# Efficient Simulation of the Outage Probability of Multihop Systems

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***Abstract*—In this paper, we present an efficient importance sampling estimator for the evaluation of the outage probability of multihop systems with amplify-and-forward channel state-information-assisted. The proposed estimator is endowed with the bounded relative error property. Simulation results show a significant reduction in terms of number of simulation runs compared to naive Monte Carlo.**

***Index Terms*—Importance sampling, multihop systems, amplify-and-forward, naive Monte Carlo.**

## I. Introduction

Multihop relaying [1] is a well-known technique to improve the coverage of communication networks. In this set-up, the signal is transmitted from the source to the destination by means of intermediate terminals. This can reduce the effect of fading or shadowing [1]-[2] and achieve high-data rate transmission [3]. The performance of communication systems using amplify-and-forward multihop relaying involves the harmonic mean of the instantaneous singal-to-noise ratio (SNR) of the hops. Only few papers were able to solve the problem exactly using some relatively complicated special functions. In [4], Hasna and Alouini derived the expressions of the probability density function (PDF) as well as the cumulative density function (CDF) of the harmonic mean of two independent and identically distributed (i.i.d) Gamma and F variates. Capitalizing on these results, they determined the outage and error performance of two-hops system with regenerative and non-regenerative relays over Nakagami-m fading channels. In [5], Tsiftsis *et al.* determined analytical expressions for the end-to-end outage probability of multihop free-space optical (FSO) wireless systems using amplify-and-forward (AF) or decode-and-forward (DF) relays over Gamma-Gamma turbulence channels. In [6], the authors studied the performance metrics, such as the outage probability, the outage capacity and the average bit error probability of amplify-and-forward multihop systems. They expressed these performance metrics in terms of the multivariable Meijer's G of Fox's H functions in the presence of generalized Gamma fadings. Most of the other works in the literature exploit the harmonic-geometric means inequality to determine a lower bound for the outage probability of multihop links (see [7], [8], and references therein).

Rare events are events with very small probabilities but their evaluation is of a paramount importance in many applications. For instance, we can encounter these events when evaluating the outage probability of wireless backhauling using FSO and millimeter wave. Standard numerical integration methods such as Gauss quadrature or sparse grids quadrature are not suitable for the estimation of rare event probabilities (see [9, Chap. 1, Sec. 1.1], [10][Chap. 4, Sec. 4.1], and [11]). In fact, numerical integration methods require the integrand to be smooth whereas rare event probabilities are often expressed using an indicator function. In addition to that, the rare event region can have a complicated geometry and thereby the application of these techniques can lead to significant errors in practice. Another reason for using simulation methods is that the numerical integration methods suffer from the curse of dimensionality, i.e. the speed of convergence of the method deteriorates rapidly if the dimension of the problem increases. For instance, trapezoidal rule with $n$ points in dimension $d$ has a speed of convergence of the order of $n^{-\frac{2}{d}}$, which becomes very slow when $d$ is large.

Interestingly enough, Monte Carlo (MC) method can be used to evaluate these outage probabilities. However, it is widely known that under the constraint of a limited number of simulation runs, naive MC estimator proves its inefficiency to estimate rare events probabilities [9]. The IS method [12] is one of the most used approaches in the evaluation of small probabilities. By introducing a clever change of probability measure, given a certain confidence interval, IS method reduces the number of required simulation runs. Involving the harmonic mean in its expression makes standard IS approaches fail when evaluating the outage probability of multihop systems. In fact, the existence of the moment generating function (MGF) is not guaranteed which makes IS based on exponential twisting not applicable for this kind of problem. Other conventional IS methods such as scaling or shifting may lead to numerical improvement but to prove that the proposed IS estimator verifies certain theoretical criteria is not a trivial task. Thereby, in this work, we use a dynamic IS approach based on [13] to study the outage probability of multihop systems with AF channel state-information-assisted. The reminder of this paper is organized as follows. We start by describing the system model in Section III. We then provide in Section II a brief description of the proposed approach in order to estimate the outage probability in our particular set-up. In Section IV, some selected numerical results are presented to show the significant improvement that the proposed approach offers compared to naive MC.

