*}
\]
These boundary conditions correspond to the exit problem. Agents have an initial distribution $\rho(x, 0)$ and seek to leave the interval $[0, 1]$ by either $x = 0$ or $x = 1$. We plot the density $\rho$ and the solution $u$ in Figure 10.1.

### 10.3 Example 2 - Hughes 1-D with a flow

The second example is the flow problem for the following initial/boundary conditions:

\[
\begin{align*}
\rho(1, t) &= 0, \\
\rho(x, 0) &= 0.4 \sin^2(3\pi x), \\
u(x, 1) &= 0.
\end{align*}
\]

In addition, at $x = 0$, we impose for $\rho$ a flow-one condition:

\[\rho(1 - \rho)^2 Du + \varepsilon \rho_x = 1,\]

and, at $x = 0$, we impose reflecting boundary condition for $u$. Here, we use a higher viscosity, $\varepsilon = 0.1$, and we present our results in Figure 10.2.

![Figure 10.1: Numerical example Hughes 1-D.](image-url)
10.4 Example 3 - 1-D forward-forward mean-field games

Here, we consider two one-dimensional forward-forward mean-field game problems, see [3, 46, 47]. The general form of such systems is

\[
\begin{aligned}
\frac{u_t}{H(u_x)} &= \varepsilon u_{xx} + g(\rho), \\
\rho_t - (H'(u_x)\rho)_x &= \varepsilon \rho_{xx},
\end{aligned}
\]  

\tag{10.6}

(10.6)

together with the initial-initial conditions

\[
\begin{aligned}
u(x, 0) &= u_0(x), \\
\rho(x, 0) &= \rho_0(x),
\end{aligned}
\]

and periodic boundary conditions.

Now, we explain how such systems were treated numerically. MFGs have built-in the adjoint structure we have considered so far, i.e., the FP equation of a MFG system is the adjoint of the linearization of the HJ equation present in the MFG system. Hence, we can use the same type of spatial schemes for the discretization of

![Figure 10.2: Numerical example for the flow problem.](image-url)
both the FP and the HJ equation. As described in the previous section, each of the
discretizations requires solving an ODE in time. Since we must handle the system
of FP coupled to a HJ equation, we treat these ODEs as a single system, and we
apply a suitable solver for the time discretization. This manner of solving is possible
because we are solving a FFMFG, where both equations evolve forward in time. In
the next two problems, we use a monotone finite-difference method for the spatial
discretization, as in section 9.4.1, and an explicit Euler scheme in time.

For the first problem, we set $H(u_x) = \frac{u_x^2}{2}$, $g(\rho) = \ln \rho$, and $\varepsilon = 0.01$. We then solve:

\begin{align*}
  u_t + \frac{u_x^2}{2} &= 0.01 \ u_{xx} + \ln \rho, \\
  \rho_t - (u_x \rho)_x &= 0.01 \ \rho_{xx},
\end{align*}

with the initial-initial conditions:

\begin{align*}
  u_0(x) &= 0.3 \cos(2\pi x), \\
  \rho_0(x) &= 1.
\end{align*}

We depict the solution of this problem in Figure 10.3. Here, we see the convergence
of the forward-forward model to a stationary solution. This form of convergence for
parabolic problems, as (10.7), was shown in [47].

Now, for the second case, we choose $H(u_x, \rho) = \frac{(p + u_x)^2}{2\rho^\alpha}$, $g(\rho) = \frac{3}{2} \rho^\alpha$, and $\varepsilon = 0$. This example is a first-order FFMFG with congestion, which is equivalent to a system
of conservation laws. Setting $v = p + u_x$, the equivalent system is

\begin{align*}
  v_t + \left( \frac{v^2}{2\rho^\alpha} - \frac{3}{2} \rho^\alpha \right)_x &= 0, \\
  \rho_t - (\rho^{1-\alpha} v)_x &= 0.
\end{align*}
For $\alpha = 1$, and for the *initial-initial conditions*

$$\begin{cases}
  u_0 = -0.5 \frac{\cos(2\pi x)}{2\pi}, \\
  \rho_0 = 1 + 0.5 \sin(2\pi x),
\end{cases}$$

the solution for the density in (10.8) is a traveling wave, depicted in Figure 10.4. Such failure of convergence is an interesting phenomenon identified in [46] for first-order forward-forward MFGs, and it illustrates an important difference between these two MFG models.

The simulations corresponding to Figure 10.3 and Figure 10.4 were produced with a spatial grid of 80 points, final time $T = 3$, and 50 points for the time sample. The simulations run in about one second.
10.5 Example 4 - Hughes model in 2-D

In this section, we illustrate the behavior of the Hughes model in a two-dimensional domain, and we verify that our solver has the expected behavior: symmetry is preserved, and density is kept below one. We solve the model with the following initial-boundary conditions:

\[
\begin{align*}
\rho(x, y, 0) &= 0.2 \sin^2(\pi x) + 0.7 \sin^2(\pi y), \\
\nu|_{\partial \Omega} &= \rho|_{\partial \Omega} = 0,
\end{align*}
\]

which corresponds to the situation of an initial density of people inside the domain followed by its evacuation through all directions.

In Figure 10.5, we present six samples of our simulation for a case without viscosity; from the initial density, \( t = 0 \), to time \( t = 0.8 \), when the evacuation of the domain is almost complete. We also plot the solution to the EK equation and the streamlines (the directions the agents follow) at two different moments. Next, in Figure 10.6, we consider a viscosity to the FP equation equal to 0.01. The smoothing effect of the Laplacian is clear on the density plots, while the Eikonal solution does not show
any apparent change. By also adding viscosity to the EK equation equal to 0.15, we observe how the coupling of the EK equation creates an extra smoothing on the density of the FP equation, in Figure 10.7.

10.6 Non-optimality on the Hughes model

Using the previous example for the two-dimensional Hughes problem, we illustrate the non-optimality in time for the evacuation of a domain using the Hughes model.

