A new smoothing technique for European basket options

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Abstract

In contrast to an option which is usually based on the price of a single asset, a basket option is based on a collection of several assets. The price of European basket options in a Black-Scholes model can be calculated in certain cases by the integral

$$C_{\mathcal{B}}(X_1,\dots,X_d) = E\left[\left(\sum_{i=1}^d w_i e^{X_i} - K\right)^+\right],$$

where $X \sim \mathcal{N}(0,\Sigma)$ with a covariance matrix Σ. The case $d=1$ refers to a standard European option. In this case, the option price $C_{\mathcal{B}}$ depends only on the log-normally distributed random variable e^{X_1} and can be calculated analytically by the famous Black-Scholes formula. Nevertheless, for $d>1$ the Black-Scholes formula cannot be applied immediately since the weighted sum of correlated log-normally distributed random variables is not log-normally distributed anymore. Hence, we have to compute an integral over the integration domain $\mathbb{R}^d$ for an integrand with a kink. We provide a simple smoothing method which produces an analytic integrand and is able to reduce the dimensionality of the integration problem by 1. Moreover, this smoothing does not introduce any approximation error. In particular, we transform the d-dimensional random variable in such a way that it is possible to apply the Black-Scholes formula with respect to a single coordinate. The resulting integration problem over $\mathbb{R}^{d-1}$ of an analytic function is then solved by an adaptive sparse grid approach. This leads, at least in considerably high dimensions, to better convergence results compared to those of standard Monte Carlo or quasi-Monte Carlo quadratures. In addition, as the numerical results demonstrate, the smoothing technique can also be used to improve the constant in the convergence of the quasi Monte-Carlo method.

1. Problem formulation

We consider $d \in \mathbb{N}$ assets with prices $S_t = \left(S_t^1,\dots,S_t^d\right)$, $t>0$. In a Black-Scholes model with risk-neutral dynamics, these prices are given by the system

(1) $$dS_t^i = \sigma_i S_t^i dW_t^i, \quad i=1,\dots,d,$$

for volatilities $\sigma_i > 0$, $i=1,\dots,d$, driven by a correlated d-dimensional Brownian motion W with

$$d\left\langle W^i, W^j\right\rangle_t = \rho_{i,j}dt, \quad i,j=1,\dots,d.$$

The system (1) has the explicit solution

(2) $$S_t^i = S_0^i \exp\left(-\frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right), \quad i=1,\dots,d,\ t>0.$$

Based on this collection of assets, the price of a standard European basket call option with strike $K>0$ and maturity $T>0$ is given by

(3) $$C_{\mathcal{B}} := E\left[\left(\sum_{i=1}^d c_i S_T^i - K\right)^+\right].$$

Since each component S_T^i in (3) is log-normally distributed, the random vector $\left(c_1 S_T^1,\dots,c_d S_T^d\right)$ can be represented as $\left(w_1 e^{X_1},\dots,w_d e^{X_d}\right)$ for scalars $w_1,\dots,w_d$ and a zero-mean Gaussian vector $X=(X_1,\dots,X_d)\sim\mathcal{N}(0,\Sigma)$. More precisely, the choice

$$w_i = c_i S_0^i e^{-\frac{1}{2}\sigma_i^2 T}, \quad i=1,\dots,d,$$
$$\Sigma_{i,j} = \sigma_i\sigma_j\rho_{i,j}T, \quad i,j=1,\dots,d,$$

is used to transform (3) into

(4) $$C_{\mathcal{B}} = E\left[\left(\sum_{i=1}^d w_i e^{X_i} - K\right)^+\right]$$

for $X\sim\mathcal{N}(0,\Sigma)$. For simplicity, we set the maturity $T=1$ in the following.

2. Smoothing of the integrand

In this section, we describe a simple technique for smoothing the integrand in (4) which, at the same time,

- produces an analytic integrand,
- does not introduce an error,
- reduces the dimensionality of the integration problem by one.

Therefore, we assume that the covariance matrix Σ is a symmetric and positive definite matrix.

The general idea is that we calculate the expectation of one Gaussian factor in (4) conditioned on the remaining $d-1$ factors. Then, the outcome is a smooth function of these remaining factors and, obviously, the dimensionality of the integration problem is reduced by one.

Problem: There is generally no closed formula for the conditional expectation of any Gaussian factor in (4) if $d>1$.