## II. System Model

In this work, we consider a non regenerative AF system. The expression of the end-to-end SNR of $N$-hop systems can be written as [7]

$$\gamma_{end} = \left( \sum_{i=1}^{N} \frac{1}{\gamma_i} \right)^{-1}. \quad (1)$$

For a given threshold $\gamma_{th}$, the outage probability of this system is given by

$$P(\gamma_{th}) = \mathbb{P}(\gamma_{end} \leq \gamma_{th}) = \mathbb{P}\left( \sum_{i=1}^{N} \frac{1}{\gamma_i} \geq \frac{1}{\gamma_{th}} \right). \quad (2)$$

Let $X_i = \frac{1}{\gamma_i}$, $i = 1, \ldots, N$ and $x_{th} = \frac{1}{\gamma_{th}}$. Then, we are interested in proposing an efficient IS estimator for the right tail of the sum of the i.i.d RVs $\{X_i\}_{i=1}^{N}$, i.e.

$$P(x_{th}) = \mathbb{P}\left( S_N = \sum_{i=1}^{N} X_i \geq x_{th} \right). \quad (3)$$

In the remainder of this paper, $\{\gamma_i\}_{i=1}^{N}$ are considered to follow one of the fading models in Table I where $\gamma(\cdot,\cdot)$ is the lower incomplete Gamma function [14, Eq. (8.350.1)], $Q_\mu(\cdot)$ is the generalized Marcum-Q function [15, Eq. (4)], $I_\nu(\cdot)$ is the $\nu$th order modified Bessel function of the first kind [14, Sec. (8.431)], and $Y_\mu(\cdot,\cdot)$ is the Yacoub's function defined in [15, Eq. (20)]. However, as we will show later on, the proposed approach is valid for any fading model that satisfies (9).

## III. Proposed approach

The outage probability in (3) can be re-written in the form $P = \mathbb{E}[\mathbb{1}_{(S_N \geq x_{th})}]$ where $\mathbb{1}_{(\cdot)}$ is the indicator function. In this case, the naive MC estimator is given by

$$\hat{P}_{MC} = \frac{1}{M} \sum_{k=1}^{M} \mathbb{1}_{(S_N(\omega_k) \geq x_{th})}, \quad (4)$$

where $M$ is the number of simulations runs of naive MC, and $\{S_N(\omega_k)\}_{k=1}^{M}$ are i.i.d. realizations of the RV $S_N$. The number of required samples should be greater than $100/P$ so that $\hat{P}_{MC}$ can estimate $P$ with $10\%$ relative error.

In order to reduce the number of simulation runs for a fixed accuracy requirement, we use IS approach in this work. To this end, IS introduces a set of biased PDFs $\{f_i^*(x)\}_{i=1}^{N}$ so that the outage probability is given by $P = \mathbb{E}^*[\mathbb{1}_{(S_N \geq x_{th})}\mathcal{L}(X_1, \ldots, X_N)]$ where $\mathbb{E}^*[\cdot]$ is the expected value w.r.t biased PDFs and the likelihood ratio is given by

$$\mathcal{L}(X_1, \ldots, X_N) = \prod_{i=1}^{N} \frac{f_i(x)}{f_i^*(x)}. \quad (5)$$

The IS estimator is thereby given by

$$\hat{P}_{IS} = \frac{1}{M^*} \sum_{k=1}^{M^*} \mathbb{1}_{(S_N(\omega_k) \geq x_{th})} \mathcal{L}(X_1(\omega_k), \ldots, X_N(\omega_k)), \quad (6)$$

where $M^*$ is the number of samples used to get the IS estimator, and for each realization, $\{X_i(\omega_k)\}_{i=1}^{N}$ are sampled independently according to its corresponding biased PDFs. It is worthy noting that the choice of the biased PDFs needs to be made with care. A good choice can lead to a significant reduction in terms of number of samples whereas a bad choice can result to worsen the quality of the naive MC estimator by increasing its variance.