In Figures 10.5, 10.6, and 10.7, we depict in two different moments the streamlines (representing the optimal direction) chosen by an agent inside the domain. We follow the dynamics of Example 4, i.e., evacuation can be performed in every direction of the 2-D plane. We observe that the optimal direction changes along time. This shows that agents do not know the best direction to follow from the beginning. Thus a delay in the evacuation can be incurred. This property of the Hughes model is due to the stationary character of the EK equation. If we were handling with a time-dependent version of the EK equation, i.e., a HJ equation, the best direction would be known from the beginning.
Figure 10.5: Evolution of Density, Eikonal Solution and Streamlines for Example 4 with no viscosity.
(a) Initial Density: $t = 0$.  
(b) Density at $t = 0.25$.  
(c) Density at $t = 0.38$.  
(d) Density at $t = 0.5$.  
(e) Density at $t = 0.65$.  
(f) Density at $t = 0.8$.  
(g) Eikonal Solution.  
(h) Streamlines at $t = 0$.  
(i) Streamlines at $t = 0.38$.  

Figure 10.6: Evolution of Density, Eikonal Solution and Streamlines for Example 4 when viscosity equal to 0.01 is added to the FP equation.
Figure 10.7: Evolution of Density, Eikonal Solution and Streamlines for Example 4 when viscosity equal 0.01 is present in the FP and equal to 0.15 in the EK.
10.7 Example 5 - The problem of evacuation of a room

In this example, we utilize the Hughes model in 2-D to study the evacuation of a room. The domain of our example is a rectangle $[0, 3] \times [0, 1]$, modeling a room with walls, and an exit on $[2.25, 3] \times \{1\}$, corresponding to a typical proportion of the size of a door in a room. We set the value of $u$ to $+\infty$ on all the boundary but on its exit, where we fix it equal to zero. The density is set equal zero on the boundary.

We perform the simulations analogously to the Hughes model in 1-D. We depict the initial condition and its evolution in Figure 10.8. The spatial grid contains 243 points, and we choose the final time $T = 1.0$. The simulation runs in 12.9 seconds. We remark that at each time iteration for the solution of the Fokker-Planck equation, one Eikonal equation in 2-D is being solved. In this example, 83 Eikonal equations were solved, and they took 78% of the total simulation time.
Figure 10.8: Evolution of the density for the Hughes model in Example 5.
10.8 Example 6 - The Kaaba model

In our last example, we present the Kaaba model formulated in Section 8.3. The main difference compared to the description of Section 10.1, on how to translate the system of PDEs into our numerical scheme, is to take into account the change of coordinates the mapping from the disc/annulus to a rectangle implies.

At first, it seems we would need to rewrite the Hughes system in new coordinates to account the domain transformation. This would then imply recalculate the HJ operator associated to the new FP equation of the system. However, it can be verified, that for the Hughes model transformed via the polar mapping, it suffices to modify only the Hamiltonians (the one associated to the HJ equation coming from the adjoint structure and the one of the EK equation) by inserting the Jacobian of the transformation accordingly. To our knowledge, this is the first example of the Hughes model in the continuous setting that is applied to a non-Cartesian domain.

In this simulation, no viscosity is present, pilgrims perform three turns around Kaaba, the grid size contains 243 points, the size of the exit is 0.33, and the final time is $T = 1.4$. The initial density is composed by four Gaussians, as in the previous example, so that we can compare the effect of the different geometry on the movement.

We present the simulation in Figure 10.9. On the top of each plot of the density, we depict the percentage of pilgrims and the maximal density inside the Kaaba domain by the respective time - described on the subtitles. We show the evolution of the density in three different ways. The first row of plots shows a 2-D plot, while the second one has 3-D plots, both in the rectangular domain. The third row depicts the density on the real geometry of the problem. The last row shows the streamlines in a particular moment and the Eikonal solution. Finally, we can observe that the maximum density reached inside the domain is not monotone in time. This phenomenon of concentrating mass is a characteristic of the Hughes model, and it indicates the difficulty in obtaining theoretical bounds for the evolution of the density in time.
We end this section by remarking that, in all simulations of this chapter, both positivity and mass preservation were observed - unless while agents are leaving through the boundary. Thus, in the Hughes model, the mass is preserved as long as no agents/pedestrians are passing through the exit.

Figure 10.9: Evolution of the density, streamlines, and Eikonal solution for the Kaaba model.
Chapter 11

Conclusions and future work

In the second part of this dissertation, we developed new a priori estimates for the Hughes model, which are an important step in understanding the well-posedness of the system for $\varepsilon > 0$. Next, we used solutions to the radial problem to prove the existence of shocks in dimension greater than one. Consequently, the Hughes model without viscosity may fail to have smooth solutions. Then, we have uncovered a new mechanism for the breakdown of classical solutions of the fixed current problem. In this case, the critical density is reached without loss of regularity in $\rho$. Moreover, we examined the dependence of the critical current on the viscosity, and we presented numerical evidence for the existence of a long-term limit.

We then introduced variations to the original model and, with the introduction of a new approach for schemes for Fokker-Planck equations, we described the approximation of the Hughes model. In fact, we developed an approach for the approximation of nonlinear Fokker-Planck equations via its adjoint Hamilton-Jacobi operator. In this way, we guarantee that the produced schemes preserve mass and positivity. The consistency of the schemes is also addressed. This methodology combines, in a novel way, monotone schemes for Hamilton-Jacobi equations with the adjoint structure of the Fokker-Planck equation.

Next, we solved systems of PDEs with a Fokker-Planck equation coupled to a Hamilton-Jacobi equation. Our methods apply to a broad range of problems where a measure-potential structure appears, including crowd and traffic models, and chemotaxis. We depict some examples of mean-field games, of the Hughes model in one
and two dimensions, and we explore properties of such models. We also present a new manner of understanding the flow of pilgrims around the Kaaba, by using a modification of the Hughes model.

