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Fast Fourier Transform Pricing Method for Exponential Lévy Processes

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Abstract

We describe a set of partial-integro-differential equations (PIDE) whose solutions represent the prices of european options when the underlying asset is driven by an exponential Lévy process. Exploiting the Lévy -Khintchine formula, we give a Fourier based method for solving this class of PIDEs. We present a novel L_∞ error bound for solving a range of PIDEs in asset pricing and use this bound to set parameters for numerical methods.

Introduction

Lévy processes form a rich field within mathematical finance. They provide a convenient generalisation of stochastic differential equations (SDEs) into the realm where the dynamics of the system under study can be discontinuous. In the realm of derivatives pricing, probably the first and best known of such examples is the model presented by Merton [10] as a generalisation to the Black-Scholes model [3]. More recently, we have seen more complex models allowing for more general dynamics of the asset price. Examples of such models include the Kou model [7], the Normal Inverse Gaussian model [2, 11], the Variance Gamma model [9], and the Carr-Geman-Madan-Yor (CGMY) [4] models.

Prices of derivatives whose underlying asset is driven by a Lévy process are given by Partial Integro-Differential Equations (PIDE) [6] that generalise the parabolic Kolmogorov-Backward equation of Black-Scholes type (cf. [3] for the seminal work) by incorporating a non-local integral term to account for the discontinuities in the asset price.

The Lévy -Khintchine representation formula (cf. eg. [12]) gives an exact and explicit expression for the characteristic function of the process. This allows computing the exact expression for the Fourier transform of the solution of the relevant PIDEs. [8] Using the inverse Fast Fourier Transform (iFFT) one may efficiently solve for the European option prices for multiple asset prices simultaneously. Furthermore, in the case of European call options, one may use the duality property presented by Dupire [5] and compute the iFFT efficiently for a wide range of strike prices at the computational cost $\mathcal{O}(N\log N)$ where N is the number of mesh points for evaluation.

Our novel contribution is a strict error bound for the discrete L_∞ error in the evaluation of the option price. We present this bound, illustrate its usage in setting the required parameters of the numerical implementation of a FFT pricing algorithm. Furthermore, we present numerical examples of this bound in the Merton framework where an explicit series expansion for the option price exists and provides a reference value.

Problem setting

We consider the logarithmic asset price $X=(X_t)\in\mathbb{R}$ to be a Lévy process with characteristic triple (γ,σ^2,ν). Given a parameter $c\in\mathbb{R}_+$ we assume the Lévy measure to be finite below 1 and to have finite variance above it

$$\int_{\mathbb{R}-\{0\}}\min\left(y^2,1\right)\nu\left(dy\right)<\infty.$$

We also assume a finite moments of the jump distribution of order c:

$$\int_{\mathbb{R}-\{0\}}e^{cy}\nu\left(dy\right)<\infty.$$

Assuming the risk-neutral dynamics, our aim is to price a derivative contract with *payoff* G depending on the terminal asset price at time T $S_T=S_0e^{X_T}$:

$$\Pi\left(s,t\right)=\underbrace{e^{-r(T-t)}}_{\equiv e^{-r\tau}}\mathrm{E}\left(G\left(S_T\right)|S_t=s\right).$$

The infinitesimal generator of the Lévy process X_t is given by the Lévy -Khinchine formula [1]

$$\begin{aligned}\mathcal{L}^Xf(x)&\equiv\lim_{h\to0}\frac{\mathrm{E}\left(f(X_{t+h})|X_t=x\right)-f(x)}{h}\\&=\gamma f'(x)+\frac{1}{2}\sigma^2f''(x)+\int_{\mathbb{R}-\{0\}}\left(f(x+y)-f(x)-y1_{|y|\le c}f'(x)\right)\nu(dy).\end{aligned}$$