Let $s_0 = 0$ and we define $s_{i-1} = X_{1,N} + \cdots + X_{i-1,N}$ where $\{X_{j,N}\}_{j=1}^{i-1}$ are samples from the biased PDFs $\{f_j^*(x)\}_{j=1}^{i-1}$. We consider the dynamic importance sampling described in [13] where the biased PDFs, for $1 \leq i < N$, are defined as

$$f_i^*(x) = \begin{cases} p_i f_X(x) + q_i \frac{f_X(x)\mathbb{1}_{(x > a(x_{th} - s_{i-1}))}}{\bar{F}_X(a(x_{th} - s_{i-1}))}, & \text{if } s_{i-1} \leq x_{th} \\ f_X(x), & \text{otherwise} \end{cases} \quad (7)$$

and for $i = N$, the biased PDF is given by

$$f_N^*(x) = \begin{cases} \frac{f_X(x)\mathbb{1}_{(x > x_{th} - s_{N-1})}}{\bar{F}_X(x_{th} - s_{N-1})}, & \text{if } s_{N-1} \leq x_{th} \\ f_X(x), & \text{otherwise} \end{cases} \quad (8)$$

where $a \in (0,1)$ and $\{(p_i, q_i)\}_{i=1}^{N-1}$ is a sequence of positive numbers such that $p_i + q_i = 1$.

**Proposition 1.** Let $X = \frac{1}{\gamma}$ be a random variable where $\gamma$ is one of the fading models presented in Table I. Then, $X$ has regularly varying tails, i.e. for some $\alpha > 0$, the complementary CDF of $X$, $\bar{F}_X(x) = \mathbb{P}(X \geq x)$, satisfies

$$\lim_{b \to \infty} \frac{\bar{F}_X(ab)}{\bar{F}_X(b)} = a^{-\alpha}, \ \forall a > 0. \quad (9)$$

*Proof.* See Appendix A. □

To sample from $F_X(\cdot | X_i > s)$, we can use the inverse transform method where the inverse CDF is given by the following proposition.

**Proposition 2.** Let $U$ be a sample from the continuous uniform distribution over the interval $(0,1)$, then $t = \frac{1}{F_\gamma^{-1}\left(F_\gamma\left(\frac{1}{s}\right)U\right)}$ is distributed according to $F_X(\cdot | X_i > s)$.

*Proof.* See Appendix B. □

**Remark 1.** If the inverse CDF of the RV $\gamma$ can not be determined analytically, Newton-Raphson method [16, Chap. 4] can be used for example to obtain numerically $F_\gamma^{-1}(\cdot)$. An alternative approach is to use the acceptance-rejection method [17, Sec. 2.2, 2.3]. It is known that, for regularly varying distributions, the distribution $f_X$ has the representation $f_X(x) = x^{-\alpha-1}L(x)$ [18] where $L$ is a slowly varying function. Thus, a potential candidate for the proposal density, when sampling from $F_X(\cdot | X_i > s)$ , is the shifted Pareto with density $g(x) = \alpha s^\alpha x^{-\alpha-1}, \ x \geq s$.