In future works, we plan to address different schemes developed for HJ equations to study FP equations. Thus, reversing the process that gave rise to effective numerical schemes for HJ equations, as Discontinuous Galerkin or ENO schemes, originally developed for conservation laws. Nevertheless, it is clear that, without monotonicity and stability properties, results for the convergence of such schemes are difficult to achieve. We also intend to extend the Hughes model in a way to study the formation of queues. Empirical studies show that, when people behave in queues, the time for the evacuation of a domain is shorter than otherwise. This is an interesting topic of study, and the comprehension of when or why people develop a queue structure is of great importance.
REFERENCES


APPENDICES

A Computational code developed

In this appendix, we present the source code we developed in the software Mathematica to generate simulations.

Listing A.1: Code for Figure 7.2

```
"profiles_simulation",
 rho01[r_] := ((ArcTan[(r/0.02 - 7)*3] + Pi/2)*0.6/Pi)*((ArcTan[(-r/0.02 + 15)*3] + Pi/2)*0.6/Pi);
 rho02[r_] := ((ArcTan[(r/0.02 - 30)*4] + Pi/2)*0.9/Pi)*((ArcTan[(-r/0.02 + 40)*4] + Pi/2)*0.9/Pi);
 rho03[r_] := (0.2 + ArcTan[(10 (r/0.1 - 4))/10])/10)*1.3;
 Plot[{rho01[r], rho02[r], rho03[r]}, {r, 0, 1}, PlotRange -> {0, 1}, AxesLabel -> {HoldForm[r], HoldForm[rho0]}, PlotLegends -> "Expressions"];

"code_simul_2D",
 R1N[rho0_?NumberQ, r0_?NumberQ, t_?NumberQ] := InverseFunction[
 1/2 (4 r0 (1 - rho0) rho0) Log[(-4 r0 (1 - rho0) rho0) + 2 (1 + Sqrt[1 - (4 r0 (1 - rho0) rho0)/#1]) #1] + Sqrt[1 - (4 r0 (1 - rho0) rho0)/#1] #1 &][t + 1/2 (2 r0 Sqrt[(r0 - 4 r0 (1 - rho0) rho0)/r0] + 4 r0 (1 - rho0) rho0 Log[(-4 r0 (1 - rho0) rho0) + 2 r0 (1 + Sqrt[1 - 4 (1 - rho0) rho0])])];
 R2N[rho0_?NumberQ, r0_?NumberQ, t_?NumberQ] := InverseFunction[1/2 (4 r0 (1 - rho0) rho0) Log[(-4 r0 (1 - rho0) rho0) + 2 (1 + Sqrt[1 - (4
```
\( Rho1N[rho0\_NumberQ, r0\_NumberQ, t\_NumberQ] := 1/2 (2 r0 + \sqrt{[1/2 (4 r0 (1 - rho0) rho0) Log[1 - (4 r0 (1 - rho0) rho0) /#1]) #1] + Sqrt[1 - (4 r0 (1 - rho0) rho0)/#1] #1 &}] + [\text{InverseFunction}[1/2 (4 r0 (1 - rho0) rho0) Log[1 - (4 r0 (1 - rho0) rho0) /#1]) #1] + Sqrt[1 - (4 r0 (1 - rho0) rho0)/#1] #1 &}] [r0 Sqrt[(1 + 2 rho0) ^2] - 2 t - 2 r0 (-1 + rho0) rho0 Log[2 r0 (1 - 2 rho0 + 2 rho0^2 + Sqrt[(1 + 2 rho0) ^ 2])])]] - Sqrt[1 + (4 r0 (-1 + rho0) rho0)/InverseFunction[1/2 (4 r0 (1 - rho0) rho0) Log[1 - (4 r0 (1 - rho0) rho0) /#1]) #1] + Sqrt[1 - (4 r0 (1 - rho0) rho0)/#1] #1 &}] [r0 Sqrt[(-1 + 2 rho0) ^2] + t - 2 r0 (-1 + rho0) rho0 Log[2 r0 (1 - 2 rho0 + 2 rho0^2 + Sqrt[(-1 + 2 rho0) ^ 2])])]]; \\
Rho2N[rho0\_NumberQ, r0\_NumberQ, t\_NumberQ] := 1/2 (2 r0 - \sqrt{[1/2 (4 r0 (1 - rho0) rho0) Log[-(4 r0 (1 - rho0) rho0) + 2 (1 + Sqrt[1 - (4 r0 (1 - rho0) rho0) /#1]) #1] + Sqrt[1 - (4 r0 (1 - rho0) rho0)/#1] #1 &}] [1/2 (2 r0 Sqrt[(-1 + 2 rho0) ^2] - 4 r0 (-1 + rho0) rho0 Log[2 r0 (1 - 2 rho0 + 2 rho0^2 + Sqrt[(-1 + 2 rho0) ^ 2])])]) + Sqrt[1 + (4 r0 (-1 + rho0) rho0)/InverseFunction[1/2 (4 r0 (1 - rho0) rho0) Log[-(4 r0 (1 - rho0) rho0) + 2 (1 + Sqrt[1 - (4 r0 (1 - rho0) rho0) /#1]) #1] + Sqrt[1 - (4 r0 (1 - rho0) rho0)/#1] #1 &}] [1/2 (2 r0 Sqrt[(-1 + 2 rho0) ^2] - 2 t - 4 r0 (-1 + rho0) rho0 Log[2 r0 (1 - 2 rho0 + 2 rho0^2 + Sqrt[(-1 + 2 rho0) ^ 2])])])]); \\
Thalf[rho0\_, r0\_] := r0 (Sqrt[(-1 + 2 rho0) ^2] - 2 (-1 + rho0) rho0 (Log[2 r0 (1 + 2 rho0) rho0 + Sqrt[(-1 + 2 rho0) ^ 2])]) - Log[4 r0 (-1 + rho0) rho0 (1 + 2/Sqrt[2] [Sqrt](1 + 2 rho0 (-1 + rho0) rho0)/InverseFunction[1/2 (4 r0 (1 - rho0) rho0) Log[-(4 r0 (1 - rho0) rho0) + 2 (1 + Sqrt[1 - (4 r0 (1 - rho0) rho0) /#1]) #1] + Sqrt[1 - (4 r0 (1 - rho0) rho0)/#1] #1 &}] [r0 Sqrt[(1 + 2 rho0) ^2] - 2 t - 2 r0 (-1 + rho0) rho0 Log[2 r0 (1 - 2 rho0 + 2 rho0^2 + Sqrt[(-1 + 2 rho0) ^ 2])])]];
\[
\left( \sqrt{(-1 + 2 \rho_0)^2} - 2 (-1 + \rho_0) \rho_0 \log\left[ 2 \rho_0 \left( 1 + 2 (-1 + \rho_0) \rho_0 + \sqrt{(-1 + 2 \rho_0)^2} \right) \right] \right) + \sqrt{\left( 1 + (4 \rho_0 (1 - \rho_0) \rho_0 + \sqrt{(-1 + 2 \rho_0)^2}) \right)} + \left( 2 \rho_0 (-1 + \rho_0) \rho_0 + 2 (1 + \sqrt{(-1 + 2 \rho_0)^2} \right) \right) \)
\[ \left( -4 r_0 (1 - \rho_0) \rho_0 \right) /\#1 \#1 \& [r_0 (\sqrt{(-1 + 2 \rho_0)^2} - 2 (-1 + \rho_0) \rho_0 \Log[2 r_0 (1 + 2 (-1 + \rho_0) \rho_0 + \sqrt{(-1 + 2 \rho_0)^2})])] \] 
\[ \text{InverseFunction}[1/2 (4 r_0 (1 - \rho_0) \rho_0) \Log[-(4 r_0 (1 - \rho_0) \rho_0) + 2 (1 + \sqrt{1 - (4 r_0 (1 - \rho_0) \rho_0) /\#1}) \#1] + \sqrt{1 - (4 r_0 (1 - \rho_0) \rho_0) /\#1} \#1 \&] [r_0 (\sqrt{(-1 + 2 \rho_0)^2} - 2 (-1 + \rho_0) \rho_0 \Log[2 r_0 (1 + 2 (-1 + \rho_0) \rho_0 + \sqrt{(-1 + 2 \rho_0)^2})])] 
\] 
\[ \left/ (16 r_0^2 (-1 + \rho_0)^2 \rho_0 + Sqrt[1 - (4 r_0 (1 - \rho_0) \rho_0) /\#1]) \right. \] 
\[ \text{InverseFunction}[1/2 (4 r_0 (1 - \rho_0) \rho_0) \Log[-(4 r_0 (1 - \rho_0) \rho_0) + 2 (1 + \sqrt{1 - (4 r_0 (1 - \rho_0) \rho_0) /\#1}) \#1] + \sqrt{1 - (4 r_0 (1 - \rho_0) \rho_0) /\#1} \#1 \&] [r_0 (\sqrt{(-1 + 2 \rho_0)^2} - 2 (-1 + \rho_0) \rho_0 \Log[2 r_0 (1 + 2 (-1 + \rho_0) \rho_0 + \sqrt{(-1 + 2 \rho_0)^2})])] 
\] 
\[ (-8 r_0 (-1 + \rho_0) \rho_0 + (-3 + 2 \rho_0) (1 + 2 \rho_0) \right) (1 + 2 \rho_0) \] 
\[ \text{InverseFunction}[1/2 (4 r_0 (1 - \rho_0) \rho_0) \Log[-(4 r_0 (1 - \rho_0) \rho_0) + 2 (1 + \sqrt{1 - (4 r_0 (1 - \rho_0) \rho_0) /\#1}) \#1] + \sqrt{1 - (4 r_0 (1 - \rho_0) \rho_0) /\#1} \#1 \&] [r_0 (\sqrt{(-1 + 2 \rho_0)^2} - 2 (-1 + \rho_0) \rho_0 \Log[2 r_0 (1 + 2 (-1 + \rho_0) \rho_0 + \sqrt{(-1 + 2 \rho_0)^2})])] 
\] 