We assume the dynamics of the asset price to be risk-neutral with constant short rate r, which fixes the drift term γ of the Lévy process:

$$\gamma=r-\frac{1}{2}\sigma^2-\int_{\mathbb{R}-\{0\}}f'\left(x\right)\left(e^y-1-y1_{|y|\le c}\right)\nu(dy)$$

and, the infinitesimal generator of X may be rewritten as

$$\begin{aligned}\mathcal{L}^Xf(x)=&\left(r-\frac{\sigma^2}{2}\right)f'(x)\\&+\frac{\sigma^2}{2}f''(x)+\int_{\mathbb{R}-\{0\}}\left(f(x+y)-f(x)-(e^y-1)f'(x)\right)\nu(dy).\end{aligned}$$

Now, take f to be defined as

$$f\left(\tau,x\right)\equiv\mathrm{E}\left(g\left(X_T\right)|X_t=x\right),$$

Then f solves the following PIDE:

$$\begin{cases}\partial_\tau f(\tau,x)&=\mathcal{L}^Xf(\tau,x)\\f(0,x)&=g(x),\qquad(\tau,x)\in[0,T]\times\mathbb{R}.\end{cases}$$

FFT method and damping

In order to accommodate for the cases, such as call options, in which $g\notin L_2\left(\mathbb{R}\right)$ we introduce the following, *damped* version of the equation: $f_\alpha(\tau,x)=e^{-\alpha x}f(\tau,x)$; we see that $\partial_\tau f_\alpha=e^{-\alpha x}\mathcal{L}^Xf(\tau,x)$. Consider the following Fourier transform: $\hat{f}(\omega)=\int_{\mathbb{R}}e^{i\omega x}f(x)dx$. Applying this transformation to f_α we get $\hat{f}_\alpha(\omega)=\hat{f}(\omega+i\alpha)$. Observe also that the Fourier transform applied to $\mathcal{L}^Xf(\tau,x)$ gives $\Psi(-i\omega)\hat{f}(\tau,\omega)$, being Ψ the characteristic exponent of the process X, which satisfies $\mathrm{E}\left(e^{zX_t}\right)=e^{t\Psi(z)}$. Explicitly,

$$\Psi\left(z\right)=\left(r-\frac{\sigma^2}{2}\right)z+\frac{\sigma^2}{2}z^2+\int_{\mathbb{R}}\left(e^{zy}-1-(e^y-1)z\right)\nu(dy).$$

From the previous considerations it can be concluded that

$$\partial_\tau\hat{f}_\alpha=\Psi(\alpha-i\omega)\hat{f}(\omega-i\alpha).$$

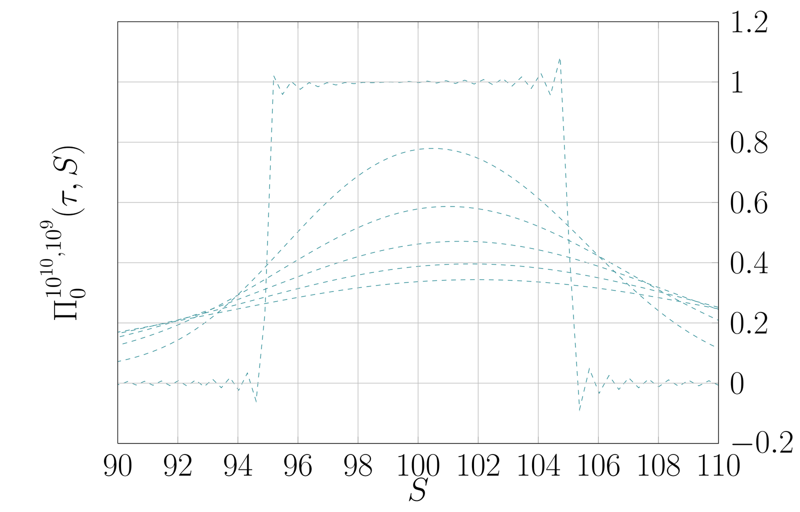
Now $\hat{f}(\omega-i\alpha)=\hat{f}_\alpha(\omega)$ so $\hat{f}_\alpha$ satisfies the following ODE

$$\begin{cases}\frac{\partial_\tau\hat{f}_\alpha(\tau,\omega)}{\hat{f}_\alpha(\tau,\omega)}&=\Psi(\alpha-i\omega)\\\hat{f}_\alpha(0,\omega)&=\hat{g}_\alpha(\omega),\end{cases}$$