In the following corollary, we characterize the goodness of the proposed IS approach

**Corollary 1.** ([13], Corollary 3.2.2.) Let $p_i^* = \frac{(N-i-1)a^{-\frac{\alpha}{2}}+1}{(N-i)a^{-\frac{\alpha}{2}}+1}$, $q_i^* = 1 - p_i^*$, and $a \in (0,1)$. Then, the

TABLE I
ASYMPTOTIC PROPERTIES FOR DIFFERENT FADING CHANNELS.

| **Fading distribution** | **PDF** | **CDF** | **Asymptotic CDF** |
|---|---|---|---|
| Weibull $(\alpha, \beta > 0)$ | $\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)$ | $1-\exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)$ | $\left(\frac{x}{\beta}\right)^{\alpha}$ |
| Gamma $(\alpha, \beta > 0)$ | $\frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}\exp(-\beta x)$ | $\frac{1}{\Gamma(\alpha)}\gamma(\alpha, \beta x)$ | $\frac{1}{\Gamma(1+\alpha)}(\beta x)^{\alpha}$ |
| Non-central $\chi^2$ $(\alpha, \beta > 0)$ | $\frac{1}{2}\exp\left(-\frac{x+\beta}{2}\right)\left(\frac{x}{\beta}\right)^{\frac{\alpha-1}{2}}I_{\alpha-1}\left(\sqrt{\beta x}\right)$ | $1-Q_{\alpha}(\sqrt{\beta}, \sqrt{x})$ | $\frac{1}{\Gamma(1+\alpha)}\exp\left(-\frac{\beta}{2}\right)\left(\frac{x}{2}\right)^{\alpha}$ |
| Gamma-Gamma $(\alpha, \beta, \Omega > 0)$ | $\frac{2(\alpha\beta)^{\frac{\alpha+\beta}{2}}x^{\frac{\alpha+\beta}{2}-1}}{\Gamma(\alpha)\Gamma(\beta)\Omega^{\frac{\alpha+\beta}{2}}}K_{\alpha-\beta}\left(2\left(\frac{\alpha\beta}{\Omega}x\right)^{\frac{1}{2}}\right)$ | $\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\ G_{1,3}^{2,1}\left[\frac{\alpha\ \beta}{\Omega}x \middle\vert \begin{matrix} 1 \\ \alpha, \beta, 0 \end{matrix}\right]$ | $\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\beta+1)}\left(\frac{\alpha\beta}{\Omega}x\right)^{\beta}$, $\alpha > \beta, \alpha-\beta \notin \mathbb{N}$ |
| $\alpha-\mu$ $(\alpha, \mu, \Omega > 0)$ | $\frac{\alpha\mu^{\mu}x^{\alpha\mu-1}}{\Gamma(\mu)\Omega^{\mu}}\exp\left(-\frac{\mu}{\Omega}x^{\alpha}\right)$ | $\frac{1}{\Gamma(\mu)}\gamma\left(\mu, \frac{\mu}{\Omega}x^{\alpha}\right)$ | $\frac{\mu^{\mu-1}}{\Omega^{\mu}\Gamma(\mu)}x^{\alpha\mu}$ |
| $\kappa-\mu$ $(\kappa, \mu, \Omega > 0)$ | $\frac{2\mu(1+\kappa)^{\frac{\mu+1}{2}}x^{\mu}\exp\left(-\frac{\mu(1+\kappa)}{\Omega}x^2\right)}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)\Omega^{\frac{\mu+1}{2}}}I_{\mu-1}\left(2\mu\sqrt{\frac{\kappa(1+\kappa)}{\Omega}}x\right)$ | $1-Q_{\mu}\left(2\sqrt{2\kappa\mu}, \sqrt{\frac{2\mu(1+\kappa)}{\Omega}x^2}\right)$ | $\frac{\exp(-\kappa\mu)}{\Gamma(\mu+1)}\left(\frac{\mu(1+\kappa)}{\Omega}x^2\right)^{\mu}$ |
| $\eta-\mu$ $(\eta, \mu, \Omega > 0)$ | $\frac{4\sqrt{\pi}\mu^{\mu+\frac{1}{2}}h^{\mu}x^{2\mu}}{\Gamma(\mu)\Omega^{\mu+\frac{1}{2}}H^{\mu-\frac{1}{2}}}\exp\left(-\frac{2\mu h}{\Omega}x^2\right)I_{\mu-\frac{1}{2}}\left(\frac{2\mu H}{\Omega}x^2\right)$ | $1-Y_{\mu}\left(\frac{H}{h}, \sqrt{\frac{2h\mu}{\Omega}x^2}\right)$ | $\frac{\sqrt{\pi}(h^2-H^2)^{\mu}\mu^{2\mu-1}}{\Gamma(\mu)\Gamma(\mu+\frac{1}{2})\Omega^{2\mu}}x^{4\mu}$ |