18 \( R1Star[\rho_0, r_0, t] := R1N[1/2, R2N[\rho_0, r_0, Re[Thalf[\rho_0, r_0]]], t - Re[Thalf[\rho_0, r_0]]] \);
19 \( R2Star[\rho_0, r_0, t] := \text{Piecewise}[\{\{R2N[\rho_0, r_0, t], t <= Re[Thalf[\rho_0, r_0]]\}\}, R1Star[\rho_0, r_0, t]] \);
20 \( R[\rho_0, r_0, t] := \text{Piecewise}[\{\{R1N[\rho_0, r_0, t], \rho_0 <= 1/2\}\}, R2Star[\rho_0, r_0, t]] \);
21 \( \text{Rho1Star}[\rho_0, r_0, t] := \text{Rho1N}[1/2, R2N[\rho_0, r_0, Re[Thalf[\rho_0, r_0]]], t - Re[Thalf[\rho_0, r_0]]] \);
22 \( \text{Rho2Star}[\rho_0, r_0, t] := \text{Piecewise}[\{\{Rho2N[\rho_0, r_0, t], t <= Re[Thalf[\rho_0, r_0]]\}\}, \text{Rho1Star}[\rho_0, r_0, t]] \);
23 \( \text{Rho}[\rho_0, r_0, t] := \text{Piecewise}[\{\{\text{Rho1N}[\rho_0, r_0, t], \rho_0 <= 1/2\}\}, \text{Rho2Star}[\rho_0, r_0, t]] \);
24 \( \text{VectorTimesSimul} = \{0, 0.1, 0.25, 0.5, 0.8, 1.0\} \);
25 \( pp = 80; \)
26 \( mr = 1; \)
Parallelize[plots2D1 = Table[ParametricPlot[{R[rho01[r0], r0, t], Rho[rho01[r0], r0, t]}, {r0, 0, 0.6}, PlotRange -> All, AxesOrigin -> {0, 0}, PlotLegends -> "time" == t, PlotPoints -> pp, MaxRecursion -> mr], {t, VectorTimesSimul}]];

plots2D2 = Table[ListPlot[Select[ParallelTable[{R[rho02[r0], r0, t], Rho[rho02[r0], r0, t]}, {r0, 0, 1, 0.0025}], NumberQ[#[[1]]] && NumberQ[#[[2]]] &], Joined -> True, PlotRange -> All, AxesOrigin -> {0, 0}, PlotLegends -> "time" == t], {t, VectorTimesSimul}];