But, a clever factorization of the covariance matrix Σ allows to overcome this obstruction. This factorization uses the rank-1 modification

(5) $$\tilde{\Sigma} = \Sigma - \lambda_1^2[1,\dots,1]\cdot[1,\dots,1]^\intercal \quad \text{with}$$
$$\lambda_1^2 = \left([1,\dots,1]\Sigma^{-1}[1,\dots,1]^\intercal\right)^{-1}.$$

It is easy to verify that $\tilde{\Sigma}$ is a symmetric and positive semidefinite matrix of rank $d-1$. We denote by (λ_i^2, v_i) for $i=2,\dots,d$ the $d-1$ eigenpairs corresponding to the $d-1$ positive eigenvalues of $\tilde{\Sigma}$. Let us further define $V := [v_1, v_2,\dots,v_d]$ with $v_1 := [1,\dots,1]^\intercal$ and $D := \mathrm{diag}(\lambda_1^2,\lambda_2^2,\dots,\lambda_d^2)$. Then, we end up with the following lemma.

Lemma 2.1. *Let Σ be a symmetric, positive definite $d\times d$ matrix. Then there is a diagonal matrix $D = \mathrm{diag}\left(\lambda_1^2,\lambda_2^2,\dots,\lambda_d^2\right)$ and an invertible matrix $V\in\mathbb{R}^{d\times d}$ with the property that $V_{i,1}\equiv 1$, $i=1,\dots,d$, such that*

$$\Sigma = VDV^\intercal.$$

The gain of Lemma 2.1 is that we are now able to separate one random variable in the sum in (4). Therefore, we replace X by $Y := V^{-1}X \sim \mathcal{N}(0,D)$ and note that the components of Y are independent. Plugging $X=VY$ into (4), we obtain that

(6) $$\begin{aligned} C_{\mathcal{B}} &= E\left[\left(\sum_{i=1}^d w_i e^{(VY)_i} - K\right)^+\right] \\ &= E\left[\left(\sum_{i=1}^d w_i \exp\left(Y_1 + \sum_{j=2}^d V_{i,j}Y_j\right) - K\right)^+\right] \\ &= E\left[\left(h(Y_2,\dots,Y_d)e^{Y_1} - K\right)^+\right] \end{aligned}$$

with

$$h(\overline{y}) := \sum_{i=1}^d w_i \exp\left(\sum_{j=2}^d V_{i,j}y_j\right), \quad \overline{y} = (y_2,\dots,y_d)\in\mathbb{R}^{d-1}.$$

The next step is to apply the Black-Scholes formula to determine the expectation with respect to Y_1 conditioned on $\overline{Y} := (Y_2,\dots,Y_d) = ((V^{-1}X)_2,\dots,(V^{-1}X)_d)$.

Lemma 2.2. *Let $\overline{D} := \mathrm{diag}(\lambda_2^2,\dots,\lambda_d^2)$. Then, it holds that $\overline{Y}\sim\mathcal{N}\left(0,\overline{D}\right)$ and*

$$E\left[\left(\sum_{i=1}^d w_i e^{X_i} - K\right)^+ \middle| \overline{Y}\right] = C_{BS}\left(h(\overline{Y})e^{\lambda_1^2/2}, K, \lambda_1\right),$$

where

$$C_{BS}(S_0,K,\sigma) := \Phi(d_1)S_0 - \Phi(d_2)K,$$
$$d_{1/2} := \frac{1}{\sigma}\left[\log\left(\frac{S_0}{K}\right) \pm \frac{\sigma^2}{2}\right],$$

is the Black-Scholes formula for $r=0$, with scaling $t=0$, $T=1$.

Hence, we are left with the Gaussian integration problem

(7) $$\begin{aligned} &C_{\mathcal{B}} = E\left[f(Z)\right],\ Z\sim\mathcal{N}\left(0, I_{d-1}\right),\ \overline{D} = \mathrm{diag}(\lambda_2^2,\dots,\lambda_d^2), \\ &f(Z) := C_{BS}\left(h\left(\sqrt{\overline{D}}Z\right)e^{\lambda_1^2/2}, K, \lambda_1\right). \end{aligned}$$

The integrand f in (7) is an analytic function, since $h\colon \mathbb{R}^{d-1}\to\mathbb{R}$ is analytic in Z and the Black-Scholes formula $C_{BS}(\cdot,K,\sigma)\colon\mathbb{R}\to\mathbb{R}$ is analytic as well.

3. Numerical results

As a numerical example, we choose the parameter in (4) as

$$\Sigma = AA^\intercal, \quad A_{i,i:d} = \sqrt{1-x_{i-1}}[1,\mathrm{cp}(x_{i:d-1})],$$
$$x_i = 0.8 + 0.2(d-i)/d \in \mathbb{R}^{d-1}, \quad \mathrm{cp}(x)_i = \prod_{k=1}^i x_k,$$
$$w_i = \frac{1}{d}\lfloor 8 + 12(d+1-i)/d\rfloor, \quad K = \sum_{i=1}^d w_i.$$