IS sampling estimator defined in (6) where the biased PDFs are given in (7) and (8) is nearly asymptotically optimal, i.e.

$$\forall\epsilon > 0, \lim_{x_{th}\to\infty}\frac{\mathbb{E}^*\left[\mathbb{1}_{(S_N\geq x_{th})}\mathcal{L}^2(X_1, \ldots, X_N)\right]}{P^2} \leq (1+\epsilon)N^2. \tag{10}$$

## IV. NUMERICAL SIMULATIONS

In this section, we show some selected numerical simulations when the fading distribution is either Weibull or Gamma. In table II, $X$ is sampled according to $F(\cdot|X_i > s)$ provided that $u$ is sampled from the uniform distribution $\mathcal{U}(0, 1)$.

TABLE II
INVERSE TRANSFORM SAMPLING FOR WEIBULL AND GAMMA FADINGS.

| **Fading distribution** | **Inverse CDF** |
|---|---|
| Weibull | $X = \frac{1}{\beta}\left[\log\left(\frac{1}{1-(1-\exp(-(\beta s)^{-\alpha}))u}\right)\right]^{-\frac{1}{\alpha}}$ |
| Gamma | $X = \frac{1}{\beta\gamma^{-1}\left(\alpha, \gamma\left(\alpha, \frac{1}{\beta s}\right)u\right)}$ |

where $\gamma^{-1}(\cdot,\cdot)$ is the inverse of the lower incomplete Gamma function.

When estimating small probabilities, an important metric to measure the performance of the proposed estimator is the relative error of the estimator. The relative error of naive MC is given by

$$\varepsilon = \frac{C_\eta}{P}\sqrt{\frac{P(1-P)}{M}}, \tag{11}$$

where $C_\eta$ is a constant that corresponds to a $(1-\eta)\%$ confidence interval. For instance, if $\eta = 5\%$ then, $C_\eta = 1.96$. The relative error of IS estimator is defined as

$$\varepsilon^* = \frac{C_\eta}{P}\sqrt{\frac{\mathbb{V}^*\left[\mathbb{1}_{(S_N\geq x_{th})}\mathcal{L}(X_1, \ldots, X_N)\right]}{M^*}}, \tag{12}$$

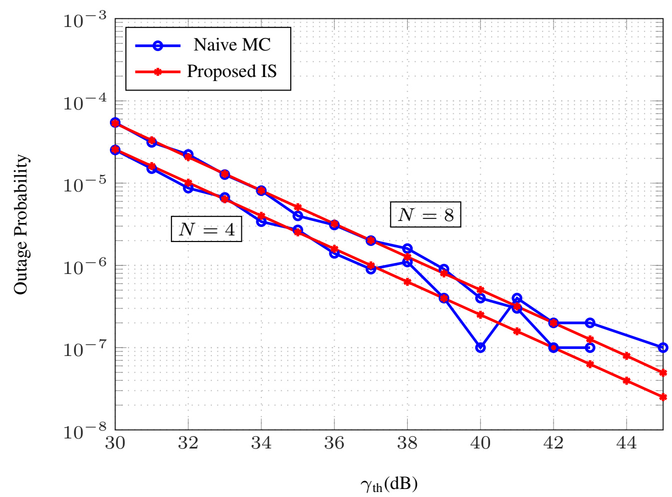


Fig. 1. Outage probability of $N$-hop system over i.i.d Weibull fading channels with $E_s/N_0 = 10$ dB, $\alpha = 2$ , and $\beta = 4$. Number of samples $M = 10^7$ and $M^* = 10^4$.