Which yields the exact solution for the fourier transform of the value function

$$\hat{f}_\alpha\left(\tau,\omega\right)=\exp\left(\tau\Psi\left(\alpha-i\omega\right)\right)\hat{g}_\alpha\left(\omega\right).$$

Figure 1: *Smoothing of the value function* $\Pi\left(\tau,S\right)$ *due to the dissipative nature of the PIDE for a range of values for time-to-maturity* $\tau\in\{0,0.1,0.2,0.3,0.4,0.5\}$. *Parameters of the model* $\sigma=\sigma_{\text{jump}}=0.1$, $r=r_{\text{jump}}=0.05$, $X_0=100$, $G=\mathbf{1}_{[95,105]}$, $\lambda=2.0$.



The value function $f\left(\tau,x\right)$ at can be then obtained with the inverse transform.

$$f_\alpha(\tau,x)=\mathcal{F}^{-1}\left[\hat{f}_\alpha\right](\tau,x)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-i\omega x}\hat{f}_\alpha(\tau,\omega)d\omega$$

As it is typically not possible to compute the inverse Fourier transform analytically, we substitute it by a numerical approximation, in which we truncate and discretise the integration domain and use the trapezoidal rule to compute an approximation of the integral for $f_\alpha(\tau,x)$. Let $N=2^q$ for an integer q. Now, in order to make use of the available FFT libraries, discretise the domain with

$$\omega_m=\frac{2\omega_{\max}m}{n}-\omega_{\max},\quad x_l=\frac{\pi l}{\omega_{\max}},$$

and similarly for the negative values of x to get

$$f_\alpha\left(\tau,x_l\right)\approx f_\alpha^{N,\omega_{\max}}\left(\tau,x_l\right)=e^{ix_l\omega_{\max}}\sum_{m=0}^{N-1}\hat{f}_\alpha\left(\omega_m\right)e^{-i\frac{2\pi ml}{N}}$$

for $l=0,1,2,...,N-1$. Finally, having computed an approximation for the relevant integral, we can recover the full value function of the option by removing the damping and adding the appropriate discount factor:

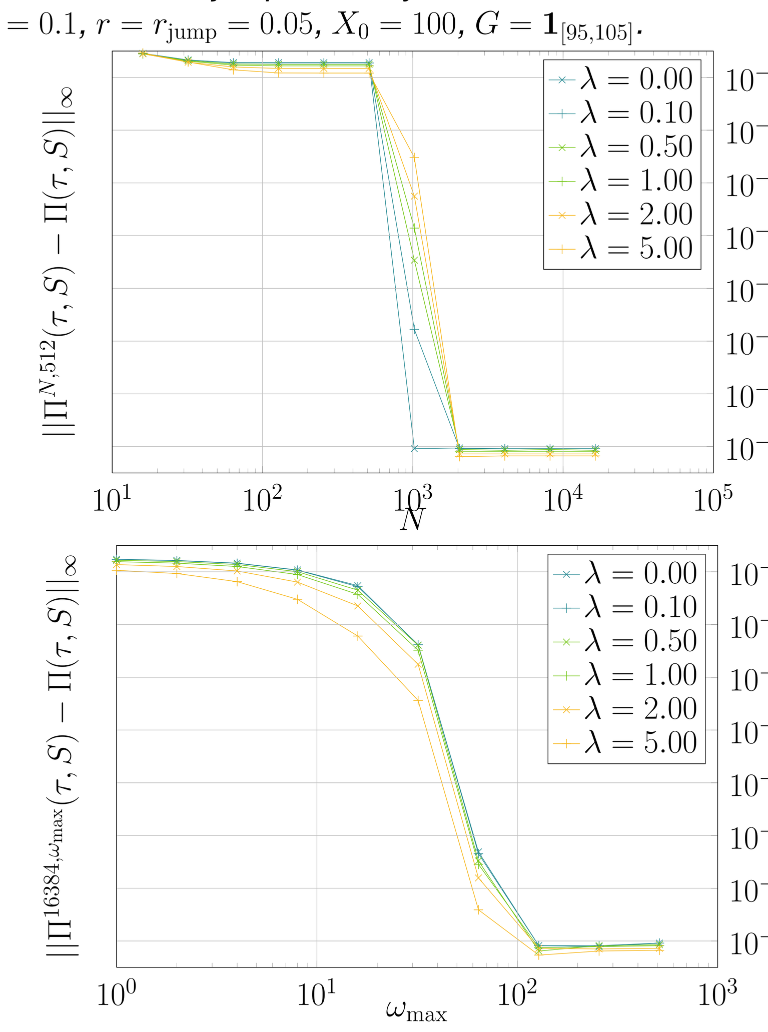
$$\Pi_\alpha^{N,\omega_{\max}}\left(t,S\right)=e^{-r(T-t)}e^{\alpha\ln(S/S_0)}f_\alpha^{N,\omega_{\max}}\left(T-t,\ln\left(S/S_0\right)\right).$$

The convergence of the inverse transform is illustrated as an example in figure 2 for the Merton model characterised by

$$\sigma\left(t,x\right)=\sigma,\quad\nu\left(y\right)=\lambda\left(2\pi\sigma_{\text{jump}}\right)^{-\frac{1}{2}}e^{\frac{\left(y-r_{\text{jump}}\right)^2}{2\sigma_{\text{jump}}^2}}.$$

For a reference value, we have an explicit series expansion for the option prices, as derived by Merton in [10].

Figure 2: *Numerical* L_∞*-error of the computation of binary option in the Merton Model. The absolute value between the truncated series expansion for different values of the jump intensity. Parameters of the model* $\tau=1.0$, $\sigma=\sigma_{\text{jump}}=0.1$, $r=r_{\text{jump}}=0.05$, $X_0=100$, $G=\mathbf{1}_{[95,105]}$.



Error estimation

In estimating the numerical error, we use theorem 6.1 in [13] to establish the spectral convergence of the numerical scheme. We decompose the error into components corresponding to the truncation and discretisation errors:

$$\begin{aligned}&\left|f_\alpha^{N,\omega_{\max}}\left(\tau,x\right)-f_\alpha\left(\tau,x\right)\right|\\\le&\left|(2\pi)^{-1}\int_{\mathbb{R}}\mathbf{1}_{\omega\ge\omega_{\max}}\left(\omega\right)e^{-i\omega x}\hat{f}_\alpha\left(\tau,\omega\right)d\omega\right|\\&+\left|(2\pi)^{-1}\sum_{m=0}^{n-1}\Delta\omega e^{-i\omega_mx}\hat{f}_\alpha\left(\tau,\omega_m\right)+\int_{|\omega|<\omega_{\max}}e^{-i\omega x}\hat{f}_\alpha\left(\tau,\omega\right)d\omega\right|=\mathcal{E}_{\mathcal{F}}+\mathcal{E}_{\mathcal{Q}}.\end{aligned}$$

We estimate the two error components separately with appropriate estimators $\overline{\mathcal{E}_{\mathcal{F}}}$ and $\overline{\mathcal{E}_{\mathcal{Q}}}$. For the boundedness of the error we assume certain regularity assumptions. Considering $h\colon\mathbb{C}\to\mathbb{C}$ such that $h(z)=e^{-izx}\hat{f}_\alpha(\tau,z)$ we need to find a strip $A=\{z\in\mathbb{C}\colon im(z)<a\}$, and a constant M, such that