Parallelize[plots2D3 = Table[ParametricPlot[{R[rho03[r0], r0, t], Rho[rho03[r0], r0, t]}, {r0, 0, 1}, PlotRange -> All, AxesOrigin -> {0, 0}, PlotLegends -> "time" == t, PlotPoints -> pp, MaxRecursion -> mr], {t, VectorTimesSimul}]];
"
"plots_simul_2D",
restylePlot2[p_, op : OptionsPattern[ListLinePlot]] := ListLinePlot[
Cases[Normal@p, Line[x__] :> x, Infinity], op, Options[p]];
"
"plot_parametrico_2D_r_rho_rho01",
restylePlot2[plots2D1, PlotRange -> All, AxesLabel -> {"r(t)", "\[CurlyRho](t)"}, PlotLegends -> VectorTimesSimul, PlotLabel -> "0<="ro <=0.6"]
"
"plot_parametrico_2D_r_rho_rho02",
restylePlot2[plots2D2, PlotRange -> All, AxesLabel -> {"r(t)", "\[CurlyRho](t)"}, PlotLegends -> VectorTimesSimul, PlotLabel -> "0<="ro <=1.0"]
"
"plot_parametrico_2D_r_rho_rho03",
restylePlot2[plots2D3, PlotRange -> All, AxesLabel -> {"r(t)", "\[CurlyRho](t)"}, PlotLegends -> VectorTimesSimul, PlotLabel -> "0<="ro <=1"}
Listing A.2: Code for Figure 7.3

```
1 "initial_profiles",
2 \frho [\text{x}, \ro, \text{t}] := (1/2) *(1 - ((\text{t} + \ro (1 - 2 \text{x})) / \text{Sqrt}[\text{t}^2 + 2 \ro (1 - 2 \text{x}) \text{t} + \ro^2]));
3 \fr [\text{x}, \ro, \text{t}] := \text{Sqrt}[\text{t}^2 + 2 \ro (1 - 2 \text{x}) \text{t} + \ro^2];
4 \text{VectorTimeSimul} = \{0, 0.1, 0.25, 0.5, 0.8, 1\};
5 \text{plotsrt1} = \text{Table}[\text{ParametricPlot}[\{\fr \rho01[\ro], \ro, \text{t}\}, \{\ro, 0, 0.6\}, \text{PlotRange} \rightarrow \text{All}, \text{AxesOrigin} \rightarrow \{0, 0\},
6 \text{PlotLegends} \rightarrow "time" \"=\" \text{t}], \{\text{t}, \text{VectorTimeSimul}\}];
7 \text{plotsrt2} = \text{Table}[\text{ParametricPlot}[\{\fr \rho02[\ro], \ro, \text{t}\}, \{\ro, 0, 1\}, \text{PlotRange} \rightarrow \text{All}, \text{AxesOrigin} \rightarrow \{0, 0\},
8 \text{PlotLegends} \rightarrow "time" \"=\" \text{t}], \{\text{t}, \text{VectorTimeSimul}\}];
9 \text{plotsrt3} = \text{Table}[\text{ParametricPlot}[\{\fr \rho03[\ro], \ro, \text{t}\}, \{\ro, 0, 1\}, \text{PlotRange} \rightarrow \text{All}, \text{AxesOrigin} \rightarrow \{0, 0\},
10 \text{PlotLegends} \rightarrow "time" \"=\" \text{t}], \{\text{t}, \text{VectorTimeSimul}\}];

11 "plot_parametrico_r_hrho01",
12 \text{restylePlot2}[\text{plotsrt1}, \text{PlotRange} \rightarrow \text{All}, \text{AxesLabel} \rightarrow \{"r(\text{t})", "\\text{CurlyRho}(\text{t})"\}, \text{PlotLegends} \rightarrow \text{VectorTimeSimul}, \text{PlotLabel} \rightarrow "0<=\ro <=0.6"\}

13 "plot_parametrico_r_hrho02",
14 \text{restylePlot2}[\text{plotsrt2}, \text{PlotRange} \rightarrow \text{All}, \text{AxesLabel} \rightarrow \{"r(\text{t})", "\\text{CurlyRho}(\text{t})"\}, \text{PlotLegends} \rightarrow \text{VectorTimeSimul}, \text{PlotLabel} \rightarrow "0<=\ro <=1.0"\}

15 "plot_parametrico_r_hrho03",
16 \text{restylePlot2}[\text{plotsrt3}, \text{PlotRange} \rightarrow \text{All}, \text{AxesLabel} \rightarrow \{"r(\text{t})", "\\text{CurlyRho}(\text{t})"\}, \text{PlotLegends} \rightarrow \text{VectorTimeSimul}, \text{PlotLabel} \rightarrow "0<=\ro <=1"\]
```
Listing A.3: Code for Figure 7.4(a)

```
Listing A.4: Code for Figure 7.4(b)

```

Listing A.5: Code for Figure 7.5(a)