where $\mathbb{V}^*[\cdot]$ is the variance w.r.t the new probability measure. For a fixed accuracy requirement $\varepsilon = \varepsilon^* = \varepsilon_0$, the required number of simulation runs of both naive MC and IS is

$$M = P(1-P)\left(\frac{C_\eta}{P\varepsilon_0}\right)^2, \tag{13}$$

$$M^* = \mathbb{V}^*\left[\mathbb{1}_{(S_N\geq x_{th})}\mathcal{L}(X_1, \ldots, X_N)\right]\left(\frac{C_\eta}{P\varepsilon_0}\right)^2. \tag{14}$$

In Fig. 1 (repectively Fig. 2), we plot the outage probability of $N$-hop system over i.i.d Weibull (respectively Gamma) fading channels against the threshold $\gamma_{th}$ using naive MC (blue curve) with $M = 10^7$ and the proposed IS (red curve) with $M^* = 10^4$. For the Weibull fading parameters, we chose $\alpha = 2$ and $\beta = 4$ and we consider two different number of hops $N \in \{4, 8\}$. As for the Gamma fading, we chose $\alpha = 1.5$ and $\beta = 3$ for $N \in \{3, 6\}$. In this section, we assume that the length of each hop is the same, that is the total distance between the transmitter and the receiver depends on

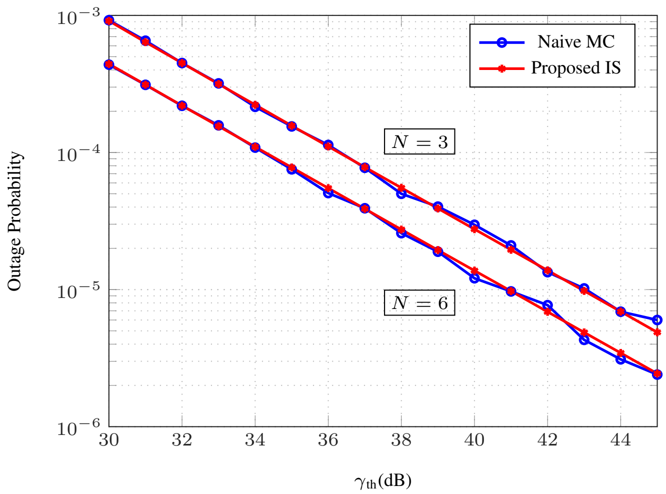


Fig. 2. Outage probability of $N$-hop system over i.i.d Gamma fading channels with $E_s/N_0 = 10$ dB, $\alpha = 1.5$ , and $\beta = 3$. Number of samples $M = 10^7$ and $M^* = 10^4$.

the number of hops in the system. We can clearly see that, unlike naive MC estimator who seems to be inaccurate as the outage probability becomes smaller, the proposed IS estimator provides an accurate estimation of the outage probability despite of using $10^3$ less samples than naive MC.

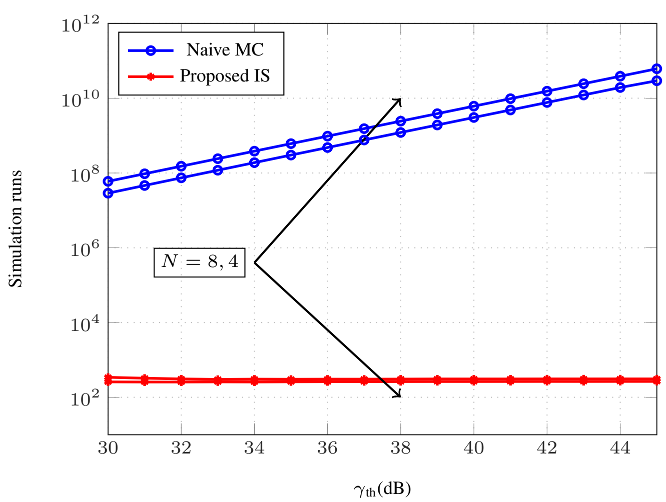


Fig. 3. Number of required simulation runs for 5% relative error for $N$-hop system over i.i.d Weibull fading channels with $E_s/N_0 = 10$ dB, $\alpha = 2$, and $\beta = 4$.