H1: h is analytic in A

H2: $h(z)\to0$ uniformly when $|z|\to\infty$ in A;

H3: $\int_{\mathbb{R}}|h(\omega+ib)|\,d\omega\le M$ for all $b\in(-a,a)$

to conclude that

$$\mathcal{E}_{\mathcal{F}}\le\frac{2M}{e^{2\pi a/\Delta\omega}-1}.$$

Assuming $\hat{g}_\alpha$ to be analytic in A and a jump measure ν such that

$$z\mapsto\int_{\mathbb{R}}\left(e^{y(\alpha-iz)}-1-\left(e^y-1\right)\left(\alpha-iz\right)\right)\nu(dy)$$

is analytic in A, we have H1 guaranteed.

For H2, assume that $|\hat{g}_\alpha(z)|\le c_1(\alpha)$ for $z\in A$, we have that

$$|h(z)|\le e^ae^{\tau c(\alpha,a)}c_1\left(\alpha\right)$$

with

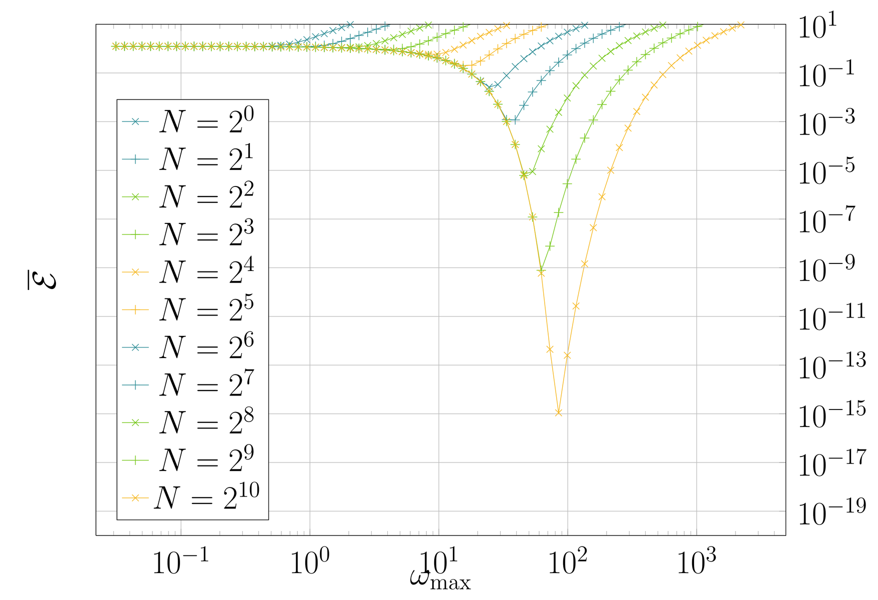
$$\begin{aligned}c\left(a,\alpha\right)=&\alpha\left(r-\frac{\sigma^2}{2}\right)+a\left|r-\frac{\sigma^2}{2}\right|+\frac{\sigma^2}{2}(\alpha+a)^2\\&+\int_{\mathbb{R}\setminus\{0\}}e^{(\alpha+\mathrm{Im}[z])y}-1-\left(\alpha+\mathrm{Im}\left[z\right]\right)\left(e^y-1\right)\nu(dy)\end{aligned}$$

Setting the M to guarantee H3:

$$M=e^ae^{\tau c(\alpha,a)}c_1(\alpha)\frac{\sqrt{2\pi}}{\sigma\sqrt{\tau}}.$$

Giving us a bound for the quadrature error.

Figure 3: *Error estimate for the binary option in the Merton framework.* $\tau=1.0$, $\sigma=\sigma_{\text{jump}}=0.1$, $r=r_{\text{jump}}=0.05$, $S_0=100$, $G=\mathbf{1}_{[95,105]}$, $\lambda=2.0$.