1 $\text{"viscosity\_and\_cong\_super"},$
2 restylePlot2[p_, op : OptionsPattern[ListLinePlot]] := ListLinePlot[
   Cases[Normal@p, Line[x_] :> x, Infinity], op, Options[p]];
3 F[v_?NumericQ] := NDSolve[v D[R[x],x] + R[x]^2 - R[x] == -0.5 && R[1] == 0, R[x], {x, 0, 1}];
5 VectorViscosities = Table[k, {k, 0.3, 1.5, 0.15}];
6 PlotCurrents = Table[Plot[RR[z, v], {z, 0, 1}, PlotRange -> {0, 1}, AxesLabel -> {x, \[CurlyRho]}, PlotLegends -> v, {v, VectorViscosities}]];
7 restylePlot2[PlotCurrents, PlotRange -> {0, 1}, AxesLabel -> {"x", "\[CurlyRho]"}, PlotLegends -> VectorViscosities, PlotLabel -> "j=0.5"]

Listing A.6: Code for Figure 7.5(b)

1 $\text{"viscosity\_and\_cong\_super\_zoom"},$
2 restylePlot2[p_, op : OptionsPattern[ListLinePlot]] := ListLinePlot[
   Cases[Normal@p, Line[x_] :> x, Infinity], op, Options[p]];
3 F[v_?NumericQ] := NDSolve[v D[R[x],x] + R[x]^2 - R[x] == -0.5 && R[1] == 0, R[x], {x, 0, 1}];
5 VectorViscositiesCritical = Table[k, {k, 0.22, 0.36, 0.02}];
6 PlotCurrents = Table[Plot[RR[z, v], {z, 0, 1}, PlotRange -> All, AxesLabel -> {x, \[CurlyRho]}, PlotLegends -> v, VectorViscositiesCritical}];
7 restylePlot2[PlotCurrents, PlotRange -> All, AxesLabel -> {"x", "\[CurlyRho]"}, PlotLegends -> VectorViscositiesCritical, PlotLabel -> "j=0.5"]
Listing A.7: Code for Figure 7.5(c)

```
"viscosity_and_cong_sub",
restylePlot2[p_, op : OptionsPattern[ListLinePlot]] := ListLinePlot[
  Cases[Normal@p, Line[x__] :> x, {Infinity}], op, Options[p]];
F[v_?NumericQ] := NDSolve[v D[R[x], x] + R[x]^2 - R[x] == -0.2 && R[1]
  == 0, R[x], {x, 0, 1}];
VectorViscosities = Table[k, {k, 0.02, 0.1, 0.02}];
PlotCurrents = Table[Plot[RR[z, v], {z, 0, 1}, PlotRange -> {0, 1},
  AxesLabel -> {x, \[Rho]}, PlotLegends -> v], {v, VectorViscosities}];
restylePlot2[PlotCurrents, PlotRange -> All, AxesLabel -> {"x", "\[CurlyRho]"}, PlotLegends -> VectorViscosities, PlotLabel -> "j=0.2"]
```
Listing A.8: Code for Figure 7.6

1. "Critical",
2. \[ F[v_\text{?NumericQ}, j_\text{?NumericQ}] := \text{NDSolve}\left[ v \frac{D[R[x], x]}{} + R[x]^2 - R[x] == j \\
    && R[1] == 0, R[x], \{x, 0, 1\}\right] ; \]
3. \[ RR[y_\text{?NumericQ}, v_\text{?NumericQ}, j_\text{?NumericQ}] := ((R[x] /. F[v, j]) /. x -> y)[[1]]; \]
4. \[ \text{Quiet}\left[ \text{Plot}\left[ -j /. \text{FindRoot}\left[ RR[0, v, j] == 1, \{j, -0.2\}\right], \{v, 0.01, 1.5\}, \right.\right. \]
    \[ \text{AxesLabel} \to \{\text{\text{Epsilon}}, j\} \right] \]
Listing A.9: Code for Figure 7.7

```
"large",

uval = NDSolve[D[u[x, t], t] - 0.05 D[u[x, t], {x, 2}] + D[u[x, t], x] 
(1 - 2 u[x, t]) == 0 && u[x, 0] == 10 x^2 (1 - x)^2 && 0.05 n! n
( n^u[n])[0, t] + u[0, t]^2 - u[0, t] == -0.2 (1 - E^(-10 t)) && u[1, t] == 0, u, {t, 0, 10}, {x, 0, 1}];

SLARGE = Plot3D[u[x, t] /. uval, {x, 0, 1}, {t, 0, 10}, PlotRange -> All, AxesLabel -> {x, t, \[Rho]}];

F[v_?NumericQ, j_?NumericQ] := NDSolve[v*D[R[x], x] + R[x]^2 - R[x] == j && R[1] == 0, R[x], {x, 0, 1}];


TAU = 50;

STAT = ListPlot3D[Flatten[Table[{j/TAU, k/TAU, RR[j/TAU, 0.05, 0.2]}, {j, 0, TAU}, {k, 0, 10*TAU}], 1], PlotRange -> All, AxesLabel -> {x, t, \[Rho]}, PlotStyle -> None, MeshStyle -> {Red, Blue}];

"small",

Show[SLARGE, STAT, PlotRange -> All, ViewPoint -> {-2.991241520671857^4, 1.4971638900187578^4, 0.51085658597365^4}]

"small",

uval = NDSolve[D[u[x, t], t] - 0.05 D[u[x, t], {x, 2}] + D[u[x, t], x] 
(1 - 2 u[x, t]) == 0 && u[x, 0] == 1 x^2 (1 - x)^2 && 0.05 n! n
( n^u[n])[0, t] + u[0, t]^2 - u[0, t] == -0.2 (1 - E^(-10 t)) && u[1, t] == 0, u, {t, 0, 10}, {x, 0, 1}];

SSMALL = Plot3D[u[x, t] /. uval, {x, 0, 1}, {t, 0, 10}, PlotRange -> All, AxesLabel -> {x, t, \[Rho]}]{*, MeshStyle -> {Red, Blue}*)];
```
Show[SSMALL, STAT, PlotRange -> All, ViewPoint -> {-2.5253927013653255, -1.9896980624420988, 1.0552219312565956}]}
Listing A.10: Code for Figure 10.1

```
1 "1 descape",
2 n = 50;
3 Clear[R];
4 T = 1; (* Final Time *)
5 U = Table[ToExpression[StringJoin["u", ToString[i]]], {i, 0, n}];
6 R = Table[ToExpression[StringJoin["r", ToString[i]]], {i, 0, n}];
7 UGUESS = Table[1/2 - Abs[1/2 - i/n], {i, 1, n - 1}];
8 rf = 0.9*Sin[3 Pi #]^2 &;
9 R0 = N@Table[rf[j/n], {j, 0, n}]; (* Initial Condition for R *)
10 UB = U /. u0 -> 0 / ToExpression[StringJoin["u", ToString[n]]] -> 0; (* Dirichlet conditions *)
11 RB = R /. r0 -> 0 / ToExpression[StringJoin["r", ToString[n]]] -> 0;
12 LD = (# - RotateLeft[#])*n &;
13 RD = (# - RotateRight[#])*n &;
14 ULB = LD[UB];
15 URB = RD[UB];
16 ham[ul_, ur_, s_] := 0.01 (ul + ur)*n + (Max[ul, 0]^2 + Max[ur, 0]^2)/2
17 - 1/((1 - s)^2);
18 ham2[ul_, ur_, s_] := (1 - s)^2 (Max[ul, 0]^2 + Max[ur, 0]^2)/2 + 0.01 (ul + ur)*n;
19 NumberVectorQ = VectorQ[#, NumberQ] &;
20 HJE = Map[# == 0 &, MapThread[ham, {ULB, URB, Rt}]][[2 ;; n]];  
21 UGS = UB /. FindRoot[HJE, Transpose[{UB[[2 ;; n]], UGUESS}]];  
22 UGUESS = UGS[[2 ;; n]];  
23 HJE = MapThread[ham2, {ULB, URB, RB}][[2 ;; n]];  
24 LHJE = Transpose@Grad[HJE, U[[2 ;; n]]];  
25 MKFC := Compile[#1, #2] &; (* COMPILED VERSION *)  
26 GG = MKFC[Flatten@{R[[2 ;; n]], U[[2 ;; n]]}, -LHJE.(R[[2 ;; n]])];  
28 GE2[A_?NumberVectorQ] := GG @@ Join[A, Ut[Join[{0}, A, {0}]]][[2 ;; n]];
```
Res = NDSolve[\[B'\[t\] == GE2[\[B\[t\]]] && \[B\[0\] == R0[[2 ;; n]]], \[B\, \{t, 0, T\}], Method -> {"BDF"}];