To have a clearer look at the performance of the IS estimator, we turn our attention to Fig. 3 where we plot the number of required simulation runs $M$ and $M^*$ of naive MC and IS, respectively, for the Weibull case. We fix the accuracy requirement to $\varepsilon_0 = 5\%$ and we take $C_\eta = 1.96$. As a consequence of the bounded relative error, we can see that the number of samples needed by IS remains almost constant, contrary to naive MC where we can see that the number of samples tends to grow rapidly. The gain in term of number of samples can be determined by taking the ratio $G = M/M^*$. We can see clearly that the use of the proposed IS results in a significant improvement compared to naive MC. For instance, if we take $N = 4$ and we consider $\gamma_{th} = 34$ dB, then using IS results in a reduction of terms of number of samples of the order of $10^6$. Similar conclusions can be drawn for the Gamma fading scenario.

## V. Conclusion

In this work, we presented an efficient IS estimator for the estimation of the outage probability of a certain class of fading channels. Being endowed with the bounded relative error property, the proposed approach results in a significant gain in terms of number of simulation runs compared to naive MC method.

## Appendix A
## Proof of Proposition 1

*Proof.* We start by writing the asymptotic expression of the CDF of the fading given in Table I in the following compact form

$$F_\gamma(x) \underset{x\to 0}{\sim} Cx^\alpha. \tag{A.1}$$

where $C$ and $\alpha$ are both non-negative constants.
We then express the CDF of $X$ as function of the CDF of $\gamma$

$$F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}\left(\gamma \ge \frac{1}{x}\right) = 1 - F_\gamma\left(\frac{1}{x}\right). \tag{A.2}$$

Therefore, $\bar{F}_X(x) = F_\gamma(\frac{1}{x})$.
Using this expression, we can write

$$\lim_{b\to\infty} \frac{\bar{F}_X(ab)}{\bar{F}_X(b)} = \lim_{b\to\infty} \frac{F_\gamma(\frac{1}{ab})}{F_\gamma(\frac{1}{b})} = \lim_{\delta\to 0} \frac{F_\gamma(\frac{\delta}{a})}{F_\gamma(\delta)} = a^{-\alpha}. \tag{A.3}$$

□

## Appendix B
## Proof of Proposition 2

Let $V$ be a sample from the continuous uniform distribution over the interval $(0, 1)$. We aim to find $t > s$ that satisfies

$$V = \int_0^t \frac{f_X(x)}{\bar{F}_X(s)} \mathbb{1}_{(x\ge s)} dx. \tag{B.1}$$

From this equation, we can write

$$\bar{F}_X(s)V = \int_s^t f_X(x)dx = F_X(t) - F_X(s). \tag{B.2}$$

Thus, the expression of $t$ is given by

$$t = F_X^{-1}\left(F_X(s) + (1 - F_X(s))V\right). \tag{B.3}$$

Sine $F_X(z) = 1 - F_\gamma(\frac{1}{z})$, we have

$$t = \frac{1}{F_\gamma^{-1}\left((1 - F_X(s))(1 - V)\right)}. \tag{B.4}$$

Since $1 - V = U$ can also be seen as a sample from the continuous uniform distribution over the interval $(0, 1)$ and $F_X(z) = 1 - F_\gamma(\frac{1}{z})$, we get

$$t = \frac{1}{F_\gamma^{-1}\left(F_\gamma(\frac{1}{s})U\right)}. \tag{B.5}$$

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