For the Truncation error, we have

$$\mathcal{E}_{\mathcal{F}}\le\frac{1}{\pi}\int_{\omega_{\max}}^\infty\left|\mathrm{Re}\left[e^{-i\omega x}\hat{f}_\alpha(\tau,\omega)\right]\right|d\omega$$

We exploit $|\hat{g}_\alpha\left(\omega\right)|\le c_1(\alpha)$ and arrive after simplifications at

$$\mathcal{E}_{\mathcal{F}}\le c_1(\alpha)e^{\tau c_2(\alpha)}\int_{\omega_{\max}}^\infty e^{-\tau\frac{\sigma^2}{2}\omega^2}d\omega=\overline{\mathcal{E}_{\mathcal{F}}},$$

with

$$c_2(\alpha)=\alpha\left(r-\frac{\sigma^2}{2}\right)+\frac{\sigma^2}{2}\alpha^2+\int_{\mathbb{R}\setminus\{0\}}\left(e^{\alpha y}-1-\alpha\left(e^y-1\right)\right)\nu(dy).$$

Combining the truncation and discretisation errors we arrive at the total error bound

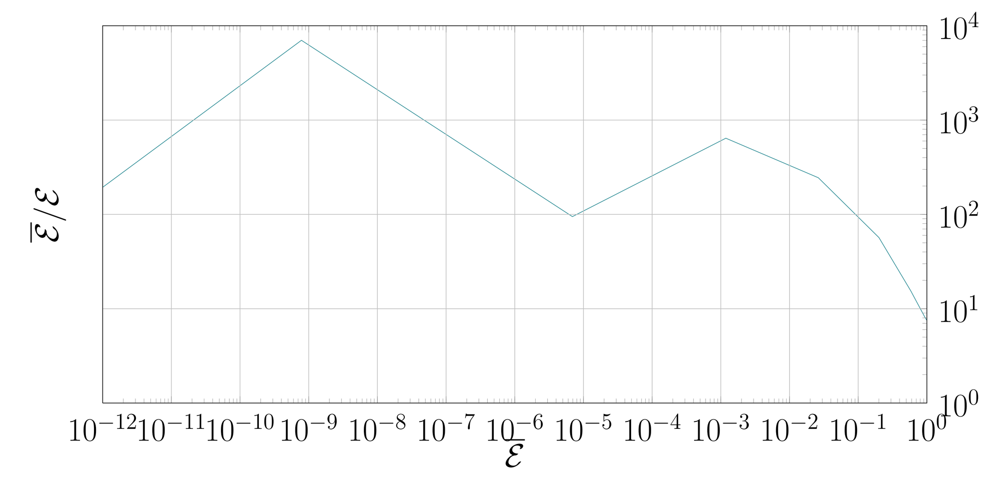
$$\overline{\mathcal{E}}=\overline{\mathcal{E}_{\mathcal{F}}}+\overline{\mathcal{E}_{\mathcal{Q}}}=\frac{2M}{e^{2\pi a/\Delta\omega}-1}+c_1(\alpha)e^{\tau c_2(\alpha)}\int_{\omega_{\max}}^\infty e^{-\tau\frac{\sigma^2}{2}\omega^2}d\omega.$$

In figure 3 we demonstrate numerically this bound for the Merton model. For evaluation of the option price up to predetermined error tolerance, we iteratively select $N_0=2$. As N dictates a computational burden, we select the numerical parameters by the following two-step procedure:

1. Minimise $\overline{\mathcal{E}}$ with respect to $\omega_{\max}$ while keeping N.
2. If $\overline{\mathcal{E}}$ exceeds the pre-defined error tolerance, increase N using $N_l=2^{l+1}$.

The tightness of the numerical bound is verified against the benchmark for the Merton model in the case illustrated in 3 is presented in 4.

Figure 4: *Efficiency of the estimator for various estimated error tolerances for the case represented in figure 3*



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Acknowledgements

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