XXR = Table[j/n, \{j, 1, n - 1\}]; (* Grid to plot R *)

NPoints = 100;

"1escaperho",

ListPlot3D[Join @@ Table[Transpose[\{XXR, Table[T j/NPoints, \{k, 1, n - 1\}], \(\[B\[t\]/.Res[[1]]\)/.t -> T j/NPoints\}], \{j, 0, NPoints\}], PlotRange -> All, AxesLabel -> \{x, t, \[Rho]\}, ColorFunction -> "Rainbow"]

"1escapeu",

ListPlot3D[Join @@ Table[Transpose[\{XXR, Table[T j/NPoints, \{k, 1, n - 1\}], \(Ut[Join[\{0\}, \[B\[t\]/.Res[[1]]/-.t -> T j/NPoints, \{0\}]][[2 ;; n]]\}], \{j, 0, NPoints\}], PlotRange -> All, AxesLabel -> \{x, t, u\}, ColorFunction -> "Rainbow"]}
Listing A.11: Code for Figure 10.2

```
1 "flow",
2 n = 50;
3 T = 1; (* Final Time *)
4 U = Table[ToExpression[StringJoin["u", ToString[i]]], {i, 0, n}];
5 R = Table[ToExpression[StringJoin["r", ToString[i]]], {i, 0, n}];
6 rf = 0.4*Sin[3 Pi #]^2 &;
7 R0 = N@Table[rf[j/n], {j, 0, n}]; (* Initial Condition for R *)
8 UGUESS = Table[4 (1 - i/n), {i, 0, 1, n}]; (* initial guess for u *)
9 UB = U /. ToExpression[StringJoin["u", ToString[n]]] -> 0; (* Dirichlet conditions at x=1 *)
10 RB = R /. ToExpression[StringJoin["r", ToString[n]]] -> 0; (* Dirichlet conditions at x=1 *)
11 LD = (# - RotateRight[#])*n &;
12 RD = (# - RotateLeft[#])*n &;
13 ULB = LD[UB];
14 URB = RD[UB];
15 epsilon = 0.1; (* Laplacian coefficient - Viscosity *)
16 ham[ul_, ur_, s_] := epsilon (ul + ur)*n + (Max[ul, 0]^2 + Max[ur, 0]^2) /2 - 1/((1 - s)^2);
17 hamleftneummann[ul_, ur_, s_] := epsilon (2*ur)*n + (Max[ur, 0]^2)/2 - 1/((1 - s)^2);
18 ham2[ul_, ur_, s_] := (1 - s)^2 (Max[ul, 0]^2 + Max[ur, 0]^2)/2 + epsilon (ul + ur)*n;
19 hamleftneummann2[ul_, ur_, s_] := epsilon (2*ur)*n + (1 - s)^2 (Max[ur, 0]^2)/2;
20 NumberVectorQ = VectorQ[#, NumberQ] &;
21 Ut[Rt_?NumberVectorQ] := Module[{HJE, UGS},
   HJE = Map[# == 0 &, Prepend[MapThread[ham, {ULB, URB, Rt}][[2 ;; n]], hamleftneummann[ULB[[1]], URB[[1]], Rt[[1]]]]];
   UGS = UB /. FindRoot[HJE, Transpose[{UB[[1 ;; n]], UGUESS}]];
   UGUESS = UGS[[1 ;; n]];]
22 UGS;
```
HJE2 = Prepend[MapThread[ham2, {ULB, URB, RB}][[2 ;; n]], hamleftneumann2[ULB[[1]], URB[[1]], RB[[1]]]]; 
LHJE2 = Transpose@Grad[HJE2, U[[1 ;; n]]]; 
MKFC := Compile[#1, #2 &; (* COMPILED VERSION *) 
Source = Map[0. &, Range[n]]; 
Source[[1]] = 0.3 n; 
GG = MKFC[Flatten@{R[[1 ;; n]], U[[1 ;; n]]}, LHJE2. (R[[1 ;; n]]) + Source]; 
GE2[A_?NumberVectorQ] := GG @@ Join[A, Ut@Join[A, 0][[1 ;; n]]]; 
Res = NDSolve[B'[t] == GE2[B[t]] && B[0] == R0[[1 ;; n]], B, {t, 0, T}, Method -> {"BDF"}]; 
XXR = Table[j/n, {j, 0, n - 1}]; (* Grid to plot R *) 
NPoints = 100; 
	d"flowrho",
ListPlot3D[Join @@ Table[Transpose[{XXR, Table[T j/NPoints, {k, 0, n - 1}], (B[t] /. Res[[1]]) /. t -> T j/NPoints}], {j, 0, NPoints}], 
PlotRange -> All, AxesLabel -> {x, t, \[Rho]}, ColorFunction -> "Rainbow"] 
	d"flowu",
ListPlot3D[Join @@ Table[Transpose[{XXR, Table[T j/NPoints, {k, 1, n}], (Ut@Join[B[t] /. Res[[1]] /. t -> T j/NPoints, {0}][[1 ;; n]])}, {j, 0, NPoints}], 
PlotRange -> All, AxesLabel -> {x, t, u}, ColorFunction -> "Rainbow"]}
Listing A.12: Code for Figure 10.3