The $(d-1)$-dimensional integral in (7) is then computed by an adaptive sparse grid approach as described in "Dimension-Adaptive Tensor-Product Quadrature" by Gerstner and Griebel

$$C_{\mathcal{B}} \approx \sum_{\alpha\in\mathcal{I}} \Delta_{\alpha_1}\otimes\ldots\otimes\Delta_{\alpha_{d-1}} f.$$

Herein, $\Delta_j := Q_j - Q_{j-1}$ and $\{Q_j\}_j$ is a sequence of quadrature rules on $\mathbb{R}$ with N_j quadrature nodes. Usually, the number of quadrature nodes is increasing, i.e. $N_{j-1}\le N_j$. We consider a sequence based on Genz-Keister points, which are nested quadrature nodes on the integration domain $\mathbb{R}$. The index set $\mathcal{I}$ is chosen adaptively in such a way that indices with a large contribution to the value of the integral are added as long as the structure of the index set is admissible. The latter one means that an index α can only be added to $\mathcal{I}$ if $\alpha - e_i$ already belongs to $\mathcal{I}$ for all unit vectors $e_1,\dots,e_{d-1}$.

We compare the results of the adaptive sparse grid approach with the Monte Carlo quadrature and with the quasi-Monte Carlo quadrature based on Sobol points. The Sobol points are constructed on the unit cube and then mapped to the real line by the inverse normal distribution. The Monte Carlo type quadratures are used to approximate the d-dimensional integral in (4) as well as the $(d-1)$-dimensional integral in (7).

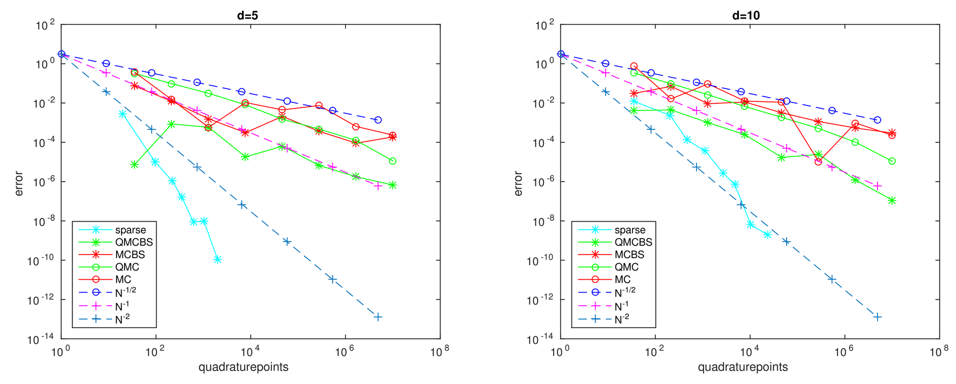

Figure 1: *Error of the different quadrature methods for $d=5$ (left) and $d=10$ (right).*

As expected, we observe from Figure 1 that the adaptive sparse grid quadrature applied to the integration problem (7) outperforms the Monte Carlo and quasi-Monte Carlo quadratures for $d=5,10$. Nevertheless, we observe as well that the smoothing of the integrand leads at least for the quasi-Monte Carlo method to a considerable improvement of the constant. For an increasing dimensionality d, we see from Figure 2 that the convergence of the adaptive sparse grid quadratures becomes worse. Nevertheless, the convergence for $d=25$ is still superior to the Monte-Carlo and quasi-Monte Carlo quadratures. Surprisingly, the dimensionality does not seem to influence the convergence behavior or the gain of the smoothing for the quasi-Monte Carlo method.

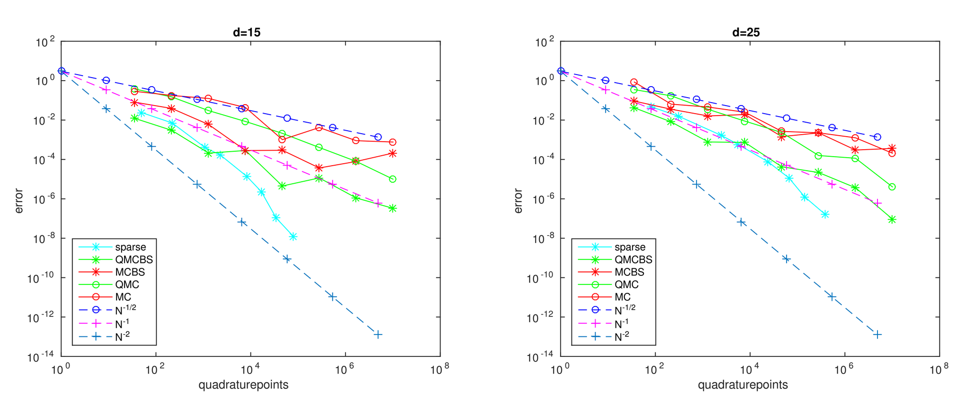

Figure 2: *Error of the different quadrature methods for $d=15$ (left) and $d=25$ (right).*