1 MyDir=NotebookDirectory [];
2 nb=NotebookOpen [MyDir~~" AuxiliaryFunctions.nb"];
3 NotebookEvaluate [nb];
4 NotebookClose [nb];
5 T=3;
6 n=80;
7 NPlots = 50;
8 times = Table [N[T/NPlots * i],{i,1,NPlots}];
9
10 U=Table [ToExpression ["u"~~ToString [i]],{i,1,n}]; (* the unknowns *)
11 M=Table [ToExpression ["m"~~ToString [i]],{i,1,n}]; (* the unknowns *)
12
13 U0=Table [0.3 Cos [2 Pi i/n]//N, {i,1,n}]//N; (* the initial condition for $u$ *)
14 M0=Table [1, {i,1,n}];
15
16 (* finite differences computed with the MONOTONICITY convention *)
17 UL=((U.RotateLeft [U]) *n);
18 UR=((U.RotateRight [U]) *n);
19 XX=Table [i/n, {i,1,n}]//N;
20
21 (* the monotone discretization of the Hamiltonian *)
22 (* standard w/o congestion *)
23 HJ[x, u1_, ur_, m_]:= (Max[u1,0]^2/2+Max[ur,0]^2/2)–Log[m]+0.01(ul+ur) n ;
24
25 (* the monotone discretization of the Fokker–Planck Hamiltonian *)
26 HJEU=MapThread [HJ, {XX,UL, UR, M}];
27 HJEUF=MapThread [HJ, {XX,UL, UR, M}];
28 MKFC=Compile[#1, #2]&; (* make compiled function operator *)
29 LHJEUF=Grad [HJEUF, U];
30 PDE=MKFC [Join [U,M], Join [HJEU, Transpose [LHJEUF].M]];
31 NumberListQ=ListQ[#]&&And@@NumberQ/@#&;
PDEF[Z_\_?NumberListQ] := PDE @@ Z;

Sol = NDSolve[Z'[t] == PDEF[Z[t]] && Z[0] == Join[U0, M0], {Z[t]}, {t, 0, T}];

PlotU = ListPlot3D[Flatten[Table[{XX[[i]], ts, (Z[t] /. Sol[[1]] /. t -> ts)[[i]]}, {i, 1, n}, {ts, times}], 1], AxesLabel -> StyleLabel3D["x", "t", "u"], ColorFunction -> Hue, LabelStyle -> Directive[FontFamily -> "Helvetica", Black, SizeNumbers]];

PlotM = ListPlot3D[Flatten[Table[{XX[[i]], ts, (Z[t] /. Sol[[1]] /. t -> ts)[[i + n]]}, {i, 1, n}, {ts, times}], 1], AxesLabel -> StyleLabel3D["x", "t", "\[Rho]"], ColorFunction -> Hue, LabelStyle -> Directive[FontFamily -> "Helvetica", Black, SizeNumbers]];
B  Published Papers and Under Preparation

Papers


Talks

- Some Numerical Aspects in Crowd Motion, in 55th Conference in Decision and Control - Las Vegas, United States - Dec. 2016;

- The Hughes Model and Numerical Aspects, in 7th European Congress of Mathematics - EMS - Berlin, Germany - July 2016;
• **Some aspects of finite-state mean-field games**, in *The 11th AIMS Conference on Dynamical Systems, Differential Equations and Applications* - Orlando, United States - July 2016;


**Poster Presentations**

• **Some Numerical Aspects on Crowd Motion - The Hughes Model**, in *The 11th AIMS Conference on Dynamical Systems, Differential Equations and Applications* - Orlando, United States - July 2016;

• **Some Numerical Aspects on Crowd Motion - The Hughes Model**, in *International Workshop in Calculus of Variations and Its Applications* - Lisbon, Portugal - Dec. 2015;

• **Finite-state mean-field games and applications**, in *International Conference on Stochastic Analysis And Related Topics* - Campinas, Brazil - Aug. 2014;

• **Socio-economic applications of finite-state mean-field games**, in *Spring School "Microscopic descriptions and mean-field equations in physics and social sciences”* - Bath, United Kingdom - May 2014.